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CONFORMAL INVARIANCE IN SUPERGRAVITY

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CONTENTS

CHAPTER I

OUTLINE

In Nature there are two basic classes of particles: the bosons, which carry an integral spin, and the fermions, with half integral spin. Supersymmetry is a symmetry between these different kinds of particles. In all existing theories of fundamental processes there is a complete division between bosons and fermions. Only supersymmetry is able to break this partition and treat both on an equal basis. In the presence of this kind of symmetry particles with integral and half integral spin are forced to occur as partners of a common supermultiplet, and are no longer treated separately. One important implication of supersymmetry is that the infinite results, which occur in a quantum mechanical formulation of conventional field theories, are often absent in supersymmetric theories. The reason of this is that bosons and fermions contribute with opposite signs to these infinities, and supersymmetry forces these contributions to cancel. In addition the invariance under supersymmetry implies that the theory must be invariant under translations. Hence if supersymmetry transformations are local then the theory must be invariant under local translations or general coordinate transformations. This implies the existence of gravitation. The gauge theory of supersymmetry is therefore called supergravity. The benefits of supersymmetric theories remain present in supergravity and may be essential in the construction of a consistent quantum theory of gravity.

The softening of ultraviolet divergencies in the quantum version of supersymmetric theories is even more striking if one considers extended supersymmetry, where one has N independent supersymmetries present. The gauge theory of N supersymmetries fused with a global internal SO(N) or SU(N) symmetry includes gravity and is called extended supergravity. Besides gravitation there are three other fundamental forces known in Nature: the electromagnetic, the weak and the strong interactions. Gauge theories of internal symmetries are very successful to describe these three types of forces. These internal symmetries can be seen as originating from subgroups of one unifying gauge group. The gauge theory of this unifying group combines the three interactions into one theory. Extended supergravity provides a unique prescription to incorporate gravitation in such a theory as well. In a natural way it implies both the existence of gravitation and the presence of internal symmetries, which may describe the other fundamental interactions. In this way extended supergravity would describe all known elementary processes in Nature.

Although extended supergravity theories are uniquely determined, they are rather complex. The field representations of all extended theories up to N=8 are known at present. Beyond N=8, supersymmetry requires the existence of massless particles with spin higher than two. It seems impossible to have a consistent description of such higher-spin particles coupled to gravity. Therefore we restrict ourselves to $N \leq 8$ extended supergravity. The existing formulations of the $N \leq 8$ theories have the disadvantage that most of them are based on fields that directly correspond to physical degrees of freedom. Such formulations are called "on-shell" and are only relevant within the context of a given action. This is in contrast to "off-shell" formulations, which make no reference to any action at all. In order to find applications of extended supergravity theories it is important to get acquainted with their off-shell formulations and to develop techniques that can make clear their structure as a classical field theory. Conformal symmetry forms an essential ingredient in studying the off-shell structure of all extended supergravity theories. The standard conformal symmetries are fused with supersymmetry into so-called superconformal transformations. The gauge theory of these conformal supersymmetries is called conformal supergravity and constitutes the backbone of all supergravity theories. In this thesis we explain the role of conformal invariance in supergravity. Furthermore we present the complete structure of extended conformal supergravity for $N \leq 4$.

The outline of this work is as follows. In chapter 2 we briefly summarize the essential properties of supersymmetry and supergravity and indicate the use of conformal invariance in supergravity. The idea that the introduction of additional symmetry transformations can make clear the structure of a field theory is not reserved to supergravity only. By means of some simple examples we show in chapter 3 how one can always introduce additional gauge transformations in a theory of massive vector fields. Moreover we show how the gauge invariant formulation sometimes explains the quantum mechanical properties of the theory. In chapter 4 we define the conformal transformations and summarize their main properties. Furthermore we explain how these conformal transformations can be used to analyse the structure of gravity. The supersymmetric extension of these results is discussed in chapter 5. Here we describe as an example how N=1 supergravity can be reformulated in a conformally invariant way. We also show that beyond N=1 the gauge fields of the superconformal symmetries do not constitute an off-shell field representation of extended conformal supergravity. Therefore we develop in chapter 6 a systematic method to construct the off-shell formulation of all extended conformal supergravity theories with $N \leq 4$. As an example we use this method to construct N=1 conformal supergravity. Finally, in chapter 7 we discuss N=4 conformal supergravity. The references can be found at the end of each chapter.

CHAPTER II

SUPERSYMMETRY

1. Introduction

Supersymmetry is a symmetry between fermions and bosons. Since bosons carry an integral number of spin and fermions carry half integral spin, the generators corresponding to this kind of symmetry are anticommuting, spinorial operators. For conventional supersymmetry the generators Q_α are Majorana (= real) *) spinors of spin 1/2. In a field theory bosons are described by fields, which carry dimension 1 (in units of mass), whereas fermions are described by fields with dimension 3/2. Therefore the generators Q_α of supersymmetry should carry dimension 1/2 to bridge the gap in canonical dimension of boson and fermion fields. As for any symmetry the commutator (or, as here, the anticommutator) of two symmetry generators should yield the generators of another symmetry. In this case the anticommutator yields generators with dimension 1/2 + 1/2 = 1, and the natural candidates for these are the (anti-hermitean) generators P_μ of translations. Indeed, the supersymmetry algebra takes the form

$$\{Q_\alpha, \bar{Q}_\beta\} = +2\, P_\mu(\gamma^\mu)_{\alpha\beta} \quad , \tag{2.1}$$

with γ^μ the Dirac γ-matrices.

The explicit form (2.1) of the supersymmetry algebra leads immediately to important consequences. For example, it implies that the Hamiltonian of a supersymmetric system is expressed by

$$H \equiv iP_0 = \frac{1}{4}\; Q_\alpha^\dagger Q_\alpha \quad . \tag{2.2}$$

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*) For notations and conventions see appendix A.

Consequently, if negative norm states are absent, the energy should be positive definite:

$$< \psi|H|\psi > = \frac{1}{4} < Q_\alpha\psi|Q_\alpha\psi > \gtrsim 0 \quad . \tag{2.3}$$

Furthermore the energy of the ground state should be zero

$$< 0|H|0 > = 0 \quad , \tag{2.4}$$

if supersymmetry is realized manifestly, i.e. if the ground state is annihilated by the operators Q:

$$Q_\alpha|0 > = Q_\alpha^\dagger|0 > = 0 \quad . \tag{2.5}$$

Let us recall that bosons and fermions have an infinite zero-point energy, which is positive for bosons and negative for fermions. More specifically, in conventional quantum field theory, a bosonic degree of freedom for a system in a finite volume yields a vacuum energy

$$< 0|H|0 >_B = \frac{1}{2} \sum_{\vec{k}} \omega_{\vec{k}} < 0|a_{\vec{k}} a_{\vec{k}}^\dagger|0 > \quad , \tag{2.6a}$$

where $\omega_{\vec{k}} = \sqrt{\vec{k}^2+m^2}$ and $a_{\vec{k}}^\dagger$, $a_{\vec{k}}$ are the bosonic creation and annihilation operator for each kind of particle. On the other hand two fermionic degrees of freedom for a system in a finite volume are described by a Majorana spinor and lead to a vacuum energy

$$< 0|H|0 >_F = - \frac{1}{2} \sum_{\vec{k},\alpha} \omega_{\vec{k}} < 0|d_{\vec{k},\alpha} d_{\vec{k},\alpha}^\dagger|0 > \quad , \tag{2.6b}$$

with $d_{\vec{k},\alpha}^\dagger$, $d_{\vec{k},\alpha}$ (α=1,2) the fermionic creation and annihilation operators of the two helicity states for the Majorana particle. Because of (2.4) the bosonic and fermionic zero-point energies should be equal in absolute size. Therefore the expressions (2.6a,b) imply that in a supersymmetric realization the numbers of bosonic and fermionic physical states are equal. This means that supersymmetric field theories must be based on multiplets containing boson and fermion fields, which describe equal numbers of boson and fermion states.

At this point we should add a word of caution. The above counting argument does only apply to physical states or <u>dynamic</u> degrees of freedom. It does not give any information about the <u>field</u> degrees of freedom (we shall also use the name spin degree of freedom), which are described by the fields contained in a supersymmetry multiplet. In section 5 we shall make it plausible that for the off-shell formulations (these we discuss in section 4) also the numbers of fermionic and bosonic field degrees of freedom are equal.

Another important consequence of the supersymmetry algebra (2.1) is that if supersymmetry is realized as a local symmetry, then the theory in question should be invariant under local translations. In other words, local supersymmetry requires invariance under general coordinate transformations and thus implies gravity. Theories with local supersymmetry are therefore called supergravity theories. The smallest supersymmetric extension of the Einstein theory of gravity is called N=1 Poincaré supergravity and describes two spin-2 and two spin-3/2 dynamic degrees of freedom. In extended supergravity, where one has N independent supersymmetries present, the underlying multiplets describe more than 2+2 (bosonic + fermionic) dynamic degrees of freedom and have a more complicated structure.

In recent years many field theories have been constructed that are invariant under supersymmetry transformations. Such theories have been shown to exhibit a number of surprising and interesting properties. One of them is that the ultraviolet divergencies in the quantum corrections to these theories are much softer than in theories without supersymmetry. The reason for this property is related to the fact that boson loops and fermion loops come with opposite signs, and because of supersymmetry this leads in many cases to direct cancellations. An example of this phenomenon is the vanishing of the zero-point energy in supersymmetric theories. Many examples of this softening of ultraviolet divergencies are known, and have inspired the hope that this property will be of crucial importance to construct a consistent quantum theory of gravity. Another interesting aspect of supersymmetric theories is that they often give rise to quite unexpected new invariance properties, such as the combined chiral-dual symmetries of field equations in extended supergravity, the USp(2N) symmetry of massive supermultiplets, and the E_7 invariance in N=8 supergravity. But perhaps the most important aspect of supersymmetry is that it provides a unique principle for the unification of elementary particles and the fundamental forces in Nature: supersymmetry by definition combines bosons and fermions, and in its local version it implies the existence of gravitation.

This chapter is organized as follows. In section 2 we consider as a simple example the Wess-Zumino model, which is based on the smallest supersymmetry multiplet. In section 3 we give the N=1 Poincaré supergravity theory. This model describes the smallest supersymmetric extension of Einstein gravity. The problem of how to find off-shell formulations of supersymmetric theories is discussed in section 4. Here we give off-shell formulations of the Wess-Zumino model and N=1 Poincaré supergravity. A counting argument concerning field degrees of freedom is derived in section 5. Finally, in section 6 we give the motivation for studying conformal invariance in supergravity theories.

2. The Wess-Zumino model

The Wess-Zumino model describes two spin-0 and two spin-1/2 dynamic degrees of freedom. In a field theory such states can be represented by a complex scalar field A and a Majorana spinor ψ_M. The Lagrangian for these fields is given by

$$\mathcal{L} = A^*\Box A - \frac{1}{2}\bar{\psi}^{\cdot}\overleftrightarrow{\partial\!\!\!/}\psi_{\cdot} - m^2A^*A - \frac{1}{2}m\,(\bar{\psi}^{\cdot}\psi^{\cdot} + c\cdot c\cdot) \quad , \qquad (2.7)$$

where m is an arbitrary mass parameter and the abbreviation c·c· indicates that the complex conjugated term is added. The symbol $\overleftrightarrow{\partial}_\mu$ denotes a differentiation to the right and to the left: $\overleftrightarrow{\partial}_\mu = \overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu$. In (2.7) we have used the chiral notation, in which chiral projections of Majorana spinors are denoted by a dot. This notation is very useful in extended supersymmetry, where N chiral spinors combine into representations of U(N) *). In that case the dot is replaced by a U(N) index i (i=1..N), which uniquely characterizes the representation.

*) For more details see appendix B.

If the positive chiral projection $1/2(1+\gamma_5)\psi_M^i$ corresponds to the N representation of U(N) we use the (consistent *)) notation

$$\psi^i = \frac{1}{2}(1+\gamma_5)\psi_M^i \qquad \bar{\psi}^i = \bar{\psi}_M^i \ \frac{1}{2}(1+\gamma_5)$$
$$\psi_i = \frac{1}{2}(1-\gamma_5)\psi_M^i \qquad \bar{\psi}_i = \bar{\psi}_M^i \ \frac{1}{2}(1-\gamma_5) \quad . \tag{2.8}$$

On the other hand, if $1/2(1+\gamma_5)\psi_M^i$ falls in the $\bar{N}$ representation we write

$$\psi^i = \frac{1}{2}(1-\gamma_5)\ \psi_M^i \qquad \bar{\psi}^i = \bar{\psi}_M^i \ \frac{1}{2}(1-\gamma_5)$$
$$\psi_i = \frac{1}{2}(1+\gamma_5)\ \psi_M^i \qquad \bar{\psi}_i = \bar{\psi}_M^i \ \frac{1}{2}(1+\gamma_5) \quad . \tag{2.9}$$

In the Wess-Zumino model we use the notation given in (2.8) with on the left-hand side the index i (i=1) replaced by a dot.

The action corresponding to the Lagrangian (2.7) is invariant under the following set of supersymmetry transformations

$$\delta A = \bar{\varepsilon}^{\cdot}\ \psi^{\cdot} \quad ,$$
$$\delta\psi^{\cdot} = \partial\!\!\!/ A\ \varepsilon_{\cdot} - m\ A^{*}\ \varepsilon^{\cdot} \quad , \tag{2.10}$$

where ε is a spacetime independent (Majorana) spinor parameter, which characterizes the supersymmetry transformation.

According to the supersymmetry algebra (2.1) the commutator of two supersymmetry transformations must yield a translation. The infinitesimal form of such a translation

$$x^\mu \to x^\mu - \xi^\mu \tag{2.11}$$

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*) For more details see appendix B.

on A (or ψ) is given by

$$\delta_P(\xi^\mu)A = + \xi^\mu \partial_\mu A \quad . \tag{2.12}$$

Indeed, calculating the commutator of two supersymmetry transformations with parameters ε_1 and ε_2 on A we find

$$[\,\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)]\; A = + \xi^\mu \partial_\mu A \quad , \tag{2.13}$$

with the translation parameter ξ^μ given by

$$\xi^\mu = \bar{\varepsilon}_2^{\boldsymbol{\cdot}}\, \gamma^\mu\, \varepsilon_{1.} + \mathrm{c.c.} \quad . \tag{2.14}$$

However a calculation of the same commutator on ψ leads to an additional term, which is proportional to the Dirac equation for ψ (by this we mean that this term vanishes after applying the Dirac equation):

$$[\,\delta_Q(\varepsilon_1),\ \delta_Q(\varepsilon_2)]\,\psi^{\boldsymbol{\cdot}} = + \xi^\mu \partial_\mu \psi^{\boldsymbol{\cdot}} - \frac{1}{2}\, \not{\xi}\, (\not{\partial}\psi^{\boldsymbol{\cdot}} + m\psi_{\boldsymbol{\cdot}}) \quad . \tag{2.15}$$

Therefore the supersymmetry algebra (2.1) is only realized on field configurations that satisfy the field equations

$$\begin{aligned} &\not{\partial}\psi^{\boldsymbol{\cdot}} + m\psi_{\boldsymbol{\cdot}} = 0 \quad , \\ &(\Box - m^2)\; A = 0 \quad . \end{aligned} \tag{2.16}$$

For this reason the multiplet (A,ψ) is called an on-shell representation of the supersymmetry algebra. Notice that the on-shell Wess-Zumino model contains 2 field degrees of freedom for the complex field A and 4 field degrees of freedom for the Majorana spinor ψ_M.

A disadvantage of on-shell representations is that the transformation rules and the commutator algebra are related to a given action. This hampers applications of such results in the context of different actions. For instance a discussion of quantization and of supersymmetric counterterms is rather difficult in the context of on-shell representations. For that purpose it is convenient to use representations of the supersymmetry algebra, which are not related to any specific action at all. A central problem in supersymmetry and supergravity is to find such representations. We will come back to this point in section 4, where we shall give an off-shell formulation of the Wess-Zumino model.

3. N=1 Poincaré supergravity

In this section we discuss N=1 Poincaré supergravity. This model describes two spin-2 and two spin-3/2 dynamic degrees of freedom and is the smallest supersymmetric extension of the Einstein theory of gravity. The physical states can be represented by boson and fermion fields respectively. We give the (nonlinear) transformation rules of these fields under spacetime-dependent supersymmetry transformations and an action, which is invariant under these transformations. Furthermore we calculate the commutator of two (spacetime-dependent) supersymmetry transformations on these fields. In analogy to the Wess-Zumino model we find that on the boson fields this commutator yields a (covariant) translation, whereas on the fermion fields the same commutator leads to additional terms, which are proportional to the field equations of the fermion fields. Before giving these results we first briefly review the theory of ordinary gravity.

In the Einstein-Cartan version of gravitation the gravitational spin-2 state is represented by a vierbein field e_μ^a, with the property that the metric tensor is given by

$$g_{\mu\nu} = e_\mu^a e_\nu^a \quad . \tag{2.17}$$

Contraction with a vierbein changes local Lorentz indices a into world indices μ and vice versa. These indices refer to the transformation character under local (internal) Lorentz and general coordinate transformations respectively.

Under these transformations the vierbein field transforms according to

$$\delta e^a_\mu = + \xi^\lambda \partial_\lambda e^a_\mu + (\partial_\mu \xi^\lambda) e^a_\lambda + \varepsilon^{ab} e^b_\mu \quad , \tag{2.18}$$

where ξ^λ and ε^{ab} are spacetime-dependent parameters, which characterize the general coordinate and local Lorentz transformations.

An inverse vierbein field e^μ_a is defined in the following way:

$$\begin{aligned} e^\mu_a \, e^b_\mu &= \delta^b_a \quad , \\ e^a_\mu \, e^\nu_a &= \delta^\nu_\mu \quad . \end{aligned} \tag{2.19}$$

Besides the vierbein, the Einstein-Cartan theory is based on a spin-connection field ω^{ab}_μ, which is not independent. It can be expressed in terms of e^a_μ in the following way *):

$$\omega^{ab}_\mu(e) = - e^\nu_{[a} (\partial_\mu e_{b]\nu} - \partial_\nu e_{b]\mu}) - e^\rho_{[a} e^\sigma_{b]} (\partial_\sigma e_{c\rho}) e^c_\mu \quad . \tag{2.20}$$

Under local Lorentz transformations with parameters ε^{ab} this field transforms according to

$$\begin{aligned} \delta\omega^{ab}_\mu &= \partial_\mu \varepsilon^{ab} - \omega^{ac}_\mu \varepsilon^{cb} + \omega^{bc}_\mu \varepsilon^{ca} \\ &\equiv D_\mu \varepsilon^{ab} \quad . \end{aligned} \tag{2.21}$$

This transformation character explains the role of the spin-connection field: it can be used as the gauge field of the local Lorentz transformations. The curvature tensor of ω^{ab}_μ is defined in the following way:

*) The antisymmetrization notation [ ] is explained in appendix A.

$$R^{ab}_{\mu\nu}(\omega) = \partial_{[\mu}\omega^{ab}_{\nu]} - \omega^{ac}_{[\mu}\,\omega^{cb}_{\nu]} \qquad . \tag{2.22}$$

This tensor transforms covariantly under Lorentz transformations and can be used for the construction of an invariant action. To that end one introduces the curvature scalar

$$R(\omega(e)) = e^{\mu}_{a}\, e^{\nu}_{b}\, R^{ab}_{\mu\nu}(\omega(e)) \qquad , \tag{2.23}$$

which is invariant under local Lorentz transformations and transforms under general coordinate transformations as a scalar. In order to construct a density one multiplies this scalar with the determinant of the vierbein field

$$e = \det e^{a}_{\mu} \qquad , \tag{2.24}$$

which transforms under general coordinate transformations as

$$\begin{aligned} \delta e &= e\, e^{\mu}_{a}\, \delta\, e^{a}_{\mu} \\ &= \partial_{\lambda}(\xi^{\lambda} e) \qquad . \end{aligned} \tag{2.25}$$

In this way one obtains the following Lagrangian density

$$\mathcal{L} = -\, e\, R(\omega(e)) \qquad . \tag{2.26}$$

Here we have taken the gravitational coupling constant κ equal to one. The field equation of e^{a}_{μ} corresponding to this Lagrangian reads:

$$2e(R^{\mu}_{a} - \tfrac{1}{2}\, e^{\mu}_{a}\, R) = 0 \qquad , \tag{2.27}$$

where $R^a_\mu = e^\nu_b R^{ab}_{\mu\nu}$ and R is the curvature scalar. In this formulation one can keep the spin-connection field as an independent field, because the Lagrangian is such that the corresponding field equations yield an algebraic equation for ω^{ab}_μ in terms of e^a_μ, which is exactly given by the defining equation (2.20).

Having thus established the Einstein-Cartan theory, we proceed by discussing its minimal supersymmetric extension. To that end we introduce an additional Majorana vector-spinor ψ^M_μ to describe the physical spin-3/2 state. The free Lagrangian for this field is given by

$$\mathcal{L} = - \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}^{\bullet}_\mu \gamma_\nu \overleftrightarrow{\partial}_\rho \psi_\sigma. \quad , \tag{2.28}$$

where we have used the chiral notation given in (2.8). The corresponding field equation reads:

$$R^{(o)\bullet}_\mu = \varepsilon^{\mu\nu\rho\sigma} \gamma_\nu \partial_{[\rho} \psi^{\bullet}_{\sigma]} = o \quad . \tag{2.29}$$

We note that the spinor $R^{(o)\bullet}_\mu$ satisfies the chiral notation given in (2.9). The Lagrangian is invariant under the following Rarita-Schwinger gauge transformations

$$\delta\psi^{\bullet}_\mu = \partial_\mu \varepsilon^{\bullet} \quad , \tag{2.30}$$

where $\varepsilon^{\bullet}$ is a spacetime-dependent spinor parameter, which characterizes the gauge transformation. Under local Lorentz and general coordinate transformations $\psi^{\bullet}_\mu$ transforms according to

$$\delta\psi^{\bullet}_\mu = + \xi^\lambda \partial_\lambda \psi^{\bullet}_\mu + (\partial_\mu \xi^\lambda)\psi^{\bullet}_\lambda + \frac{1}{2} \varepsilon^{ab} \sigma_{ab} \psi^{\bullet}_\mu \quad . \tag{2.31}$$

In the N=1 Poincaré supergravity model this spin-3/2 field, the gravitino field, is coupled to the vierbein field of ordinary gravitation in the following way:

$$\mathcal{L} = - e\, R(\omega(e,\psi)) - \varepsilon^{\mu\nu\rho\sigma}\, \bar{\psi}^{\cdot}_{\mu}\, \gamma_{\nu} \overleftrightarrow{D}_{\rho} \psi_{\sigma} . \tag{2.32}$$

Here $\omega_{\mu}^{ab}(e,\psi)$ is the following expression in terms of the vierbein and gravitino field:

$$\omega_{\mu}^{ab}(e,\psi) = \omega_{\mu}^{ab}(e) - \frac{1}{2}\left\{ (2\bar{\psi}^{\cdot}_{\mu}\, \gamma^{[a}\psi^{b]}_{\cdot} + \bar{\psi}^{\cdot\,[a}\gamma_{\mu}\psi^{b]}_{\cdot}) + c\cdot c\cdot \right\} . \tag{2.33}$$

Furthermore we have used the Lorentz-covariant derivative

$$D_{\rho}\psi^{\cdot}_{\sigma} = (\partial_{\rho} - \frac{1}{2}\,\omega_{\rho}^{ab}\sigma_{ab})\, \psi^{\cdot}_{\sigma} . \tag{2.34}$$

The field equations of e_{μ}^{a} and ψ_{μ} corresponding to this Lagrangian read

$$\begin{aligned} &2e(R^{\mu}_{a} - \frac{1}{2}\, e^{\mu}_{a}\, R) + (\varepsilon^{\mu\nu\rho\sigma}\, \bar{\psi}^{\cdot}_{\nu}\gamma_{a} D_{\rho}\psi_{\sigma.} + c\cdot c\cdot) = 0 , \\ &R^{\cdot}_{\mu} = \frac{1}{e}\, \varepsilon^{\mu\nu\rho\sigma}\, \gamma_{\nu} D_{[\rho}\psi^{\cdot}_{\sigma]} = 0 . \end{aligned} \tag{2.35}$$

Again one can keep the spin-connection field ω_{μ}^{ab} as an independent field in the Lagrangian, because its corresponding field equation yields an algebraic expression in terms of e_{μ}^{a} and ψ_{μ}, which is exactly given by (2.33).

The action corresponding to (2.32) is invariant under the following set of local supersymmetry transformations:

$$\begin{aligned} &\delta e_{\mu}^{a} = \bar{\varepsilon}^{\cdot}\, \gamma^{a}\psi_{\mu.} + c\cdot c\cdot , \\ &\delta\psi^{\cdot}_{\mu} = D_{\mu}\varepsilon^{\cdot} , \end{aligned} \tag{2.36}$$

where D_{μ} is defined as in (2.34) and $\varepsilon^{\cdot}$ is a spacetime-dependent parameter characterizing the supersymmetry transformation. These transformation rules clarify the structure of $\omega_{\mu}^{ab}(e,\psi)$: it is the supersymmetric covariantization of $\omega_\mu^{ab}(e)$. To show the invariance of the action defined by (2.32) one need not vary $\omega_\mu^{ab}(e,\psi)$ according to the chain rule, since this variation is always multiplied by

$$\frac{\delta I}{\delta\omega_\mu^{ab}}(e,\psi) \equiv 0 \quad , \tag{2.37}$$

where I denotes the action. In this 3/2-order formalism, as it is called, one only varies the vierbein and gravitino fields that occur explicitly in the action but not the ones, which result from expanding $\omega_\mu^{ab}(e,\psi)$ in terms of e and ψ. The variation of the gravitino field yields the following term:

$$\begin{aligned} &-2\,\varepsilon^{\mu\nu\rho\sigma}\,\bar{\psi}_\mu^{\cdot}\,\gamma_\nu D_\rho D_\sigma \varepsilon_{.} + c.c. \\ &= +\,\varepsilon^{\mu\nu\rho\sigma}\,\bar{\psi}_\mu^{\cdot}\gamma_\nu\,\sigma_{ab}\varepsilon_{.}\,R_{\rho\sigma}^{ab}(\omega) + c.c. \quad . \end{aligned} \tag{2.38}$$

The commutator part of the γ-matrices leads to an expression, which, together with the vierbein variation of the second term in (2.32), is proportional to the Bianchi identity

$$\varepsilon^{\mu\nu\rho\sigma}\left\{ R_{\nu\rho,\sigma}{}^{a}(\omega) + (\bar{\psi}_\nu^{\cdot}\,\gamma^a D_\rho \psi_{\sigma.} + c.c.)\right\} = 0 \quad . \tag{2.39}$$

The anticommutator part yields:

$$\begin{aligned} &+\frac{1}{2}e\;\delta_{abc}^{\mu\rho\sigma}\,(\bar{\varepsilon}_{.}\gamma_c\psi_\mu^{\cdot})\,R_{\rho\sigma}^{ab}(\omega) + c.c. \\ &= -2e\,(\bar{\varepsilon}_{.}\gamma_c\psi_\mu^{\cdot})\,(R_c^\mu - \frac{1}{2}e_c^\mu R) + c.c. \quad , \end{aligned} \tag{2.40}$$

which cancels against the vierbein variation of the first term in (2.32).

We finally consider the commutator of two supersymmetry transformations with spacetime-dependent parameters ε_1 and ε_2 on the graviton and gravitino field. Calculating first this commutator on e_μ^a we find

$$[\delta_Q(\varepsilon_1),\ \delta_Q(\varepsilon_2)]\ e_\mu^a = + \xi^\lambda\partial_\lambda e_\mu^a + (\partial_\mu\xi^\lambda)\ e_\lambda^a - \xi^\lambda\omega_\lambda^{ab} e_\mu^b$$

$$- \xi^\lambda(\bar\psi_\lambda^{\cdot}\gamma^a\psi_{\mu.} + c.c.) \qquad , \qquad (2.41)$$

with ξ^λ defined in (2.14). We recognize the first two terms on the right-hand side as a general coordinate transformation of e_μ^a (cp. eq.(2.18)). These terms are therefore in accordance with the supersymmetry algebra (2.1). The last two terms can be identified as field-dependent Lorentz and supersymmetry transformations with parameters $\varepsilon^{ab} = -\xi^\lambda\omega_\lambda^{ab}$ and $\varepsilon^{\cdot} = -\xi^\lambda\psi_\lambda^{\cdot}$ respectively. These terms are nonlinear modifications of the supersymmetry algebra. Such modifications are to be expected, since also the algebra of spacetime transformations changes. For example, two local translations do in general not commute, while global translations do. The nonlinear terms in (2.41) can be viewed as the Lorentz and supersymmetry covariantizations of the first two terms. Therefore by definition the right-hand side of (2.41) is called a covariant translation

$$\delta^{cov}_{g.c.t.}(\xi^\lambda)\ e_\mu^a = + \xi^\lambda\hat{D}_\lambda e_\mu^a + (\partial_\mu\xi^\lambda)e_\lambda^a$$

$$= \left\{ \delta_{g.c.t.}(\xi^\lambda) + \delta_M(-\xi^\lambda\omega_\lambda^{ab}) + \delta_Q(-\xi^\lambda\psi_\lambda^{\cdot}) \right\} e_\mu^a \quad , \qquad (2.42)$$

where $\delta_{g.c.t.}$, δ_M and δ_Q represent general coordinate, local Lorentz and supersymmetry transformations respectively and the derivative $\hat{D}_\mu$ is covariant with respect to local Lorentz and supersymmetry transformations. A calculation of the same commutator on ψ_μ leads, in analogy to the Wess-Zumino model, to additional terms, which are proportional to the field equation of ψ_μ (see (2.35)):

$$[\delta_Q(\varepsilon_1),\ \delta_Q(\varepsilon_2)]\psi_\mu^{\cdot} = 2\ \xi^\lambda\ D_{[\lambda}\psi_{\mu]}^{\cdot} - \frac{1}{4}\ \bar\varepsilon_2^{\cdot}\ \sigma_{\rho\sigma}\varepsilon_1^{\cdot}\ (\sigma_{\rho\sigma}\ R_{\mu.} + 4\delta_{\mu\rho}\ R_{\sigma.})$$

$$+ \frac{1}{8}\ \xi^\lambda \left\{ 3\gamma_\lambda\ R_\mu^{\cdot} - 2\delta_{\lambda\mu}\ \ \gamma\cdot R^{\cdot} - 2\gamma_\mu\ (R_\lambda^{\cdot} - \gamma_\lambda\gamma\cdot R^{\cdot}) \right\} \qquad . \qquad (2.43)$$

The first term on the right-hand side corresponds to a covariant translation with parameter ξ^λ, whereas the remaining terms vanish upon use of the gravitino field equation $R^{\cdot}_\mu = 0$. For this reason the multiplet (e^a_μ, ψ_μ) is called an on-shell supergravity multiplet. In the next section we shall give an off-shell formulation of the N=1 Poincaré supergravity theory. The field degrees of freedom of the on-shell model are 12 degrees of freedom for ψ^M_μ and 6 degrees of freedom for e^a_μ. These numbers are obtained as follows. The Rarita-Schwinger field has a priori 16 degrees of freedom, which are reduced to 12 through the gauge invariance (2.30). Similarly the vierbein field has 16 degrees of freedom. The gauge freedom implied by (2.18) reduces that number to 6 degrees of freedom (corresponding to 4 general coordinate transformations and 6 internal Lorentz transformations).

4. Off-shell formulations

In order to find specific applications of supersymmetric theories and to acquire an understanding of their dynamical properties, it is an obvious requisite to clarify their structure. An essential element in such a clarification is the construction of representations of the supersymmetry algebra in terms of fields, which do not necessarily satisfy field equations and hence are not related to a given action. Such representations are called off-shell representations. So far there exists no fixed procedure to construct these off-shell representations. In fact most extended supergravity theories are at present only known on the basis of an on-shell formulation. The disadvantage of such a formulation has been explained in section 2. In this section we shall give the off-shell formulations of the Wess-Zumino model and the N=1 Poincaré supergravity theory.

We first reconsider the on-shell Wess-Zumino model discussed in section 2. In this model the commutator of two supersymmetry transformations on ψ yields a translation together with a term, which vanishes upon use of the field equation of ψ (see eq. (2.15)). Therefore the supersymmetry algebra (2.1) is only valid for field configurations that satisfy the field equations (2.16). One can however envisage a new variation of ψ, which circumvents this requirement. The price one has to pay is that a new field H must be added to the on-shell model. More specifically, the field equation term $-\frac{1}{2}\xi\!\!/ (\partial\!\!/\psi^{\cdot} + m\psi_{\cdot})$ in (2.15) can be cancelled by the following variation of ψ

$$\delta\psi^{\cdot} = H\varepsilon^{\cdot} \qquad , \tag{2.44}$$

where the variation of H to $(\partial\!\!\!/\psi^{\cdot} + m\psi_{\cdot})$ must be

$$\delta H = \bar{\varepsilon}_{\cdot}(\partial\!\!\!/\psi^{\cdot} + m\psi_{\cdot}) \qquad . \tag{2.45}$$

It should be noted that this cancellation can only by achieved by the introduction of a complex scalar field. One can now verify that on all fields the commutator of two supersymmetry transformations has the form (2.13) independent of any field equation. For this reason the multiplet (A,ψ,H) is called an off-shell multiplet. One may wonder whether the presence of the mass parameter m in the transformation rule of H is necessary. This parameter can be eliminated by redefining H in the following way:

$$F = H - mA^{*} \qquad . \tag{2.46}$$

In terms of A, ψ and F the transformation rules are

$$\begin{aligned} \delta A &= \bar{\varepsilon}^{\cdot}\psi^{\cdot} \qquad , \\ \delta\psi^{\cdot} &= \partial\!\!\!/ A\varepsilon_{\cdot} + F\varepsilon^{\cdot} \qquad , \\ \delta F &= \bar{\varepsilon}_{\cdot}\,\partial\!\!\!/\psi^{\cdot} \qquad . \end{aligned} \tag{2.47}$$

A Lagrangian for these fields is given by

$$\mathcal{L} = A^{*}\Box A - \frac{1}{2}\,\bar{\psi}^{\cdot}\overleftrightarrow{\partial\!\!\!/}\psi_{\cdot} + m(AF + c.c.) - \frac{1}{2}\,m\,(\bar{\psi}^{\cdot}\psi^{\cdot} + c.c.) + F^{*}F \qquad . \tag{2.48}$$

An unusual feature is that the scalar field F describes no dynamical degree of freedom. This field can be eliminated from the action by means of its field equation

$$F = - mA^* \tag{2.49}$$

without disturbing the invariance of the action. For this reason F is called an auxiliary field. Its only purpose is to obtain a representation of the supersymmetry algebra in terms of fields, which are not related to a specific action. Upon substitution of the field equations (2.16) and (2.49) the off-shell multiplet (A,ψ,F) reduces to its on-shell version (A,ψ).

Having thus explained how one can obtain an off-shell formulation of the Wess-Zumino model, we proceed by showing how the on-shell supergravity multiplet (e_μ^a, ψ_μ) can be extended to such a representation. To that end one must again introduce additional fields in order to cancel all field equation terms in the commutator on ψ_μ (see (2.43)). It appears that these terms can be cancelled by a transformation of ψ_μ into a complex scalar field F and a vector field A_a:

$$\delta\psi_\mu^{\cdot} = - \frac{1}{3}\gamma_\mu F\epsilon_{\cdot} + iA_\mu\epsilon^{\cdot} - \frac{1}{3}i\gamma_\mu \not{A}\epsilon^{\cdot} \quad , \tag{2.50}$$

where the transformations of F and A_a must be

$$\begin{aligned} \delta F &= \frac{1}{2}\bar{\epsilon}^{\cdot}\gamma\cdot\hat{R}^{\cdot} \quad , \\ \delta A_a &= - \frac{3}{4}i\bar{\epsilon}^{\cdot}\left(\hat{R}_{a.} - \frac{1}{3}\gamma_a\gamma\cdot\hat{R}_{.}\right) + c.c.\, . \end{aligned} \tag{2.51}$$

Here $\hat{R}_\mu^{\cdot} = 0$ is the supercovariantized field equation of ψ_μ:

$$\begin{aligned} \hat{R}_\mu^{\cdot} &= \frac{1}{e}\epsilon_{\mu\nu\rho\sigma}\gamma_\nu\hat{D}_\rho\psi_\sigma^{\cdot} \\ &= \frac{1}{e}\epsilon_{\mu\nu\rho\sigma}\gamma_\nu\left\{(D_\rho + iA_\rho)\psi_\sigma^{\cdot} + \frac{1}{3}\gamma_\rho(F\psi_{\sigma.} + i\not{A}\psi_\sigma^{\cdot})\right\} = 0 \quad . \end{aligned} \tag{2.52}$$

We note that in the presence of F and A_a the contribution of ω_μ^{ab} to the commutator on ψ_μ is given by (2.43), but with $D_{[\lambda}\psi_{\mu]}$ and $R_\mu^{\cdot}$ replaced by $\hat{D}_{[\lambda}\psi_{\mu]}$ and $\hat{R}_\mu^{\cdot}$ respectively. A nontrivial feature is that in the presence of F and A_a the commutator algebra (2.41) gets nonlinear modifications depending on these fields. Besides a covariant translation the algebra contains an additional Lorentz transformation:

$$[\delta_Q(\varepsilon_1),\ \delta_Q(\varepsilon_2)] = \delta^{cov}_{g.c.t.}(\xi^\mu) + \delta_M(\varepsilon^{ab}) \quad , \tag{2.53}$$

with the parameter ε^{ab} given by

$$\varepsilon^{ab} = 2(\bar{\varepsilon}_1^{\cdot}\ \sigma^{ab}\ \eta_{2.} - \bar{\varepsilon}_{2.}\ \sigma^{ab}\ \eta_1^{\cdot}) + c.c. \quad . \tag{2.54}$$

In this expression $\eta^{\cdot}$ stands for

$$\eta^{\cdot} = \frac{1}{3}\,(F\varepsilon_{.} + i\not{A}\varepsilon^{\cdot}) \quad . \tag{2.55}$$

After the introduction of F and A_a the commutator of two supersymmetry transformations has on all fields the form (2.53) independent of any field equation. Therefore the multiplet $(e_\mu^a, \psi_\mu, A_a, F)$ is the off-shell version of the supergravity multiplet $(e_\mu^a,\ \psi_\mu)$. The transformation rules of the fields are

$$\begin{aligned}
\delta e_\mu^a &= \bar{\varepsilon}^{\cdot}\ \gamma^a\ \psi_{\mu.} + c.c. \quad &,\\
\delta\psi_\mu^{\cdot} &= (D_\mu + iA_\mu)\varepsilon^{\cdot} - \gamma_\mu\eta^{\cdot} \quad &,\\
\delta F &= \frac{1}{2}\,\bar{\varepsilon}^{\cdot}\ \gamma\cdot\hat{R}^{\cdot} \quad &,\\
\delta A_a &= -\frac{3}{4}\,i\ \bar{\varepsilon}^{\cdot}\ (\hat{R}_{a.} - \frac{1}{3}\,\gamma_a\gamma\cdot\hat{R}_{.}) + c.c.. &
\end{aligned} \tag{2.56}$$

A Lagrangian for these fields is given by

$$\mathcal{L} = - e\, R(\omega(e,\psi)) - \varepsilon^{\mu\nu\rho\sigma}\, \bar{\psi}_\mu \gamma_\nu \overleftrightarrow{D}_\rho \psi_\sigma - \frac{4}{3}\, e\, (F^* F - A_a^2) \quad . \tag{2.57}$$

In analogy to the Wess-Zumino model, the fields F and A_a describe no dynamical degrees of freedom. They are auxiliary fields. Upon substitution of their field equations

$$F = 0 \ , \quad A_a = 0 \tag{2.58}$$

and those of e_μ^a and ψ_μ, the off-shell supergravity multiplet $(e_\mu^a, \psi_\mu, A_a, F)$ reduces to its on-shell version (e_μ^a, ψ_μ) (cp. eq.(2.36)).

A closer study of the off-shell formulations presented in this section shows that they are based on multiplets of fields describing an equal number of bosonic and fermionic field degrees of freedom (d.o.f.). This is in contrast to the corresponding on-shell multiplets, which contain different numbers. More specifically the off-shell Wess-Zumino multiplet (A, ψ, F) contains 4 + 4 (fermionic + bosonic) field d.o.f. and the off-shell supergravity multiplet $(e_\mu^a, \psi_\mu, A_a, F)$ contains 12 + 12 field d.o.f.. In the next section we shall show that this equality of bosonic and fermionic field degrees of freedom is a general property of off-shell formulations. This off-shell counting, as it is called, plays an important role in the discussion of auxiliary field formulations of supersymmetric theories.

We finally note that off-shell formulations are often not unique. For instance there exists another off-shell formulation of the N=1 supergravity model, which also contains 12 + 12 field degrees of freedom. More specifically, one can obtain the same cancellation of field equation terms in the commutator on ψ_μ as before by a transformation of ψ_μ into an axial gauge (i.e. having a gauge invariance) vector A_μ and an axial vector E_a, which is divergence-free:

$$\delta A_\mu = \partial_\mu \Lambda \qquad , \qquad D.E = 0 \quad . \tag{2.59}$$

Here D_μ is the Lorentz-covariant derivative. This then leads to another off-shell supergravity multiplet (e_μ^a, ψ_μ, A_μ, E_a) with the same field degrees of freedom. The transformation rules of these fields are

$$\begin{aligned}
\delta e_\mu^a &= \bar{\varepsilon}^{\cdot}\, \gamma^a\, \psi_{\mu.} + c{\cdot}c{\cdot} \quad, \\
\delta \psi_\mu^{\cdot} &= (D_\mu + iA_\mu)\varepsilon^{\cdot} - i\, \gamma_\mu\, \not{E}\, \varepsilon^{\cdot} \quad, \\
\delta E_a &= -\frac{1}{4}\, i\, \bar{\varepsilon}^{\cdot}\, \hat{R}_{a.} + c{\cdot}c{\cdot} \quad, \\
\delta A_\mu &= -\frac{3}{4}\, i\, (\hat{R}_{\mu.} - \frac{1}{3}\gamma_\mu\, \gamma{\cdot}\, \hat{R}_{.}) + c{\cdot}c{\cdot}\ .
\end{aligned} \tag{2.60}$$

These transformations are an invariance of the action corresponding to the Lagrangian

$$\mathcal{L} = -\, e\, R(\omega(e,\psi)) - \varepsilon^{\mu\nu\rho\sigma}\, \bar{\psi}_\mu^{\cdot}\, \gamma_\nu\, \overleftrightarrow{D}_\rho\, \psi_{\sigma.} - 4e(3E_a^2 - 2A.E). \tag{2.61}$$

There even exist more off-shell formulations of the N=1 supergravity model. However, these formulations are non-minimal in the sense that they contain more than 12 + 12 field degrees of freedom. In extended supergravity the situation is even worse: many auxiliary fields are needed to close the commutator algebra without using field equations and many inequivalent off-shell formulations do exist.

5. Off-shell counting

As we remarked before, off-shell formulations of supersymmetric theories are based on multiplets of fields describing an equal number of bosonic and fermionic field degrees of freedom. In this section we show the underlying idea of this off-shell counting, as it is called. To that end we first give a counting argument concerning the field degrees of freedom, which are described by a massless spin-1 vector field. After that we extend this argument to supersymmetry multiplets as a whole.

The Maxwell theory of a massless spin-1 particle is based on a vector field $A_\mu(x)$. This field describes not 4 but 3 field degrees of freedom only because of the Maxwell gauge transformations $\delta A_\mu = \partial_\mu \Lambda$. One can view $A_\mu(x)$ as a (reducible) so-called induced representation of the Poincaré algebra of Lorentz transformations M and translations P:

$$[M_{\mu\nu}, M_{\rho\sigma}] = M_{\nu\rho}\delta_{\mu\sigma} - M_{\mu\rho}\delta_{\nu\sigma} - M_{\nu\sigma}\delta_{\mu\rho} + M_{\mu\sigma}\delta_{\nu\rho} \equiv 4M_{[\nu}{}^{[\rho}\delta_{\mu]}{}^{\sigma]} \quad ,$$

$$[M_{\mu\nu}, P_\rho] = 2\, P_{[\mu}\delta_{\nu]\rho} \quad , \qquad (2.62)$$

$$[P_\mu, P_\nu] = 0 \quad .$$

For more details on induced representations we refer to chapter 4, where we will consider the theory of induced representations in the context of the conformal algebra. Here we only give this as a result. To indicate which representations re contained in $A_\mu(x)$ it is convenient to go to momentum space, and to decompose A_μ into the independent vecors $p_\mu = (\vec{p}, p_4)$, $\bar{p}_\mu = (\vec{p}, -p_4)$ and the two transverse polarization vectors e^i_μ, i=1,2. Of course $p\cdot e^i = \bar{p}\cdot e^i = 0$, but $p\cdot\bar{p} \neq 0$. In this decomposition A_μ can be written as

$$A_\mu(p) = a^i(p^2)e^i_\mu + b(p^2)\bar{p}_\mu + c(p^2)p_\mu \quad . \qquad (2.63)$$

Each representation of the Poincaré algebra is characterized by the value of the energy-momentum squared p^2 and the spin s of the corresponding field d.o.f.. If $p^2=0$ the representation is called massless and the spin s contains two helicities. On the other hand if $p^2\neq 0$ (but fixed) the representation is called massive and each spin s contains 2s + 1 helicities. In the following we identify "p^2 arbitrary" with a massive representation. Furthermore we call the representation $(p^2=m^2(o), s)$ a massive (massless) spin-s representation (or multiplet) of the Poincaré algebra. More specifically, in (2.63) the components a^i carry helicity $\pm$ 1, whereas b and c correspond to helicity o. Because of the Maxwell gauge transformations the component c along p_μ corresponds to a gauge degree of freedom. We can choose c arbitrary, but by taking c=b we see explicitly that A_μ corresponds to a spin-1 object. In an <u>off-shell</u> formulation A_μ is not subject to a wave equation, but the gauge transformation is left. Therefore the energy-momentum squared p^2 of the components a^i and b is left arbitrary. This means that these field degrees of freedom should form massive representations of the Poincaré algebra. Indeed, the components a^i and b correspond to the 3 degrees of freedom of a massive spin-1 representation. On the other hand in an on-shell formulation A_μ is subject to the field equation

$$\Box A_\mu - \partial_\mu \partial \cdot A = 0 \quad \text{or} \quad p^2 A_\mu(p) - p_\mu p \cdot A(p) = 0 \qquad , \tag{2.64}$$

which implies for a^i and b

$$(p^2 a^i) e^i_\mu + (p^2 b) \bar{p}_\mu - (b p \cdot \bar{p}) p_\mu = 0 \qquad . \tag{2.65}$$

From the independence of the vectors e^i_μ, $\bar{p}_\mu$ and p_μ we deduce that b must vanish and that the value of p^2 of the remaining components $a^i(p^2)$ is restricted to light-like values only. Therefore in this on-shell formulation the relevant counting is based on massless representations which implies 2 degrees of freedom corresponding to a massless spin-1 multiplet.

For the same reason a spin-s massless field should at least describe 2s + 1 field degrees of freedom, which should form a massive spin-s multiplet, if we don't invoke field equations. It is known that for s > 1 this is not yet sufficient for a Lagrangian description of high-spin fields. We have already seen an example of this. The Einstein-Cartan description of a massless spin-2 particle (see section 3) is based on the vierbein field e^a_μ. This field describes 6 degrees of freedom, which form a massive spin-2 and spin-0 representation of the Poincaré algebra.

From the above we see that off-shell fields have in general more degrees of freedom than on-shell fields. These degrees of freedom are at least those of massive representations, but possibly combinations thereof. A similar situation arises for off-shell supersymmetry representations. They will be composed of various massive on-shell supersymmetry representations.

We extend the above counting arguments to supersymmetry multiplets as a whole. To that end we first have to know what the massive and massless representations of the super-Poincaré algebra are. By this we mean the algebra of Poincaré transformations (Lorentz transformations M and translations P) and supersymmetry transformations Q, which is given by (2.62) together with:

$$\begin{aligned} \{Q_\alpha, \bar{Q}_\beta\} &= +2P_\mu(\gamma^\mu)_{\alpha\beta} \quad , \\ [M_{\mu\nu}, Q_\alpha] &= -(\sigma_{\mu\nu})_{\alpha\beta}Q_\beta \quad , \\ [P_\mu, Q_\alpha] &= 0 \quad . \end{aligned} \tag{2.66}$$

The representations of this super algebra can be classified in the same way as this is done for the Poincaré algebra. For more details we refer the reader to the literature (see references). To indicate which representations are described by the field components of a supersymmetry multiplet it is again convenient to perform a Fourier transformation on these fields. Each representation is characterized by the value of the energy-momentum squared p^2 and the so-called superspin S. Each superspin S corresponds to a number of integer and half-integer (ordinary) spins s in such a way that the numbers of integer and half-integer helicities (or spin d.o.f.) are equal. We have listed some representations in the table.

spin-s (super)multiplet	N=1 massive representations	dynamic d.o.f.	N=1 massless representations	dynamic d.o.f.
2	(2,3/2,3/2,1)	8 + 8	(2,3/2)	2 + 2
3/2	(3/2,1,1,1/2)	6 + 6	(3/2,1)	2 + 2
1	(1,1/2,1/2,0)	4 + 4	(1,1/2)	2 + 2
1/2	(1/2,0,0)	2 + 2	(1/2,0,0)	2 + 2

table. N=1 massive and massless representations of the super-Poincaré algebra. The numbers between brackets denote the spins contained in each representation. Furthermore we have indicated the number of (bosonic + fermionic) dynamic d.o.f. described by the representation.

We may now use the same counting arguments as before, but now in the context of a whole supersymmetry multiplet of fields. In an off-shell formulation the field components of such a multiplet are not subject to generalized wave equations. Therefore the energy-momentum squared p^2 is left arbitrary and all field degrees of freedom should form massive representations of the super-Poincaré algebra. On the other hand, if the fields satisfy field equations, it is possible that the value of p^2 is restricted to light-like values $p^2=0$, as we have seen before. In that case these field components form massless representations of the super-Poincaré algebra. If an arbitrary mass parameter m is present the value of p^2 can be fixed by $p^2=m^2$. Of course, the corresponding fields should then form massive representations. One can now understand why an off-shell multiplet contains an equal number of bosonic and fermionic field degrees of freedom. This is a direct consequence of the fact that these field degrees of freedom form massive representations of the super-Poincaré algebra, which contain equal numbers of integer and half-integer spin degrees of freedom.

To clarify the above arguments we conclude this section by giving some explicit examples. First we consider the Wess-Zumino model. An off-shell formulation of this model is based on the fields A, F and ψ, which describe $4 + 4$ field degrees of freedom. The table shows that these fields fall in two massive spin-1/2 multiplets. The reason that we have two of these multiplets is that the multiplet (A,ψ,F) is a complex representation. This is not special for supersymmetry and is related to the fact that the spinor ψ is a complex representation of the Poincaré algebra: in addition to $\psi^{\cdot}$ we have the complex conjugate spinor $\psi_{\cdot}$ of opposite chirality. A reality condition is given by the massive Dirac equation, which relates $\psi^{\cdot}$ to $\psi_{\cdot}$. However this condition puts the multiplet on-shell, since by supersymmetry the following equations are related:

$$
\begin{aligned}
&\partial\psi^{\cdot} + m\psi_{\cdot} = 0 \quad &, \\
&(\Box - m^2)A = 0 \quad &, \qquad (2.67)\\
&F = - mA^* \quad &.
\end{aligned}
$$

Therefore the on-shell version of the Wess-Zumino model is based on A and ψ only. The field equations restrict the value of p^2 of these components to $p^2=m^2$ and restrict the a priori 4 degrees of freedom of ψ to two (the latter step was not taken in section 2 when we counted the degrees of freedom of ψ). This means that A and ψ form a massive spin-1/2 multiplet. Another example is provided by the N=1 supergravity model. The two minimal off-shell formulations $(e_\mu^a, \psi_\mu, A_a, F)$ and $(e_\mu^a, \psi_\mu, A_\mu, E_a)$ are both based on 12 + 12 field degrees of freedom. The table shows that the field components of the first multiplet fall in a massive spin-2 and two massive spin-1/2 multiplets, whereas those of the second one fall in a massive spin-2 and spin-1 multiplet. On the other hand in the on-shell formulation (e_μ^a, ψ_μ) there are 2 + 2 field degrees of freedom, which form a massless spin-2 multiplet.

6. Conformal supersymmetry

Even if we have the possibility of choosing an off-shell formulation of a supergravity multiplet, there are substantial complications caused by the nonlinearities present in invariant actions and transformation rules. These nonlinear modifications, which also occur in ordinary gravity, induce corresponding modifications in the commutator algebra. We have seen an example of this in section 4: the commutator algebra of the N=1 supergravity multiplet contains besides a covariant translation an additional Lorentz transformation depending on the auxiliary fields F and A_a. In extended supergravity those nonlinear modifications become more and more complicated because of the large numbers of auxiliary fields present in these theories.

Another disadvantage is that there exist different off-shell formulations of one on-shell supersymmetry multiplet. For instance, we have seen that there exist two minimal off-shell formulations of N=1 Poincaré supergravity, which are inequivalent, whereas there exist non-minimal off-shell formulations as well.

To explain the structure of all these inequivalent off-shell formulations it is very useful to introduce the idea of conformal supergravity or Weyl multiplet. By definition the Weyl multiplet is the smallest irreducible (with respect to the Poincaré algebra) multiplet, which contains the gravitational spin-2 degree of freedom. In ordinary gravity the vierbein field describes 6 degrees of freedom, which fall in a massive spin-2 and spin-0 representation of the Poincaré algebra. In chapter 4 we shall show how this vierbein field can be decomposed into its irreducible components by introducing the conformal transformations. In this case the irreducible part containing the spin-2 degree of freedom is called the conformal gravity multiplet. In chapter 5 we shall show how in the same way each Poincaré supergravity multiplet can be decomposed into its irreducible components by introducing the superconformal transformations, which are the supersymmetric generalization of the conformal symmetries. In this way each off-shell Poincaré supergravity multiplet decomposes into the Weyl multiplet and a number of additional so-called compensating supermultiplets. Inequivalent off-shell formulations of Poincaré supergravity differ in the choice of compensating multiplets but always contain the Weyl multiplet. Hence the Weyl multiplet constitutes the backbone of all inequivalent off-shell formulations. For instance the two minimal 12 + 12 off-shell formulations of N=1 Poincaré supergravity both decompose into the 8 + 8 N=1 Weyl multiplet containing the gravitational spin-2 degree of freedom, but they differ in the 4 + 4 compensating supermultiplets.

By introducing conformal invariance in extended supergravity one also makes clear the structure of the nonlinear modifications in the transformation rules and commutator algebra. The reason of this is that the presence of the conformal symmetries put stringent conditions on these nonlinear modifications. Consequently the superconformal invariants for the multiplets have a simpler form.

In a superconformal formulation one still keeps the option of discussing Poincaré supergravity, since this is based on a subcase of the superconformal symmetry. The transition form superconformal to Poincaré theories is achieved by making appropriate gauge choices in the superconformal formulation thereby reducing the gauge invariance to those of the super-Poincaré theory. In this sense the superconformal field representation is gauge equivalent to that of Poincaré supergravity.

It is the purpose of this thesis to explain how the above ideas are applied in supergravity. As a first step we shall therefore explain in the next chapter how one can always achieve the irreducibility of a multiplet by introducing additional gauge invariances. More specifically, we shall show in detail how this can be done for massive vector fields.

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CHAPTER III

GAUGE EQUIVALENT FORMULATIONS

1. Introduction

In the previous chapter we have argued that the complexity of supergravity makes it advantageous to use superconformally invariant formulations. The superconformal symmetries are introduced in order to separate the irreducible part of a supergravity multiplet containing the spin-2 degree of freedom. This part is called the Weyl multiplet and constitutes the backbone of all inequivalent off-shell formulations of extended Poincaré supergravity theories.

In order to explain how one can always decompose a multiplet into its irreducible components by introducing additional gauge invariances, we show in this chapter how this can be done for vector fields. A massive vector field W_μ (the Proca field) describes four field degrees of freedom, which form a massive spin-1 and spin-0 multiplet. One may introduce gauge invariance to separate this field into its irreducible components. In this way one obtains a gauge invariant reformulation of the theory, which is related to a description of a massive vector field given by Stueckelberg forty years ago. On the other hand a massless vector field A_μ is already irreducible. This is so because A_μ describes not 4 but 3 field degrees of freedom owing to the Maxwell gauge transformations $\delta A_\mu = \partial_\mu \Lambda$. These degrees of freedom form a massive spin-1 multiplet.

This chapter is organized as follows. In section 2 we explain how the Proca field may be decomposed into its independent constituents by introducing gauge invariance. In section 3 we show how the same can be done for massive vector fields in the adjoint representation of any group G. In particular we discuss an example with G = SU(2). Finally, in section 4 we consider an example with massive vector fields in the adjoint representation of SU(2) interacting with a scalar field ρ. Furthermore we show under which conditions this example is related to the usual Brout-Englert-Higgs mechanism.

2. The Stueckelberg model

In this section we consider a theory of a massive vector field W_μ:

$$\mathcal{L} = -\frac{1}{4} G_{\mu\nu}^2(W) - \frac{1}{2} m^2 W_\mu^2 \quad . \tag{3.1}$$

Here m is an arbitrary mass parameter and $G_{\mu\nu}$ has the form

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu \quad . \tag{3.2}$$

Because of the presence of the mass term the Lagrangian (3.1) is not invariant under gauge transformations of the form $\delta W_\mu = \partial_\mu \Lambda$. However we may obtain a reformulation of this Lagrangian, which is gauge invariant. To that end we introduce a scalar field $\phi(x)$, which transforms under a gauge transformation with an inhomogeneous term

$$\delta\phi(x) = m\Lambda(x) \tag{3.3}$$

and therefore does not describe a new degree of freedom. Furthermore we introduce a gauge field A_μ, which transforms according to

$$\delta A_\mu = \partial_\mu \Lambda(x) \quad . \tag{3.4}$$

This field can be used to define a covariant derivative of ϕ:

$$D_\mu \phi = \partial_\mu \phi - m A_\mu \quad . \tag{3.5}$$

To avoid new degrees of freedom we may relate the gauge field A_μ to the massive vector field W_μ, which does not transform under the gauge transformation.

To this end we impose the following gauge invariant condition on A_μ:

$$W_\mu = -\frac{1}{m} D_\mu \phi = A_\mu - \frac{1}{m} \partial_\mu \phi \quad . \tag{3.6}$$

This relation between A_μ and W_μ allows us to reformulate the Lagrangian (3.1) in the following way:

$$\mathcal{L} = -\frac{1}{4} G^2_{\mu\nu}(A) - \frac{1}{2} (D_\mu \phi)^2 \quad . \tag{3.7}$$

This Lagrangian is invariant under the gauge transformations specified by (3.3) and (3.4).

By performing the above manipulations we have related the Lagrangians (3.1) and (3.7) to each other. The first theory is based on a massive vector field W_μ, which describes four field degrees of freedom. On the other hand the second formulation has a gauge invariance and is based on a gauge field A_μ, which describes not 4 but 3 field degrees of freedom, and a scalar ϕ, which describes one field degree of freedom. The gauge field A_μ forms a massive spin-1 multiplet, whereas ϕ corresponds to a massive spin-0 multiplet. Hence in (3.6) we have effectively decomposed W_μ into its independent components A_μ and ϕ. In this formula we recognize the second term on the right-hand side as a gauge transformation of A_μ parametrized by ϕ. This gauge transformation term is essential to relate the gauge invariant field W_μ to the noninvariant field A_μ: it compensates for the gauge transformation of A_μ such that the right-hand side of (3.6) is gauge invariant. For this reason the scalar ϕ is called a <u>compensating</u> field. In the next section we shall explain how this procedure of decomposing W_μ into its independent constituents by introducing a gauge invariance and a compensating field may easily be extended to include massive vector fields in the adjoint representation of any group G as well. Before doing this we shall first discuss in this section the relation between the Lagrangians (3.1) and (3.7) in more detail.

The equivalence of the Lagrangian (3.7) to the original formulation (3.1) can be seen by reabsorbing the dependence on ϕ into the definition of A_μ through a ϕ-dependent gauge transformation. This is simply a reversion of the argument that led to (3.7). Alternatively one could make a choice of gauge. A suitable condition is

$$\phi(x) = 1 \quad . \tag{3.8}$$

With this choice we find

$$D_\mu \phi = - mA_\mu = - mW_\mu \tag{3.9}$$

and the Lagrangian (3.7) reduces to the form (3.1). In this gauge it is easy to give an interpretation of the particle spectrum of the theory, because the gauge degree of freedom $\phi(x)$ is no longer present. The condition (3.8) is called the unitary gauge. One easily deduces from the form of the massive propagator

$$\Delta_{\mu\nu}(W) \propto \frac{\delta_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}}{k^2 + m^2} \tag{3.10}$$

that its residue vanishes upon contraction with k_μ. Hence 3 physical degrees of freedom are propagating corresponding to a massive spin-1 particle.

The reformulation (3.7) was first used by Stueckelberg in order to improve the high-energy behaviour of the massive propagator $\Delta_{\mu\nu}(W)$. In the limit for large momenta this propagator behaves as a constant and therefore leads to divergent loop diagrams. To obtain a propagator with an improved behaviour it is convenient to impose in the reformulation (3.7) the Lorentz condition

$$\partial_\mu A_\mu = 0 \quad . \tag{3.11}$$

To calculate the propagators in this gauge we add the following gauge fixing term to the Lagrangian (3.7):

$$\mathcal{L}^{\text{fix.}} = - \frac{1}{2} (\alpha \partial_\mu A_\mu - \frac{1}{\alpha} m\phi)^2 \quad . \tag{3.12}$$

The propagators corresponding to the modified Lagrangian are given by

$$\Delta_{\mu\nu}(A) \propto \frac{\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}}{k^2+m^2} + \frac{1}{\alpha^2}\frac{1}{k^2 + \left(\frac{m}{\alpha}\right)^2}\frac{k_\mu k_\nu}{k^2} \quad , \tag{3.13}$$

$$\Delta_{\mu\nu}(\phi) \propto \frac{1}{k^2 + \left(\frac{m}{\alpha}\right)^2} \quad . \tag{3.14}$$

In the limit $\alpha \to \infty$ these propagators are called the propagators in the Landau gauge. In the limit for large momenta the expressions (3.13) and (3.14) both behave as $\frac{1}{k^2}$. In the gauge (3.11) one can apply the standard methods of renormalization theory. On the other hand this gauge is complicated in that the fields A_μ and ϕ do not individually have a direct physical meaning. In particular the gauge-dependent poles at $k^2 = -\left(\frac{m}{\alpha}\right)^2$ in (3.13) and (3.14) are unphysical and should cancel out in any actual calculation of observables. To obtain a correct interpretation of the particle spectrum one must calculate the propagation of A_μ and ϕ between physical sources. In doing so the gauge-dependent propagators $\Delta_{\mu\nu}(A)$ and $\Delta(\phi)$ recombine into the gauge-independent massive propagator $\Delta_{\mu\nu}(W)$ in the following way:

$$\Delta_{\mu\nu}(W) = \Delta_{\mu\nu}(A) + \frac{k_\mu k_\nu}{m^2}\Delta(\phi) \quad . \tag{3.15}$$

Therefore only A_μ and ϕ together represent the three polarizations of a massive spin-1 state.

An important benefit of a gauge invariant reformulation is that it presents a convenient way to relate different field representations to each other. For instance the inequivalent formulations (3.1) and (3.12) can both be viewed as constructed out of the gauge invariant reformulation (3.7) after imposing different gauge choices. Of course, the relation between these inequivalent formulations can also be made explicit by making field-dependent redefinitions, but in the general case these redefinitions can be quite complicated.

One could ask oneself the question whether also the Maxwell theory of a massless vector field can be viewed as the gauge invariant reformulation of a theory without gauge invariance. If one restricts oneself to work within the framework of a local field theory, one cannot impose gauge conditions like the Lorentz gauge $\partial . A = 0$ *). With this restriction, a formulation without gauge invariance does exist, but such a formulation has no manifest Lorentz covariance. A convenient gauge condition is the light-cone gauge, in which one of the light-cone coordinates

$$A^{\pm} = \frac{1}{\sqrt{2}} (A^0 \pm A^3) \tag{3.16}$$

is set equal to zero. Here 0 and 3 label the time and longitudinal directions respectively. This gauge has been used by Dirac fifty years ago, when discussing canonical quantization on a lightlike plane. An advantage of this gauge is that gauge degrees of freedom are no longer present, but it has the obvious drawback of losing manifest Lorentz covariance.

3. Massive vector fields in the adjoint representation of a group G

The procedure outlined in the previous section can easily be adjusted to massive vector fields in the adjoint representation of any group G. In this section we first indicate how this can be done. We next work out in detail an example with G = SU(2).

In the general case we consider a theory of massive vector fields W_μ in the adjoint representation of G (we may think of G as a group of n x n matrices):

$$W_\mu \rightarrow W'_\mu = V W_\mu V^{-1} \quad , \tag{3.17}$$

with V a group element of G and W_μ a Lie algebra valued field. Because of the presence of a mass term $\mathrm{Tr}(m^2 W_\mu^2)$ the Lagrangian for such fields is not invariant under gauge transformations of the form

$$W_\mu \rightarrow W'_\mu = U(x)\, W_\mu U^{-1}(x) + (\partial_\mu U(x)) U^{-1}(x) \quad , \tag{3.18}$$

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*) See for instance C.Itzykson and J.B.Zuber, Quantum Field Theory, McGraw-Hill, 1980.

where $U(x)$ forms for every point in spacetime a group element of G. However we may obtain a reformulation of the Lagrangian, which is invariant under such transformations. These local G transformations are not the spacetime-dependent extension of the original rigid transformations (3.17) and act on a different kind of indices. (It means that by taking constant parameters we get a global G transformation, which is different from the original one.) Therefore we call this group of gauge transformations local G to distinguish it from the group rigid G of global G transformations, which act on W_μ according to (3.17). To obtain this local G invariant reformulation we introduce, in analogy to the Stueckelberg model, scalars $\Phi(x)$ which form spacetime dependent group elements of G. Under local G transformations these scalars are multiplied to the left by a group element $U(x)$ of G:

$$\Phi(x) \to \Phi'(x) = U(x)\Phi(x) \qquad \text{(local G)} \tag{3.19}$$

and therefore they do not describe new degrees of freedom. Under rigid G transformations they are multiplied to the right by an element Z of G according to

$$\Phi(x) \to \Phi'(x) = \Phi(x)Z^{-1} \quad . \quad \text{(rigid G)} \tag{3.20}$$

Moreover we introduce Lie algebra valued gauge fields A_μ of the local G transformations. The transformation character of these gauge fields is given by (3.18) with W_μ replaced by A_μ. These gauge fields enable us to define a covariant derivative of Φ in the following way:

$$D_\mu\Phi = (\partial_\mu - A_\mu)\Phi \quad . \tag{3.21}$$

This covariant derivative transforms homogeneously under both local and rigid G transformations if we take the gauge fields A_μ inert under rigid G:

$$\begin{aligned} D_\mu\Phi &\to (D_\mu\Phi)' = U(x)(D_\mu\Phi) \quad , \quad \text{(local G)} \\ D_\mu\Phi &\to (D_\mu\Phi)' = (D_\mu\Phi)\, Z^{-1} \quad . \quad \text{(rigid G)} \end{aligned} \tag{3.22}$$

To avoid new degrees of freedom we want to relate the gauge fields A_μ to the massive vector fields W_μ. However, this is not trivial because of the different transformation character of these fields under local and rigid G. An essential role in relating A_μ to W_μ is played by the covariant derivative of Φ. The transformation character of this covariant derivative is given in (3.22). Out of this derivative we may construct a local G invariant by multiplying it with the inverse group element Φ^{-1} of Φ to the left. The resulting expression $\Phi^{-1}D_\mu\Phi$ has the same transformation character as W_μ under rigid G if we identify the group elements V and Z in the transformations (3.17) of W_μ and (3.20) of Φ respectively. Therefore we may relate A_μ to W_μ in the following way:

$$W_\mu = - \Phi^{-1}D_\mu\Phi = \Phi^{-1}A_\mu\Phi + (\partial_\mu\Phi^{-1})\Phi \qquad . \tag{3.23}$$

Substituting this expression into the Lagrangian for W_μ one obtains a reformulated Lagrangian in terms of A_μ and Φ which is invariant under both local and rigid G transformations specified by (3.18) (with W_μ replaced by A_μ), (3.19) and (3.20).

The relation (3.23) may be interpreted in the same way as eq.(3.6) in the Stueckelberg model. On the one hand the massive vector fields W_μ describe 4N field degrees of freedom (N is the dimension of G), which form N massive spin-1 and N massive spin-0 multiplets. On the other hand the gauge fields A_μ describe not 4N but 3N field degrees of freedom owing to their transformations under local G. These degrees of freedom form N massive spin-1 multiplets, whereas the remaining N spin-0 multiplets are represented by the scalars $\Phi(x)$. Therefore eq.(3.23) can be viewed as an effective decomposition of the fields W_μ into their independent components A_μ and Φ. The second term on the right-hand side of this equation plays the same role as the $(\partial_\mu\phi)$-term in eq.(3.6). This term is a gauge transformation with parameter Φ^{-1} of the A_μ-dependent term in (3.23). It compensates for the gauge transformations of this A_μ-term such that the right-hand side of (3.23) is invariant under local G. Hence the scalars Φ play - just as the scalar ϕ in the previous section - the role of compensating fields.

We will now give an explicit example with G=SU(2). We consider a theory of massive vector fields in the adjoint representation of SU(2) corresponding to the Lagrangian

$$\mathcal{L} = 2\,\mathrm{Tr}\,\left(\frac{1}{4}\, G^2_{\mu\nu}(W) + \frac{1}{2}\, m^2\, W^2_\mu \right) \qquad . \qquad (3.24)$$

Such vector fields can be written as a linear combination of the three generators T_i of the Lie algebra of SU(2):

$$W_\mu = W^i_\mu\, T_i \qquad (i=1,2,3) \qquad . \qquad (3.25)$$

The generators are antihermitean traceless matrices, which satisfy the commutation relations

$$[T_i,\, T_j] = -\,\epsilon_{ijk}\, T_k \qquad (3.26)$$

and may be normalized according to

$$\mathrm{Tr}(T_i T_j) = -\frac{1}{2}\,\delta_{ij} \qquad . \qquad (3.27)$$

An explicit representation is provided by the three standard Pauli matrices τ_i:

$$T_i = \frac{i}{2}\,\tau_i \qquad . \qquad (3.28)$$

The tensor $G_{\mu\nu}$ in (3.24) has the form

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - [W_\mu, W_\nu] \qquad . \qquad (3.29)$$

In terms of $\vec{W}_\mu = (W^1_\mu, W^2_\mu, W^3_\mu)$ the Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} G^2_{\mu\nu}(\vec{W}) - \frac{1}{2} m^2 \vec{W}^2_\mu \quad , \tag{3.30}$$

where the curvature tensor $G_{\mu\nu}(\vec{W})$ has the form

$$G_{\mu\nu}(\vec{W}) = \partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu + \vec{W}_\mu \times \vec{W}_\nu \quad . \tag{3.31}$$

The Lagrangian (3.24) is invariant under rigid SU(2) transformations specified by (3.17) with V a group element of SU(2), i.e. V is a unitary matrix with unit determinant. To obtain a local SU(2) invariant reformulation of this Lagrangian we introduce 3 compensating scalars $\Phi(x)$, which form for every point in spacetime a group element of SU(2). This means that $\Phi(x)$ is a unitary matrix with unit determinant and spacetime-dependent matrix elements. Such a matrix satisfies a nonlinear constraint

$$\Phi^\dagger(x)\, \Phi(x) = \mathbb{1} \quad . \tag{3.32}$$

This is in contrast to the Stueckelberg model, where the compensating scalar $\phi(x)$ satisfies no constraint at all. At the end of this section we show that this constraint implies non-polynomial interactions.

Under local and rigid SU(2) transformations the scalars $\Phi(x)$ transform according to (3.19) and (3.20). We may introduce gauge fields A_μ of the local SU(2) transformations as before. These gauge fields take values in the Lie algebra of SU(2). Therefore they are 2x2 antihermitean traceless matrices, which can be written as

$$A_\mu = \frac{i}{2} \vec{A}_\mu \cdot \vec{\tau} \quad . \tag{3.33}$$

They can be related to the vector fields W_μ by eq.(3.23). Substitution of this relation into the Lagrangian (3.24) leads to a reformulated Lagrangian in terms of A_μ and Φ given by

$$\mathcal{L} = 2\,\mathrm{Tr}\left(\frac{1}{4} G^2_{\mu\nu}(A) - \frac{1}{2} m^2 (D_\mu\Phi)^\dagger(D_\mu\Phi)\right) \quad . \tag{3.34}$$

This Lagrangian is invariant under local SU(2) transformations specified by (3.18) (with A_μ instead of W_μ), (3.19) and (3.20).

The equivalence of the reformulation (3.34) to the original form (3.24) can be made explicit by making a choice of gauge. A suitable condition is the unitary gauge

$$\Phi(x) = \mathbb{1} \quad . \tag{3.35}$$

After imposing this choice we find

$$D_\mu\Phi = -A_\mu = -W_\mu \tag{3.36}$$

and the Lagrangian (3.34) directly reduces to the form (3.24). Besides the SU(2) gauge transformations the Lagrangian (3.34) is invariant under a set of global transformations, which include the original rigid SU(2) transformations. More specifically these global transformations are given by the original rigid SU(2) transformation with parameter V together with a special local SU(2) transformation with constant parameter $U(x)=W$. These combined global transformations act on A_μ and Φ according to

$$\begin{aligned} A_\mu &\to A'_\mu = W A_\mu W^{-1} \quad , \\ \Phi &\to \Phi' = W\Phi V^{-1} \quad . \end{aligned} \tag{3.37}$$

Clearly only the special global transformations characterized by $W=V$ leave the gauge condition $\Phi = \mathbb{1}$ invariant and hence are also an invariance of the resulting Lagrangian (3.24). These special global transformations are different from the original rigid SU(2) transformations. They are specified by a so-called decomposition rule, which holds on $A_\mu \equiv W_\mu$:

$$(\text{rigid } SU(2))'(V) = (\text{rigid } SU(2))(V) \otimes (\text{local } SU(2))(U(x) = V) \quad . \tag{3.38}$$

Here we use a notation, where (rigid SU(2))(V) denotes a rigid SU(2) transformation with parameter V and (local SU(2))(U(x)=V) denotes a special local SU(2) transformation with parameter U(x)=V. A decomposition rule like (3.38) is typical for theories in which a symmetry has been broken down to a smaller one by means of a gauge choice. For instance in the standard Weinberg-Salam model a SU(2) $\otimes$ U(1) gauge symmetry is broken down to a U(1) symmetry. After imposing the SU(2) gauge condition, the remaining U(1) symmetry is not given by the U(1) factor in SU(2) $\otimes$ U(1) but by a combination of this U(1) factor and the U(1) subgroup of SU(2).

To make the non-polynomial interactions defined by the Lagrangian (3.34) explicit, we write the scalars $\Phi(x)$, which are 2x2 unitary matrices with unit determinant, in the following way:

$$\Phi(x) = \sigma(x)\, \mathbb{1} + \frac{i}{2}\, \vec{\phi}(x)\cdot\vec{\tau} \quad , \tag{3.39}$$

where the real functions $\sigma(x)$ and $\vec{\phi}(x)$ are restricted by the relation

$$\sigma^2(x) + \frac{1}{4}\vec{\phi}^2(x) = 1 \quad . \tag{3.40}$$

In terms of σ and $\vec{\phi}$ the Lagrangian (3.34) reads

$$\mathcal{L} = -\frac{1}{4} G^2_{\mu\nu}(\vec{A}) - \frac{1}{2} m^2 \left((D_\mu\vec{\phi})^2 + 4(D_\mu\sigma)^2\right) \quad . \tag{3.41}$$

Here we have defined

$$D_\mu \Phi \equiv D_\mu \sigma \, \mathbb{1} + \frac{i}{2} D_\mu \vec{\phi} \cdot \vec{\tau} \qquad . \qquad (3.42)$$

The structure of the SU(2) covariant derivative D_μ follows from the transformation rules of $\sigma, \vec{\phi}$ and $\vec{A}_\mu$ under infinitesimal local SU(2) transformations:

$$\begin{aligned} \delta\sigma &= -\frac{1}{4} \vec{\omega}(x) \cdot \vec{\phi} \qquad , \\ \delta\vec{\phi} &= -\frac{1}{2} \vec{\omega}(x) \times \vec{\phi} + \sigma\vec{\omega}(x) \qquad , \\ \delta\vec{A}_\mu &= -\vec{\omega}(x) \times \vec{A}_\mu + \partial_\mu \vec{\omega}(x) \qquad , \end{aligned} \qquad (3.43)$$

where $\vec{\omega}(x)$ are the parameters characterizing these transformations. We now solve σ in terms of $\vec{\phi}$ and perform the following field redefinition:

$$\vec{\phi} = \left(\frac{1}{1 + \frac{1}{16} \vec{\psi}^2} \right) \vec{\psi} \qquad . \qquad (3.44)$$

The Lagrangian then takes on the following form

$$\mathcal{L} = -\frac{1}{4} G^2_{\mu\nu}(\vec{A}) - \frac{1}{2} m^2 \left(\frac{1}{(1 + \frac{1}{16} \psi^2)^2} \right) (D_\mu \vec{\psi})^2 \qquad , \qquad (3.45)$$

with the SU(2) covariant derivative $D_\mu \vec{\psi}$ of $\vec{\psi}$ given by

$$D_\mu \vec{\psi} = \partial_\mu \vec{\psi} + \frac{1}{2} \vec{A}_\mu \times \vec{\psi} - (1 - \frac{1}{16} \vec{\psi}^2) \vec{A}_\mu - \frac{1}{8} (\vec{A}_\mu . \vec{\psi}) \vec{\psi} \qquad . \qquad (3.46)$$

In this form the non-polynomial interactions are manifest. In the next section we discuss a set of models, where such interactions are avoided.

4. The Brout-Englert-Higgs mechanism

The non-polynomial structure of the massive SU(2) model considered in the previous section shows the lack of renormalizability of this example. We stress that independent of the fact whether this theory is renormalizable or not one can always achieve the irreducibility of the massive vector fields W_μ by introducing SU(2) gauge transformations. The reason that the local SU(2) invariant Lagrangian (3.45) contains non-polynomial terms is that the compensating scalars $\Phi(x)$ satisfy a nonlinear constraint. In this section we show how such a constraint can be avoided by extending the massive SU(2) model to include an additional scalar field $\rho(x)$ as well. This field ρ shall be absorbed into the definition of Φ such that the compensating scalars satisfy no constraint anymore. More specifically, we shall construct a whole set of models describing massive vector fields W_μ in the adjoint representation of SU(2) and a real scalar field ρ, which can be reformulated in a SU(2) gauge invariant way without introducing non-polynomial terms. Moreover we shall indicate the subset of these models, which correspond to renormalizable theories.

We consider the following extension of the massive SU(2) model (cp. (3.24)):

$$\mathcal{L} = 2 \operatorname{Tr} \left(\frac{1}{4} G^2_{\mu\nu}(W) + f(\rho^2) W^2_\mu \right) + g(\rho^2)(\partial_\mu \rho)^2 + V(\rho^2) \ . \tag{3.47}$$

Here we have used the same notations as in the previous section. The function $g(\rho^2)$ is arbitrary. We assume that the potential function $V(\rho^2)$ reaches its minimum for $\rho=\rho_{min}$. The vector fields W_μ acquire their mass through their interaction with ρ. Therefore we require that the function $f(\rho^2)$ satisfies the relation

$$f(\rho^2_{min}) = \frac{1}{2} m^2 \tag{3.48}$$

To separate the vector fields W_μ into their irreducible components we may perform the same manipulations as in the massive SU(2) model. Using the same definitions the Lagrangian (3.47) takes on the following gauge equivalent form:

$$\mathcal{L} = 2 \operatorname{Tr} \left(\frac{1}{4} G^2_{\mu\nu}(A) - f(\rho^2)(D_\mu \Phi)^\dagger (D_\mu \Phi) \right) + g(\rho^2)(\partial_\mu \rho)^2 + V(\rho^2) \ . \tag{3.49}$$

At this point we have already succeeded in decomposing W_μ into the independent components A_μ and ϕ.

The 3 compensating scalars $\Phi(x)$, which are present in the Lagrangian (3.49) satisfy the nonlinear constraint (3.32). Therefore this Lagrangian would contain non-polynomial terms if ρ were absent. However the presence of this scalar ρ enables us to avoid the constraint on Φ. An obvious way to achieve this is to redefine $\Phi(x)$ in terms of unrestricted fields $H(x)$ according to

$$H(x) = \rho(x)\Phi(x) \qquad . \qquad (3.50)$$

After this redefinition ρ is identified as the norm of H:

$$H^\dagger(x)H(x) = \rho^2(x)\ \mathbb{1} \qquad (3.51)$$

and in this way a constraint of the form $H^\dagger H = \mathbb{1}$ is avoided. The general form of the matrices H is given by

$$\begin{aligned} H(x) &= \frac{1}{2}(H_o + i\vec{H}\cdot\vec{\tau}) \\ &= \frac{1}{2}\begin{pmatrix} H_o + iH_3 & iH_1 + H_2 \\ iH_1 - H_2 & H_o - iH_3 \end{pmatrix} \qquad . \qquad (3.52) \end{aligned}$$

A SU(2) covariant derivative of H is given by

$$D_\mu H = (\partial_\mu - \frac{i}{2}\vec{A}_\mu\cdot\vec{\tau})H \qquad . \qquad (3.53)$$

In terms of these unrestricted fields the Lagrangian (3.49) takes on the following form:

$$\mathcal{L} = \frac{1}{2} \langle G^2_{\mu\nu}(A) \rangle - 4 \frac{f(\frac{1}{2}\langle H^\dagger H \rangle)}{\langle H^\dagger H \rangle} \langle (D_\mu H)^\dagger (D_\mu H) \rangle$$
$$+ \frac{1}{\langle H^\dagger H \rangle} \left\{ \frac{f(\frac{1}{2}\langle H^\dagger H \rangle)}{\langle H^\dagger H \rangle} + \frac{1}{8} g(\tfrac{1}{2}\langle H^\dagger H \rangle) \right\} \left\{ \langle H^\dagger (D_\mu H) \rangle + \langle H (D_\mu H)^\dagger \rangle \right\}^2$$
$$+ V(\tfrac{1}{2}\langle H^\dagger H \rangle) \quad . \tag{3.54}$$

Here the trace of the matrix $H^\dagger H$ is denoted by $\langle H^\dagger H \rangle$ and the same for all other 2x2 matrices. The gauge equivalence of this Lagrangian to its original form (3.47) can be made explicit by imposing the unitary gauge

$$H(x) = \rho(x) \ \mathbb{1} \quad . \tag{3.55}$$

Substituting this condition into (3.53) we find

$$D_\mu H = (\partial_\mu \rho) \mathbb{1} - \frac{i}{2} \rho \ \vec{W}_\mu \cdot \vec{\tau} \tag{3.56}$$

and (3.54) reduces to the form (3.47). In this way we have reformulated a whole set of models describing massive vector and scalar fields in a gauge invariant way without introducing non-polynomial terms.

In the formulation (3.54) we may now apply the standard methods of renormalization theory to investigate which restrictions must be imposed on the functions f, g and V in order to obtain a renormalizable theory. This subset of renormalizable models is characterized by the property that the corresponding Lagrangian has no coupling constants with negative dimensions. Here the dimension of the Lagrangian is equal to four, while each derivative is counted as a dimension 1 object. This implies that the dimension of A_μ is equal to one. If we also take the dimension of H equal to one, the functions f, g and V are restricted to the form

$$\begin{aligned} f(\langle H^\dagger H \rangle) &= \alpha + \beta \langle H^\dagger H \rangle \quad , \\ g(\langle H^\dagger H \rangle) &= \gamma \quad , \\ V(\langle H^\dagger H \rangle) &= \delta \langle H^\dagger H \rangle + \varepsilon \langle H^\dagger H \rangle^2 \quad , \end{aligned} \tag{3.57}$$

where α, β, γ, δ and ε are constant parameters. Because of (3.48) these parameters are restricted by the relation

$$\alpha - \frac{\beta\delta}{2\varepsilon} = \frac{1}{2} m^2 \qquad (3.58)$$

In this restricted form the mechanism of gauge equivalent formulations (for non-abelian groups) has first been used by Brout, Englert and Higgs in order to prove the renormalizability of this subset of models.

5. Conclusions

By means of some simple examples we have shown how one can always achieve the irreducibility of massive vector fields by introducing additional gauge transformations. In the gauge invariant reformulation one still keeps the option of discussing a theory of massive vector fields since one can always remove the gauge invariances by imposing a number of gauge conditions. In this way one constructs theories within the context of a higher symmetry which are gauge equivalent to theories with less symmetry.

For massive vector fields in the adjoint representation of a non-abelian group G the gauge invariant reformulation contains non-polynomial terms. Such terms can be avoided by introducing additional scalar fields into the model. Once these terms are absent one may apply the standard methods of renormalization theory and investigate which restrictions must be imposed to obtain a renormalizable model. In this thesis we shall use gauge equivalent formulations only within the context of classical field representations. We shall not bother us about the quantum properties of the theory.

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CHAPTER IV

CONFORMAL SYMMETRY

1. Introduction

In order to apply the ideas presented in the previous chapter to massless spin-2 fields it is necessary to introduce conformal gauge transformations. The gravitational spin-2 state is represented by a metric tensor field $g_{\mu\nu}(x)$. This tensor transforms under an infinitesimal general coordinate transformation (g.c.t.)

$$x^{\mu} \to (x^{\mu})' = x^{\mu} - \xi^{\mu}(x) \quad , \tag{4.1}$$

with parameters $\xi^{\mu}(x)$, according to

$$\begin{aligned}\delta_{g.c.t.}\, g_{\mu\nu}(x) &\equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x) \\ &= (\partial_{\mu}\xi^{\lambda})g_{\lambda\nu}(x) + (\partial_{\nu}\xi^{\lambda})g_{\lambda\mu}(x) + \xi^{\lambda}\partial_{\lambda}g_{\mu\nu}(x) \ .\end{aligned} \tag{4.2}$$

Using the definition of the Christoffel symbols

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\, g^{\rho\sigma}(\partial_{\mu}g_{\sigma\nu} + \partial_{\nu}g_{\sigma\mu} - \partial_{\sigma}g_{\mu\nu}) \tag{4.3}$$

it is easy to rewrite this transformation in the following way:

$$\delta_{g.c.t.}\, g_{\mu\nu} = D_{\mu}\xi_{\nu} + D_{\nu}\xi_{\mu} \quad , \tag{4.4}$$

where $\xi_{\mu} \equiv g_{\mu\nu}\xi^{\nu}$ and the covariant derivative is given by

$$D_{\mu}\xi_{\nu} = \partial_{\mu}\xi_{\nu} - \Gamma^{\rho}_{\mu\nu}\xi_{\rho} \quad . \tag{4.5}$$

The general coordinate transformations (4.4) are of the same type as the Maxwell gauge transformations $\delta A_\mu = \partial_\mu \Lambda$. Owing to these transformations the symmetric tensor $g_{\mu\nu}$ does not describe 10 but (10-4)=6 field degrees of freedom, which form a massive spin-2 and spin-0 representation of the Poincaré algebra. In the conformally invariant reformulation $g_{\mu\nu}$ is replaced by a redefined field $(g_{\mu\nu})^c$, which transforms under an additional gauge transformation with parameter $\sigma(x)$ according to

$$g^{c\prime}_{\mu\nu}(x) = \sigma(x) g^{c}_{\mu\nu}(x) \qquad . \tag{4.6}$$

Therefore $(g_{\mu\nu})^c$ describes (10-4-1)=5 field degrees of freedom only, corresponding to a massive spin-2 representation of the Poincaré algebra.

Before showing explicitly how one may reformulate the theory of gravity in a conformally invariant way, we consider in the first part of this chapter the conformal transformations in a more general context. We recall that in special relativity a particle is described by a point moving along a world line $x^\mu = x^\mu(\tau)$. Here x^μ is a local coordinate system and τ is the proper time, which is defined by (we use the Pauli metric $\delta_{\mu\nu} \equiv \mathrm{diag}(+,+,+,+)$):

$$d\tau^2 = \delta_{\mu\nu}\, dx^\mu\, dx^\nu \qquad . \tag{4.7}$$

Under a general coordinate transformation (4.1) we have $d\tau'^2 = \delta'_{\mu\nu} dx'^\mu dx'^\nu$, where $\delta'_{\mu\nu}$ is obtained from (4.2) by replacing $g_{\mu\nu}$ by $\delta_{\mu\nu}$. Covariance implies $d\tau'^2 = d\tau^2$. There exists a subclass of general coordinate transformations, which leaves the form of the proper time invariant, i.e.:

$$\delta(\delta_{\mu\nu}) \equiv \delta'_{\mu\nu} - \delta_{\mu\nu} = 0 \qquad . \tag{4.8}$$

That happens to be the case for the Poincaré spacetime transformations

$$x'^\mu = x^\mu - a^\mu + L^\mu{}_\nu x^\nu \qquad , \tag{4.9}$$

where a^μ is a constant four vector and L is a Lorentz rotation matrix. One may investigate whether the Poincaré transformations can be extended to a larger group of spacetime transformations, which leave the form of the "proper time" $d\tau^2\neq 0$ (massive particles) or $d\tau^2=0$ (massless particles) invariant. It appears that for massless particles this is indeed possible. In that case equation (4.8), which is satisfied by Poincaré transformations only, may be replaced by the weaker condition

$$\delta(\delta_{\mu\nu}) = (\sigma(x)-1)\delta_{\mu\nu} \qquad , \qquad (4.10)$$

where $\sigma(x)$ is an arbitrary function of the coordinates. This is equivalent to the requirement that $\delta_{\mu\nu}$ transforms according to (4.6). One easily verifies that the transformations characterized by (4.10) leave only "angles" invariant. Therefore they are referred to as <u>conformal</u> transformations and the corresponding group is called the conformal group. In this way one can view conformal invariance as the highest degree of spacetime symmetry that a theory without a mass parameter (or more precisely without a dimensionful parameter) can have.

This chapter is organized as follows. In section 2 we derive the explicit form of the conformal transformations and discuss some of its properties. In section 3 we use the technique of induced representations to construct representations of the conformal algebra. The coupling of matter fields to conformal gravity is discussed in section 4. In section 5 we show in detail how one may introduce conformal invariance in the Einstein-Cartan version of gravitation. Finally, in section 6 we give the maximal set of constraints, which may be imposed on the conformal curvatures. To keep everything as general as possible we work in this chapter in d spacetime dimensions.

2. The conformal algebra

Any general coordinate transformation in a d-dimensional space that satisfies (4.8)(with $\delta_{\mu\nu}$ replaced by the metric tensor $g_{\mu\nu}(x)$) is called an isometry. For an infinitesimal transformation (4.1) this equation is equivalent to the requirement that (cf. eq.(4.2) and (4.4))

$$\delta_{g.c.t.}\, g_{\mu\nu}(x) = D_\mu\xi_\nu(x) + D_\nu\xi_\mu(x) = 0 \ . \qquad (4.11)$$

Hence an infinitesimal isometry leaves the metric locally, i.e. in a point, invariant. Any vector field $\xi_\mu(x)$ that satisfies (4.11) is called a Killing vector associated with the metric $g_{\mu\nu}(x)$. The problem of determining all infinitesimal isometries of a given metric is related to the problem of determining all Killing vectors associated with that metric. To see, what the maximum number of independent solutions of (4.11) is, we recall that the commutator of two covariant derivatives is given by

$$(D_\mu D_\nu - D_\nu D_\mu)\xi_\rho \equiv R_{\mu\nu\rho}{}^{\lambda}\xi_\lambda \qquad , \tag{4.12}$$

with $R_{\mu\nu\rho}{}^{\lambda}$ the Riemann curvature tensor, which satisfies the Bianchi identity

$$R_{\mu\nu\rho}{}^{\lambda} + R_{\rho\mu\nu}{}^{\lambda} + R_{\nu\rho\mu}{}^{\lambda} = 0 \qquad . \tag{4.13}$$

By combining (4.12) and (4.13), we find that any vector field $\xi_\mu(x)$ must satisfy the relation

$$(D_\mu D_\nu - D_\nu D_\mu)\xi_\rho + (D_\rho D_\mu - D_\mu D_\rho)\xi_\nu + (D_\nu D_\rho - D_\rho D_\nu)\xi_\mu = 0 \qquad . \tag{4.14}$$

For a Killing vector, (4.11) and (4.14) give

$$(D_\mu D_\nu - D_\nu D_\mu)\xi_\rho + D_\rho D_\mu \xi_\nu = 0 \tag{4.15}$$

and thus (4.12) becomes

$$D_\rho D_\mu \xi_\nu = - R_{\mu\nu\rho}{}^{\lambda}\xi_\lambda \qquad . \tag{4.16}$$

This equation implies that, given ξ_μ and $D_\mu \xi_\nu$ at some point X, we can determine the second derivative of ξ_μ at X. Furthermore we can find successively higher derivatives of ξ_μ at X by taking derivatives of eq.(4.16). The function $\xi_\mu(x)$ can then be constructed as a Taylor series in (x-X) within some finite neighborhood of X. Therefore the Killing vector $\xi_\mu(x)$ of a given metric is uniquely specified by the values of $\xi_\mu(X)$ and $D_\mu\xi_\nu(X)$ at any particular point X. Because there are d independent quantities $\xi_\mu(X)$ and $\frac{1}{2}d(d-1)$ independent quantities $D_\mu\xi_\nu(X)$ (recall eq.(4.11)), a d-dimensional space can have at most $\frac{1}{2}d(d+1)$ independent Killing vectors.

In d-dimensional Minkowski space we can choose Cartesian coordinates with vanishing Christoffel symbols. In that case eq.(4.11) reduces to

$$\partial_{(\mu}\xi_{\nu)} = 0 \quad . \tag{4.17}$$

Expanding $\xi_\mu(x)$ as a power series in x_μ

$$\xi_\mu(x) = \Lambda^{(o)}_\mu + \Lambda^{(1)}_{\mu\nu}x_\nu + \Lambda^{(2)}_{\mu\nu\rho}x_\nu x_\rho + \ldots \quad , \tag{4.18}$$

with $\Lambda^{(o)}_\mu$, $\Lambda^{(1)}_{\mu\nu}$,... constant parameters, we find

$$\Lambda^{(o)}_\mu \text{ arbitrary}, \quad \Lambda^{(1)}_{(\mu\nu)} = 0 \quad , \tag{4.19}$$

whereas the remaining parameters are zero. Hence we are left with d parameters $\xi_\mu = \Lambda^{(o)}_\mu$ and $\frac{1}{2}d(d-1)$ parameters $\varepsilon_{\mu\nu} = -\Lambda^{(1)}_{[\mu\nu]}$, which characterize the $\frac{1}{2}d(d+1)$ Poincaré transformations:

$$\xi_\mu(x) = \xi_\mu - \varepsilon_{\mu\nu}x_\nu \quad . \tag{4.20}$$

We now consider equation (4.6) (or (4.10) with $\delta_{\mu\nu}$ replaced by $g_{\mu\nu}(x)$), which is the defining equation of the conformal transformations in a d-dimensional space. This equation is clearly less restrictive than eq.(4.11). For infinitesimal transformations $x \to x'$ we have instead of (4.11):

$$\delta_{g.c.t.}g_{\mu\nu}(x) = D_\mu\xi_\nu(x) + D_\nu\xi_\mu(x) = (\sigma(x) - 1)\, g_{\mu\nu}(x) \quad , \tag{4.21}$$

with $\sigma(x)$ given in (4.6). This equation immediately implies that

$$D_\mu \xi_\nu + D_\nu \xi_\mu - \frac{2}{d} g_{\mu\nu}(g^{\rho\sigma} D_\rho \xi_\sigma) = 0 \qquad . \tag{4.22}$$

To see, how many independent solutions this equation can have, we substitute (4.22) into (4.14)

$$(D_\mu D_\nu - D_\nu D_\mu)\xi_\rho + D_\rho D_\mu \xi_\nu = \frac{1}{d}\,(\delta_{\rho\mu} D_\nu - \delta_{\rho\nu} D_\mu - \delta_{\mu\nu} D_\rho) D\cdot\xi \tag{4.23}$$

and thus for infinitesimal conformal transformations (4.12) becomes

$$D_\rho D_\mu \xi_\nu = - R_{\mu\nu\rho}{}^{\lambda}\xi_\lambda + \frac{1}{d}(\delta_{\rho\mu} D_\nu - \delta_{\rho\nu} D_\mu - \delta_{\mu\nu} D_\rho) D\cdot\xi \qquad . \tag{4.24}$$

This equation implies that in order to construct $\xi_\mu(x)$ in some neighborhood of a point X as a Taylor series in x-X we need to know the value of ξ_μ, $D_\mu \xi_\nu$ <u>and</u> $D_\mu D.\xi$ at X. Because there are d independent quantities $\xi_\mu(X)$, $\frac{1}{2}\,d(d-1)+1$ independent quantities $D_\mu \xi_\nu(X)$ (recall (4.22)) and d independent quantities $D_\mu D.\xi(X)$, a d-dimensional space can have at most $\frac{1}{2}\,(d+1)(d+2)$ independent conformal transformations.

In d-dimensional Minkowski space (4.22) reduces to

$$\partial_{(\mu}\xi_{\nu)} - \frac{1}{d}\,\delta_{\mu\nu}\partial\cdot\xi = 0 \qquad . \tag{4.25}$$

Substituting the expansion (4.18) we find

$$\begin{aligned} &\Lambda^{(1)}_{(\mu\nu)} - \frac{1}{d}\,\delta_{\mu\nu}\Lambda^{(1)}_{\rho\rho} = 0 \qquad , \\ &\Lambda^{(2)}_{(\mu\nu)\rho} - \frac{1}{d}\,\delta_{\mu\nu}\Lambda^{(2)}_{\sigma\sigma\rho} = 0 \qquad , \\ &\Lambda^{(3)}_{(\mu\nu)\rho\sigma} - \frac{1}{d}\,\delta_{\mu\nu}\Lambda^{(3)}_{\tau\tau\rho\sigma} = 0 \qquad , \text{ etc.} \qquad . \end{aligned} \tag{4.26}$$

The first two equations can be solved in the following way

$$\Lambda^{(1)}_{\mu\nu} = \Lambda^{(1)}_{[\mu\nu]} + \frac{1}{d}\,\delta_{\mu\nu}\Lambda^{(1)}_{\rho\rho} \quad ,$$
$$\Lambda^{(2)}_{\mu\nu\rho} = \frac{1}{2d}\left\{ 2\,\delta_{\mu(\nu}\Lambda^{(2)}_{\sigma\sigma\rho)} - \delta_{\nu\rho}\Lambda^{(2)}_{\sigma\sigma\mu} \right\} \quad , \tag{4.27}$$

whereas the remaining equations give more restrictions on the parameters $\Lambda^{(n)}$ $(n > 3)$ than the independent components contained in each $\Lambda^{(n)}$. Therefore we have

$$\Lambda^{(n)}_{\mu\nu\rho\ldots} = 0 \qquad (n > 3) \tag{4.28}$$

and we are left with d parameters $\xi_\mu = \Lambda^{(o)}_\mu$, $\frac{1}{2}d(d-1)$ parameters $\varepsilon_{\mu\nu} = -\Lambda^{(1)}_{[\mu\nu]}$, one parameter $\varepsilon = +\frac{1}{d}\Lambda^{(1)}_{\rho\rho}$ and d parameters $\varepsilon_\mu = +\frac{1}{2d}\Lambda^{(2)}_{\rho\rho\mu}$. They correspond to translations, Lorentz rotations, dilatations and special conformal transformations respectively. The corresponding most general solution of (4.25) is given by

$$\xi_\mu(x) = \xi_\mu - \varepsilon_{\mu\nu}x_\nu + \varepsilon x_\mu + 2x_\mu \varepsilon . x - \varepsilon_\mu x^2 \quad . \tag{4.29}$$

We denote the generators of these infinitesimal conformal transformations by P_μ (translations), $M_{\mu\nu}$ (Lorentz transformations), D (dilatations) and K_μ (special conformal transformations). They obey the following commutation relations between each other:

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= 4\, M_{[\nu}{}^{[\rho}\delta^{\sigma]}_{\mu]} \quad , \\
[M_{\mu\nu}, P_\rho] &= 2\, P_{[\mu}\delta_{\nu]\rho} \quad , \\
[M_{\mu\nu}, K_\rho] &= 2\, K_{[\mu}\delta_{\nu]\rho} \quad , \\
[P_\mu, K_\nu] &= 2\,(\delta_{\mu\nu}D - M_{\mu\nu}) \quad , \\
[P_\mu, D] &= P_\mu \ \text{ and } [K_\mu , D] = -K_\mu \quad ,
\end{aligned} \tag{4.30}$$

while the remaining commutators are zero. The conformal algebra (4.30) is isomorphic to the SO(d,2) algebra. In terms of the generators L_{ab} of SO(d,2) the correspondence is given by:

$$M_{\mu\nu} = L_{\mu\nu} \quad , \quad D = L_{d+2,d+1} \quad ,$$
$$P_\mu = L_{\mu,d+2} + L_{\mu,d+1} \quad , \quad K_\mu = L_{\mu,d+2} - L_{\mu,d+1} \quad . \tag{4.31}$$

By projecting the finite linear SO(d,2) transformations in (d+2)-dimensional space onto d-dimensional Minkowski space one arrives at the following nonlinear realization of the group SO(d,2) in d spacetime dimensions:

$$x'^\mu = - a^\mu + L^\mu{}_\nu x^\nu - bx^\mu + \frac{1}{\tau(x)} (+ x^\mu + c^\mu x^2) \quad , \tag{4.32}$$

where b is a constant parameter, a^μ and c^μ are constant d-vectors and L is a Lorentz rotation matrix. The function $\tau(x)$ is given by

$$\tau(x) = 1 + 2\, c\cdot x - c^2 x^2 \quad . \tag{4.33}$$

For infinitesimal parameters (4.32) is equivalent to (4.29).

Now we have derived the explicit form (4.29) of the conformal symmetries, our next task is to define what we mean by a conformal transformation on a field. This will be done in the next section.

3. Representations of the conformal algebra

Consider a field $\phi_\alpha(x)$, where α stands for a collection of internal indices (i.e. they do not refer to general coordinate transformations). A conformal transformation

$$x' = gx \tag{4.34}$$

specified by (4.32), with g an element of the conformal group G, can be represented by a transformation matrix $T(g)$ acting on $\phi_\alpha(x)$ in the following way:

$$(T(g)\phi)_\alpha(x) = h_{\alpha\beta}(g,x)\phi_\beta(g^{-1}x) \quad . \tag{4.35}$$

Using the group properties of the representation matrices $T(g)$:

$$(T(g_1)T(g_2)\phi)_\alpha(x) = (T(g_1g_2)\phi)_\alpha(x) \quad \forall\, g_1,g_2\in G \ . \tag{4.36}$$

one can show that the matrices $h_{\alpha\beta}(g,x)$ must satisfy the following relations:

$$h_{\alpha\beta}(g_1g_2,x)= h_{\alpha\gamma}(g_1,x)h_{\gamma\beta}(g_2,g_1^{-1}\, x) \quad \forall\, g_1,g_2\in G \ . \tag{4.37}$$

The stability subgroup $H\subset G$ of $x=0$ is defined by

$$x=0 \rightarrow gx = 0 \quad \forall\, g\in H \quad . \tag{4.38}$$

The relations (4.37) imply that the matrices $h_{\alpha\beta}(g,0)$, which act on internal indices only, are restricted by

$$h_{\alpha\beta}(g_1g_2,0) = h_{\alpha\gamma}(g_1,0)h_{\gamma\beta}(g_2,0) \qquad \forall\, g_1,g_2\in H \ . \tag{4.39}$$

Consequently these matrices constitute a representation of H in the internal index space $\phi_\alpha(0)$. From (4.29) and (4.38) we derive that the algebra of H is isomorphic to the algebra generated by the conformal operators M,D and K. We denote the generators of H by $\Sigma_{\mu\nu}$, Δ and κ_μ. They satisfy the same commutation relations as the conformal generators $M_{\mu\nu}$, D and K_μ given in (4.30). One can show that the x-space is isomorphic to the coset space G/H.

The theory of induced representations gives a prescription how to extend every representation of H on $\phi_\alpha(0)$ to a representation of G on $\phi_\alpha(x)$. For this purpose we define for every spacetime point x a translation $\alpha(x)$ such that

$$\alpha(x)0 \equiv x \quad \forall\, x \qquad . \tag{4.40}$$

In addition we choose the basis $\{\alpha\}$ in index space in such a way that spacetime translations do not act on internal indices, i.e. $h_{\alpha\beta}(\alpha(x),x) = \delta_{\alpha\beta}$. This implies that under translations P the transformed field $(T(g)\phi)_\alpha(x)$ is given by

$$(T(g)\phi)_\alpha(x) \equiv (\exp\, \xi^\lambda P_\lambda \phi)_\alpha(x) = \exp(\xi^\lambda \partial_\lambda)\ \phi_\alpha(x) \qquad . \tag{4.41}$$

Given a representation (Σ,Δ,κ) of H on $\phi(0)$ a representation (P,M,D,K) of G on $\phi(x)$ is given by

$$(T(g)\phi)_\alpha(x) \equiv h_{\alpha\beta}(\alpha(x)^{-1}\ g\alpha(g^{-1}x),0)\ \phi_\beta(g^{-1}x) \qquad . \tag{4.42}$$

For translations P this formula reduces to the form (4.41). We note that $\alpha(x)^{-1}g\alpha(g^{-1}x)$ is an element of H:

$$\alpha(x)^{-1}g\alpha(g^{-1}x)\ 0 \equiv \alpha(x)^{-1}\ g\ g^{-1}x = \alpha(x)^{-1}x \equiv 0 \qquad . \tag{4.43}$$

One can easily verify by using (4.39) that the functions $h_{\alpha\beta}(g,x)$ defined by (4.42) satisfy the relations (4.37), i.e. (4.42) defines a representation of G on $\phi(x)$.

Taking the following representation for $\alpha(x)$

$$\alpha(x) = \exp(-\, x_\lambda P_\lambda) \qquad , \tag{4.44}$$

where P_μ now indicates a representation of the translation generators in x-space $(\xi^\lambda P_\lambda x_\mu \equiv -\,\xi_\mu)$, we can calculate the group elements $\alpha(x)^{-1}g\alpha(g^{-1}x)$ for infinitesimal transformations g. Using the Baker-Campbell-Hausdorff formula

$$\exp(\alpha_i T_i)\cdot\exp(\beta_j T_j) = \exp\left\{ \alpha_i T_i + \beta_j T_j + \frac{1}{2}\alpha_i\beta_j[T_i,T_j] \right.$$
$$+ \frac{1}{12}\alpha_i\alpha_j\beta_k\,[T_i,[T_j,T_k]] + \frac{1}{12}\alpha_i\beta_j\beta_k\,[[T_i,T_j],T_k]$$
$$\left. + \text{repeated commutators of the T's} \right\} \tag{4.45}$$

and the commutation relations (4.30) we find

$$\begin{aligned}
&\exp(x_\lambda P_\lambda)\exp(\tfrac{1}{2}\varepsilon_{\rho\sigma}M_{\rho\sigma})\exp - (x_\rho - \varepsilon_{\rho\sigma}x_\sigma)P_\rho &&= \exp(\tfrac{1}{2}\varepsilon_{\rho\sigma}M_{\rho\sigma}) ,\\
&\exp(x_\lambda P_\lambda)\exp(\varepsilon D)\exp - (x_\rho + \varepsilon x_\rho)P_\rho &&= \exp(\varepsilon D) ,\\
&\exp(x_\lambda P_\lambda)\exp(\varepsilon_\mu K_\mu)\exp - (x_\rho + 2x_\rho\varepsilon.x - \varepsilon_\rho x^2)P_\rho = \\
&\qquad = \exp(\varepsilon_\mu K_\mu + 2\varepsilon.xD + 2\varepsilon_{[\rho}x_{\sigma]}M_{\rho\sigma}) ,
\end{aligned} \tag{4.46}$$

up to terms of order $\varepsilon^2_{\rho\sigma}$, ε^2 and ε^2_μ respectively. Substituting these results into (4.42) we deduce the following transformation rules (we omit internal indices):

$$\begin{aligned}
\delta_P\phi(x) &= \xi^\lambda\partial_\lambda\,\phi(x) ,\\
\delta_M\phi(x) &= \tfrac{1}{2}\varepsilon^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)\,\phi(x) + \delta_\Sigma(\varepsilon_{\mu\nu})\phi(x) ,\\
\delta_D\phi(x) &= \varepsilon(x_\lambda\partial_\lambda)\,\phi(x) + \delta_\Delta(\varepsilon)\,\phi(x) ,\\
\delta_K\phi(x) &= \varepsilon^\mu(-x^2\partial_\mu + 2x_\mu x_\lambda\partial_\lambda)\,\phi(x) + (\delta_\Delta(2\varepsilon_\lambda x_\lambda) + \delta_\Sigma(4\varepsilon_{[\mu}x_{\nu]}) + \delta_\kappa(-2\varepsilon_\mu))\phi(x) .
\end{aligned} \tag{4.47}$$

Here ξ_λ, $\varepsilon_{\mu\nu}$, ε and ε_μ are constant parameters, which characterize an infinitesimal P, M, D and K transformation respectively. In (4.47) we use a notation, where $\delta_P\phi(x) \equiv \xi^\lambda P_\lambda\phi(x)$, $\delta_M\phi(x) \equiv \frac{1}{2}\varepsilon^{\rho\sigma}M_{\rho\sigma}\phi(x)$, $\delta_D\phi(x) \equiv \varepsilon D\phi(x)$, $\delta_K\phi(x) \equiv \varepsilon^\mu K_\mu\phi(x)$ and the same for δ_Σ, δ_Δ and δ_κ. For later convenience we have made the redefinition $\kappa_\mu \to -\frac{1}{2}\kappa_\mu$ in (4.47). We note that the general form of (4.47) is a conformal transformation in x-space accompanied by a rotation in internal H-space.

Consider a matter field $\phi(x)$ transforming according to (4.47). Then the derivative $\partial_\mu\phi(x)$ of $\phi(x)$ transforms under a conformal transformation as:

$$\delta_G(\partial_\mu\phi(x)) = \text{as in (4.47)} + \partial_\mu\xi^\lambda(x)(\partial_\lambda\phi(x)) \quad , \qquad (4.48)$$

where $\xi^\lambda(x)$ is the parameter of a conformal transformation (see (4.29)). The last term on the r.h.s. of (4.48) represents a rotation in the tangent space to x. We denote vectors in this space by greek indices $\mu, \nu, \ldots$. They are called world indices. To construct a derivative, which transforms according to (4.47) we need a field that is able to convert world indices into internal H indices and vice versa. This leads to the introduction of the <u>inverse d-bein field</u> $e^\mu_{\{H\}}(x)$, where {H} denotes a collection of internal H indices, which determine the transformation character of $e^\mu_{\{H\}}$ under H. Under a conformal transformation $e^\mu_{\{H\}}$ transforms as:

$$\delta_G\, e^\mu_{\{H\}}(x) = \text{as in (4.47)} - \partial_\lambda\xi^\mu(x)\, e^\lambda_{\{H\}}(x) \quad , \qquad (4.49)$$

with $\xi^\mu(x)$ given by (4.29). Using this transformation one can verify that the derivative

$$\partial_{\{H\}}\phi(x) \equiv e^\mu_{\{H\}}(x)\partial_\mu\phi(x) \qquad (4.50)$$

transforms according to (4.47). A <u>d-bein field</u> $e_\mu^{\{H\}}(x)$ is defined through the relations

$$\begin{aligned} e^\mu_{\{G\}}(x)e_\mu^{\{H\}}(x) &= \delta^{\{H\}}_{\{G\}} \quad , \\ e_\mu^{\{H\}}(x)e^\nu_{\{H\}}(x) &= \delta^\nu_\mu \quad . \end{aligned} \qquad (4.51)$$

Under a conformal transformation $e_\mu^{\{H\}}$ transforms according to

$$\delta_G e_\mu^{\{H\}}(x) = \text{as in } (4.47) + \partial_\mu \xi^\lambda(x) e_\lambda^{\{H\}}(x) \quad . \tag{4.52}$$

We now fix the d-bein field uniquely by the requirement that under a conformal transformation the derivative $\partial_{\{H\}}\phi(x)$ rotates in internal H-space in the same way as $\partial_\mu\phi(x)$ does in the tangent space to x. This is equivalent to the requirement that the d-bein field is invariant under a conformal transformation:

$$\delta_G e_\mu^{\{H\}}(x) = 0 \quad . \tag{4.53}$$

This condition has a unique solution given by

$$e_\mu^{\{H\}}(x) \to \delta_\mu^a \quad , \tag{4.54}$$

where δ_μ^a is assigned to the following representation of H:

$$\delta_H \delta_\mu^a = \varepsilon^{ab}\delta_\mu^b - \varepsilon\delta_\mu^a \quad . \tag{4.55}$$

Here we use a notation where Latin indices a,b,...(a,b=1...d) denote vectors in internal H-space.

Now we have derived the transformation character of a field under conformal transformations and defined a suitable derivative, we are able to construct invariant actions. We will do this for matter fields, which carry spin 0, $\frac{1}{2}$ and 1. These spins are specified by the transformation of the fields under internal Lorentz transformations. More specifically, we have

$$\begin{array}{llll} \text{spin } 0 & : \text{ scalar field } \phi(x) \;, & \delta_H\phi & = w\varepsilon\phi \quad , \\ \text{spin } \frac{1}{2} & : \text{ spinor field } \psi(x) \;, & \delta_H\psi & = \frac{1}{2}\varepsilon_{ab}\sigma_{ab}\psi + w\varepsilon\psi \quad , \\ \text{spin } 1 & : \text{ vector field } A_a(x) \;, & \delta_H A_a & = \varepsilon_{ab}A_b + w\varepsilon A_a \quad . \end{array} \tag{4.56}$$

It is convenient to take ϕ, ψ and A_a inert under internal κ transformations. The transformations under internal dilatations are characterized by the Weyl weight w. The derivatives $\partial_a\phi$, $\partial_a\psi$ and $\partial_{[a}A_{b]} \equiv F_{ab}$ again transform as induced representations of G. However they form representations (Σ, Δ, κ) of H which differ from the representations formed by the original matter fields. Using the transformation properties of δ^μ_a and ϕ, ψ, A_a one can verify that under internal H transformations the derivatives transform according to

$$
\begin{aligned}
\delta_H(\partial_a\phi) &= \varepsilon_{ab}\partial_b\phi + (w+1)\varepsilon\partial_a\phi - w\varepsilon_a\phi , \\
\delta_H(\partial_a\psi) &= \varepsilon_{ab}\partial_b\psi + \frac{1}{2}\varepsilon_{cd}\sigma_{cd}(\partial_a\psi) + (w+1)\varepsilon\partial_a\psi - w\varepsilon_a\psi - \varepsilon_c\sigma_{ca}\psi , \\
\delta_H(F_{ab}) &= 2\varepsilon_{[ac}F_{cb]} + (w+1)\varepsilon F_{ab} - (w-1)\varepsilon_{[a}A_{b]} .
\end{aligned}
\tag{4.57}
$$

With the aid of (4.57) one can calculate the transformations of the d'Alembertian $\Box\phi \equiv \partial^a\partial_a\phi$, the Dirac operator $\partial\!\!\!/\psi \equiv \gamma^a\partial_a\psi$ on ψ and the squared of F_{ab} under internal H. They are given by

$$
\begin{aligned}
\delta_H(\Box\phi) &= (w+2)\varepsilon\Box\phi + (d-2-2w)\varepsilon_a\partial_a\phi , \\
\delta_H(\partial\!\!\!/\psi) &= \frac{1}{2}\varepsilon_{cd}\sigma_{cd}\partial\!\!\!/\psi + (w+1)\varepsilon\partial\!\!\!/\psi + \left(\frac{1}{2}d - \frac{1}{2} - w\right)\varepsilon\!\!/\psi , \\
\delta_H(F^2_{ab}) &= 2(w+1)\varepsilon F^2_{ab} - 2(w-1)F_{ab}\varepsilon_a A_b .
\end{aligned}
\tag{4.58}
$$

From these transformations one deduces that the actions defined by the following Lagrangians are invariant under conformal transformations:

$$
\begin{aligned}
\mathcal{L}(\phi) &\propto \delta\,\phi\Box\phi \quad \text{with} \quad w(\phi) = \frac{1}{2}(d-2) , \\
\mathcal{L}(\psi) &\propto \delta\,\bar{\psi}\partial\!\!\!/\psi \quad \text{with} \quad w(\psi) = \frac{1}{2}(d-1) , \\
\mathcal{L}(A_a) &\propto \delta\,F^2_{ab} \quad \text{with} \quad w(A_a) = 1 \text{ and } d=4 .
\end{aligned}
\tag{4.59}
$$

Here δ is the determinant of the matrix δ^a_μ:

$$
\delta \equiv \det \delta^a_\mu \quad , \quad \delta_H(\delta) = -d\varepsilon\delta . \tag{4.60}
$$

If we only consider the G invariance of the above actions (and not the internal H invariance) we can take $\delta=1$ (note that this is consistent with $\delta_G(\delta)=0$). From now on we will therefore omit this δ in the action. The last equation in (4.59) tells us that the Maxwell action is invariant in d=4 dimensions only. The Weyl weight of the photon gauge field $A_\mu=\delta^a{}_\mu A_a$ is zero. This is consistent with the Maxwell gauge transformations $\delta_{Ma}A_\mu=\partial_\mu\Lambda$, which commute with internal dilatations:

$$[\delta_\Delta(\epsilon), \delta_{Ma}(\Lambda)] A_\mu = 0 \qquad . \tag{4.61}$$

4. Matter and conformal gravity

In order to construct an action for matter fields, which is invariant under conformal transformations with spacetime-dependent parameters, we first have to extend the group G of rigid conformal transformations to a group of local transformations. Taking spacetime-dependent parameters in the rigid transformation rules (4.32) of the coordinates x^μ, it is no longer meaningful to distinguish translations, Lorentz transformations, dilatations and special conformal transformations. Local translations automatically include all these transformations. Hence the local version of (4.32) is given by general coordinate transformations (G.C.T.)

$$x^\mu \to (x^\mu)' = x^\mu - \xi^\mu(x) \qquad , \tag{4.62}$$

with $\xi^\mu(x)$ an arbitrary function of the coordinates. Since G.C.T. do not explicitly contain Lorentz transformations, dilatations and special conformal transformations it is now not possible to accompany these transformations, when acting on a field, with internal H transformations. Hence we have to separately extend the rigid H rotations to spacetime-dependent H transformations. This means that the intrinsic coupling of coordinate transformations and internal H transformations in the rigid transformation rules (4.47) is no longer present in the spacetime-dependent version of the conformal symmetries. The local version of (4.47) is given by

$$\delta\phi(x) = \xi^\lambda(x)\partial_\lambda\phi(x) + \left(\delta_\Sigma(\epsilon^{ab}(x)) + \delta_\Delta(\epsilon(x)) + \delta_K(\epsilon^a(x))\right)\phi(x) \quad , \tag{4.63}$$

where $\xi^\lambda(x)$ parametrizes a local translation and $\varepsilon^{ab}(x)$, $\varepsilon(x)$, $\varepsilon^a(x)$ are arbitrary functions of the coordinates, which characterize the local internal H transformations. From (4.63) we deduce that the local version of the group G of global transformations is given by:

$$G_{global} \rightarrow (G.C.T. \otimes H)_{local} \quad . \tag{4.64}$$

Consequently we have to use different kinds of indices for the G.C.T. and H symmetries. In analogy to the previous section we denote the components of space and time by greek indices μ, ν,.... These indices are called world indices. On the other hand tensors in internal H-space are denoted by Latin indices a, b,.... Such indices are called local H indices.

A next step in the construction of a locally invariant action is the definition of a suitable covariant derivative for the matter fields. It is straightforward to define a derivative which is covariant with respect to H. For that purpose we introduce gauge fields ω_μ^{ab} (spin connection field), b_μ (dilatation field) and f_μ^a (conformal boost field) for Lorentz rotations, dilatations and conformal boosts respectively and define:

$$D_\mu^H\phi(x) \equiv \partial_\mu\phi(x) - (\delta_\Sigma(\omega_\mu^{ab}) + \delta_\Delta(b_\mu) + \delta_K(f_\mu^a))\phi(x) \quad . \tag{4.65}$$

As in conventional gauge theories the transformations of these gauge fields follow from the structure constants of H. They are given by

$$\begin{aligned} \delta_H\omega_\mu^{ab} &= D_\mu\varepsilon^{ab} \quad , \\ \delta_H b_\mu &= \partial_\mu\Lambda_D \quad , \\ \delta_H f_\mu^a &= D_\mu\Lambda_K^a + \varepsilon^{ab}f_\mu^b + \Lambda_D f_\mu^a \quad , \end{aligned} \tag{4.66}$$

where ε^{ab}, Λ_D and Λ_K^a are spacetime-dependent parameters characterizing the Lorentz rotations, dilatations and conformal boosts respectively and D_μ is covariant with respect to Lorentz transformations only. Under general coordinate transformations the gauge fields transform as a covariant vector X_μ:

$$\delta_{g.c.t.} X_\mu = + \xi^\lambda(x)\partial_\lambda X_\mu + \partial_\mu \xi^\lambda(x) X_\lambda \quad . \tag{4.67}$$

The derivative $D^H_\mu \phi$ is not yet fully covariant in the sense that its variation under a general coordinate transformation contains derivatives of the parameters $\xi_\mu(x)$:

$$\delta_{g.c.t.} D^H_\mu \phi = + \xi^\lambda(x)\partial_\lambda (D^H_\mu \phi) + \partial_\mu \xi^\lambda(x)(D^H_\lambda \phi) \quad . \tag{4.68}$$

To get rid of the second term on the r.h.s. of (4.68) we introduce the following spacetime-dependent version of the inverse d-bein field δ^μ_a:

$$(\delta^\mu_a)_{global} \rightarrow (e^\mu_a(x))_{local} \quad , \tag{4.69}$$

with

$$\begin{aligned} \delta_H e^\mu_a(x) &= \varepsilon_{ab} e^\mu_b(x) + \Lambda_D e^\mu_a(x) \quad , \\ \delta_{g.c.t.} e^\mu_a(x) &= + \xi^\lambda(x)\partial_\lambda e^\mu_a(x) - \partial_\lambda \xi^\mu(x) e^\lambda_a(x) \quad . \end{aligned} \tag{4.70}$$

This inverse d-bein field $e^\mu_a(x)$ enables us to define a fully covariant derivative in the following way:

$$D^c_a \phi(x) \equiv e^\mu_a(x) D^H_\mu \phi(x) \quad . \tag{4.71}$$

Under general coordinate transformations this derivative transforms as a general matter field:

$$\delta_{g.c.t.} (D^c_a \phi) = + \xi^\lambda(x)\partial_\lambda (D^c_a \phi) \quad . \tag{4.72}$$

Its variation under local H transformations follows from the variations of e^μ_a and ϕ under local H. A d-bein field $e^a_\mu(x)$ is defined through the relations:

$$
\begin{aligned}
e^\mu_b(x)e^a_\mu(x) &= \delta^a_b \quad , \\
e^a_\mu(x)e^\nu_a(x) &= \delta^\nu_\mu \quad .
\end{aligned}
\tag{4.73}
$$

As an example we give the covariant derivatives of matter fields, ϕ, ψ and A_a, which carry spin 0, $\frac{1}{2}$ and 1 respectively. Under local H transformations these fields transform according to (4.56) with ε and ε_{ab} replaced by spacetime-dependent parameters $\Lambda_D(x)$ and $\varepsilon_{ab}(x)$. The covariant derivatives $D^c_a\phi$, $D^c_a\psi$ and $D^c_{[a}A_{b]} \equiv F^c_{ab}$ of these fields are given by:

$$
\begin{aligned}
D^c_a\phi &= e^\mu_a(\partial_\mu - wb_\mu)\phi \quad , \\
D^c_a\psi &= e^\mu_a(\partial_\mu - \tfrac{1}{2}\omega^{ab}_\mu\sigma_{ab} - wb_\mu)\psi \quad , \\
F^c_{ab} &= e^\mu_{[a}(\partial_\mu - wb_\mu)A_{b]} - e^\mu_{[a}\omega_{\mu,b]c}A_c \quad .
\end{aligned}
\tag{4.74}
$$

Under local H transformations the derivatives transform according to:

$$
\begin{aligned}
\delta_H D^c_a\phi &= \varepsilon_{ab}D^c_b\phi + (w+1)\varepsilon D^c_a\phi \quad , \\
\delta_H D^c_a\psi &= \varepsilon_{ab}D^c_b\psi + \tfrac{1}{2}\varepsilon_{cd}\sigma_{cd}(D^c_a\psi) + (w+1)\varepsilon D^c_a\psi \quad , \\
\delta_H F^c_{ab} &= 2\,\varepsilon_{[ac}F^c_{cb]} + (w+1)\varepsilon F^c_{ab} \quad .
\end{aligned}
\tag{4.75}
$$

These local H variations are not the spacetime-dependent version of the rigid transformation rules(4.57). The reason of this is that in constructing a gauge theory of $(G.C.T. \otimes H)_{local}$ we have lost any information about the rigid conformal subgroup G_{global} we started with. In fact many other rigid subgroups of $(G.C.T. \otimes H)_{local}$ lead to the same gauge theory. The missing terms in (4.75), which are present in (4.57), are internal κ transformations. These rigid internal κ transformations follow from the third and fourth term on the r.h.s. of the last equation in (4.47). These two terms originate from (4.46) and the following commutator, which is not present in the algebra of H:

$$[K_\mu, P_\nu] = 2(\delta_{\mu\nu}D - M_{\mu\nu}) \quad . \tag{4.76}$$

The same internal κ transformations can be generated on the r.h.s. of (4.75) by the following variations of b_μ and ω_μ^{ab}:

$$\begin{aligned} \delta_\kappa \omega_\mu^{ab} &= 2\,\Lambda_K^{[a} e_\mu^{b]} \quad , \\ \delta_\kappa b_\mu &= \Lambda_K^a e_\mu^a \quad , \end{aligned} \tag{4.77}$$

which are in accordance to the above commutator. Taking into account these additional transformations the r.h.s. of (4.75) is given by the spacetime-dependent version of (4.57). In this way we are able to define a flat spacetime limit, which is invariant under the rigid conformal transformations specified by (4.47). There is a unique ground state field configuration ($e_\mu^a, \omega_\mu^{ab}, b_\mu$, f_μ^a) which is invariant under these rigid transformations. It is given by

$$e_\mu^a(x) = \delta_\mu^a \quad ; \quad \omega_\mu^{ab} = b_\mu = f_\mu^a = 0 \quad . \tag{4.78}$$

Combining equations (4.66), (4.70) and (4.77) we find that the transformation rules of the gauge fields $e_\mu^a, \omega_\mu^{ab}, b_\mu$ and f_μ^a under local H are given by

$$\begin{aligned} \delta_H e_\mu^a &= \varepsilon^{ab} e_\mu^b - \Lambda_D e_\mu^a \quad , \\ \delta_H \omega_\mu^{ab} &= D_\mu \varepsilon^{ab} + 2\,\Lambda_K^{[a} e_\mu^{b]} \quad , \\ \delta_H b_\mu &= \partial_\mu \Lambda_D + \Lambda_K^a e_\mu^a \quad , \\ \delta_H f_\mu^a &= D_\mu \Lambda_K^a + \varepsilon^{ab} f_\mu^b + \Lambda_D f_\mu^a \quad . \end{aligned} \tag{4.79}$$

It is now straightforward to construct locally invariant actions for ϕ, ψ and A_a. With the aid of (4.75) and (4.77) one can verify that the conformal d'Alembertian $\Box^c\phi \equiv D^{ac}D^c_a\phi$ of ϕ is given by

$$\Box^c\phi = e^\mu_a \left\{ (\partial_\mu - (w+1)b_\mu)D^c_a\phi - \omega_\mu^{ab}D^c_b\phi + w f^a_\mu\phi \right\} \quad . \tag{4.80}$$

The transformations of $\Box^c\phi$, the Dirac operator $\not{D}^c\psi \equiv \gamma^a D^c_a\psi$ on ψ and the squared of F^c_{ab} are given by the r.h.s. of (4.58) with ε^{ab}, ε and ε^a replaced by the spacetime-dependent parameters $\varepsilon^{ab}(x)$, $\Lambda_D(x)$ and $\Lambda^a_K(x)$. From this we deduce that the actions defined by the following Lagrangians are invariant under G.C.T. and local H transformations:

$$\begin{aligned}
\mathcal{L}(\phi) &\propto e\,\phi\Box^c\phi && \text{with } w(\phi) = \tfrac{1}{2}(d-2) \quad , \\
\mathcal{L}(\psi) &\propto e\,\bar{\psi}\not{D}^c\psi && \text{with } w(\psi) = \tfrac{1}{2}(d-1) \quad , \\
\mathcal{L}(A_a) &\propto e\,(F^c_{ab})^2 && \text{with } w(A_a) = 1 \text{ and } d=4 \quad .
\end{aligned} \tag{4.81}$$

Here e is the determinant of the d-bein field $e^a_\mu(x)$:

$$e \equiv \det e^a_\mu \quad . \tag{4.82}$$

In the flat spacetime limit (4.78) the actions defined by (4.81) reduce to those given by (4.59). As we expect, the above Weyl weights of ϕ, ψ and A_a are equal to the flat spacetime values given by (4.59). In analogy to (4.59) the Maxwell action in (4.81) is invariant in d=4 dimensions only.

5. The Poincaré gauge

Now we have established the main properties of the conformal symmetries, we are able to show explicitly how the d-bein field can be decomposed into its irreducible components by means of these symmetries. We consider Einstein gravity in d dimensions:

$$\mathcal{L} = - e\, R(\omega(e)) \quad . \tag{4.83}$$

Here $R(\omega(e))$ is the standard curvature scalar. For our notations we refer to section (2.3). The action defined by (4.83) is invariant under general coordinate transformations and internal local Lorentz rotations $\delta e_\mu^a = \varepsilon^{ab} e_\mu^b$. However, invariance of this action under internal (finite) dilatations of the form

$$e_\mu^a \to (e_\mu^a)' = \exp(w\Lambda_D) e_\mu^a \tag{4.84}$$

requires w to be zero. To obtain invariance under such scale transformations we introduce a compensating scalar $\phi(x)$, which transforms under dilatations according to

$$\phi(x) \to \phi'(x) = \exp\left(\frac{1}{2}(d-2)\Lambda_D\right)\phi(x) \quad . \tag{4.85}$$

To construct a covariant derivative of ϕ we introduce a gauge field b_μ:

$$\delta b_\mu = \partial_\mu \Lambda_D \quad . \tag{4.86}$$

We next express the scale invariant d-bein field e_μ^a into a redefined d-bein field $(e_\mu^a)^c$, which does transform under the local dilatations (4.84) (we choose w=-1):

$$e_\mu^a = \phi^{\frac{2}{d-2}} (e_\mu^a)^c \quad , \tag{4.87}$$

$$(e_\mu^a)^c \to (e_\mu^a)^{c'} = \exp(-\Lambda_D)(e_\mu^a)^c \quad . \tag{4.88}$$

The right-hand side of (4.87) is invariant under dilatations, because the scale transformation of ϕ compensates for the scale transformation of $(e_\mu^a)^c$. In terms of $(e_\mu^a)^c$ the Lagrangian (4.83) is given by (we omit the index c):

$$\mathcal{L} = - e\, \phi^2 R(\omega(e)) + 2e\, \frac{(d-1)}{(d-2)} (\partial_\mu \phi)^2 \quad . \tag{4.89}$$

This Lagrangian can be rewritten (up to a total derivative) in a scale covariant way:

$$\mathcal{L} = -e\phi^2 R(\omega(e,b)) - 2e\,\frac{(d-1)}{(d-2)}\,\phi(\partial_a - \tfrac{1}{2}\,db_a - \omega_{b,ba}(e,b))(\partial_a - \tfrac{1}{2}\,(d-2)b_a)\phi\ , \tag{4.90}$$

with

$$\omega_\mu^{ab}(e,b) = \omega_\mu^{ab}(e) + 2b^{[a}e_\mu^{b]} \quad . \tag{4.91}$$

This expression is invariant under local scale transformations specified by (4.85), (4.86) and (4.88).

The new degrees of freedom described by the dilatation gauge field b_μ can be eliminated by introducing a symmetry under shift transformations:

$$b_\mu \rightarrow b_\mu + \Lambda_\mu^K \quad . \tag{4.92}$$

This is equivalent to the statement that (4.90) is independent of b_μ. One can verify that under these shifts $R(\omega(e,b))$ transforms according to

$$R(\omega(e,b)) \rightarrow R(\omega(e,b)) + (d-1)D_a\Lambda_K^a \quad , \tag{4.93}$$

with D_μ a Lorentz-covariant derivative. Hence $R(\omega(e,b))$ can be identified with the trace of the gauge field f_μ^a of κ transformations:

$$R(\omega(e,b)) = (d-1)f_\lambda{}^\lambda \quad . \tag{4.94}$$

After making this identification the Lagrangian (4.90) is given by:

$$\mathcal{L} = -2e\,\frac{(d-1)}{(d-2)}\,\phi\left\{(\partial_a - \tfrac{1}{2}\,db_a - \omega_{b,ba})D_a^c\phi + \tfrac{1}{2}\,(d-2)f_a{}^a\phi\right\}\ , \tag{4.95}$$

with $D^c_a\phi$ defined in (4.74). The expression between curled brackets is exactly the conformal d'Alembertian of ϕ defined in eq.(4.80)(taken with $w = \frac{1}{2}(d-2)$). Therefore we can write (4.95) as:

$$\mathcal{L} = -2e\,\frac{(d-1)}{(d-2)}\,\phi\Box^c\phi \qquad . \tag{4.96}$$

The action defined by this Lagrangian is invariant under general coordinate transformations and local H transformations, i.e. Lorentz rotations, dilatations and special conformal transformations. Hence we have succeeded in reformulating Einstein gravity in d dimensions in a conformally invariant way. The d-bein field $(e^a_\mu)^c$ present in this reformulation describes 5 field degrees of freedom, which form a massive spin-2 representation of the Poincaré group. This field is called the conformal gravity multiplet (or conformal d-bein field).

The gauge equivalence of the Lagrangian (4.96) to the original formulation (4.83) can be made explicit by imposing a consistent set of gauge conditions. To break the invariance under κ transformations one may set the dilatation gauge field b_μ equal to zero, whereas the invariance under dilatations can be broken by adjusting ϕ to a constant. The <u>Poincaré gauge</u> is thus defined by:

$$b_\mu = 0 \quad , \quad \phi = 1 \qquad . \tag{4.97}$$

After imposing these conditions the Lagrangian (4.96) reduces straightforwardly to the form (4.83).

6. <u>Conventional constraints</u>

In the previous section we constructed a conformally invariant theory in which the Lorentz gauge field ω^{ab}_μ could be expressed in terms of derivatives of the d-bein field e^a_μ and the dilatation gauge field b_μ as given in (4.91) (see also (2.20)). This expression is the solution of the following conformal curvature constraint:

$$R^a_{\mu\nu}(P) \equiv D_{[\mu}e^a_{\nu]} = 0 \qquad . \tag{4.98}$$

Here the derivative D_μ is covariant with respect to Lorentz rotations and dilatations. In addition the trace of the conformal boost gauge field f_μ^a could be related to e_μ^a and b_μ as indicated in (4.94). One can verify that this expression for $f_\lambda{}^\lambda$ is the solution of the conformal curvature constraint

$$e_a^\mu\, e_b^\nu\, R_{\mu\nu}^{ab}(M) = 0 \quad , \tag{4.99}$$

with $R_{\mu\nu}^{ab}(M)$ given by

$$R_{\mu\nu}^{ab}(M) \equiv \partial_{[\mu}\omega_{\nu]}^{ab} - \omega_{[\mu}^{ac}\omega_{\nu]}^{cb} - 2f_{[\mu}^{[a}e_{\nu]}^{b]} \quad . \tag{4.100}$$

In this section we discuss these curvature constraints in a more systematic way in the context of conformal gravity viewed as the gauge theory of the conformal algebra SO(d,2). This approach of conformal invariance is slightly different from the one presented in section 4 and enables us to discuss the above constraints in a more transparent way.

The explicit form of the conformal curvatures follows from the structure constants of the SO(d,2) algebra (see eq.(4.30)) and is given by

$$\begin{aligned}
R_{\mu\nu}^{a}(P) &= D_{[\mu}e_{\nu]}^{a} \quad ,\\
R_{\mu\nu}^{ab}(M) &= \partial_{[\mu}\omega_{\nu]}^{ab} - \omega_{[\mu}^{ac}\omega_{\nu]}^{cb} - 2f_{[\mu}^{[a}e_{\nu]}^{b]} \quad ,\\
R_{\mu\nu}(D) &= \partial_{[\mu}b_{\nu]} - f_{[\mu}^{a}e_{\nu]}^{a} \quad ,\\
R_{\mu\nu}^{a}(K) &= D_{[\mu}f_{\nu]}^{a} \quad .
\end{aligned} \tag{4.101}$$

These expressions transform covariantly under the gauge field transformation rules (4.79) and the P gauge transformations of the SO(d,2) algebra:

$$\delta_P e_\mu^a = D_\mu \xi_P^a \quad ,$$
$$\delta_P \omega_\mu^{ab} = 2\, \xi_P^{[a} f_\mu^{b]} \quad , \qquad (4.102)$$
$$\delta_P b_\mu = - \xi_P^a f_\mu^a \quad .$$

They satisfy the following Bianchi identities:

$$\sum_{(abc)} (D_a R_{bc}^d(P) + R_{ab}{}^d{}_c(M) - R_{ab}(D)\delta_c^d) = 0 \quad ,$$
$$\sum_{(abc)} (D_a R_{bc}{}^{de}(M) + 2\, R_{ab}^d(P) f_c^e + 2\, R_{ab}^d(K)\delta_c^e) = 0 \quad ,$$
$$\sum_{(abc)} (\partial_a R_{bc}(D) - R_{ab}^d(P) f_c^d + R_{ab}^c(K)) = 0 \quad , \qquad (4.103)$$
$$\sum_{(abc)} (D_a R_{bc}^d(K) + R_{ab}(D) f_c^d + R_{ab}{}^{de}(M) f_c^e) = 0 \quad ,$$

where $\sum_{(abc)}$ denotes the cyclic sum over Lorentz indices a,b and c. Besides the SO(d,2) gauge symmetries all gauge fields transform as covariant vectors under general coordinate transformations.

In gauging the SO(d,2) transformations one is led to introduce $\underline{\frac{1}{2}(d+1)(d+2)x(d-1)}$ field degrees of freedom, which are described by the gauge fields e_μ^a, ω_μ^{ab}, b_μ and f_μ^a. To convert a gauge theory of SO(d,2) into a gauge theory of spacetime transformations we want to express the P gauge transformations into general coordinate transformations and the remaining SO(d,2) symmetries, after which we end up with a gauge theory of the type considered in section 4. More specifically, we want to make the following truncation:

$$\text{G.C.T.} \oplus SO(d,2) \rightarrow \text{G.C.T.} \oplus H \quad , \qquad (4.104)$$

where H is the subalgebra of M,D and K transformations. A convenient way to achieve such a truncation is by imposing a set of so-called <u>conventional</u> constraints on the SO(d,2) curvatures. To explain the underlying idea we rewrite a general coordinate transformation on e_μ^a in the following way:

$$\begin{aligned}\delta_{g.c.t.}(\xi^\lambda)e_\mu^a &\equiv +\xi^\lambda\partial_\lambda e_\mu^a + (\partial_\mu\xi^\lambda)e_\lambda^a \\ &= \xi^\lambda R_{\lambda\mu}^a(P) + (\delta_P(\xi^\lambda e_\lambda^a) + \delta_M(\xi^\lambda\omega_\lambda^{ab}) + \delta_D(\xi^\lambda b_\lambda))e_\mu^a\end{aligned} \tag{4.105}$$

or

$$\delta_{g.c.t.}^{cov}(\xi^\lambda)e_\mu^a = \xi^\lambda R_{\lambda\mu}^a(P) + \delta_P(\xi^\lambda e_\lambda^a)e_\mu^a \quad , \tag{4.106}$$

where $\delta_{g.c.t.}^{cov}$ denotes a general coordinate transformation, which is covariant with respect to Lorentz rotations and dilatations. We call this a covariant translation. Equation (4.106) shows that after imposing the curvature constraint $R_{\mu\nu}^a(P)=0$ the P gauge transformations on e_μ^a are no longer independent and become identical to covariant translations. Furthermore the gauge field ω_μ^{ab} is no longer independent and can be expressed in terms of e_μ^a and b_μ. The transformation of this field under H remains the same since the constraint $R_{\mu\nu}^a(P)=0$ is invariant under H. This constraint eliminates $\underline{\frac{1}{2}d^2(d-1)}$ field degrees of freedom: ω_μ^{ab} is dependent ($\frac{1}{2}d(d-1)^2$ field d.o.f.) and e_μ^a describes not $d(d-1)$ but $\frac{1}{2}d(d-1)$ field d.o.f. owing to its transformation under Lorentz gauge transformations.

It turns out that in order to obtain the same truncation on the remaining gauge fields one needs to impose an additional curvature constraint:

$$R_{\mu\nu}^{ab}(M)e_b^\nu = 0 \quad . \tag{4.107}$$

This constraint is an extension of (4.99). After imposing (4.107) the gauge field f_μ^a is no longer independent and can be expressed in terms of derivatives of e_μ^a and b_μ according to

$$f_\mu^a = \frac{2}{(d-2)}\left(R'^a_\mu(M) - \frac{1}{2(d-1)}e_\mu^a R'(M)\right) \quad . \tag{4.108}$$

Here $R'^a_\mu = R'^{ab}_{\mu\nu}e_b^\nu$, $R' = R'^a_\mu e_a^\mu$ and the prime indicates that in the corresponding expressions f_μ^a is set equal to zero. The trace of (4.108) yields equation (4.94).

The dependent gauge field f_μ^a transforms under H as before, because the constraint (4.107) is invariant under H. This constraint eliminates $\underline{d^2}$ field degrees of freedom: f_μ^a is dependent (d(d-1) field d.o.f.), b_μ describes no d.o.f. at all owing to its transformation under K ((d-1) field d.o.f.) and e_μ^a describes not $\frac{1}{2}$ d(d-1) but $\frac{1}{2}$ (d+1)(d-2) field d.o.f. because of its transformation under dilatations.

The constraints (4.98) and (4.107) are called conventional because they both enable us to express algebraically some of the gauge fields in terms of the others. Furthermore they both preserve all H transformations but not the P symmetries. After imposing the constraints these P transformations are identical to covariant translations. In this way the gauge theory of SO(d,2) reduces to a gauge theory of spacetime transformations in which the P gauge field e_μ^a can be identified as the d-bein field. The conventional constraints form a consistent truncation of the gauge group G.C.T. ⊗ SO(d,2), which brings us back to the formulation presented in section 4 based on the gauge group G.C.T. ⊗ H.

Another effect of the conventional constraints is that they achieve a maximal irreducibility of the gauge field configuration (e_μ^a, ω_μ^{ab}, b_μ, f_μ^a). In the presence of these constraints the number of field degrees of freedom described by the gauge fields is reduced from $\frac{1}{2}$ (d+1)(d+2)x(d-1) to $\frac{1}{2}$ (d+1)(d+2)x(d-1) - $\frac{1}{2}$ d^2(d-1) - d^2 = $\underline{\frac{1}{2}\ (d+1)(d-2)}$. These field degrees of freedom are entirely described by the d-bein field e_μ^a and form a massive spin-2 representation of the Poincaré group. Indeed massive spin-2 states have $\frac{1}{2}$ (d+1)(d-2) helicity components in d dimensions.

We conclude this section by showing how the above irreducibility is achieved in terms of the conformal curvatures. Substitution of the conventional constraints into the Bianchi identities (4.103) leads to further curvature restrictions:

$$\begin{aligned} &R_{ab}(D) = 0 \quad , \\ &R_{ab}^{c}(K) = (d-3)\, D_e\, R_{ab}{}^{ec}(M) \quad . \end{aligned} \qquad (4.109)$$

These equations show that R(M) is the only independent curvature. The a priori $(\frac{1}{2}\ d(d-1))^2$ components of this curvature are restricted to $\frac{1}{2}$(d+1)(d-2) independent ones by the following algebraic and differential identities:

$$
\begin{aligned}
&R_{\mu\nu,ab}(M) &&= R_{ab,\mu\nu}(M) &&, \quad (\tfrac{1}{8}d(d+1)(d-1)(d-2)) \\
&R_{[\mu\nu,ab]}(M) &&= 0 &&, \quad (\tfrac{1}{24}d(d-1)(d-2)(d-3)) \\
&R_{\mu\nu}{}^{ab}(M)e_b^\nu &&= 0 &&, \quad (\tfrac{1}{2}d(d+1)) \qquad (4.110)\\
&D_bD_{[d}R_{ef]}{}^{ab}(M) &&= 0 &&, \quad (\tfrac{1}{3}(d+1)(d-1)(d-3)) \\
&D^{[a}D_{[d}R_{ef]}{}^{bc]}(M) &&= 0 &&. \quad (\tfrac{1}{12}d(d+1)(d-1)(d-4))
\end{aligned}
$$

We have indicated the number of independent constraints between brackets. These identities follow from combining the Bianchi identities with the conventional constraints.

References

For earlier discussions of conformal invariance and further references, see

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CHAPTER V

CONFORMAL SUPERSYMMETRY

1. Introduction

In this chapter we show how one can decompose the N=1 supergravity multiplet (e_μ^a, ψ_μ, A_a, F) into its irreducible pieces by introducing the superconformal transformations. These transformations are the supersymmetric generalization of the conformal symmetries. In chapter 2 we have shown that the multiplet (e_μ^a, ψ_μ, A_a, F) describes 12+12 field degrees of freedom, which form a massive spin-2 and two massive spin-1/2 representations of the super-Poincaré algebra (cf. section (2.5)). In the superconformally invariant formulation the fields e_μ^a, ψ_μ, and A_a are related to redefined fields e_μ^a , ψ_μ and A_μ , which transform under additional gauge transformations:

$$
\begin{aligned}
\delta\, e_\mu^a &= D_\mu \xi^a + \varepsilon^{ab} e_\mu^b - \Lambda_D e_\mu^a \quad , \\
\delta\, \psi_\mu^{\cdot} &= D_\mu \varepsilon^{\cdot} - \gamma_\mu \eta^{\cdot} - \frac{1}{2} \Lambda_D \psi_\mu^{\cdot} - \frac{3}{4}\, i\Lambda_{U(1)} \psi_\mu^{\cdot} \quad , \\
\delta\, A_\mu &= \partial_\mu \Lambda_{U(1)} \quad .
\end{aligned}
\tag{5.1}
$$

Here the parameters ξ^a, ε^{ab}, Λ_D, $\Lambda_{U(1)}$, $\varepsilon^{\cdot}$ and $\eta^{\cdot}$ characterize covariant translations, Lorentz rotations, dilatations, chiral U(1) transformations, supersymmetry transformations and a new kind of supersymmetry transformations, called S supersymmetry, respectively. Owing to these gauge transformations the multiplet (e_μ^a, $\psi_\mu^{\cdot}$, A_a) describes precisely 8+8 field degrees of freedom, which form a massive spin-2 representation of the super-Poincaré algebra. This multiplet is called the N=1 conformal supergravity or Weyl multiplet and constitutes the backbone of all inequivalent off-shell formulations of N=1 Poincaré supergravity. The remaining 4+4 field degrees of freedom form a chiral multiplet. We shall see that this multiplet is the supersymmetric generalization of the scalar ϕ in the previous chapter (cf.eq.(4.87)) and plays the role of a compensating supermultiplet.

Although we have not restricted ourselves in the previous chapter to any particular spacetime dimension d, we will consider in this thesis (extended) supersymmetry in d=4 dimensions only. Of course one can consider an implementation of the above ideas in the context of supergravity in higher dimensions. The fact that one supersymmetry generator Q in d>4 dimensions reduces to more supersymmetry generators $Q^i (i>1)$ in d=4 dimensions indicates that there is a relationship between N=1 supersymmetry in d>4 dimensions and N>1 supersymmetry in d=4 dimensions. In particular the limit N=8 in d=4 dimensions corresponds to N=1 in d=11 dimensions. Recently the superconformal ideas have been applied in d=10 dimensions. It was shown that reduction of the N=1, d=10 conformal supergravity multiplet to four dimensions leads to the N=4, d=4 Weyl multiplet. For more details about this see the references at the end of this chapter.

The outline of this chapter is as follows. In section 2 we derive the explicit form of the rigid superconformal transformations. In section 3 we consider the problem of constructing field representations of the superconformal algebra. In section 4 we construct the gauge theory of the superconformal algebra and address to the problem of how to find the Weyl multiplets for extended supersymmetry. In particular we give a counting argument, which proves that the gauge fields of the superconformal symmetries do not form for N>1 a complete field representation of conformal supergravity. Hence one needs additional matter fields. In the next chapter we will develop a systematic method to find these matter fields. The coupling of matter supermultiplets to conformal supergravity is discussed in section 5. Having thus established the main properties of the superconformal transformations, we finally show in section 6 how one can decompose the N=1 supergravity multiplet $(e_\mu^a, \psi_\mu, A_a, F)$ into its irreducible submultiplets by introducing these transformations.

2. The superconformal algebra

In the previous chapter we have defined the conformal transformations as the subset of general coordinate transformations that leave the traceless part of the metric tensor $g_{\mu\nu}(x)$ locally invariant (cf.eqs.(4.21) and (4.22)):

$$\delta_{g.c.t.} g_{\mu\nu}(x) = (\sigma(x)-1) g_{\mu\nu}(x) + D_{(\mu}\xi_{\nu)} - \frac{1}{4} g_{\mu\nu} g^{\rho\sigma} D_\rho \xi_\sigma = 0 \quad . \tag{5.2}$$

The explicit solution of the above equation for flat Minkowski space is given in eq.(4.29).

In the same way one can define conformal supersymmetry transformations as the subset of local supersymmetry transformations that leave the gamma-traceless part of the gravitino field $\psi^{\cdot}_{\mu}(x)$ locally invariant, i.e.

$$\delta_Q \psi^{\cdot}_{\mu}(x) = \lambda(x)\gamma_{\mu}\gamma^{\rho}\psi^{\cdot}_{\rho}(x) \quad , \tag{5.3}$$

where $\lambda(x)$ is an arbitrary function of the coordinates x. In a space with metric tensor $g_{\mu\nu}(x)=e^{a}_{\mu}(x)e^{a}_{\nu}(x)$ the gravitino field transforms under local Q transformations according to

$$\delta_Q \psi^{\cdot}_{\mu}(x) = \mathcal{D}_{\mu}\varepsilon^{\cdot}(x) \equiv (\partial_{\mu} - \frac{1}{2}\omega^{ab}_{\mu}\sigma_{ab})\varepsilon^{\cdot}(x) \quad , \tag{5.4}$$

where the spin connection field ω^{ab}_{μ} is related to the Christoffel symbols $\Gamma^{\rho}_{\mu\nu}$ as follows:

$$\Gamma^{\rho}_{\mu\nu} = (D_{\mu}e^{a}_{\nu})e^{\rho}_{a} \quad . \tag{5.5}$$

In this expression D_{μ} denotes a supercovariant derivative. Substitution of (5.4) into (5.3) leads to the following defining equation for the conformal supersymmetries:

$$\mathcal{D}_{\mu}\varepsilon^{\cdot} - \frac{1}{4}\gamma_{\mu}\not{\mathcal{D}}\varepsilon^{\cdot} = 0 \quad . \tag{5.6}$$

In the definition of the covariant derivative $\mathcal{D}_{\mu}$ given in (5.4) we have not considered terms, which are proportional to $\gamma_{\mu}\varepsilon^{\cdot}$ or $\varepsilon^{\cdot}$. We note that in (5.6) a $\gamma_{\mu}\varepsilon^{\cdot}$ term would drop out, while an $\varepsilon^{\cdot}$ term does not lead to inconsistencies.

Of course, the commutator of two supersymmetry transformations should always yield another symmetry of the theory in question. In the general case the spacetime part of this commutator is given by

$$[\bar{\varepsilon}^{\cdot}_{1}(x)Q_{\cdot}, \bar{\varepsilon}^{\cdot}_{2}(x)Q_{\cdot}] \rightarrow \xi^{\lambda}(x)P_{\lambda} \quad , \tag{5.7}$$

with

$$\xi^{\lambda}(x) = \bar{\varepsilon}_2^{\cdot}(x)\gamma^{\mu}\varepsilon_{1\cdot}(x) + c\cdot c\cdot \qquad . \tag{5.8}$$

Here $\varepsilon_1^{\cdot}$ and $\varepsilon_2^{\cdot}$ are the parameters of two local supersymmetry transformations and ξ_λ parametrizes a general coordinate transformation. In a theory with global conformal invariance the parameters ξ_λ should describe a conformal transformation. Hence they must satisfy the differential identity given in (5.2). Substituting (5.8) into this identity leads to a differential constraint on the spinor parameters $\varepsilon^{\cdot}(x)$. One can verify that this constraint is equivalent to equation (5.6). This shows that the definitions of conformal transformations (see (5.2)) and conformal supersymmetries (see (5.6)) are consistent with the supersymmetry algebra.

In flat Minkowski space we can choose Cartesian coordinates with vanishing Christoffel symbols. In that case eq.(5.6) reduces to

$$(\partial_\mu - \frac{1}{4}\gamma_\mu \not{\partial})\varepsilon^{\cdot}(x) = 0 \qquad . \tag{5.9}$$

Expanding $\varepsilon^{\cdot}(x)$ as a power series in x_μ

$$\varepsilon^{\cdot}(x) = \varepsilon^{(o)\cdot} + \varepsilon_\mu^{(1)\cdot}x_\mu + \varepsilon_{\mu\nu}^{(2)\cdot}x_\mu x_\nu + \ldots \qquad , \tag{5.10}$$

with $\varepsilon^{(o)\cdot}$, $\varepsilon_\mu^{(1)\cdot}$, ... constant spinor parameters, we find

$$\varepsilon^{(o)\cdot} \text{ arbitrary}, \quad \varepsilon_\mu^{(1)\cdot} = \frac{1}{4}\gamma_\mu \not{\varepsilon}^{(1)\cdot} \qquad , \tag{5.11}$$

whereas the remaining parameters are zero. Hence a rigid conformal supersymmetry is characterized by two constant spinor parameters $\varepsilon^{\cdot}=\varepsilon^{(o)\cdot}$ and $\eta^{\cdot}=\frac{1}{4}\not{\varepsilon}^{(1)\cdot}$. The parameter $\varepsilon^{\cdot}$ corresponds to ordinary or Q supersymmetry, while the parameter $\eta^{\cdot}$ describes a different kind of supersymmetry, called special or S supersymmetry. We note that for $\varepsilon^{\cdot}$ and $\eta^{\cdot}$ we use the chiral notation given in (2.8) and (2.9) respectively. In terms of $\varepsilon^{\cdot}$ and $\eta^{\cdot}$ the most general solution of (5.9) is given by

$$\varepsilon^{\cdot}(x) = \varepsilon^{\cdot} + \not{x}\eta^{\cdot} \qquad . \tag{5.12}$$

Substitution of (5.12) into (5.8) shows that the spacetime part of a $[\bar{\varepsilon}_2^{\cdot}Q_{.}, \bar{\varepsilon}_1^{\cdot}Q_{.}]$, $[\bar{\varepsilon}^{\cdot}Q_{.}, \bar{\eta}^{\cdot}S_{.}]$ and a $[\bar{\eta}_2^{\cdot}S_{.}, \bar{\eta}_1^{\cdot}S_{.}]$ commutator yields a P, (M+D) and K transformation respectively. For instance the commutator of two S transformations with parameters $\eta_1^{\cdot}$ and $\eta_2^{\cdot}$ yields a special conformal transformation with parameter $\varepsilon^a = -\bar{\eta}_2^{\cdot}\gamma^a\eta_{1.} + c\cdot c\cdot$ (cf.eq.(4.29)):

$$\begin{aligned}\xi_\lambda(x) &= -\bar{\eta}_2^{\cdot}\, \not{x}\gamma_\lambda\not{x}\eta_{1.} + c\cdot c\cdot \\ &= (-\bar{\eta}_2^{\cdot}\gamma_\rho\eta_{1.} + c\cdot c\cdot)(2x_\lambda x_\rho - x^2\delta_{\lambda\rho}) \qquad . \end{aligned} \tag{5.13}$$

To close the algebra of conformal supersymmetries one needs to include additional internal transformations as well. More specifically, to close the [Q,S] commutator one needs an internal chiral U(1) transformation. In extended supersymmetry, where we have N independent Q and N independent S supersymmetries present, this U(1) transformation generalizes to a chiral U(N) transformation (except for the case N=4 where SU(4) is sufficient). These internal symmetries can be derived by using the Jacobi identities, which the superconformal generators must satisfy. In this way one finds the supersymmetric generalization of the conformal algebra $SO(4,2)\cong SU(2,2)$ given in (4.30). This superalgebra is denoted by $SU(2,2|N)$. Below we give the nonvanishing (anti)commutators of $SU(2,2|N)$. The conformal subalgebra is given by (4.30), while the following (anti)commutators enter in its supersymmetric extension:

$$\begin{aligned}
&\{Q^i_\alpha, Q_{\beta j}\} = -2(\not{P}C)_{\alpha\beta}\delta^i_j \quad , \quad \{S^i_\alpha, S_{\beta j}\} = +2(\not{K}C)_{\dot\alpha\beta}\delta^i_j \quad ,\\
&\{S^i_\alpha, Q_{\beta j}\} = (2DC + 2(\sigma_{\mu\nu}C)M_{\mu\nu} + AC)_{\dot\alpha\beta}\delta^i_j + 4\, B^i_j C_{\dot\alpha\beta} \quad ,\\
&[P_\mu, S^i_\alpha] = -(\gamma_\mu Q)^i_\alpha \quad , \quad [K_\mu, Q^i_\alpha] = +(\gamma_\mu S)^i_\alpha \quad ,\\
&[M_{\mu\nu}, Q^i_\alpha] = -(\sigma_{\mu\nu}Q)^i_\alpha \quad , \quad [M_{\mu\nu}, S^i_\alpha] = -(\sigma_{\mu\nu}S)^i_\alpha \quad ,\\
&[D, Q^i_\alpha] = -\tfrac{1}{2}Q^i_\alpha \quad , \quad [D, S^i_\alpha] = +\tfrac{1}{2}S^i_\alpha \quad ,\\
&[A, Q^i_\alpha] = -(\tfrac{4}{N}-1)Q^i_\alpha \quad , \quad [A, S^i_\alpha] = -(\tfrac{4}{N}-1)S^i_\alpha \quad ,\\
&[B^i_j, Q^k_\alpha] = -\delta^k_j Q^i_\alpha + \tfrac{1}{N}\delta^i_j Q^k_\alpha, \quad [B^i_j, S^k_\alpha] = -\delta^k_j S^i_\alpha + \tfrac{1}{N}\delta^i_j S^k_\alpha \quad ,\\
&[B^i_j, B^k_\ell] = -\delta^k_j B^i_\ell + \delta^i_\ell B^k_j \quad .
\end{aligned} \tag{5.14}$$

Here A is the U(1) and B^i_j the SU(N) generator. The generators Q^i_α and S^i_α satisfy the chiral notations given in (2.9) and (2.8) respectively. In (5.14) we have used a shorthand notation to denote chiral projections of Dirac matrices, e.g.

$$(\gamma_\mu)_{\dot\alpha\beta} \equiv \left(\tfrac{1}{2}(1+\gamma_5)\gamma_\mu\right)_{\alpha\beta} \quad , \quad (\gamma_\mu)_{\alpha\dot\beta} \equiv \left(\tfrac{1}{2}(1-\gamma_5)\gamma_\mu\right)_{\alpha\beta} \quad , \tag{5.15}$$

and the same for all other matrices. The superconformal generators satisfy generalized Jacobi identities. As an example we prov·· the (Q,Q,S) Jacobi identities:

$$\begin{aligned}
&[Q_{\alpha i},\{Q_{\beta j},S^k_\gamma\}] + [S^k_\gamma,\{Q_{\alpha i},Q_{\beta j}\}] + [Q_{\beta j},\{S^k_\gamma,Q_{\alpha i}\}] \\
&= 2C_{\dot\gamma\beta}Q_{\alpha i}\delta^k_j + 2(\sigma_{\mu\nu}C)_{\dot\gamma\beta}(\sigma_{\mu\nu}Q)_{\alpha i}\delta^k_j - 4\, C_{\dot\gamma\beta}Q_{\alpha j}\delta^k_i + (\alpha i \leftrightarrow \beta j) \\
&= -\,12\, C_{[\dot\alpha\beta}Q_{\gamma]i}\delta^k_j + (\alpha i \leftrightarrow \beta j) = 0
\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}
&[Q^i_\alpha,\{Q_{\beta j},S^k_\gamma\}] + [S^k_\gamma,\{Q^i_\alpha,Q_{\beta j}\}] + [Q_{\beta j},\{S^k_\gamma,Q^i_\alpha\}] \\
&= 4\, C_{\dot\gamma\beta}Q^k_\alpha\delta^i_j - 2(\gamma_\mu C)_{\alpha\dot\beta}(\gamma_\mu Q)^k_\gamma\delta^i_j = 0 \quad .
\end{aligned} \tag{5.17}$$

In (5.16) and (5.17) we have used the following identities, which follow from the completeness relation given in (A.15):

$$\begin{aligned}
(\sigma_{\mu\nu}C)_{\dot\gamma\beta}(\sigma_{\mu\nu}Q)_{\alpha i} &= C_{\dot\gamma\beta}Q_{\alpha i} - 2\, C_{\dot\alpha\beta}Q_{\gamma i} \quad , \\
(\gamma_\mu C)_{\alpha\dot\beta}(\gamma_\mu Q)^k_\gamma &= 2\, C_{\dot\gamma\beta}Q^k_\alpha \quad .
\end{aligned} \tag{5.18}$$

Now we have defined the superconformal algebra we will consider in the next section what the rigid transformation rules of a multiplet of fields under these symmetries are. In particular we will give an explicit example for N=1.

3. Representations of the superconformal algebra

To construct field representations of the superconformal algebra one can again apply the method of induced representations. In section (4.3) we have shown how the conformal group can be represented on functions $\phi_\alpha(x)$. Here $\{x_\mu\}$ are the coordinates of the coset space $SU(2,2)/H$, where H is the stability subgroup of $x_\mu=0$ (cf.eq.(4.38)). The index α in $\phi_\alpha(x)$ denotes an internal H index. In the x-space the P transformations act as translations $x_\mu \to x_\mu - \xi_\mu$. In the superconformal case these results generalize as follows. From the explicit form of the superconformal algebra (see (5.14)) we deduce that if $M_{\mu\nu},D,K_\mu=0$ and $P_\mu \neq 0$ when acting on a fixed point we also must have $A,B^i_j,S^i_\alpha=0$ and $Q^i_\alpha \neq 0$ for the same point. This leads to the definition of a coset space $SU(2,2|N)/H$, where H represents the algebra generated by the operators $M_{\mu\nu},D,K_\mu,A,B^i_j$ and S^i_α. This coset space can be parametrized by commuting coordinates x_μ and anticommuting variables θ^i_α. In this way we obtain a so-called superspace. The superconformal transformations can be represented as coordinate transformations in this superspace. The H transformations form the stability subgroup of $(x_\mu,\theta^i_\alpha)=0$, whereas the P and Q transformations correspond to translations:

$$
\begin{aligned}
x_\mu &\to x_\mu - \xi_\mu + (\bar{\varepsilon}^i\gamma_\mu\theta_i + \text{c.c.}) \quad ,\\
\theta^i_\alpha &\to \theta^i_\alpha - \varepsilon^i_\alpha \quad .
\end{aligned}
\tag{5.19}
$$

The form (5.19) of the P and Q transformations depend on the parametrization (x_μ,θ^i_α) of the superspace. Another useful parametrization is the following one:

$$
(z_\mu,\theta^i_\alpha) \equiv (x_\mu + \bar{\theta}^i\gamma_\mu\theta_i,\theta^i_\alpha) \quad , \tag{5.20}
$$

where z_μ is a complex spacetime parameter. In terms of these coordinates the P and Q transformations take the following form:

$$
\begin{aligned}
z_\mu &\to z_\mu - \xi_\mu + 2\,\bar{\varepsilon}_i\gamma_\mu\theta^i \quad ,\\
\theta^i_\alpha &\to \theta^i_\alpha - \varepsilon^i_\alpha \quad .
\end{aligned}
\tag{5.21}
$$

A field $\phi_h(x,\theta)$ (the index h denotes an internal H index) defined over the superspace (x_μ,θ^i_α) is called a superfield. Assuming that $\phi_h(x_\mu=0,\ \theta^i_\alpha=0)$ transforms according to an irreducible representation of H one can establish a formula similar to (4.42), which defines a representation of $SU(2,2|N)$ on $\phi_h(x,\theta)$. Such a formula enables one to calculate the global superconformal transformation rules of $\phi_h(x,\theta)$. In analogy to section (4.3) the P and Q transformations do not affect the internal index h. In the parametrization (5.19) the generators of these transformations are given by the following differential operators acting on the superfield:

$$Q^i_\alpha = \frac{\partial}{\partial\bar\theta_{\alpha i}} - (\gamma_\mu\theta)^i_\alpha \frac{\partial}{\partial x_\mu} \quad , \quad P_\mu = \frac{\partial}{\partial x_\mu} \quad , \tag{5.22}$$

$$Q_{\alpha i} = \frac{\partial}{\partial\bar\theta^i_\alpha} - (\gamma_\mu\theta)_{\alpha i} \frac{\partial}{\partial x_\mu} \quad .$$

Using these expressions one can verify the $\{Q,Q\}$ anticommutator:

$$\{Q^i_\alpha, Q_{\beta j}\} = \left\{ \frac{\partial}{\partial\bar\theta_{\alpha i}} - (\gamma_\mu\theta)^i_\alpha \frac{\partial}{\partial x_\mu} \, , \, \frac{\partial}{\partial\bar\theta^j_\beta} - (\gamma_\nu\partial)_{\beta j} \frac{\partial}{\partial x_\nu} \right\}$$

$$= -\,2(\gamma_\mu C)_{\alpha\beta} \frac{\partial}{\partial x_\mu}\, \delta^i_j = -\,2(\not{P} C)_{\alpha\beta}\delta^i_j \quad . \tag{5.23}$$

Of course we can also use the parametrization (5.20). In that case we define superfields $\phi_h(z,\theta)$. On these superfields the P and Q generators are represented in the following way:

$$Q'^i_\alpha = \frac{\partial}{\partial\bar\theta_{\alpha i}} - 2(\gamma_\mu\theta)^i_\alpha \frac{\partial}{\partial z_\mu} \quad , \quad P_\mu = \frac{\partial}{\partial z_\mu} \quad , \tag{5.24}$$

$$Q'_{\alpha i} = \frac{\partial}{\partial\bar\theta^i_\alpha} \quad .$$

The Q' generators (5.24) are related to the Q generators (5.22) by means of a similarity transformation:

$$Q'^{i}_{\alpha} = \exp(+U)Q^{i}_{\alpha} \exp(-U) \quad ,$$
$$Q'_{\alpha i} = \exp(+U)Q_{\alpha i} \exp(-U) \quad , \tag{5.25}$$

with the operator U acting on a superfield given by

$$U \equiv -\bar{\theta}^{i}\gamma_{\mu}\theta_{i} \frac{\partial}{\partial x_{\mu}} \quad . \tag{5.26}$$

This operator relates the parametrizations $(x_{\mu},\theta^{i}_{\alpha})$ and $(z_{\mu},\theta^{i}_{\alpha})$:

$$(\exp U)x_{\mu} = x_{\mu} + \bar{\theta}^{i}\gamma_{\mu}\theta_{i} = z_{\mu} \quad . \tag{5.27}$$

An important difference with the purely conformal case is that a scalar superfield $\phi(x,\theta)$ does not define an irreducible representation of SU(2,2|N). Explicitly, one can impose an invariant constraint

$$D^{i}\phi(x,\theta) \equiv \left(\frac{\partial}{\partial\bar{\theta}_{i}} + (\gamma_{\mu}\theta)^{i} \frac{\partial}{\partial x_{\mu}} \right) \phi(x,\theta) = 0 \tag{5.28}$$

and find nontrivial solutions. To solve this equation it is convenient to use the parametrization $(z_{\mu},\theta^{i}_{\alpha})$. In that case the constraint is given by

$$\left(\exp(+U)D^{i} \exp(-U)\right)\left(\exp(+U)\phi(x,\theta)\right) = 0 \quad \forall x \quad . \tag{5.29}$$

or

$$\frac{\partial}{\partial\bar{\theta}_{i}} \phi(z_{\mu},\theta) = 0 \qquad \forall z \quad . \tag{5.30}$$

This equation is solved by any superfield ϕ^{+}, which does not explicitly depend on θ_{i}. Such a superfield is called a left-handed chiral superfield. The complex conjugate of ϕ^{+} is precisely a right-handed chiral superfield, satisfying

$$D_{i}\phi^{-}(x,\theta) \equiv \left(\frac{\partial}{\partial\bar{\theta}^{i}} + (\gamma_{\mu}\theta)_{i} \frac{\partial}{\partial x_{\mu}} \right) \phi^{-}(x,\theta) = 0 \tag{5.31}$$

or

$$\frac{\partial}{\partial\bar\theta^{i}}\,\phi^{-}(z^{*}_{\mu},\theta) = 0 \qquad . \tag{5.32}$$

Hence a right-handed chiral superfield ϕ^{-} depends only on θ^{i} through the space-time parameter z_{μ}. Up to now there exists no fixed procedure to find the constraints for a general superfield $\phi_{h}(x,\theta)$.

The finite number of components occurring in the Taylor expansion of a superfield $\phi_{h}(x,\theta)$ to the anticommuting variables θ^{i}_{α} are ordinary fields defined over x-space only. They form the field components of a whole multiplet of bose and fermi fields, which under Q supersymmetry are transformed into each other. In this thesis we will not follow the superfield approach but rather work in terms of the field components of the supermultiplet. As an example we give the field components of a N=1 left-handed chiral superfield:

$$\phi^{+}(z_{\mu},\theta^{\cdot}) = A(z) + \bar\theta^{\cdot}\psi^{\cdot}(z) + \frac{1}{2}\bar\theta^{\cdot}\theta^{\cdot}F(z) \qquad . \tag{5.33}$$

This superfield can be re-expressed in terms of functions of x^{μ} by means of

$$\phi^{+}(z_{\mu},\theta^{\cdot}) = \exp(-U)\phi^{+}(x,\theta^{\cdot}) \qquad . \tag{5.34}$$

The Q transformation of $\phi^{+}(z_{\mu},\theta^{\cdot})$ is given by

$$\begin{aligned}\delta_{Q}\phi^{+}(z_{\mu},\theta^{\cdot}) &= (\bar\varepsilon^{\cdot}Q'_{\cdot} + \bar\varepsilon Q'^{\cdot}_{\cdot})\phi^{+}(z_{\mu},\theta^{\cdot})\\ &= \left(\bar\varepsilon^{\cdot}\,\frac{\partial}{\partial\bar\theta^{\cdot}} - 2\,\bar\varepsilon_{\cdot}\gamma_{\mu}\theta^{\cdot}\,\frac{\partial}{\partial z_{\mu}}\right)\phi^{+}(z_{\mu},\theta^{\cdot}) \qquad .\end{aligned} \tag{5.35}$$

Substituting the expansion (5.33) we find for the field components:

$$\begin{aligned}\delta_{Q}\phi^{+}(z_{\mu},\theta^{\cdot}) &\equiv \delta A(z) + \bar\theta^{\cdot}\delta\psi^{\cdot}(z) + \frac{1}{2}\bar\theta^{\cdot}\theta^{\cdot}\delta F(z)\\ &= \bar\varepsilon^{\cdot}\psi^{\cdot}(z) + \bar\theta^{\cdot}(2\not\partial A(z)\varepsilon_{\cdot} + F(z)\varepsilon^{\cdot}) + \frac{1}{2}\bar\theta^{\cdot}\theta^{\cdot}(2\bar\varepsilon_{\cdot}\not\partial\psi^{\cdot}(z)) .\end{aligned} \tag{5.36}$$

Taking $z_\mu = 2x_\mu$ in (5.36) we derive the following transformation rules for the chiral multiplet:

$$\begin{aligned}
\delta A(x) &= \bar{\varepsilon}^{\cdot}\psi^{\cdot}(x) \quad ,\\
\delta\psi^{\cdot}(x) &= \partial\!\!\!/ A(x)\varepsilon_{\cdot} + F(x)\varepsilon^{\cdot} \quad ,\\
\delta F(x) &= \bar{\varepsilon}_{\cdot}\,\partial\!\!\!/\psi^{\cdot}(x) \quad .
\end{aligned} \tag{5.37}$$

We now give the H transformations of this chiral superfield without proof. These transformations can be derived in analogy to the pure conformal case. For more details we refer to the references at the end of this chapter. The M,D and K transformations of the field components of ϕ^+ are the same as in (4.47), i.e. the rigid spacetime transformations are accompanied by internal Σ,Δ or κ transformations. In the same way we find that a global s transformation or, equivalently, a Q transformation with spacetime-dependent parameter $\varepsilon^{\cdot}(x) = x\!\!\!/\eta^{\cdot}$ (cf.eq.(5.12)) in superspace is accompanied by an internal S transformation:

$$\delta_s(\eta^{\cdot})\phi^+(x,\theta^{\cdot}) = (\delta_Q(\varepsilon^{\cdot}(x) = x\!\!\!/\eta^{\cdot}) + \delta_S(\eta^{\cdot}))\phi^+(x,\theta^{\cdot}) \quad . \tag{5.38}$$

We now give the <u>internal</u> H transformations of $\phi^+(z,\theta^{\cdot})$ (in the context of field components we mean by internal that these transformations do not act on x). Note that these are not the complete transformations. The full M,D and K transformations are defined in eq.(4.47) (we only give the internal Σ,Δ,κ part of these transformations), while the complete s transformation is given in eq.(5.38). The internal H transformations read as follows:

$$\begin{aligned}
\delta_\Sigma\phi^+(z,\theta^{\cdot}) &= \left(\frac{1}{2}\varepsilon_{ab}\bar{\theta}^{\cdot}\sigma_{ab}\frac{\partial}{\partial\bar{\theta}^{\cdot}}\right)\phi^+(z,\theta^{\cdot}) \quad ,\\
\delta_\Delta\phi^+(z,\theta^{\cdot}) &= \left(w\Lambda_D + \frac{1}{2}\Lambda_D\bar{\theta}^{\cdot}\frac{\partial}{\partial\bar{\theta}^{\cdot}}\right)\phi^+(z,\theta^{\cdot}) \quad ,\\
\delta_\kappa\phi^+(z,\theta^{\cdot}) &= 0 \quad ,
\end{aligned} \tag{5.39}$$

$$\delta_S\phi^+(z,\theta^\cdot) = \left((\bar\theta^\cdot\theta^\cdot)\ \bar\eta_\cdot \frac{\partial}{\partial\bar\theta^\cdot} + 2w\bar\eta_\cdot\theta^\cdot\right)\phi^+(z,\theta^\cdot) \qquad ,$$

$$\delta_{U(1)}\phi^+(z,\theta^\cdot) = \left(-\frac{1}{2}\,iw\Lambda_{U(1)} + \frac{3}{4}\,i\Lambda_{U(1)}\bar\theta^\cdot \frac{\partial}{\partial\bar\theta^\cdot}\right)\phi^+(z,\theta^\cdot) \ .$$

Here we have given the chiral superfield a Weyl weight w. It follows from the full superconformal algebra, that the chiral weight c of ϕ^+ is related to w as given in the last equation of (5.39).

Substituting the expansion (5.33) into (5.39) and taking $z_\mu=2x_\mu$ we find that the Q and internal H transformations of the field components $(A,\psi^\cdot,F)$ of the N=1 chiral multiplet are given by:

$$\begin{aligned}
\delta A &= \bar\varepsilon^\cdot\psi^\cdot + w\Lambda_D A - \frac{i}{2}\,w\Lambda_{U(1)}A \qquad , \\
\delta\psi^\cdot &= \partial\!\!\!/ A\varepsilon_\cdot + F\varepsilon^\cdot + 2wA\eta_\cdot + (w+1/2)\Lambda_D\psi^\cdot - \frac{1}{2}(w-\frac{3}{2})i\Lambda_{U(1)}\psi^\cdot + \frac{1}{2}\,\varepsilon_{ab}\sigma_{ab}\psi^\cdot \ , \\
\delta F &= \bar\varepsilon_\cdot\partial\!\!\!/\psi^\cdot - 2(w-1)\bar\eta_\cdot\psi^\cdot + (w+1)\Lambda_D F - \frac{1}{2}\,(w-3)i\Lambda_{U(1)}F \ .
\end{aligned} \qquad (5.40)$$

It is instructive to verify that the commutator of two rigid superconformal transformations on $(A,\psi^\cdot,F)$ coincides with the superconformal algebra (5.14). In particular the commutator of two Q or s transformations and of a Q and s transformation are given by the following expressions:

$$\begin{aligned}
[\delta_Q(\varepsilon_1),\delta_Q(\varepsilon_2)] &= \delta_P(\xi^\mu) \qquad , \\
[\delta_Q(\varepsilon),\delta_s(\eta)] &= \delta_D(\varepsilon) + \delta_M(\varepsilon^{ab}) + \delta_{U(1)}(\Lambda_{U(1)}) \qquad , \\
[\delta_s(\eta_1),\delta_s(\eta_2)] &= \delta_K(\varepsilon^a) \qquad ,
\end{aligned} \qquad (5.41)$$

with the parameters on the r.h.s. of (5.41) given by

$$
\begin{aligned}
\xi^{\mu} &= \bar{\varepsilon}_2^{\cdot}\gamma^{\mu}\varepsilon_1 + c.c. \,, \\
\varepsilon &= -\bar{\varepsilon}^{\cdot}\eta + c.c. \,, \\
\varepsilon^{ab} &= +2\bar{\varepsilon}^{\cdot}\sigma^{ab}\eta + c.c. \,, \\
\Lambda_{U(1)} &= -2i\bar{\varepsilon}^{\cdot}\eta + c.c. \,, \\
\varepsilon^{a} &= -\bar{\eta}_2^{\cdot}\gamma^{a}\eta_1 + c.c. \,.
\end{aligned} \tag{5.42}
$$

In order to construct an action for $(A,\psi^{\cdot},F)$, which is invariant under global superconformal transformations, we define a d'Alembertian $\Box A \equiv \partial^a\partial_a A$ of A and a Dirac operator $\not{\partial}\psi^{\cdot} \equiv \gamma^a\partial_a\psi^{\cdot}$ on $\psi^{\cdot}$ as in section (4.3). The vierbein field δ_μ^a occurring in these definitions transforms under the internal H transformations according to (4.55) such that δ_μ^a is invariant under rigid superconformal transformations. Using these definitions one can verify that for w=1 the action corresponding to the following Lagrangian is invariant under all superconformal transformations:

$$
\mathcal{L} = A^*\Box A - \frac{1}{2}\bar{\psi}^{\cdot}\overleftrightarrow{\not{\partial}}\psi + F^*F \,. \tag{5.43}
$$

In section 5 we will construct an extension of this action, which is invariant under local superconformal transformations. This action describes the coupling of a chiral matter multiplet with N=1 conformal supergravity. Before doing this we first discuss in the next section the gauge theory of the superconformal algebra.

4. Gauge theory of the superconformal algebra

In order to gauge the $SU(2,2|N)$ symmetries we introduce the conformal gauge fields $e_\mu^a, \omega_\mu^{ab}, b_\mu, f_\mu^a$ (see chapter 4) and gauge fields $\psi_\mu^i, \phi_\mu^i, V_{\mu j}^i$ (i=1..N) and A_μ for the Q, S, SU(N) and U(1) transformations respectively. These gauge fields describe $\underline{(45+3N^2) + (24N)}$ (bosonic + fermionic) field degrees of freedom. The bosonic degrees of freedom are described by the gauge fields $e_\mu^a(12)$, $\omega_\mu^{ab}(18)$, $b_\mu(3)$, $f_\mu^a(12)$, $V_{\mu}{}^i{}_j(3(N^2-1))$ and $A_\mu(3)$ (except for N=4 where A_μ is absent). The fermionic degrees of freedom are described by the gauge fields $\psi_\mu^i(12N)$ and $\phi_\mu^i(12N)$. We have listed the generators of $SU(2,2|N)$ with their corresponding gauge fields in table 1.

superconformal gauge symmetries		gauge fields	
translations	P	e_μ^a	12
Lorentz rotations	M	ω_μ^{ab}	18
dilatation	D	b_μ	3
conformal boosts	K	f_μ^a	12
supersymmetries	Q	ψ_μ^i	12N
special supersymmetries	S	ϕ_μ^i	12N
chiral SU(N)	B	$V_{\mu}{}^i{}_j$	$3(N^2-1)$
chiral U(1)	A	A_μ	3

table 1. Generators and corresponding gauge fields of the superconformal group. The numbers in the right column denote the field degrees of freedom represented by the gauge fields. The U(1) symmetry is absent in the case of N=4.

The transformations of the superconformal gauge fields follow from the structure constants of the superconformal algebra. The transformations under Q and S supersymmetry, dilatations, conformal boosts, P and chiral U(1) transformations are given below; the assignments with respect to the remaining symmetries follow directly from the index structure of the fields:

$$
\begin{aligned}
\delta e_\mu^a &= -\Lambda_D e_\mu^a + (\bar\varepsilon^i \gamma^a \psi_{\mu i} + c.c.) + D_\mu \xi_P^a &, \\
\delta \omega_\mu^{ab} &= 2\Lambda_K^{[a} e_\mu^{b]} + (-2\,\bar\varepsilon^i \sigma^{ab} \phi_{\mu i} + 2\,\bar\psi_\mu^i \sigma^{ab} \eta_i + c.c.) + 2\xi_P^{[a} f_\mu^{b]} &, \\
\delta b_\mu &= \partial_\mu \Lambda_D + \Lambda_K^a e_\mu^a + (\bar\varepsilon^i \phi_{\mu i} - \bar\psi_\mu^i \eta_i + c.c.) - \xi_P^a f_\mu^a &, \\
\delta f_\mu^a &= D_\mu \Lambda_K^a + \Lambda_D f_\mu^a + (2\,\bar\eta^i \gamma^a \phi_{\mu i} + c.c.) &, \\
\delta \psi_\mu^i &= D_\mu \varepsilon^i - \tfrac{1}{2}\Lambda_D \psi_\mu^i - \gamma_\mu \eta^i - i\,\frac{(4-N)}{4N}\Lambda_{U(1)}\psi_\mu^i + \xi_P^a \gamma_a \phi_\mu^i &, \\
\delta \phi_\mu^i &= D_\mu \eta^i + \tfrac{1}{2}\Lambda_D \phi_\mu^i - \tfrac{1}{2} f_\mu^a \gamma_a \varepsilon^i + \tfrac{1}{2}\Lambda_K^a \gamma_a \psi_\mu^i + i\,\frac{(4-N)}{4N}\Lambda_{U(1)}\phi_\mu^i &, \\
\delta V_{\mu j}^i &= \bar\varepsilon^i \phi_{\mu j} - \bar\psi_\mu^i \eta_j - \frac{1}{N}\delta_j^i\,(\bar\varepsilon^k \phi_{\mu k} - \bar\psi_\mu^k \eta_k) - h.c. &, \\
\delta A_\mu &= (2i\,\bar\varepsilon^i \phi_{\mu i} - 2i\,\bar\psi_\mu^i \eta_i + c.c.) + \partial_\mu \Lambda_{U(1)} &.
\end{aligned}
\qquad (5.44)
$$

In (5.44) the spinors ψ_μ^i and ε^i satisfy the chiral notation given in (2.8), whereas for ϕ_μ^i and η^i we use the chiral notation given in (2.9). The derivatives D_μ are covariant with respect to M,D and (S) U(N) transformations. Notice that the chiral charge of ψ_μ^i and ϕ_μ^i vanishes for N=4. The abbreviation h·c· denotes a hermitian conjugation.

From the transformations (5.44) it is straightforward to define the SU(2,2|N) curvature tensors. They are given by:

$$
\begin{aligned}
R_{\mu\nu}^a(P) &= D_{[\mu} e_{\nu]}^a - \bar\psi_{[\mu}^i \gamma^a \psi_{\nu]i} &, \\
R_{\mu\nu}^{ab}(M) &= \partial_{[\mu} \omega_{\nu]}^{ab} - \omega_{[\mu}^{ac} \omega_{\nu]}^{cb} - 2f_{[\mu}^{[a} e_{\nu]}^{b]} + 2(\bar\psi_{[\mu}^i \sigma^{ab} \phi_{\nu]i} + c.c.) &, \\
R_{\mu\nu}(D) &= \partial_{[\mu} b_{\nu]} - f_{[\mu}^a e_{\nu]}^a - (\bar\psi_{[\mu}^i \phi_{\nu]i} + c.c.) &, \\
R_{\mu\nu}^a(K) &= D_{[\mu} f_{\nu]}^a - 4\,\bar\phi_{[\mu}^i \gamma^a \phi_{\nu]i} &, \\
R_{\mu\nu}^i(Q) &= D_{[\mu} \psi_{\nu]}^i - \gamma_{[\mu} \phi_{\nu]}^i &, \\
R_{\mu\nu}^i(S) &= D_{[\mu} \phi_{\nu]}^i + f_{[\mu}^a \gamma_a \psi_{\nu]}^i &,
\end{aligned}
\qquad (5.45)
$$

$$R_{\mu\nu}(A) = \partial_{[\mu}A_{\nu]} - 2i(\bar{\psi}^i_{[\mu}\,\phi_{\nu]\,i} - c\cdot c\cdot) \quad ,$$

$$R^{i}_{\mu\nu j}(V) = \partial_{[\mu}V^{i}_{\nu]\,j} - V^{i}_{[\mu k}V^{k}_{\nu]\,j} - (\bar{\psi}^{i}_{[\mu}\phi_{\nu]\,j} - \frac{1}{N}\,\delta^{i}_{j}\bar{\psi}^{k}_{[\mu}\phi_{\nu]\,k} - h\cdot c\cdot) \ .$$

Here the derivative D_μ is covariant with respect to Lorentz rotations and dilatations. The expressions (5.45) are the superconformal extension of the pure conformal curvatures given in (4.101). They transform covariantly under the gauge field transformation rules (5.44). In addition they satisfy Bianchi identities which we do not list here.

We now proceed in analogy to the treatment of the gauge theory of the conformal algebra, which was discussed in section (4.6). To achieve a maximal irreducibility of the superconformal gauge field configuration we impose a maximal set of conventional constraints on the superconformal curvatures. Inspection of the explicit form of these curvatures shows that R(P), R(M), R(D) and R(Q) contain terms proportional to a connection field, multiplied by a vierbein field. Hence these connections, ω^{ab}_μ, f^a_μ and ϕ^i_μ, can be expressed in terms of other fields by imposing curvature constraints. For this purpose the following set of constraints suffices:

$$\begin{aligned} R^{a}_{\mu\nu}(P) &= 0 \quad , \\ \hat{R}^{ab}_{\mu\nu}(M)e^{\nu}_{b} &= 0 \quad , \\ \gamma^{\mu}R^{i}_{\mu\nu}(Q) &= 0 \quad . \end{aligned} \tag{5.46}$$

At first sight it seems that one can also restrict R(D), but in the presence of the first constraint of (5.46) one can show that R(D) is no longer independent by virtue of an SU(2,2|N) Bianchi identity. The notation $\hat{R}(M)$ indicates that we have included certain modifications which are required for supercovariance. We explain this below.

As we have mentioned above the constraints (5.46) determine the gauge fields ω^{ab}_μ, f^a_μ and ϕ^i_μ in terms of the other fields. Since (5.46) is invariant under M,D,K and (S)U(N) transformations the corresponding transformations of ω^{ab}_μ, f^a_μ and ϕ^i_μ implied by the SU(2,2|N) algebra remain unaffected. However, the constraints are not invariant under Q and S supersymmetry, and therefore the Q and S transformations of ω_μ^{ab}, f_μ^a and ϕ_μ^i change by extra terms proportional to the covariant curvatures. As an example we show how one can determine the extra term in the Q transformation of ω_μ^{ab} by requiring invariance of the constraint R(P)=0:

$$\begin{aligned}\delta_Q R^a_{\mu\nu}(P) &= \delta_Q^{gauge} R^a_{\mu\nu}(P) - (\delta\omega^{ab}_{[\mu})^{add} e^b_{\nu]} \\ &= (\bar{\varepsilon}^i \gamma^a R_{\mu\nu}(Q)_i + c.c.) - (\delta\omega^{ab}_{[\mu})^{add} e^b_{\nu]} = 0 \qquad . \end{aligned} \tag{5.47}$$

Here δ_Q^{gauge} denotes the variation of R(P) according to the SU(2,2|N) algebra, while the second term denotes the variation of R(P) owing to the extra term $(\delta\omega_\mu^{ab})^{add}$ in the Q transformation of ω_μ^{ab}. From (5.47) we deduce that this term is given by

$$(\delta\omega_\mu^{ab})^{add} = 2\bar{\varepsilon}^i \gamma_\mu R^{ab}(Q)_i + c.c. \qquad . \tag{5.48}$$

Because of these new variations the covariant curvatures for M,K and S require extra terms which do not follow from the SU(2,2|N) structure. The presence of these modifications, which we refrain from giving explicitly, is indicated by using the notation $\hat{R}(M)$, $\hat{R}(S)$ and $\hat{R}(K)$. We should mention that the detailed form of the conventional constraints is not crucial, as long as they fully restrict the gauge fields in question. As it turns out Q supersymmetry is necessarily affected by the presence of the constraints, but for N=1 and 2 it is possible to construct a set of S invariant constraints. One can obtain an S invariant constraint on R(M) by adding a term $R_{\mu a}(A)$ to this equation.

The expressions for ω_μ^{ab}, f_μ^a and ϕ_μ^i, which follow from (5.46), can now be given. The results are expressed by

$$\begin{aligned}\omega_\mu^{ab}(e,\psi,b) &= \omega_\mu^{ab}(e,\psi) + 2b^{[a}e_\mu^{b]} \quad , \\ f_\mu^a &= \tfrac{1}{2}\,(\hat{R}_\mu^{a\prime}(M) - \tfrac{1}{6}\, e_\mu^a\, \hat{R}'(M)) \quad , \\ \phi_\mu^i &= \tfrac{1}{2}\,(\sigma_{ab}\gamma_\mu - \tfrac{1}{3}\,\gamma_\mu\sigma_{ab})R'^{\,i}_{ab}(Q) . \end{aligned} \tag{5.49}$$

The notation $\hat{R}'(M)$ indicates that we have omitted the f_μ^a-dependent term that occurs in $\hat{R}(M)$, while $R'(Q)$ indicates that we have omitted the ϕ_μ^i-dependent term.

When combined with the constraints (5.46) the $SU(2,2|N)$ Bianchi identities lead to further relations among the superconformal curvatures. As it turns out the only independent bosonic curvatures are $\hat{R}(M)$, which satisfies the same identities (4.110) as the pure conformal curvature $R(M)$, and $R(V)$, $R(A)$, which satisfy the following Bianchi identities:

$$\sum_{(abc)} D_a R_{bc}(V)^i_j = \sum_{(abc)} D_a R_{bc}(A) = 0 \qquad . \tag{5.50}$$

Here (abc) indicates a cyclic permutation of the indices a,b and c and D_a is a supercovariant derivative. The only independent fermionic curvature is $R(Q)$ because $\hat{R}(S)$ satisfies

$$\begin{aligned} \hat{R}^{-i}_{ab}(S) &= \not{D}\, R^{-i}_{ab}(Q) \qquad , \\ \hat{R}^{+i}_{ab}(S) &= 0 \qquad . \end{aligned} \tag{5.51}$$

Here we use the notation

$$\hat{R}^{\pm i}_{ab}(S) \equiv \frac{1}{2}\,(\hat{R}^i_{ab}(S) \pm \frac{1}{2}\,\varepsilon_{abcd}\hat{R}^i_{cd}(S)) \qquad . \tag{5.52}$$

The a priori 24N components of $R(Q)$ are restricted to 8N independent ones by the following identities:

$$\begin{aligned} R^{+i}_{ab}(Q) &= 0 \qquad , \\ \sigma_{ab}R^{-i}_{ab}(Q) &= 0 \qquad . \end{aligned} \tag{5.53}$$

Although we have now achieved a maximal irreducibility of the superconformal gauge field configuration the above procedure does not guarantee that for general N the superconformal gauge fields constitute a complete field representation of the superconformal algebra. The following counting argument proves that for $N\neq 1$

this is indeed not the case. The constraints (5.46) eliminate 40+16N (bosonic + fermionic) field degrees of freedom. Hence after imposing these constraints we are left with $(5+3N^2)+8N$ (or $(5+3N^2-3)+8N$ for N=4) field degrees of freedom. The bosonic degrees of freedom are described by the independent gauge fields $e_\mu^a(5)$, $V_{\mu j}^{i}(3(N^2-1))$ and $A_\mu(3)$, while the fermionic degrees of freedom are described by $\psi_\mu^i(8N)$. From the Bianchi identities, which are satisfied by the curvatures corresponding to these gauge fields (see eqs.(4.110), (5.50) and (5.53)) we deduce that the gauge field degrees of freedom form one massive spin-2, 2N massive spin-3/2 and N^2 (or N^2-1 for N=4) massive spin-1 representations of the Poincaré algebra. In table 2 we have indicated these representations together with the spins, which are contained in a massive spin-2 representation of the super-Poincaré algebra for N=1,..,4. From this table we immediately see that only for N=1 the

spin s	N = 1	N = 2	N = 3	N = 4	gauge fields
2	1	1	1	1	1
3/2	2	4	6	8	2N
1	1	6	15	27	N^2 (or N^2-1 for N=4)
1/2		4	20	48	
0		1	14	42	

table 2. Massive spin-2 representations of the super-Poincaré algebra. The numbers in the centre column denote the spins contained in each representation. The numbers in the right column indicate the massive spin states, which are described by the superconformal gauge fields.

superconformal gauge fields form a massive spin-2 representation of the super-Poincaré algebra. For N>1 additional matter fields must be added to the gauge fields in order to obtain such a massive spin-2 representation. Hence only N=1 conformal supergravity is based on the gauge fields presented in this section, whereas for higher N the theory is still incomplete. In the next chapter we shall develop a systematic method to construct complete field representations of the superconformal algebras with $N \leqslant 4$.

5. Matter and conformal supergravity

In this section we consider the coupling of a matter chiral multiplet with N=1 conformal supergravity and construct an action for the chiral multiplet, which is invariant under local superconformal transformations. Before doing this we first discuss the N=1 Weyl multiplet.

In the presence of the conventional constraints (5.46) (for N=1) the only independent superconformal gauge fields are $(e_\mu^a, \psi_\mu^{\cdot}, A_\mu, b_\mu)$. These fields describe 8+8 field degrees of freedom, which form a massive spin-2 representation of the super-Poincaré algebra (cf. table 2 in section 4). Their transformation rules are given in eq.(5.44). Here we give some of these transformations:

$$
\begin{aligned}
\delta e_\mu^a &= \bar{\varepsilon}^{\cdot}\gamma^a\psi_{\mu\cdot} + c.c. \quad ,\\
\delta\psi_\mu^{\cdot} &= D_\mu\varepsilon^{\cdot} - \gamma_\mu\eta^{\cdot} \quad ,\\
\delta A_\mu &= (2i\bar{\varepsilon}^{\cdot}\phi_{\mu\cdot} - 2i\bar{\psi}_\mu^{\cdot}\eta_{\cdot} + c.c.) + \partial_\mu\Lambda_{U(1)} \quad ,\\
\delta b_\mu &= (\bar{\varepsilon}^{\cdot}\phi_{\mu\cdot} - \bar{\psi}_\mu^{\cdot}\eta_{\cdot} + c.c.) + \partial_\mu\Lambda_D \quad .
\end{aligned}
\tag{5.54}
$$

The expressions of the dependent gauge fields ω_μ^{ab}, f_μ^a and $\phi_\mu^{\cdot}$ are given in eq.(5.49). Their Q and S transformations do not coincide with the transformations implied by the SU(2,2|1) algebra given in (5.44). Because the constraints (5.46) are not invariant under Q and S supersymmetry the Q and S transformations of ω_μ^{ab}, f_μ^a and $\phi_\mu^{\cdot}$ change by extra terms proportional to the covariant curvatures. In eq.(5.47) we have shown how one can derive these extra terms in the transformations. Explicitly, these terms are given by

$$
\begin{aligned}
(\delta\omega_\mu^{ab})^{add} &= 2\,\bar{\varepsilon}^{\cdot}\gamma_\mu R^{ab}(Q)_{\cdot} + c.c. \quad ,\\
(\delta f_\mu^a)^{add} &= -\,2\,\bar{\varepsilon}^{\cdot}\sigma^{ab}\not{D}R_{\mu b}(Q)_{\cdot} + 2\,\bar{R}_{\mu b}(Q)^{\cdot}\sigma^{ab}\eta_{\cdot} + c.c. \quad ,\\
(\delta\phi_\mu^{\cdot})^{add} &= \frac{3}{8}\,i\,(\sigma_{ab}\gamma_\mu - \frac{1}{3}\,\gamma_\mu\sigma_{ab})R_{ab}(A)\varepsilon^{\cdot} \quad .
\end{aligned}
\tag{5.55}
$$

The superconformal theory that we have now defined is the gauge theory of the superconformal transformations. The algebra no longer coincides completely with the $SU(2,2|1)$ algebra, because we have imposed a number of constraints. Since the independent gauge fields describe 8+8 field degrees of freedom, which form a massive spin-2 supermultiplet, we expect that in analogy to the pure conformal case (cf. section (4.6)) the commutator of two Q transformations yields a superconformally covariant translation instead of a P gauge transformation, while the remaining commutators still coincide with corresponding ones of the $SU(2,2|1)$ algebra. We give the most relevant commutators below:

$$
\begin{aligned}
[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] &= (\bar{\epsilon}_2^{\cdot}\gamma^a\epsilon_{1.} + c\cdot c\cdot)D_a \quad , \\
[\delta_Q(\epsilon), \delta_S(\eta)] &= \delta_\Delta(\Lambda_D) + \delta_\Sigma(\epsilon^{ab}) + \delta_{U(1)}(\Lambda_{U(1)}) \quad , \\
[\delta_S(\eta_1), \delta_S(\eta_2)] &= \delta_\kappa(\Lambda_K^a) \quad .
\end{aligned} \tag{5.56}
$$

The parameters on the right-hand side of (5.56) are given by

$$
\begin{aligned}
\Lambda_D &= -\bar{\epsilon}^{\cdot}\eta_{.} + c\cdot c\cdot \quad , \\
\epsilon^{ab} &= 2\,\bar{\epsilon}^{\cdot}\sigma^{ab}\eta_{.} + c\cdot c\cdot \quad , \\
\Lambda_{U(1)} &= -2i\bar{\epsilon}^{\cdot}\eta_{.} + c\cdot c\cdot \quad , \\
\Lambda_K^a &= 2\bar{\eta}_2^{\cdot}\gamma^a\eta_{1.} + c\cdot c\cdot \quad .
\end{aligned} \tag{5.57}
$$

The (spacetime-dependent) parameters of the internal Δ, Σ, $U(1)$ transformations have the same value as the (constant) parameters of the rigid D, M, $U(1)$ transformations which result from the commutator of two global supersymmetries on the field components of the chiral multiplet (see eq.(5.42)). The κ parameter differs from the K parameter by a factor -2. This is a consequence of the redefinition $\kappa_\mu \rightarrow -1/2\,\kappa_\mu$ which we made after eq.(4.47).

In coupling the chiral multiplet to N=1 conformal supergravity, the rigid transformation rules (5.40) will get nonlinear modifications. These modifications are basically of the same structure as the nonlinearities present in conventional gravity theories. To find these nonlinear terms we impose the commutator algebra (5.56), (5.57) on the field components A, $\psi^{\bullet}$ and F of the chiral multiplet. This is done by induction. One first calculates the commutator of two supersymmetry transformations on the basis of the linearized results but with space-time-dependent parameters. To realize the commutator algebra (5.56), (5.57) requires the addition of terms of second order in the fields to the transformation rules. One then repeats the calculation on the basis of these new transformation rules. This in turn will introduce terms of higher order in the fields in the transformation rules etc.. In the general case this iterative procedure can be rather complicated if many nonlinear terms are consistent with superconformal invariance, because usually all possible terms do indeed appear. For the N=1 chiral multiplet the resulting nonlinear Q and S transformation rules are rather simple:

$$\begin{aligned} \delta A &= \bar{\varepsilon}^{\bullet}\psi^{\bullet} \quad , \\ \delta\psi^{\bullet} &= \not{D}^{c}A\varepsilon_{\bullet} + F\varepsilon^{\bullet} + 2wA\eta_{\bullet} \quad , \\ \delta F &= \bar{\varepsilon}_{\bullet}\not{D}^{c}\psi^{\bullet} - 2(w-1)\bar{\eta}_{\bullet}\psi^{\bullet} \quad . \end{aligned} \tag{5.58}$$

The supercovariant derivative D_a^c in (5.58) is defined by

$$\begin{aligned} D_a^c\phi(x) \equiv e_a^{\mu}(x)\Big\{ &\partial_{\mu}\phi(x) - (\delta_{\Sigma}(\omega_{\mu}^{ab}) + \delta_{\Delta}(b_{\mu}) + \delta_{K}(f_{\mu}^{a}) \\ &+ \delta_{U(1)}(A_{\mu}) + \delta_{Q}(\psi_{\mu}^{\bullet}) + \delta_{S}(\phi_{\mu}^{\bullet}))\,\phi(x)\Big\} \quad . \end{aligned} \tag{5.59}$$

Inserting the transformations of A and $\psi^{\bullet}$ into (5.59) we find that the supercovariant derivatives D_a^cA and $D_a^c\psi^{\bullet}$ are given by:

$$\begin{aligned} D_a^cA &\equiv e_a^{\mu}(\partial_{\mu}A - wb_{\mu}A + \tfrac{i}{2}wA_{\mu}A - \bar{\psi}_{\mu}^{\bullet}\psi^{\bullet}) \quad , \\ D_a^c\psi^{\bullet} &\equiv e_a^{\mu}(\partial_{\mu}\psi^{\bullet} - \tfrac{1}{2}\omega_{\mu}^{bc}(e,\psi,b)\sigma_{bc}\psi^{\bullet} - (w + \tfrac{1}{2})b_{\mu}\psi^{\bullet} \\ &\quad + \tfrac{1}{2}(w - \tfrac{3}{2})\, iA_{\mu}\psi^{\bullet} - \not{D}^{c}A\psi_{\mu\bullet} - F\psi_{\mu}^{\bullet} - 2wA\phi_{\mu\bullet}) \quad . \end{aligned} \tag{5.60}$$

We conclude this section with the construction of an action for the chiral multiplet (5.58), which is invariant under local superconformal transformations. For this purpose we use the following results, which one can easily verify:

(1) Two chiral multiplets $(A_1, \psi_1^{\cdot}, F_1)$ and $(A_2, \psi_2^{\cdot}, F_2)$ with Weyl weights w_1 and w_2 respectively can be multiplied in the following way to yield again a chiral multiplet with Weyl weight w_1+w_2:

$$(A_1, \psi_1^{\cdot}, F_1) \otimes (A_2, \psi_2^{\cdot}, F_2) \equiv (A_1A_2, A_1\psi_2^{\cdot} + A_2\psi_1^{\cdot}, A_1F_2 + A_2F_1 - \bar{\psi}_1^{\cdot}\psi_2^{\cdot}) \quad . \tag{5.61}$$

(2) A locally superconformal invariant action for a chiral multiplet $(A, \psi^{\cdot}, F)$ with Weyl weight w=3 is defined by the following Lagrangian:

$$\mathcal{L} = \frac{e}{2} (F + \bar{\psi}_{\mu\cdot}\gamma^{\mu}\psi^{\cdot} + 2\bar{\psi}_{\mu\cdot}\sigma_{\mu\nu}\psi_{\nu\cdot}A) + c\cdot c\cdot \quad . \tag{5.62}$$

Using these results it is straightforward to construct an invariant action for a chiral multiplet $(A, \psi^{\cdot}, F)$ with Weyl weight w=1. The F component of such a chiral multiplet is inert under S supersymmetry (see eq.(5.58)). Therefore F^* defines again a chiral multiplet, which is called the kinetic chiral multiplet. Its components are given by

$$(A, \psi^{\cdot}, F)^{kin} = (F^*, \not{D}^c\psi_{\cdot}, \Box^c A^*) \tag{5.63}$$

and transform according to (5.58) with weight w=2. The explicit form of the superconformal d'Alembertian $\Box^c A \equiv D^{ac}D^c_a A$ follows from the superconformal transformations of the derivative $D^c_a A$ (see (5.60) for w=1):

$$\delta(D^c_a A) = \bar{\varepsilon}^{\cdot} D^c_a \psi^{\cdot} - \bar{\eta}^{\cdot}\gamma_a\psi^{\cdot} + 2\Lambda_D(D^c_a A) - \frac{i}{2}\Lambda_{U(1)}(D^c_a A) + \varepsilon_{ab}D^c_b A - \Lambda^a_K A \quad . \tag{5.64}$$

Application of a second derivative now gives according to (5.59):

$$\Box^c A = e^{\mu}_a \left\{ (\partial_\mu - 2b_\mu + \frac{i}{2} A_\mu)D^c_a A - \omega^{ab}_\mu D^c_b A + f^a_\mu A - \bar{\psi}^{\cdot}_\mu D^c_a\psi^{\cdot} + \bar{\phi}^{\cdot}_\mu\gamma_a\psi^{\cdot} \right\} . \tag{5.65}$$

We now multiply a w=1 chiral multiplet with its associated kinetic chiral multiplet according to (5.61), viz.

$$(A,\ \psi^{\cdot},\ F) \otimes (A,\ \psi^{\cdot},\ F)^{\mathrm{kin}} = (AF^{*},\ A\cancel{D}^{c}\psi_{.} + F^{*}\psi^{\cdot},\ A\Box^{c}A^{*} + F^{*}F - \bar{\psi}^{\cdot}\cancel{D}^{c}\psi_{.})\ . \tag{5.66}$$

This defines a chiral multiplet with w=3 from which one can directly construct a superconformally invariant by means of (5.62). The Lagrangian takes the form

$$\mathcal{L} = \frac{e}{2}\,(A^{*}\Box^{c}A - \frac{1}{2}\,\bar{\psi}^{\cdot}\overleftrightarrow{\cancel{D}}^{c}\psi_{.} + F^{*}F)$$
$$+ \frac{e}{2}\,(\bar{\psi}_{\mu.}\gamma^{\mu}\,(A\cancel{D}^{c}\psi_{.} + F^{*}\psi^{\cdot}) + 2\,\bar{\psi}_{\mu.}\sigma_{\mu\nu}\psi_{\nu.}AF^{*}) + \mathrm{c.c.}\ . \tag{5.67}$$

The action defined by this Lagrangian is the superconformal extension of the action given in (4.96), which describes the coupling of a scalar field to conformal gravity.

6. Decomposition of N=1 Poincaré supergravity

We are now able to show explicitly how one can decompose the N=1 Poincaré supergravity multiplet (e_{μ}^{a}, $\psi_{\mu}^{\cdot}$, A_{a}, F) into its irreducible pieces by introducing the superconformal transformations. To this end we apply the same procedure described in section (4.5), where we have decomposed the d-bein field e_{μ}^{a} into its irreducible components by means of the conformal transformations.

We consider the N=1 Poincaré multiplet (e_{μ}^{a}, $\psi_{\mu}^{\cdot}$, A_{a}, F). The Q transformations of the field components are given in eq.(2.56), while a Lagrangian for these fields is given in eq.(2.57). The action defined by this Lagrangian is invariant under general coordinate transformations, local Q supersymmetry transformations and local internal Lorentz rotations, but not under local internal dilatations, special conformal transformations, U(1) chiral rotations and S supersymmetry transformations. To obtain invariance under these transformations we introduce a compensating chiral multiplet (A, $\psi^{\cdot}$, F) (we choose the weight w=1). The field components of this multiplet do transform under the full superconformal group. To construct covariant derivatives we introduce the N=1 Weyl multiplet (e_{μ}^{a}, $\psi_{\mu}^{\cdot}$, A_{μ}, b_{μ}). The field components of this multiplet are the gauge fields of the superconformal transformations. The Q and S transformations of the chiral multiplet are given in eq.(5.58), while the Q and S transformations of the Weyl components are given in eq.(5.54).

We now express the field components of the Poincaré multiplet (e_μ^a, $\psi_\mu^\cdot$, A_a, F) in terms of the field components of the chiral multiplet (A, $\psi^\cdot$, F) and the Weyl multiplet (e_μ^a, $\psi_\mu^\cdot$, A_μ, b_μ):

$$
\begin{aligned}
(e_\mu^a)^P &= (A^*A)^{1/2} e_\mu^a \quad , \\
(\psi_\mu^\cdot)^P &= A^{-1/2} A^* \psi_\mu^\cdot + \frac{1}{2} A^{-1/2} e_\mu^a \gamma_a \psi_\cdot \quad , \\
(A_a)^P &= -\frac{3}{4} i \, (A^*A)^{-3/2} (A^* D_a^c A - \frac{1}{4} \bar{\psi}^\cdot \gamma_a \psi_\cdot) + c \cdot c \cdot \quad , \\
(F)^P &= -\frac{3}{2} A^{-2} F^* \quad .
\end{aligned}
\tag{5.68}
$$

The r.h.s. of (5.68) is uniquely determined by the requirement that it is invariant under local internal dilatations, U(1) chiral rotations and S supersymmetry transformations. The field components of (A, $\psi^\cdot$, F) occur such that they compensate for the transformations of (e_μ^a, $\psi_\mu^\cdot$, A_μ, b_μ). The overall multiplicative factors on the r.h.s. of (5.68) are chosen such that the Q transformations of the expressions on the right-hand side are equal to the Q transformations of the Poincaré fields on the left-hand side modulo a field-dependent Lorentz transformation:

$$
\delta_Q^P(\varepsilon^\cdot) = \delta_Q^C(A^{1/2} A^{*-1} \varepsilon^\cdot) + \delta_\Sigma^C(\varepsilon_{ab} = A^{-1/2} A^{*-1} \bar{\varepsilon}^\cdot \sigma_{ab} \psi^\cdot + c \cdot c \cdot) \quad . \tag{5.69}
$$

By substituting the redefinitions (5.68) into the Lagrangian $\mathcal{L}^P$ for the Poincaré multiplet (see eq.(2.57)) we obtain a Lagrangian in terms of (A, $\psi^\cdot$, F) and (e_μ^a, $\psi_\mu^\cdot$, A_μ, b_μ), which is invariant under the full superconformal group. This Lagrangian is proportional to the Lagrangian $\mathcal{L}^{SC}$ given in eq.(5.67), which describes the coupling of the chiral multiplet with the N=1 Weyl multiplet. Explicitly, we have:

$$
\mathcal{L}^P \to -3 \, \mathcal{L}^{SC} \quad . \tag{5.70}
$$

As an example we show how one obtains the $A^*\Box A$ term in $\mathcal{L}^{SC}$. Using eq.(4.89) we immediately see that the Einstein term in $\mathcal{L}^P$ yields the following contribution:

$$3e(\partial_\mu (A^*A)^{1/2})^2 = \frac{3}{4}\, e(A^*A)^{-1}(A^*\partial_\mu A + A\partial_\mu A^*)^2 \qquad . \tag{5.71}$$

In addition the A_a^2 term in $\mathcal{L}^P$ leads to the following contribution:

$$-\frac{3}{4}\, e(A^*A)^{-1}(A^*\partial_\mu A - A\partial_\mu A^*)^2 \qquad . \tag{5.72}$$

Adding (5.71) and (5.72) we obtain for the $A^*\Box A$ term

$$-\,3eA^*\Box A \qquad , \tag{5.73}$$

which is in accordance to eq.(5.70) (see also eq.(5.67)).

We have now succeeded in rewriting N=1 Poincaré supergravity in a superconformally invariant way. In this procedure we have decomposed the N=1 Poincaré multiplet $(e_\mu^a,\ \psi_\mu^{\cdot},\ A_a,\ F)$ into its irreducible submultiplets $(e_\mu^a,\ \psi_\mu^{\cdot},\ A_\mu,\ b_\mu)$ and $(A,\ \psi^{\cdot},\ F)$, which describe 8+8 and 4+4 field degrees of freedom respectively. The conformal fields form a massive spin-2 representation of the super-Poincaré algebra, while the chiral fields form two massive spin-1/2 representations.

The gauge equivalence of the Lagrangian $-3\mathcal{L}^{SC}$ to the original formulation $\mathcal{L}^P$ can be made explicit by imposing a consistent set of gauge conditions. To break the invariance under K and D transformations we impose the Poincaré gauge conditions $b_\mu=0$ and $|A|=1$ (cf.eq.(4.97)). The invariance under U(1) and S transformations are broken by adjusting A^*/A to a constant and by taking the spinor $\psi^{\cdot}$ equal to zero. The <u>super-Poincaré gauge</u> is thus defined by:

$$b_\mu = 0, \quad A = 1, \quad \psi^{\cdot} = 0 \qquad . \tag{5.74}$$

After imposing these conditions (5.68) becomes

$$\begin{aligned} (e_\mu^a)^P &= e_\mu^a \quad , & (A_\mu)^P &= \frac{3}{4}\, A_\mu \quad , \\ (\psi_\mu^{\cdot})^P &= \psi_\mu^{\cdot} \quad , & (F)^P &= -\,\frac{3}{2}\, F^* \end{aligned} \tag{5.75}$$

and the Lagrangian $-3\mathcal{L}^{SC}$ directly reduces to the form $\mathcal{L}^P$.

In the presence of the super-Poincaré gauge the decomposition rule (5.69) is no longer valid (note that the second term on the r.h.s. of (5.69) vanishes in this gauge). Instead we have a decomposition rule in analogy to eq.(3.38), which reads as follows:

$$\delta_Q^P(\varepsilon^\cdot) = \delta_Q(\varepsilon^\cdot) + \delta_S(\eta^\cdot = -\tfrac{1}{2}F^*\varepsilon_\cdot + \tfrac{i}{4}\not{A}\varepsilon^\cdot) + \delta_K(\Lambda_\mu^K = -\bar{\varepsilon}^\cdot\phi_{\mu\cdot} + \bar{\psi}_\mu^\cdot\eta_\cdot + c\cdot c\cdot). \tag{5.76}$$

The second term on the right-hand side of (5.76) is added to keep $\psi^\cdot$ zero after a Q transformation, while the third term is added to keep b_μ equal to zero after such a transformation. As an example we show how the Q transformation of $(\psi_\mu^\cdot)^P$ is obtained from the superconformal transformations of $\psi_\mu^\cdot$:

$$\begin{aligned}\delta_Q^P(\varepsilon^\cdot)(\psi_\mu^\cdot)^P &= \delta_Q(\varepsilon^\cdot)\,\psi_\mu^\cdot + \delta_S(\eta^\cdot = -\tfrac{1}{2}F^*\varepsilon_\cdot + \tfrac{i}{4}\not{A}\varepsilon^\cdot)\,\psi_\mu^\cdot \\ &= (D_\mu + i(A_\mu)^P)\,\varepsilon^\cdot - \tfrac{1}{3}\gamma_\mu((F)^P\varepsilon_\cdot + i(\not{A})^P\varepsilon^\cdot)\quad .\end{aligned} \tag{5.77}$$

The derivative D_μ is covariant with respect to Lorentz transformations only. The transformation (5.77) is exactly the Q transformation of $(\psi_\mu^\cdot)^P$ given in (2.55) and (2.56). We note that the Q transformations of the Poincaré fields are much more complicated than the Q transformations of the superconformal and chiral fields. This is one of the advantages of working within the superconformal context. To keep all transformation rules as transparent as possible we shall therefore always postpone the super-Poincaré gauge until the very end.

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CHAPTER VI

THE SUPERCURRENT

1. Introduction

In the preceding chapter we have shown that for $N > 1$ the gauge fields of the superconformal symmetries do not constitute a field representation of conformal supergravity. In section (5.4) we have proven that these superconformal gauge fields do not form for $N > 1$ a massive spin-2 representation of the super-Poincaré algebra. Therefore only $N = 1$ conformal supergravity can be described in terms of these gauge fields, whereas for $N > 1$ additional matter fields are required to form complete field representations. In this chapter we develop a systematic method to construct these field representations for $N \leq 4$.

In this method we consider the coupling of matter to the superconformal gauge fields. To describe the lowest-order coupling we expand the superconformal gauge fields about their flat spacetime values. By doing so we can distinguish in the Lagrangian a part $\mathcal{L}^{(o)}_{\text{matter}}$, which is independent of the superconformal gauge fields. This term denotes the matter Lagrangian in flat space. The next-order part $\mathcal{L}^{(1)}$, which is linear in the superconformal gauge fields, defines the currents: each gauge field in $\mathcal{L}^{(1)}$ couples to a current. Using the explicit form of these currents one can construct successive supersymmetry variations of them and find the complete so-called multiplet of currents (or supercurrent) for any given theory. The known currents form a part of the field components of this current multiplet. To obtain invariance of the gauge field × current coupling terms in $\mathcal{L}^{(1)}$ we are forced to associate to each remaining component of the current multiplet a corresponding matter field in the Weyl multiplet. The starting point in the method is the construction of the current multiplet. By using invariance of the gauge field × current coupling terms we can then derive the linearized transformation rules of all field components (i.e. gauge fields and matter fields) of the Weyl multiplet.

In four dimensions matter multiplets do not exist for $N > 4$. The reason for this is that if $N > 4$ the multiplet must contain a spin-3/2 field. This field transforms under Rarita-Schwinger gauge transformations according to $\delta\psi_\mu = \partial_\mu\varepsilon$.

Hence in the coupling to supergravity ψ_μ will transform linearly to the vierbein field and therefore has to be identified with the gravitino field. Consequently the natural limit of the construction procedure is the N = 4 Weyl multiplet. To illustrate the method we reconstruct the N = 1 Weyl multiplet. In the next chapter we present the complete structure of the N = 4 Weyl multiplet.

This chapter is organized as follows. In section 2 we explain the method by means of a simple example. In the presence of conventional constraints the superconformal gauge fields are not mutually independent. In section 3 we show that the independent gauge fields couple to modified currents, which satisfy additional algebraic constraints. We will illustrate this by giving three examples. Finally, in section 4 we apply the method to rederive the N = 1 Weyl multiplet.

2. The Yang-Mills current

To explain the ideas outlined in the introduction we consider N Majorana matter spinor fields $\vec{\psi} = (\psi_1, \ldots, \psi_N)$, which transform according to a representation of some r-dimensional Lie group G:

$$\begin{aligned} \delta\psi_i(x) &= (\Lambda^a T_a)_{ij}\, \psi_j(x) , && i, j = (1, \ldots, N) \\ [T_a, T_b] &= f^c_{ab}\, T_c . && a, b, c = (1, \ldots, r) \end{aligned} \tag{6.1}$$

Here Λ^a are the r spacetime-dependent parameters of the (infinitesimal) G transformation. A Lagrangian for $\vec{\psi}$, which is invariant under global G transformations is given by

$$\mathcal{L}^{(o)}_{\text{matter}} = -\tfrac{1}{2}\, \bar{\vec{\psi}}\cdot \not{\partial}\vec{\psi} \;\equiv -\tfrac{1}{2}\, g_{ij}\bar{\psi}^i \not{\partial}\psi^j , \tag{6.2}$$

where g_{ij} is a symmetric invariant metric in the representation considered, i.e.

$$(\Lambda\cdot T)_{ki}g_{kj} + (\Lambda\cdot T)_{kj}g_{ki} = 0 . \tag{6.3}$$

When the Λ^a become spacetime-dependent, the Lagrangian (6.2) is no longer invariant:

$$\delta\mathcal{L}^{(o)}_{matter} = -\frac{1}{2}\,(\partial_\mu\Lambda^a)\,(\vec{\bar{\psi}}\cdot\gamma_\mu\,T_a\,\vec{\psi}) \qquad . \qquad (6.4)$$

In the Noether procedure this term is cancelled by adding a term $\mathcal{L}^{(1)}$ to $\mathcal{L}^{(o)}_{matter}$, which is linear in the gauge field V_μ of the G transformations. This gauge field takes values in the Lie algebra g of G:

$$V_\mu(x) = V^a_\mu(x)\,T_a \qquad . \qquad (6.5)$$

In addition, its transformation rule under local G transformations contains an inhomogeneous term:

$$\delta V^a_\mu = \partial_\mu\Lambda^a \qquad . \qquad (6.6)$$

This enables us to cancel the variation (6.4) by means of the following term:

$$\mathcal{L}^{(1)} = V^a_\mu\,(\frac{1}{2}\,\vec{\bar{\psi}}\cdot\gamma_\mu\,T_a\,\vec{\psi}) \equiv V^a_\mu J_{\mu a} \qquad . \qquad (6.7)$$

This term defines the Noether currents $J_{\mu a}$ $(a = 1,..,r)$:

$$J_{\mu a} = \frac{1}{2}\,\vec{\bar{\psi}}\cdot\gamma_\mu\,T_a\,\vec{\psi} \qquad . \qquad (6.8)$$

If the matter fields $\vec{\psi}$ satisfy the free field equations $\partial\!\!\!/\psi = 0$, the variation of $\mathcal{L}^{(o)}_{matter}$ vanishes up to a total derivative. This implies that up to the order considered also the variation of the $V^a_\mu J_{\mu a}$ term must vanish up to a total derivative. From this we deduce that the Noether currents $J_{\mu a}$ are conserved:

$$\int d^4x\,(\partial_\mu\Lambda^a)\,J_{\mu a} \equiv 0 \;\rightarrow\; \partial_\mu J_{\mu a} = 0 \qquad . \tag{6.9}$$

Using the explicit expression (6.8) one can verify that this is indeed the case.

Because of the transformation of the Noether currents $J_{\mu a}$ the Lagrangian

$$\mathcal{L} = \mathcal{L}^{(o)}_{\text{matter}} + V^a_\mu J_{\mu a} \tag{6.10}$$

is not invariant under the local G transformations specified by (6.1) and (6.6). We now show how the variation caused by the transformation of $J_{\mu a}$ can be cancelled by adding linear terms to the transformation rule of V^a_μ. To that end we first calculate the G transformation of $J_{\mu a}$ using the explicit expression (6.8) of $J_{\mu a}$ in terms of the matter fields $\vec{\psi}$. This yields the following result:

$$\begin{aligned}\delta J_{\mu a} &= \tfrac{1}{2}\,(\Lambda\cdot T\vec{\bar{\psi}})\cdot\;\gamma_\mu T_a\vec{\psi} + \tfrac{1}{2}\,\vec{\bar{\psi}}\cdot(\gamma_\mu T_a\;\Lambda\cdot T\vec{\psi}) \\ &= \tfrac{1}{2}\,\vec{\bar{\psi}}\cdot(\gamma_\mu[T_a,\Lambda\cdot T]\;\vec{\psi}) = f^c_{ab}\Lambda^b J_{\mu c} \qquad .\end{aligned} \tag{6.11}$$

The variation of the $V^a_\mu J_{\mu a}$ term caused by (6.11) must be cancelled by a corresponding transformation of V^a_μ, because no other terms in the Lagrangian contribute to this order. Hence we have:

$$\int d^4x\,\{V^a_\mu\,(f^c_{ab}\Lambda^b J_{\mu c}) + (\delta V^a_\mu)J_{\mu a}\} = 0 \qquad . \tag{6.12}$$

This enables us to calculate the linear terms in the transformation of V^a_μ:

$$\delta^{\text{lin}}\,V^a_\mu = -f^a_{bc}V^b_\mu\Lambda^c \qquad . \tag{6.13}$$

In the presence of these terms the gauge fields V^a_μ form a representation of the Lie algebra g of G:

$$[\delta(\Lambda_1)\ ,\delta(\Lambda_2)]\ V_\mu^a = \delta(\Lambda_3)\ V_\mu^a \qquad , \qquad (6.14)$$

with

$$\Lambda_3^a = f^a_{bc}\Lambda_2^b\Lambda_1^c \qquad . \qquad (6.15)$$

To derive the linearized transformation rules of the extended Weyl multiplets we can apply the same procedure as outlined above. To that end we consider a matter theory, which is invariant under rigid superconformal transformations. We first construct the energy-momentum tensor $\theta_{\mu\nu}$, which is the Noether current of translations. Since the matter fields satisfy their free field equations, it suffices to start from an on-shell formulation of the matter theory considered. We next apply in analogy to eq. (6.11) successive supersymmetry transformations of $\theta_{\mu\nu}$. This leads to the remaining Noether currents of the superconformal symmetries. For $N > 1$ these Noether currents do not constitute a complete multiplet under supersymmetry. This follows from the counting argument given in section (5.4), which is again applicable since the gauge fields and their corresponding currents contain the same numbers of degrees of freedom. However, in this case it is easy to find the missing components. Namely, one can obtain them by constructing successive variations of the known currents, up to the point where one encounters only derivatives of bilinears that have been found before. This will occur after at most 4N supersymmetry variations, because of the anticommuting nature of the supersymmetry generators. We note that, although the matter fields are on-shell, the multiplet of currents is a genuine off-shell multiplet. We have already mentioned in the introduction that the remaining components of the current multiplet couple to the matter fields of the Weyl multiplet.

We now proceed in analogy to the example above. Once the linear transformation rules of the full current multiplet are known, one can find the linear transformation rules of all field components (i.e. gauge fields and matter fields) of the Weyl multiplet by requiring invariance of the gauge field × current coupling terms (cf. eq. (6.12)). From these linear transformations one can derive the complete nonlinear transformation rules and corresponding superconformal algebra by means of an iterative procedure. We will describe this procedure in section 4.

3. Improved currents

To obtain a maximal irreducibility of the superconformal gauge fields we have imposed the conventional constraints on the superconformal curvatures (see section (5.4)). In the presence of these constraints the superconformal gauge fields are not mutually independent. When we consider the coupling of matter to these gauge fields we can only define the currents, which couple to the independent gauge fields. In this section we show that these currents are modifications of the currents, which couple to the same gauge fields in the absence of conventional constraints. They differ by so-called improvement terms. These terms are generated by the dependence of the remaining gauge fields on the independent ones. The modified currents satisfy additional algebraic constraints, which are generated by the improvement terms. Such modified currents are called improved currents. Successive supersymmetry transformations of these improved currents lead to the current multiplet, which is relevant for the construction of the (extended) Weyl multiplets.

To explain the above ideas we give three examples of matter coupled to gravitational gauge fields both in the absence and in the presence of conventional constraints. The first example describes the coupling of a matter spinor field to Einstein gravity in d dimensions. The second example concerns the coupling of a complex scalar field to conformal gravity in d dimensions. In the last example we consider the coupling of a chiral multiplet to $N = 1$ conformal supergravity.

Example 1

Consider a Majorana matter spinor field ψ coupled to Einstein gravity in d dimensions:

$$\mathcal{L} = -\frac{1}{2}\, e\, e_a{}^{\mu}\, \bar{\psi}^{\cdot}\Gamma_a \left(\partial_\mu - \frac{1}{4}\,\omega_\mu{}^{bc}\Gamma_{bc}\right) \psi_{\cdot} + c.c. \qquad . \qquad (6.16)$$

Here e_μ^a and ω_μ^{ab} are the gravitational gauge fields (see section (2.3)) in the absence of conventional constraints. This means that e_μ^a and ω_μ^{ab} are independent. The gamma matrices Γ_μ in d dimensions are defined by the relation

$$\Gamma_\mu\Gamma_\nu + \Gamma_\nu\Gamma_\mu = 2\,\delta_{\mu\nu} \qquad (\mu,\nu = 1..d) \qquad . \qquad (6.17)$$

For these matrices we use the notation $\Gamma_{\mu\nu} \equiv \Gamma_{[\mu}\Gamma_{\nu]}$ and $\Gamma_{\mu\nu\rho} \equiv \Gamma_{[\mu}\Gamma_\nu\Gamma_{\rho]}$. The expression between brackets in (6.16) is the Lorentz-covariant derivative D_μ.

To describe the lowest-order coupling between $\psi^{\cdot}$ and e_μ^a, ω_μ^{ab} we expand the gravitational gauge fields about their flat spacetime values $e_\mu^a(x) = \delta_\mu^a$ and $\omega_\mu^{ab}(x) = 0$ (see eq. (4.78)). We define

$$e_\mu^a \equiv \delta_\mu^a + h_{a\mu} \qquad , \; e_a{}^{\mu} \equiv \delta_a^\mu - h_{\mu a} \qquad . \qquad (6.18)$$

Substitution of (6.18) into (6.16) leads in lowest order to (up to a total derivative):

$$\mathcal{L} = -\frac{1}{2}\,\bar{\psi}^{\cdot}\not{\partial}\psi_{\cdot} + h_{a\mu}\left(\frac{1}{2}\,\bar{\psi}^{\cdot}\Gamma_\mu\partial_a\psi_{\cdot} + c.c.\right) + \omega_\mu^{ab}\left(\frac{1}{8}\,\bar{\psi}^{\cdot}\Gamma_\mu\Gamma^{ab}\psi_{\cdot} + c.c.\right)$$

$$\equiv \mathcal{L}^{(o)}_{\text{matter}} + h_{a\mu}\theta_{\mu a} + \omega_\mu^{ab}\, S_\mu^{ab} \qquad . \qquad (6.19)$$

In (6.19) we have omitted terms, which vanish upon use of the free Dirac equation $\not{\partial}\psi^{\cdot} = 0$. The currents $\theta_{\mu a}$ and S_μ^{ab} are called the energy-momentum tensor and the internal Lorentz current respectively.

In the previous section we have shown how the inhomogeneous terms in the transformations of the gauge fields lead to constraints on the currents, which couple to these gauge fields. One can find these inhomogeneous terms by substituting (6.18) into (2.18) and using (2.21):

(i) general coordinate transformations : $\delta h_{a\mu} = \partial_\mu \xi_a$,

(ii) internal spin rotations Σ : $\delta\omega_\mu{}^{ab} = \partial_\mu \varepsilon^{ab}$, $\delta h_{a\mu} = \varepsilon_{a\mu}$. (6.20)

Using the conjectured invariance of the full action under general coordinate transformations and internal Lorentz rotations we find in analogy to eq. (6.9), that if the matter field ψ satisfies the free field equation $\partial\!\!\!/\psi' = 0$, the following identity must hold:

$$\int d^4x \, \{(\partial_\mu \xi_a + \varepsilon_{a\mu})\theta_{\mu a} + (\partial_\mu \varepsilon^{ab}) S_\mu^{ab}\} \equiv 0 \quad . \tag{6.21}$$

Since the parameters ε_{ab} and ξ_μ are arbitrary eq. (6.21) leads to the following constraints on $\theta_{\mu a}$ and S_μ^{ab}:

$$\begin{aligned} &\text{(i)} \quad \partial_\mu \theta_{\mu a} = 0 \qquad , \text{(general coordinate transformations)} \\ &\text{(ii)} \quad \partial_\mu S_\mu^{ab} + \theta^{[ab]} = 0 \qquad . \text{(internal Lorentz rotations } \Sigma) \end{aligned} \tag{6.22}$$

We note that to derive (6.22) we did not use the explicit form of $\mathcal{L}^{(o)}_{\text{matter}}$. For another matter theory invariant under the symmetries (6.20) we would obtain the same result. One can verify the constraints (6.22) for our example by substituting the expressions for $\theta_{\mu a}$ and S_μ^{ab} in terms of ψ' given in (6.19).

From (6.22) we deduce that the current S_μ^{ab} is not conserved. However, we can define a modified current S'^{ab}_μ, which is conserved. Namely, by combining the above identities we find

$$\text{(ii)}' \quad \partial_\mu S'^{ab}_\mu \equiv \partial_\mu (S_\mu^{ab} - \theta_\mu{}^{[a}x^{b]}) = 0 . \quad \text{(Lorentz transformations M)} \tag{6.23}$$

The modified current S'^{ab}_μ corresponds to the flat-spacetime Lorentz transformations M, which have been defined in chapter 4 (see eq. (4.47)). The inhomogeneous terms in the M transformations of $h_{a\mu}$ and ω_μ^{ab} (taken with spacetime-dependent parameters) are given by:

(ii)' Lorentz transformations M:
$$\delta h_{a\mu} = \partial_\mu(-\varepsilon_{ab}\, x_b) + \varepsilon_{a\mu} = -(\partial_\mu \varepsilon_{ab})x_b \quad ,$$
$$\delta\omega_\mu^{ab} = \partial_\mu \varepsilon^{ab} \quad . \qquad (6.24)$$

Substituting these transformations into the variation of the gauge field × current coupling terms (cf. eq. (6.21)) we indeed obtain the conserved current S'^{ab}_μ.

We now reconsider the invariance of the action defined by (6.16) in the presence of the conventional constraint

$$R^a_{\mu\nu}(P) \equiv \partial_{[\mu}\, e^a_{\nu]} - \omega_{[\mu}{}^{ab} e^b_{\nu]} = 0 \quad . \qquad (6.25)$$

From this constraint we can solve the Lorentz gauge field ω_μ^{ab} in terms of derivatives of the d- bein field according to (2.20). Substituting the expansion (6.18) into (2.20) leads in lowest order to

$$\omega_\mu^{ab}(h) \equiv \partial_\mu h^{[ab]} + \partial^{[a} h^{b]\mu} + \partial^{[a} h^{\mu b]} \quad . \qquad (6.26)$$

Consequently, instead of two currents $\theta_{\mu a}$ and S_μ^{ab}, we only have one current, which is a modification of $\theta_{\mu a}$:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}^{(o)}_{matter} + h_{a\mu}\theta_{\mu a} + (\partial_\mu h^{[ab]} + \partial^{[a} h^{b]\mu} + \partial^{[a} h^{\mu b]})\, S_\mu^{ab} \\ &= \mathcal{L}^{(o)}_{matter} + h_{a\mu}(\theta_{\mu a} + \partial_\lambda\, S_{a,\mu\lambda} - 2\partial_\lambda S_{[\mu,\lambda]\, a}) + \text{total derivative} \\ &\equiv \mathcal{L}^{(o)}_{matter} + h_{a\mu}\theta^{imp}_{\mu a} + \text{total derivative} \quad . \end{aligned} \qquad (6.27)$$

The modified current $\theta^{imp}_{\mu a}$ differs from $\theta_{\mu a}$ by so-called improvement terms which are generated by the dependence of ω^{ab}_{μ} on e^{a}_{μ}. Improvement terms are conserved because of their form: they can generally be written as the divergence of an anti-symmetric tensor. That this is the case follows from the curl structure present in the explicit solution (6.26) for the dependent gauge field ω^{ab}_{μ}.

Owing to the presence of the improvement terms the modified current $\theta^{imp}_{\mu a}$ satisfies an additional algebraic constraint. For this reason $\theta^{imp}_{\mu a}$ is called an improved current. To derive this constraint we consider the variation of the gauge field × current coupling term $h_{a\mu}\theta^{imp}_{\mu a}$ under one of the symmetries (6.20). In analogy to eq. (6.9) we find:

$$\int d^4x \; (\partial_\mu \xi_a + \epsilon_{a\mu}) \; \theta^{imp}_{\mu a} \equiv 0 \qquad . \qquad (6.28)$$

From this we deduce the following differential and algebraic constraint on $\theta^{imp}_{\mu a}$:

$$\begin{aligned} &\partial_\mu \theta^{imp}_{\mu a} = 0 && \text{, (general coordinate transformations)} \\ &\theta^{imp}_{\mu a} = \theta^{imp}_{a\mu} && \text{. (internal Lorentz rotations } \Sigma) \end{aligned} \qquad (6.29)$$

In deriving (6.29) we have implicitly assumed that the transformations (6.20) have not been affected by the presence of the conventional constraints. One can easily verify that this is the case. The construction of the modified current $\theta^{imp}_{\mu a}$, which satisfies the conditions above is thus intrinsically related to the possibility for choosing a conventional constraint.

In order to derive (6.29) we did not use the explicit form of $\mathcal{L}^{(o)}_{matter}$. Of course we can verify the constraints (6.29) for this example. Using the explicit expressions of $\theta_{\mu a}$ and S^{ab}_{μ} in terms of ψ' given in (6.19) we find for the improved energy-momentum tensor $\theta^{imp}_{\mu a}$ (see eq. (6.27)):

$$\theta^{imp}_{\mu a} = \frac{1}{2}\,\bar{\psi}^{\cdot}\Gamma_{\mu}\partial_{a}\psi_{\cdot} - \frac{1}{8}\,\partial_{\lambda}\,(\bar{\psi}^{\cdot}\Gamma_{a\mu\lambda}\psi_{\cdot}) + c\cdot c\cdot$$
$$= \frac{1}{2}\,\bar{\psi}^{\cdot}\Gamma_{(\mu}\partial_{a)}\psi_{\cdot} + c\cdot c\cdot \qquad (6.30)$$

It is obvious that this expression satisfies the identities given in (6.29).

Example 2

In this example we consider a complex scalar field A (with Weyl weight $w(A) = \frac{1}{2}\,(d-2)$) coupled to conformal gravity in d dimensions:

$$\mathcal{L} = eA^{*}\square^{c}A$$
$$\equiv eA^{*}\,\{(\partial_{a} - \frac{1}{2}\,d\,b_{a} - \omega_{b,ba})\,(\partial_{a} - \frac{1}{2}\,(d-2)\,b_{a})\,A + \frac{1}{2}\,(d-2)\,f^{a}_{a}A\}. \qquad (6.31)$$

The conformal d'Alembertian is defined in chapter 4 (cf. eq. (4.80)). To describe the lowest-order coupling between A and $(e^{a}_{\mu},\ \omega^{ab}_{\mu},\ b_{\mu},\ f^{a}_{\mu})$ we expand the conformal gauge fields about the flat spacetime configuration (4.78). Using definition (6.18) we find in lowest order (up to a total derivative):

$$\mathcal{L} = \mathcal{L}^{(o)}_{matter} + h_{a\mu}\theta_{\mu a} + \omega^{ab}_{\mu\cdot}S^{ab}_{\mu} + b_{\mu}T_{\mu} + f^{a}_{\mu}U^{a}_{\mu} \qquad , \qquad (6.32)$$

with the currents $\theta_{\mu a}$, S^{ab}_{μ}, T_{μ} and U^{a}_{μ} given by

$$\theta_{\mu a} = (\partial_{\mu}A^{*})\,(\partial_{a}A) - A^{*}\partial_{\mu}\partial_{a}A \qquad ,$$
$$S^{ab}_{\mu} = -\delta^{[a}_{\mu}A^{*}(\partial^{b]}A) \qquad ,$$
$$T_{\mu} = -\frac{1}{2}d\,A^{*}\overleftrightarrow{\partial}_{\mu}A - (\partial_{\mu}A^{*})A \qquad , \qquad (6.33)$$
$$U^{a}_{\mu} = \frac{1}{2}\,(d-2)\delta^{a}_{\mu}A^{*}A \qquad .$$

In (6.33) we have omitted terms, which vanish upon use of the free Klein-Gordon equation $\Box A = 0$. The free matter term $\mathcal{L}^{(o)}_{\text{matter}} = A^{*}\Box A$ has been discussed in chapter 4 (cf. eq. (4.59)).

In the absence of conventional constraints the conformal gauge fields are mutually independent. Together they describe $\frac{1}{2}$ (d+1) (d+2) × (d-1) field degrees of freedom (cf. section (4.6)). The inhomogeneous terms in the transformations of these fields are given by eq. (6.20), together with (cf. eq. (4.79)):

$$
\begin{aligned}
&\text{(iii)} \quad \text{internal dilatations } \Delta \quad : \delta b_\mu = \partial_\mu \Lambda_D,\ \delta h_{a\mu} = -\delta_{a\mu}\Lambda_D \quad , \\
&\text{(iv)} \quad \text{internal conformal boosts } \kappa \quad : \delta f^a_\mu = \partial_\mu \Lambda^a_K,\ \delta\omega^{ab}_\mu = 2\,\Lambda^{[a}_K \delta^{b]}_\mu ,\ \delta b_\mu = \Lambda_{K\mu} \ .
\end{aligned}
\tag{6.34}
$$

Substituting these transformations into the variation of the gauge field × current coupling terms (cf. eq. (6.21)), we find the constraints (6.22) and the following ones:

$$
\begin{aligned}
&\text{(iii)} \quad \partial_\mu T_\mu + \theta_{\mu\mu} = 0 \qquad , \text{(internal dilatations } \Delta) \\
&\text{(iv)} \quad \partial_\mu U^a_\mu - 2\,S^{a\mu}_\mu - T^a = 0 \qquad . \text{(internal conformal boosts } \kappa)
\end{aligned}
\tag{6.35}
$$

To derive these constraints we did not use the explicit form of $\mathcal{L}^{(o)}_{\text{matter}}$. One can verify them for this example by substituting the explicit expressions of the conformal currents in terms of A given in (6.33).

In analogy to the previous example we can find the conserved currents S'^{ab}_μ, T'_μ and U'^a_μ corresponding to flat spacetime M, D and K transformations respectively by combining the relations (6.22) and (6.35). For S'^{ab}_μ we find the same result (6.23) as before, while T'_μ and U'^a_μ are given by:

$$
\text{(iii)}' \quad \partial_\mu T'_\mu \equiv \partial_\mu\,(T_\mu + \theta_{\mu a} x^a) = 0 \qquad , \qquad \text{(dilatations D)} \tag{6.36}
$$

$$
\text{(iv)}' \quad \partial_\mu U'^a_\mu \equiv \partial_\mu\,(U^a_\mu - 2\,S^{ab}_\mu x_b - T_\mu x^a + \theta_{\mu b}\,(\tfrac{1}{2}x^2\delta^{ab} - x^a x^b)) = 0 \ .
$$

(conformal boosts K)

One can derive these currents immediately by using the inhomogeneous terms in the M, D and K transformations (taken with spacetime-dependent parameters) of $(h_{a\mu}, \omega_\mu^{ab}, b_\mu, f_\mu^a)$. These transformations have been defined in eq. (4.47). For the M transformations the inhomogeneous terms are given in eq. (6.24), while for D and K we have:

$$\begin{aligned}
&\text{(iii)'} \quad \text{dilatations D} &&: \delta h_{a\mu} = (\partial_\mu \varepsilon) x_a \;,\; \delta b_\mu = \partial_\mu \varepsilon \;, \\
&\text{(iv)'} \quad \text{conformal boosts K} &&: \delta h_{a\mu} = (\partial_\mu \varepsilon_b)\,(2\, x_a x_b - \delta_{ab} x^2),\; \delta b_\mu = 2(\partial_\mu \varepsilon_b)\, x_b, \\
& && \quad \delta\omega_\mu^{ab} = 4\,(\partial_\mu \varepsilon^{[a})\, x^{b]} \;,\; \delta f_\mu^a = -2\, \partial_\mu \varepsilon^a \;.
\end{aligned} \tag{6.37}$$

In the presence of the conventional constraints (4.98) and (4.107) the conformal gauge fields describe $\frac{1}{2}(d+1)(d-2)$ field degrees of freedom, which form a massive spin-2 representation of the Poincaré algebra. In this case the only independent gauge fields are the d-bein field e_μ^a and the dilaton field b_μ. Since the remaining gauge fields are in lowest order given by (these expressions can be found by substituting definition (6.18) into eqs. (4.91) and (4.108)):

$$\begin{aligned}
\omega_\mu^{ab}(h,b) &\equiv \partial_\mu h^{[ab]} + \partial^{[a} h^{b]\mu} + \partial^{[a} h^{\mu b]} + 2\, b^{[a} \delta_\mu^{b]} \;, \\
f_\mu^a(h,b) &\equiv \frac{1}{(d-2)} \{\partial_\mu \partial^a h_{\lambda\lambda} + \Box h_{(\mu a)} - \partial_\lambda \partial_{(\mu} h_{a)\lambda} - \partial_\lambda \partial_{(\mu} h_{\lambda a)} \\
&\quad - \frac{1}{(d-1)}\, \delta_\mu^a\, (\Box h_{\lambda\lambda} - \partial_\rho \partial_\sigma h_{\rho\sigma})\} + \partial_\mu b^a \;.
\end{aligned} \tag{6.38}$$

We only have two currents, which are modifications of $\theta_{\mu a}$ and T_μ:

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}^{(o)}_{\text{matter}} + h_{a\mu}\theta_{\mu a} + \omega_\mu^{ab}(h,b)\, S_\mu^{ab} + b_\mu T_\mu + f_\mu^a(h,b)\, U_\mu^a \\
&= \mathcal{L}^{(o)}_{\text{matter}} + h_{a\mu}\,\{\theta_{\mu a} + \partial_\lambda S_{a,\mu\lambda} - 2\partial_\lambda S_{[\mu,\lambda]\, a}
\end{aligned}$$

$$+ \frac{1}{(d-2)} \left((\delta_{a\mu} \partial_\rho \partial_c - \delta_{\mu(\rho} \partial_{c)} \partial_a) U^c_\rho + (\Box \delta_{\rho(\mu} \delta_{a)c} - \partial_\mu \partial_{(c} \delta_{\rho)a}) U^c_\rho \right.$$

$$\left. - \frac{1}{(d-1)} (\delta_{a\mu} \Box - \partial_a \partial_\mu) U_{\rho\rho} \right) \} + b_a (T_a + 2 S^{a\mu}_\mu - \partial_\mu U^a_\mu)$$

$$+ \text{total derivative}$$

$$\equiv \mathcal{L}^{(o)}_{\text{matter}} + h_{a\mu} \theta^{\text{imp}}_{\mu a} + b_\mu T^{\text{imp}}_\mu + \text{total derivative} \qquad . \qquad (6.39)$$

Because b_μ is the only field in $\mathcal{L}$, which transforms under internal κ transformations, we immediately deduce from the invariance of $\mathcal{L}$ under these transformations that T^{imp}_μ is identically zero, i.e. b_μ decouples from the matter field A. Indeed, using the second relation of (6.35) we find

$$T^{\text{imp}}_a = T_a + 2 S^{a\mu}_\mu - \partial_\mu U^a_\mu \equiv 0 \qquad . \qquad (6.40)$$

To derive the constraints on $\theta^{\text{imp}}_{\mu a}$ we consider the variation of the gauge field × current coupling term $h_{a\mu} \theta^{\text{imp}}_{\mu a}$ under one of the symmetries (6.20) and (6.34). In analogy to eq. (6.28) we find:

$$\int d^4x \, (\partial_\mu \xi_a + \varepsilon_{a\mu} - \delta_{a\mu} \Lambda_D) \, \theta^{\text{imp}}_{\mu a} \equiv 0 \qquad . \qquad (6.41)$$

From this we deduce the following differential and algebraic constraints on $\theta^{\text{imp}}_{\mu a}$:

$$\begin{aligned} &\partial_\mu \theta^{\text{imp}}_{\mu a} = 0 && \text{, (general coordinate transformations)} \\ &\theta^{\text{imp}}_{\mu a} = \theta^{\text{imp}}_{a\mu} && \text{, (internal Lorentz rotations } \Sigma\text{)} \qquad (6.42) \\ &\theta^{\text{imp}}_{\mu\mu} = 0 && \text{. (internal dilatations } \Delta\text{)} \end{aligned}$$

Hence the current $\theta^{\text{imp}}_{\mu a}$ is an improved current, which satisfies additional algebraic constraints compared to $\theta_{\mu a}$. Owing to these constraints the a priori d^2 components of $\theta^{imp}_{\mu a}$ are restricted to $\frac{1}{2}(d+1)(d-2)$ independent ones, which is the same number as the independent components described by the d-bein field e^a_μ.

In order to derive (6.42) we did not use the explicit form of $\mathcal{L}^{(o)}_{matter}$. We can verify the constraints (6.42) for this example. Using the explicit expression of $\theta_{\mu a}$, S^{ab}_μ and U^a_μ in terms of A given in (6.33) we find for the improved energy-momentum tensor $\theta^{imp}_{\mu\nu}$ (see eq. (6.39)):

$$\begin{aligned}\theta^{imp}_{\mu\nu} &= (\partial_\mu A^*)(\partial_\nu A) - A^*\partial_\mu\partial_\nu A \\ &+ 2\,\partial_\lambda\,(\delta_{\nu[\lambda}A^*\partial_{\mu]}A) + \frac{1}{2}\frac{(d-2)}{(d-1)}(\Box\,\delta_{\mu\nu} - \partial_\mu\partial_\nu)(A^*A) \\ &= 2\,(\partial_{(\mu}A^*)(\partial_{\nu)}A) - \delta_{\mu\nu}(\partial_\rho A^*)(\partial_\rho A) + \frac{1}{2}\frac{(d-2)}{(d-1)}(\Box\,\delta_{\mu\nu} - \partial_\mu\partial_\nu)(A^*A).\end{aligned} \tag{6.43}$$

It is easy to see that this expression satisfies the identities given in (6.42).

Example 3

As a last example we consider the coupling of a (on-shell) N = 1 chiral multiplet $(A,\psi^\cdot)$ to the N = 1 superconformal gauge fields in four dimensions. The action which describes this coupling is given in eq. (5.67). Substituting the expansion (6.18) into this expression we find in lowest order:

$$\begin{aligned}\mathcal{L} = \mathcal{L}^{(o)}_{matter} &+ h_{a\mu}\theta_{\mu a} + \omega^{ab}_\mu S^{ab}_\mu + b_\mu T_\mu + f^a_\mu U^a_\mu + A_\mu a_\mu \\ &+ (\bar\psi^\cdot_\mu J^\cdot_\mu + \bar\phi^\cdot_\mu G^\cdot_\mu + c.c.)\,,\end{aligned} \tag{6.44}$$

with the currents $(\theta_{\mu a}, S^{ab}_\mu, T_\mu, U^a_\mu, a_\mu, J^\cdot_\mu, G^\cdot_\mu)$ given by

$$
\begin{aligned}
\theta_{\mu a} &= (\partial_\mu A^*)(\partial_a A) - A^*\partial_\mu\partial_a A &&+ \tfrac{1}{2}(\bar{\psi}^{\cdot}\gamma_\mu\partial_a\psi_{.} + c.c.) \quad, \\
S_\mu^{ab} &= -\delta_\mu^{[a}A^*(\partial^{b]}A) &&+ \tfrac{1}{4}(\bar{\psi}^{\cdot}\gamma_\mu\sigma^{ab}\psi_{.} + c.c.) \quad, \\
T_\mu &= (\partial_\mu A^*)A - 2A^*(\partial_\mu A) && \quad, \\
U_\mu^a &= \delta_\mu^a A^* A && \quad, \\
a_\mu &= \tfrac{1}{2} i A^*\overleftrightarrow{\partial}_\mu A - \tfrac{1}{4} i \bar{\psi}^{\cdot}\gamma_\mu\psi_{.} && \quad, \\
J_\mu^{\cdot} &= \tfrac{1}{2}(\not{\partial}A^*)\gamma_\mu\psi^{\cdot} - \tfrac{1}{2}A^*\overleftrightarrow{\partial}_\mu\psi^{\cdot} && \quad, \\
G_\mu^{\cdot} &= -\tfrac{1}{2}A^*\gamma_\mu\psi^{\cdot} && \quad.
\end{aligned}
\tag{6.45}
$$

In (6.45) we have omitted terms which vanish upon use of the free field equations $\Box A = 0$ and $\not{\partial}\psi^{\cdot} = 0$. The free matter term

$$
\mathcal{L}^{(o)}_{\text{matter}} = A^*\Box A - \tfrac{1}{2}\bar{\psi}^{\cdot}\overleftrightarrow{\not{\partial}}\psi_{.} \tag{6.46}
$$

has been discussed before (see eq. (5.43) with $F = 0$).

Of course $N = 1$ conformal supergravity is only defined in the presence of the conventional constraints (5.46). Nevertheless it is instructive to first consider the currents (6.45) in the absence of these constraints. Again these currents satisfy differential constraints which follow from the inhomogeneous terms in the transformation rules of the superconformal gauge fields. These inhomogeneous terms are given by (6.20), (6.34) and

(v)	Q supersymmetry	: $\delta\psi_\mu^{\cdot} = \partial_\mu\varepsilon^{\cdot}$,
(vi)	internal S supersymmetry	: $\delta\phi_\mu^{\cdot} = \partial_\mu\eta^{\cdot}$, $\delta\psi_\mu^{\cdot} = -\gamma_\mu\eta^{\cdot}$, (6.47)
(vii)	chiral U(1) symmetry	: $\delta A_\mu = \partial_\mu\Lambda_{U(1)}$.

They lead to the identities (6.22), (6.35), together with

(v)	$\partial_\mu J_\mu^{\cdot} = 0$	, (Q supersymmetry)
(vi)	$\partial_\mu G_\mu^{\cdot} - \gamma_\mu J_\mu^{\cdot} = 0$	, (internal S supersymmetry) (6.48)
(vii)	$\partial_\mu a_\mu = 0$	. (chiral U(1) symmetry)

By combining these constraints one can derive the currents corresponding to the flat-spacetime transformations. This leads to the definitions (6.23), (6.36) and the next one:

$$\text{(vi)}' \quad \partial_\mu G_\mu^{\cdot\,'} \equiv \partial_\mu \left(G_\mu^{\cdot} - \not{\partial}\, J_\mu^{\cdot} \right) = 0. \text{(rigid s supersymmetry)} \tag{6.49}$$

This definition is consistent with the inhomogeneous terms in the rigid s supersymmetry transformations of $\psi_\mu^{\cdot}$ and $\phi_\mu^{\cdot}$, taken with spacetime-dependent parameter, (cf. eq. (5.38)):

$$\text{(vi)}' \quad \text{rigid s supersymmetry} \quad : \delta\psi_\mu^{\cdot} = \not{\partial}\partial_\mu \eta^{\cdot}, \delta\phi_\mu^{\cdot} = \partial_\mu \eta^{\cdot} \quad . \tag{6.50}$$

In the presence of the conventional constraints (5.46) the only independent superconformal gauge fields are $(e_\mu^a, \psi_\mu^{\cdot}, A_\mu, b_\mu)$. Since the remaining gauge fields are in lowest order given by eqs. (6.38) and

$$\phi_\mu^{\cdot}(\psi) \equiv \frac{1}{2} \left(\sigma_{ab}\gamma_\mu - \frac{1}{3} \gamma_\mu \sigma_{ab} \right) \partial_{[a}\psi^{\cdot}{}_{b]} \quad , \tag{6.51}$$

we only have four independent currents, which are modifications of $\theta_{\mu a}$, T_μ, a_μ and $J_\mu^{\cdot}$:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}^{(o)}_{\text{matter}} + h_{a\mu}\theta_{\mu a} + \omega_\mu^{ab}(h,b)\, S_\mu^{ab} + b_\mu T_\mu + f_\mu^a(h,b)\, U_\mu^a \\ &\qquad + A_\mu a_\mu + \left(\bar{\psi}_\mu^{\cdot} J_\mu^{\cdot} + \bar{\phi}_\mu^{\cdot}(\psi)\, G_\mu^{\cdot} + \text{c.c.} \right) \\ &\equiv \mathcal{L}^{(o)}_{\text{matter}} + h_{a\mu}\theta_{\mu a}^{\text{imp}} + A_\mu a_\mu^{\text{imp}} + \left(\bar{\psi}_\mu^{\cdot} J_\mu^{\cdot\,\text{imp}} + \text{c.c.} \right) + b_\mu T_\mu^{\text{imp}} \quad . \end{aligned} \tag{6.52}$$

In analogy to the second example the gauge field b_μ decouples from the theory, i.e. $T_\mu^{\text{imp}} \equiv 0$. In addition we have $a_\mu^{\text{imp}} = a_\mu$. The modified current $J_\mu^{\cdot\,\text{imp}}$ is given by:

$$J^{\cdot\,imp}_{\mu} = J^{\cdot}_{\mu} + \frac{1}{2}\,(\,\gamma_{\rho}\sigma_{\mu\lambda} - \frac{1}{3}\,\sigma_{\mu\lambda}\gamma_{\rho})\;\partial_{\lambda}G^{\cdot}_{\rho} \qquad , \tag{6.53}$$

while the expression for $\theta^{imp}_{\mu a}$ is given in eq. (6.39).

To derive the constraints on the improved currents ($\theta^{imp}_{\mu a}$, $J^{\cdot\,imp}_{\mu}$, a_{μ}) we consider the variation of the gauge field × current coupling terms in (6.52). We thus find the restrictions (6.42), (6.48) (vii) and

$$\partial_{\mu}J^{\cdot\,imp}_{\mu} = 0 \quad , \text{(Q supersymmetry)} \tag{6.54}$$

$$\gamma_{\mu}J^{\cdot\,imp}_{\mu} = 0 \quad . \text{(internal S supersymmetry)}$$

Owing to these conditions the currents ($\theta^{imp}_{\mu a}$, $J^{\cdot\,imp}_{\mu}$, a_{μ}) describe 8 + 8 field degrees of freedom which couple to the 8 + 8 independent gauge fields (e^{a}_{μ}, $\psi^{\cdot}_{\mu}$, A_{μ}).

Substituting the expressions (6.45) of the Noether currents into the definitions of the improved currents ($\theta^{imp}_{\mu a}$, $J^{\cdot\,imp}_{\mu}$, a_{μ}) (see eqs. (6.39) and (6.53)) we find that these currents are given by

$$\begin{aligned}
\theta^{imp}_{\mu\nu} &= 2\,(\partial_{(\mu}A^{*})\;(\partial_{\nu)}A) - \delta_{\mu\nu}(\partial_{\rho}A^{*})\;(\partial_{\rho}A) + \frac{1}{3}\,(\Box\,\delta_{\mu\nu} - \partial_{\mu}\partial_{\nu})\;(A^{*}A) \\
&\qquad + \frac{1}{2}\,\bar{\psi}^{\cdot}\gamma_{(\mu}\overleftrightarrow{\partial}_{\nu)}\psi^{\cdot} \quad , \\
J^{\cdot\,imp}_{\mu} &= (\not{\partial}A^{*})\;\gamma_{\mu}\psi^{\cdot} + \frac{4}{3}\,\sigma_{\mu\lambda}\partial_{\lambda}\,(A^{*}\psi^{\cdot}) \quad , \\
a_{\mu} &= \frac{1}{2}\,i\,A^{*}\overleftrightarrow{\partial}_{\mu}A - \frac{1}{4}\,i\,\bar{\psi}^{\cdot}\gamma_{\mu}\psi^{\cdot} \quad .
\end{aligned} \tag{6.55}$$

One can verify that they indeed satisfy the identities given in eqs. (6.42), (6.48) (vii) and (6.54), viz.

$$\partial_\mu \theta^{imp}_{\mu\nu} = 0 \quad , \quad \partial_\mu J^{\cdot imp}_\mu = 0 \quad ,$$

$$\theta^{imp}_{\mu\nu} = \theta^{imp}_{\nu\mu} \quad , \quad \gamma_\mu J^{\cdot imp}_\mu = 0 \quad , \tag{6.56}$$

$$\theta^{imp}_{\mu\mu} = 0 \quad , \quad \partial_\mu a_\mu = 0 \quad .$$

4. The N = 1, d = 4 supercurrent

In order to illustrate the ideas presented in the previous sections we reconstruct the N = 1 Weyl multiplet. In the next chapter we will apply the same procedure to derive the complete structure of the N = 4 Weyl multiplet.

The starting point is the construction of the N = 1, d = 4 multiplet of (improved) currents (or supercurrent) corresponding to the coupling of a (on-shell) N = 1 chiral multiplet $(A,\psi^\cdot)$ to the N = 1 superconformal gauge fields $(e^a_\mu, \psi^\cdot_\mu, A_\mu)$. To derive the field components of this current multiplet we first consider the improved energy-momentum tensor $\theta^{imp}_{\mu\nu}$ (see eq. (6.55)). This current corresponds to the translation invariance of the free matter term given in (6.46) and is a bilinear in the field components of the chiral multiplet (see eq. (6.55)). We next apply a supersymmetry transformation on $\theta^{imp}_{\mu\nu}$. In calculating this transformation we always use the field equations $\Box A = 0$ and $\partial\!\!\!/\psi^\cdot = 0$ of A and $\psi^\cdot$ respectively. The transformation of $\theta^{imp}_{\mu\nu}$ leads to another field component of the current multiplet, namely the (improved) supersymmetry current $J^\cdot_\mu$. In his turn a supersymmetry transformation of $J^\cdot_\mu$ again leads to a new field component of the current multiplet, namely the chiral U(1) current a_μ (see eq. (6.55)). Finally, a transformation of a_μ only leads back to $J^\cdot_\mu$. This means that at this point we have found the complete N = 1, d = 4 supercurrent. This could already be guessed from the fact that the currents $(\theta^{imp}_{\mu\nu}, J^{\cdot imp}_\mu, a_\mu)$ describe 8 + 8 field degrees of freedom. We now give the transformation rules of the current multiplet.

$$\delta\theta^{imp}_{\mu\nu} = - \bar{\varepsilon}^\cdot\, \sigma_{(\mu\lambda}\partial_\lambda J^{\cdot imp}_{\nu)} + c.c. \quad ,$$

$$\delta J^{\cdot imp}_\mu = \gamma_\lambda \theta^{imp}_{\mu\lambda}\, \varepsilon_\cdot + i\, (\gamma_\rho\, \sigma_{\mu\lambda} - \tfrac{1}{3}\, \sigma_{\mu\lambda}\gamma_\rho)\ \partial_\lambda a_\rho\, \varepsilon_\cdot \quad , \tag{6.57}$$

$$\delta a_\mu = - \tfrac{3}{4}\, i\, \bar{\varepsilon}^\cdot J^{\cdot imp}_\mu + c.c. \quad .$$

From the N = 1, d = 4 supercurrent $(\theta_{\mu\nu}^{\text{imp}}, J_\mu^{\cdot\text{imp}}, a_\mu)$ one can derive a corresponding multiplet of fields $(h_{a\mu}, \psi_\mu^{\cdot}, A_\mu)$ by requiring invariance of the gauge field × current coupling terms:

$$\int d^4x \, (h_{a\mu}\theta_{\mu a}^{\text{imp}} + (\bar{\psi}_\mu^{\cdot} J_\mu^{\cdot} + \text{c.c.}) + A_\mu a_\mu) \quad . \tag{6.58}$$

The transformation rules of $(h_{a\mu}, \psi_\mu^{\cdot}, A_\mu)$ are given by

$$\begin{aligned}
\delta h_{a\mu} &= (\bar{\varepsilon}^{\cdot}\gamma_{(a}\psi_{\mu).} + \text{c.c.} - \text{trace}) + \partial_\mu\xi_a + \varepsilon_{a\mu} - \delta_{a\mu}\Lambda_D \quad , \\
\delta\psi_\mu^{\cdot} &= (-\partial_\lambda(h_{(a\mu)} - \tfrac{1}{4}\delta_{a\mu}h_{\lambda\lambda})\,\sigma_{\lambda a}\varepsilon^{\cdot} + \tfrac{3}{4}\, i\, A_\mu\varepsilon^{\cdot} - \gamma - \text{trace}) + \partial_\mu\varepsilon^{\cdot} - \gamma_\mu\eta^{\cdot} , \\
\delta A_\mu &= (i\,\bar{\varepsilon}^{\cdot}\,(\sigma_{ab}\gamma_\mu - \tfrac{1}{3}\gamma_\mu\sigma_{ab})\,(\partial_{[a}\psi_{b]}^{\cdot} - \tfrac{1}{4}\partial_{[a}\gamma_{b]}\,\gamma\cdot\psi^{\cdot}) + \text{c.c.}) + \partial_\mu\Lambda_{U(1)} \quad ,
\end{aligned} \tag{6.59}$$

where the parameters $\varepsilon^{\cdot}$, ξ_μ, $\eta^{\cdot}$, $\varepsilon_{\mu\nu}$, Λ_D and $\Lambda_{U(1)}$ characterive Q, g.c.t. and internal S, Σ, Δ and U(1) transformations respectively. All inhomogeneous terms in the transformations (6.59) are in correspondence to the constraints (6.56) on the improved currents. The Q transformations in (6.59) are determined up to field-dependent gauge transformations. By applying such transformations we can bring the Q transformations in the following equivalent form:

$$\begin{aligned}
\delta h_{a\mu} &= \bar{\varepsilon}^{\cdot}\gamma_a\psi_{\mu.} + \text{c.c.} \quad , \\
\delta\psi_\mu^{\cdot} &= (\partial_\mu - \tfrac{1}{2}\omega_\mu^{ab}(h)\,\sigma_{ab} + \tfrac{3}{4}\, i\, A_\mu)\,\varepsilon^{\cdot} \quad , \\
\delta A_\mu &= 2\, i\,\bar{\varepsilon}^{\cdot}\,\phi_{\mu.}(\psi) + \text{c.c.} \quad ,
\end{aligned} \tag{6.60}$$

with $\omega_\mu^{ab}(h)$ and $\phi_\mu^{\cdot}(\psi)$ defined in eq. (6.26) and (6.51) respectively.

The derivation of the complete nonlinear transformations proceeds by induction. This procedure resembles the method described in chapter 5 (see after eq. (5.57)) to calculate the nonlinear transformation rules of matter fields which are coupled to supergravity. We now briefly describe this iterative procedure. We first replace in the linearized transformation rules (6.60) $h_{a\mu}$ everywhere by e_μ^a.

We then assign Weyl and chiral weights to all fields in the multiplet. After choosing the standard Weyl weight $w = -1$ for e_μ^a (we must take $c = 0$ for the chiral weight of e_μ^a) all other weights are determined by the $[Q,D]$ and $[Q,A]$ commutators. We then make the gauge field b_μ explicit in the covariant derivatives. The transformations of b_μ are determined up to field-dependent $\kappa_\cdot$ transformations. We choose them to concide with the S U $(2,2|1)$ transformation rules given in eq. (5.44):

$$\delta b_\mu = (\bar{\epsilon}^\cdot\ \phi_{\mu\cdot} - \bar{\psi}_\mu^\cdot \eta_\cdot + c.c.) + \partial_\mu \Lambda_D + \Lambda_{K\mu} \qquad . \qquad (6.61)$$

We next covariantize all derivatives with respect to internal Σ, κ and S transformations by introducing the gauge fields ω_μ^{ab}, f_μ^a and $\phi_\mu^\cdot$. These gauge fields are not independent. They are completely determined by the conventional constraints which also fix their transformation rules (cf. section (5.5)). We now calculate the commutator algebra on the basis of the linearized transformation rules, but with spacetime-dependent parameters and ordinary derivatives replaced by covariant derivatives. We then impose this algebra on all field components. This requires the addition of terms of second order in the fields to the transformation rules. One now repeats the calculation on the basis of the new transformation rules. This may lead to terms of higher order in the fields in the transformation rules etc..

In the next chapter we apply this procedure to calculate the complete nonlinear transformation rules of the $N = 4$ Weyl multiplet. For $N = 1$ the results are rather simple. We find that the complete nonlinear transformations of (e_μ^a, $\psi_\mu^\cdot$, A_μ) are given by eq. (5.54). These transformations are just the covariantizations of the linearized transformations rules (6.60).

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CHAPTER VII

EXTENDED CONFORMAL SUPERGRAVITY

1. Introduction

In this chapter we study extended conformal supergravity with $N \leq 4$. We take the limit $N = 4$, because this is the natural limit for the construction procedure described in the previous chapter (cf. section (6.1.)). Presently these exists no method to construct conformal supergravity for $N > 4$. There are indications that drastic changes take place beyond $N = 4$ (see the references at the end of this chapter).

The first step in the construction of extended conformal supergravity is to find the multiplet of currents. In this chapter we present the irreducible $N = 4$, $d = 4$ multiplet of currents, which contains the gravitational spin-2 degree of freedom. This supercurrent contains all corresponding $d = 4$ supercurrents for lower N. Once the supercurrent is known, it is straightforward to derive the linearized transformation rules of $N = 4$ conformal supergravity. These linearized results can be extended to the full theory by means of iteration.

To present the complete nonlinear transformations and corresponding algebra it is advantageous to use a formulation which exhibits the highest possible degree of invariance. Therefore we first construct a new version of $N = 4$ conformal supergravity which is manifestly symmetric under an extra local U(1) and rigid SU(1,1) group. In this formulation the complete nonlinear results are obtained after a finite number of iterations. We thus find the superconformal transformations and the corresponding algebra which are given in the text.

This chapter is organized as follows. We present the $N = 4$ multiplet of currents in section 2. Here we also give the linearized transformation rules of the $N = 4$ Weyl multiplet. The formulation with an extra local U(1) and rigid SU (1,1) invariance is discussed in section 3, while the full nonlinear transformations are given in section 4. Finally, in section 5 we give our conclusions.

2. The $N = 4$, $d = 4$ supercurrent

To construct the largest gravitational multiplet of currents, we consider an (on-shell) $N = 4$ supersymmetric matter theory. The only known candidate for this is the supersymmetric Yang-Mills theory, and for our purpose the abelian version suffices. It is based on a gauge field V_μ, a quartet of Majorana spinors ψ^i, for which we use the chiral notation given in (2.8) and a Lorentz scalar ϕ^{ij}, antisymmetric in the SU (4) indices i, j and subject to an SU (4)-covariant reality constraint:

$$\phi^{ij} = (\phi_{ij})^* = \frac{1}{2}\,\varepsilon^{ijkl}\phi_{kl} \qquad . \qquad (7.1)$$

The field strength corresponding to V_μ is denoted by $F_{\mu\nu}(\equiv \partial_\mu V_\nu - \partial_\nu V_\mu)$, and we define (anti-) selfdual components by

$$F^{\pm}_{\mu\nu} \equiv \frac{1}{2}\,(F_{\mu\nu} \pm \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}) \qquad . \qquad (7.2)$$

The fields transform under four independent supersymmetries with parameters ε^i according to:

$$\begin{aligned}
\delta V_\mu &= \bar{\varepsilon}^i\,\gamma_\mu\psi_i + \text{c.c.} \qquad , \\
\delta F^-_{\mu\nu} &= \bar{\varepsilon}_i\,\partial\!\!\!/\sigma_{\mu\nu}\psi^i \qquad , \\
\delta\psi^i &= -\sigma.F^-\varepsilon^i - 2\,i\,\partial\!\!\!/\,\phi^{ij}\varepsilon_j \qquad , \\
\delta\phi_{ij} &= 2i\bar{\varepsilon}_{[i}\psi_{j]} - i\,\varepsilon_{ijkl}\,\bar{\varepsilon}^k\psi^l \qquad .
\end{aligned} \qquad (7.3)$$

These transformations are an invariance of the action corresponding to the following Lagrangian:

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2}\bar{\psi}^i \overleftrightarrow{\partial\!\!\!/}\, \psi_i - \frac{1}{2}(\partial_\mu \phi^{ij})(\partial_\mu \phi_{ij}) \quad . \tag{7.4}$$

Furthermore the action defined by (7.4) is invariant under translations and chiral SU (4) transformations. Note that the transformations (7.3) do not allow for an extra chiral U(1) invariance: the transformation of V_μ implies that the chiral weights of ϵ^i and ψ_i are opposite, whereas the transformation of ϕ_{ij} implies that those of ϵ_i and ψ_j are opposite. From this we deduce that all chiral weights are zero, i.e. there is no chiral U(1) invariance present. The commutator of two Q transformations on $(V_\mu, \psi^i, \phi_{ij})$ yields a general coordinate transformation with parameter $\xi^\mu = 2\,\bar{\epsilon}_1^i \gamma_\mu \epsilon_{2i} + \mathrm{c.c.}$.

The improved Noether currents of translations, supersymmetry and chiral SU (4) transformations can be constructed in the standard way (see chapter 6). The explicit expressions are given by

$$\begin{aligned}
\theta_{\mu\nu}^{imp} &= -4\, F^+_{\mu\lambda} F^-_{\nu\lambda} - \bar{\psi}^i \gamma_{(\mu} \overleftrightarrow{\partial}_{\nu)} \psi_i + \delta_{\mu\nu}(\partial_\rho \phi^{ij})(\partial_\rho \phi_{ij}) \\
&\quad - 2(\partial_\mu \phi^{ij})(\partial_\nu \phi_{ij}) - \frac{1}{3}(\Box\, \delta_{\mu\nu} - \partial_\mu \partial_\nu)(\phi^{ij}\phi_{ij}) \quad , \\
J_{\mu i}^{imp} &= -\sigma\cdot F^- \gamma_\mu \psi_i + 2\, i\, \phi_{ij} \overleftrightarrow{\partial}_\mu \psi^j + \frac{4}{3}\, i\, \sigma_{\mu\lambda} \partial_\lambda (\phi_{ij}\psi^j) \quad , \\
v_{\mu j}^{\,i} &= \phi^{ik} \overleftrightarrow{\partial}_\mu \phi_{kj} + \bar{\psi}^i \gamma_\mu \psi_j - \frac{1}{4}\, \delta^i_j\, \bar{\psi}^k \gamma_\mu \psi_k \quad ,
\end{aligned} \tag{7.5}$$

where the SU (4) current $v_{\mu j}^{\,i}$ is antihermitean and traceless. In (7.5) we have omitted terms, which vanish upon use of the free field equations

$$\partial_\mu F^+_{\mu\nu} = 0 \quad , \quad \partial\!\!\!/ \psi^i = 0 \quad , \quad \Box\, \phi^{ij} = 0 \tag{7.6}$$

of V_μ, ψ^i and ϕ^{ij} respectively. Using these field equations one can verify that the Noether currents (7.5) satisfy the following differential and algebraic identities:

$$\partial_\mu \theta^{imp}_{\mu\nu} = 0 \quad , \quad \partial_\mu J^{imp}_{\mu i} = 0 \quad ,$$

$$\theta^{imp}_{\mu\nu} = \theta^{imp}_{\nu\mu} \quad , \quad \gamma_\mu J^{imp}_{\mu i} = 0 \quad , \qquad (7.7)$$

$$\theta^{imp}_{\mu\mu} = 0 \quad , \quad \partial_\mu v^{i}_{\mu j} = 0 \quad .$$

We immediately deduce from (7.7) that the Noether currents $(\theta^{imp}_{\mu\nu}, J^{imp}_{\mu i}, v^{i}_{\mu j})$ do not constitute a massive spin-2 representation of the super-Poincaré algebra. More specifically, they describe 50 + 32 (bosonic + fermionic) field degrees of freedom, whereas a N = 4 massive spin-2 representation describes 128 + 128 dynamic degrees of freedom (see table 2 in chapter 5). Therefore 78 + 96 components are still missing. To derive the additional quantities, which are needed to describe these degrees of freedom we apply successive Q transformations on the Noether currents (7.5), always using the free field equations(7.6). We thus find that the remaining components of the supercurrent are given by:

$$\begin{aligned}
c &= (F^{-}_{ab})^2 && , \quad (2) \\
\lambda_i &= \sigma \cdot F^{+} \psi_i && , \quad (16) \\
e_{ij} &= \bar{\psi}_i \psi_j && , \quad (20) \\
t^{ij}_{ab} &= \bar{\psi}^i \sigma_{ab} \psi^j + 2\, i\, \phi^{ij} F^{-}_{ab} && , \quad (36) \qquad (7.8) \\
\xi^{ij}_{k} &= \frac{1}{2} \varepsilon^{ijmn} (\phi_{mn} \psi_k + \phi_{kn} \psi_m) && , \quad (80) \\
d^{ij}_{kl} &= \phi^{ij} \phi_{kl} - \frac{1}{6} \delta^{[i}_{k} \delta^{j]}_{l} \phi^{mn} \phi_{mn} && , \quad (20)
\end{aligned}$$

where the number between brackets denotes the number of independent components. In this way we find that besides the Noether currents (7.5) the supercurrent contains a complex scalar c, a symmetric scalar e_{ij}, an antisymmetric tensor (both in Lorentz and SU (4) indices) t^{ij}_{ab}, which is antiselfdual in the indices a,b, a scalar d^{ij}_{kl} in the 20-dimensional real representation of SU (4), and two spinors λ_i and ξ^{ij}_k, the latter in a complex 20-dimensional SU (4) representation. One easily verifies that the Noether currents (7.5) together with the quantities (7.8) indeed describe 128 + 128 field degrees of freedom.

The supersymmetry transformations of the currents (7.5) and the quantities (7.8) are as follows:

$$
\begin{aligned}
\delta c &= 2\,\bar\varepsilon_i \not\partial \lambda^i \,, \\
\delta\lambda_i &= c^* \varepsilon_i + \not\partial e_{ik}\varepsilon^k - \sigma\cdot t_{ik}\not\partial\varepsilon^k \,, \\
\delta e_{ij} &= 2\,\bar\varepsilon_{(i}\lambda_{j)} + \frac{4}{3}\, i\, \varepsilon_{mnk(i}\,\bar\varepsilon^k \not\partial \xi^{mn}_{j)} \,, \\
\delta t^{ij}_{ab} &= -\frac{1}{2}\,\varepsilon^{ijkl}\bar\varepsilon_k \gamma^\rho\,\sigma_{ab}\,J_{\rho l} - 2\,\bar\varepsilon^{[i}\sigma_{ab}\lambda^{j]} \\
&\qquad + \frac{2}{3}\, i\,\varepsilon^{ijkl}\bar\varepsilon_n \not\partial\,\sigma_{ab}\,\xi^n{}_{kl} \,, \\
\delta\xi^{ij}_k &= \frac{3}{4}\, i\,\varepsilon^{ijmn}\,\sigma\cdot t_{km}\varepsilon_n + \frac{3}{4}\, i\,\varepsilon^{ijmn} e_{mk}\,\varepsilon_n \\
&\qquad - \frac{3}{2}\, i\,\gamma\cdot v^{[i}{}_k\,\varepsilon^{j]} + \frac{3}{2}\, i \not\partial d^{ij}{}_{kl}\,\varepsilon^l - (\text{trace}) \,, && (7.9) \\
\delta d^{ij}_{kl} &= \frac{8}{3}\, i\,\bar\varepsilon_{[k}\xi^{ij}{}_{l]} - \frac{8}{3}\, i\,\delta^{[i}_{[k}\bar\varepsilon_n \xi^{nj]}{}_{l]} + \text{h.c.} \,, \\
\delta\theta^{imp}_{\mu\nu} &= 2\bar\varepsilon^k\,\sigma_{(\mu\lambda}\partial_\lambda\, J^{imp}_{\nu)k} + \text{c.c.} \,, \\
\delta J^{imp}_{\mu i} &= -\gamma_\nu\theta^{imp}_{\mu\nu}\,\varepsilon_i - 2\,(\gamma_\rho\sigma_{\mu\lambda} - \frac{1}{3}\sigma_{\mu\lambda}\gamma_\rho)\,\partial_\lambda v^k_{\rho i}\,\varepsilon_k \\
&\qquad - (\sigma_{ab}\sigma_{\mu\lambda} + \frac{1}{3}\,\sigma_{\mu\lambda}\sigma_{ab})\,\varepsilon_{iklm}\partial_\lambda t^{kl}_{ab}\,\varepsilon^m \,, \\
\delta v^i_{\mu j} &= -\,\bar\varepsilon^i J^{imp}_{\mu j} + \frac{1}{4}\,\delta^i_j\,\bar\varepsilon^k\, J^{imp}_{\mu k} + \frac{8}{3}\, i\,\bar\varepsilon^k\sigma_{\mu\lambda}\partial_\lambda\xi^i_{kj} - \text{h.c.} \,.
\end{aligned}
$$

From the multiplet of currents (7.9) one can derive a corresponding multiplet of fields by requiring invariance of the gauge field × current coupling terms:

$$\int d^4x \, \{h_{\mu\nu}\theta^{imp}_{\mu\nu} + V^i_{\mu j}\, v^j_{\mu i} + D^{ij}_{kl}\, d^{kl}_{ij} + (\bar{\psi}^i_\mu J^{imp}_{\mu i} + C\, c$$
$$+ \bar{\Lambda}^i \lambda_i + E^{ij} e_{ij} + T^{ij}_{ab}\, t_{ab\; ij} + \bar{\chi}^{ij}_k \xi^k_{ij} + c \cdot c \cdot)\} \qquad . \tag{7.10}$$

Here we use the obvious notation in which every component of the current multiplet couples to a corresponding field. In table 1 we have listed some details about these fields. Their transformations are given by:

$$\delta C = \bar{\varepsilon}^i \Lambda_i \qquad ,$$
$$\delta \Lambda_i = 2 \not{\partial} C \varepsilon_i + E_{ij}\varepsilon^j + \varepsilon_{ijkl}\sigma\cdot T^{kl}\varepsilon^j \qquad ,$$
$$\delta E_{ij} = 2\,\bar{\varepsilon}_{(i}\not{\partial}\Lambda_{j)} - 2\,\bar{\varepsilon}^k \chi^{mn}{}_{(i}\varepsilon_{j)kmn} \qquad ,$$
$$\delta T^{ij}_{ab} = 4\,\bar{\varepsilon}^{[i}(\partial_{[a}\psi^{j]}_{b]} - \tfrac{1}{2}\gamma_{[a}\phi^{j]}_{b]}(\psi)) + \bar{\varepsilon}^k\sigma_{ab}\chi^{ij}_k + \tfrac{1}{2}\varepsilon^{ijkl}\bar{\varepsilon}_k\not{\partial}\sigma_{ab}\Lambda_l ,$$
$$\begin{aligned}\delta\chi^{ij}_k = {} & -\sigma\cdot T^{ij} \overleftrightarrow{\not{\partial}}\, \varepsilon_k - \tfrac{2}{3}\,\delta^{[i}_k \sigma\cdot T^{j]l} \overleftrightarrow{\not{\partial}}\, \varepsilon_l \\ & -2\,\sigma\cdot R^{[i}_k(V)\,\varepsilon^{j]} - \tfrac{2}{3}\,\delta^{[i}_k\,\sigma\cdot R^{j]}_l(V)\,\varepsilon^l \\ & -\tfrac{1}{2}\,\varepsilon^{ijlm}\not{\partial}E_{kl}\,\varepsilon_m + D^{ij}_{kl}\,\varepsilon^l \qquad ,\end{aligned} \tag{7.11}$$
$$\delta D^{ij}_{kl} = -4\,\bar{\varepsilon}^{[i}\not{\partial}\,\chi^{j]}_{kl} + 4\,\delta^{[i}_{[k}\,\bar{\varepsilon}^m\not{\partial}\chi^{j]}_{ml]} + h\cdot c\cdot \qquad ,$$
$$\delta h_{a\mu} = \bar{\varepsilon}^i\,\gamma_a\psi_{\mu i} + c\cdot c\cdot \qquad ,$$
$$\delta\psi^i_\mu = 2\,(\partial_\mu\varepsilon^i - \tfrac{1}{2}\,\omega^{ab}_\mu(h)\,\sigma_{ab}\,\varepsilon^i - V^i_{\mu j}\,\varepsilon^j) - \sigma\cdot T^{ij}\,\gamma_\mu\varepsilon_j \;,$$
$$\delta V^i_{\mu j} = \bar{\varepsilon}^i\,\phi_{\mu j}(\psi) + \bar{\varepsilon}^k\,\gamma_\mu\chi^i_{kj} - \tfrac{1}{4}\,\delta^i_j\,\bar{\varepsilon}^k\phi_{\mu k}(\psi) - h\cdot c\cdot \qquad .$$

Here $R^i_{\mu\nu j}(V) \equiv \partial_\mu V^i_{\nu j} - \partial_\nu V^i_{\mu j}$ is the field strength of $V^i_{\mu j}$ and $\omega^{ab}_\mu(h)$ is defined in eq. (6.26). The lowest-order expression $\phi^i_\mu(\psi)$ is given by

$$\phi^i_\mu(\psi) \equiv (\sigma_{ab}\gamma_\mu - \tfrac{1}{3}\gamma_\mu\sigma_{ab})\,\partial_{[a}\psi^i_{b]} \quad . \tag{7.12}$$

In this chapter our conventions for the transformations of the superconformal gauge fields (and corresponding algebra) slightly differ from the ones in chapter 5 (cf. eq. (5.44)).

In principle it is now straightforward to derive the complete nonlinear transformation rules and the corresponding superconformal algebra. For this purpose one can apply the iterative procedure, which is described in section (6.4). Choosing for e^a_μ the standard weights $w = -1$ (Weyl weight) and $c = 0$ (chiral weight) we find for the weights of the other fields the values given in table 1. If one now proceeds with the iterative procedure described in section (6.4), one

Field	Type	Restrictions	SU (4)	w	c
C	boson	complex	1	0	-2
Λ_i	fermion	$\gamma_5\Lambda_i = \Lambda_i$	4	$\frac{1}{2}$	$-\frac{3}{2}$
E_{ij}	boson	$E_{ij} = E_{ji}$; complex	10	1	-1
$T_{ab}{}^{ij}$	boson	$T_{ab}{}^{ij} = -T_{ba}{}^{ij} = -T_{ab}{}^{ji}$; $\frac{1}{2}\varepsilon_{ab}{}^{cd}T_{cd}{}^{ij} = -T_{ab}{}^{ij}$	6	1	-1
χ^{ij}_k	fermion	$\gamma_5\chi^{ij}_k = \chi^{ij}_k$; $\chi^{ij}_k = -\chi^{ji}_k$; $\chi^{ij}_j = 0$	20	$\frac{3}{2}$	$-\frac{1}{2}$
$D^{ij}{}_{kl}$	boson	$D^{ij}{}_{kl} = \frac{1}{4}\varepsilon^{ijmn}\varepsilon_{klpq}D^{pq}{}_{mn}$; $D_{kl}{}^{ij} = (D^{kl}{}_{ij})^* = D^{ij}{}_{kl}$; $D^{ij}{}_{kl} = -D^{ji}{}_{kl} = -D^{ij}{}_{lk}$; $D^{ij}{}_{kj} = 0$	20	2	0
e^a_μ	boson	vierbein	1	-1	0
ψ^i_μ	fermion	$\gamma_5\psi^i_\mu = \psi^i_\mu$; gravitino	4	$-\frac{1}{2}$	$-\frac{1}{2}$
$V^i_{\mu j}$	boson	$V_{\mu i}{}^j = (V^i_{\mu j})^* = -V^j_{\mu i}$; $V^i_{\mu i} = 0$; SU (4) gauge field	15	0	0
b_μ	boson	dilatational gauge field	1	0	0

<u>table</u> 1. Fields of N = 4 conformal supergravity. We have indicated the various algebraic restrictions on the fields, their representation assignments, and Weyl and chiral weight factors.

recognizes after a few steps that the scalar field C (see eq. (7.11)) occurs in a non-polynomial fashion in the Q-transformation rules. This is possible because C is inert under internal dilatations Δ. Therefore one can expect that the complete transformation rules contain a priori arbitrary functions of $|C|^2$. However, it appears that these functions have a remarkable systematic structure. The linearized transformations (7.11) are known to be consistent with a rigid chiral U(1) symmetry, and it turns out that the nonlinear Q transformations contain precisely such a U(1) transformation with a field-dependent coefficient as a uniform component. Furthermore, all derivatives are augmented by $C^* \overleftrightarrow{\partial}_\mu C$ terms in such a way that this quantity can be interpreted as a new gauge field that makes the derivative covariant with respect to local U(1) transformations. These facts can be viewed as an indication that the theory can be reformulated in a form which is manifestly symmetric under local chiral U(1) transformations. It turns out that this reformulation also has a rigid SU (1,1) invariance. More specifically, it appears that the scalar C in the original formulation occurs as a parametrization of the coset space SU(1,1)/U(1). It has been known for some time that SU (1,1) invariance plays a role in Poincaré supergravity (see the references at the end of this chapter), but this symmetry was never linked to the superconformal sector of the theory.

Before giving in section 4 the explicit construction of the full theory with a manifest rigid SU (1,1) and local U(1) invariance, we first review in the next section some properties of SU (1,1) and its coset decomposition into SU (1,1)/U(1) and U(1). Furthermore we indicate how the relation between the reformulated theory and the original formulation in terms of a complex scalar C can be made explicit by imposing a gauge condition that breaks the local U(1) and rigid SU (1,1) invariance.

3. Rigid SU (1,1) and local U(1) invariance

By definition SU (1,1) ($\cong$ SO (2,1)) is the group of complex 2x2 matrices with unit determinant that leave the metric η = diag (+1, -1) invariant. Therefore elements U of SU (1,1) satisfy (to compare we also give the corresponding more known relation for SU (2) ($\cong$ SO (3)) :

$$U^\dagger \eta U = \eta \quad , \qquad (SU\,(1,1))$$

$$U^\dagger \delta U = \delta \text{ or } U^\dagger U = \mathbb{1} \quad . \qquad (SU\,(2)) \tag{7.13}$$

Here δ is the SU (2) invariant metric $\delta \equiv \mathrm{diag}\,(+1,\,+1)$ and $U^\dagger$ is the hermitean conjugate of U. An arbitrary element of the group SU (1,1) can be written as (again we give the corresponding expression for SU (2), cf. section (3.3)):

$$U = \begin{pmatrix} \phi_1 & \phi_2^* \\ \phi_2 & \phi_1^* \end{pmatrix} \quad \text{with} \quad \phi_1^*\phi_1 - \phi_2^*\phi_2 \equiv \phi^\alpha\phi_\alpha = 1 \;,(SU\,(1,1)) \tag{7.14}$$

$$U = \begin{pmatrix} \phi_1 & -\phi_2^* \\ \phi_2 & \phi_1^* \end{pmatrix} \quad \text{with} \quad \phi_1^*\phi_1 + \phi_2^*\phi_2 = 1 \qquad .(SU\,(2)) \tag{7.15}$$

In (7.14) we use a notation for the doublet ϕ_α, in which the metric η raises and lowers indices according to

$$\phi^\alpha \equiv \eta^{\alpha\beta}\,(\phi_\beta^*) = (\phi_1^*,\, -\,\phi_2^*) \quad . \tag{7.16}$$

One can verify that both the SU (1,1) transformations (7.14) and the SU (2) transformations (7.15) leave the two-index Levi-Civita tensor $\varepsilon_{\alpha\beta}$ invariant:

$$U^T \varepsilon U = \varepsilon \quad . \tag{7.17}$$

Here U^T is the transpose of U.

We recall that each group elementof SU (2) can be written as the exponent of a linear combination of the three SU (2) generators $T_i = \frac{i}{2}\,\tau_i$ (i = 1,2,3) (τ_i are the standard Pauli matrices, cf. eq. (3.28)) in the following way:

$$U = \begin{pmatrix} \phi_1 & -\phi_2^* \\ \phi_2 & \phi_1^* \end{pmatrix} = \exp \frac{i}{2}\, \vec{\alpha}\cdot\vec{\tau} \quad , \tag{7.18}$$

with $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ defined by

$$\begin{aligned} \Phi_1 &= \cos\left(\tfrac{1}{2}\alpha\right) + i \sin\left(\tfrac{1}{2}\alpha\right) \hat{\alpha}_3 \quad &, \quad \alpha^2 \equiv \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \, , \\ \Phi_2 &= \sin\left(\tfrac{1}{2}\alpha\right)\left(i\hat{\alpha}_1 - \hat{\alpha}_2\right) \quad &, \quad \hat{\alpha}_i \equiv \frac{\alpha_i}{\alpha}\ (i = 1,2,3) \, . \end{aligned} \tag{7.19}$$

This is not true for SU (1,1). The generators of the Lie algebra su(1,1) of SU (1,1) are given by:

$$T_1 = \frac{1}{2}\tau_1 \quad , \quad T_2 = \frac{1}{2}\tau_2 \quad , \quad T_3 = \frac{i}{2}\tau_3 \quad . \tag{7.20}$$

Exponentiation of these generators gives:

$$\exp \frac{1}{2}\left(\alpha_1\tau_1 + \alpha_2\tau_2 + i\,\alpha_3\tau_3\right) = \begin{pmatrix} \Phi_1 & \Phi_2^* \\ \Phi_2 & \Phi_1^* \end{pmatrix} \quad , \tag{7.21}$$

with Φ_1 and Φ_2 given by

$$\begin{aligned} \Phi_1 &= \cosh\left(\tfrac{1}{2}\alpha\right) + i \sinh\left(\tfrac{1}{2}\alpha\right) \hat{\alpha}_3 \quad &, \quad \alpha^2 \equiv \alpha_1^2 + \alpha_2^2 - \alpha_3^2, \\ \Phi_2 &= \sinh\left(\tfrac{1}{2}\alpha\right)\left(\hat{\alpha}_1 + i\,\hat{\alpha}_2\right) \quad &, \quad \hat{\alpha}_i \equiv \frac{\alpha_i}{\alpha}\ (i = 1,2,3). \end{aligned} \tag{7.22}$$

This does not lead to all possible group elements. For instance, the matrix

$$U = \begin{pmatrix} -\cosh\rho & \sinh\rho \\ \sinh\rho & -\cosh\rho \end{pmatrix} \quad , \quad \rho \neq 0 \tag{7.23}$$

is an element of SU (1,1), but it is not possible to write it in the form (7.21), (7.22), because (7.22) implies that $(\Phi_1 + \Phi_1^*) \gtrsim -1$.

The vector space of the noncompact Lie algebra su (1,1) decomposes quite naturally into two vector subspaces:

$$\mathrm{su}\,(1,1) = \mathrm{u}\,(1) \oplus (\mathrm{su}\,(1,1) \bmod \mathrm{u}\,(1)) \quad , \qquad (7.24)$$

where u (1) is the maximal compact Lie subalgebra generated by the compact generator $T_3 = \frac{i}{2}\tau_3$ and (su (1,1) mod u (1)) is the vector subspace consisting of the remaining generators $T_1 = \frac{1}{2}\tau_1$ and $T_2 = \frac{1}{2}\tau_2$, both noncompact. The decomposition (7.24) of the Lie algebra su (1,1) has the following counterpart in the Lie group SU (1,1):

$$\mathrm{SU}\,(1,1) = \mathrm{U}\,(1)\,.\,\mathrm{SU}\,(1,1)/\mathrm{U}\,(1) \quad . \qquad (7.25)$$

The left coset SU (1,1)/U (1) is defined as the set of group elements $c_0, c_1, \ldots \in$ SU (1,1) with the property that

$$\mathrm{U}\,(1)\cdot c_0 + \mathrm{U}\,(1)\cdot c_1 + \ldots = \mathrm{SU}\,(1,1) \qquad (7.26)$$

and, furthermore, no element $g \in$ SU (1,1) is contained more than once in the sum on the left. In other words, the c_i are chosen in such a way that every group element $g \in$ SU (1,1) can be written uniquely as the product of an element $h \in$ U (1) with an element c in the left coset SU (1,1)/U (1):

$$g = h\cdot c \qquad h \in \mathrm{U}\,(1)\ ,\ c \in \mathrm{SU}\,(1,1)/\mathrm{U}\,(1) \quad . \qquad (7.27)$$

A natural choice for the coset representatives is given by

$$U = \exp\frac{1}{2}(\alpha_1\tau_1 + \alpha_2\tau_2) = \begin{pmatrix} \Phi_1 & \Phi_2^* \\ \Phi_2 & \Phi_1 \end{pmatrix} \quad , \quad \Phi_1^2 - \Phi_2^*\Phi_2 = 1, \qquad (7.28)$$

with Φ_1 (is real and ≥ 1) and Φ_2 given by

$$\Phi_1 = \cosh\left(\frac{1}{2}\alpha\right) \quad , \quad \alpha^2 \equiv \alpha_1^2 + \alpha_2^2 \quad , \tag{7.29}$$
$$\Phi_2 = \sinh\left(\frac{1}{2}\alpha\right)(\hat{\alpha}_1 + i\,\hat{\alpha}_2) \quad , \quad \hat{\alpha}_i \equiv \frac{\alpha_i}{\alpha}\ (i = 1,2) \quad .$$

On the other hand the general form of an element of the U(1) subgroup of SU (1,1) is given by:

$$U = \exp \frac{i}{2}\Lambda\tau_3 = \begin{pmatrix} \exp\left(\frac{i}{2}\Lambda\right) & 0 \\ 0 & \exp\left(-\frac{i}{2}\Lambda\right) \end{pmatrix} \quad . \tag{7.30}$$

Hence each group element U of SU (1,1) can be written as the product of an element in U (1) with an element in the coset SU(1,1)/U (1), which both can be written into the exponential form. For instance, the group element U given in eq. (7.23) can be written as

$$U = \exp(i\pi\tau_3) \cdot \exp(-\rho\tau_1) \quad . \tag{7.31}$$

Because of the constraint $\phi^\alpha\phi_\alpha = 1$, a general SU (1,1) doublet Φ_α describes 3 field degrees of freedom. Now assume that Φ transforms both under rigid SU (1,1) transformations and local chiral U (1) transformations according to

$$\begin{aligned} (U(\phi))' &= S\, U(\phi) \quad , \qquad &&(\text{rigid SU }(1,1)) \\ \phi'_\alpha &= \exp(-i\Lambda)\,\Phi_\alpha \quad , \qquad &&(\text{local U }(1)) \end{aligned} \tag{7.32}$$

with S an element of SU (1,1) and U (Φ) the matrix given in (7.14). The extra local U (1) invariance allows us to remove one further degree of freedom by a choice of gauge. For instance, one may impose the following gauge condition (notice that we always have $|\Phi_1| \geq 1$):

$$\Phi_1^* = \Phi_1 \quad . \tag{7.33}$$

In that case Φ describes 2 field degrees of freedom. After imposing the gauge choice (7.33) Φ corresponds to a representation of the coset space SU (1,1)/U (1) (see eq. (7.28)). Such a coset representative can be parametrized, modulo local U (1) gauge transformations, in terms of a singe complex variable. A possible parametrization consistent with (7.33) is

$$\Phi_\alpha = \frac{1}{(1 - |C|^2)^{\frac{1}{2}}} (1,C) \qquad , \quad |C| \leq 1 \qquad . \qquad (7.34)$$

The variable $C = \Phi_2/\Phi_1$ is called the projective coordinate of the coset space SU (1,1)/U (1).

After imposing the gauge condition (7.33) both the local U (1) and rigid SU (1,1) invariance are broken and we are left with a rigid U (1) symmetry. This symmetry consists of the previous local U (1)(see eq. (7.32)) but now restricted to spacetime-independent transformations, combined with the U (1) subgroup of SU (1,1) (see eq. (7.30)) in such a way that (7.33) remains unaffected. This is specified by the following decomposition rule (cf. eq. (3.38)):

$$\begin{aligned}(\text{rigid U }(1))' \, (\Lambda_{U\,(1)}) &= (\text{rigid U }(1)) \, (\Lambda = 2\,\Lambda_{U\,(1)}) \\ &\otimes (\text{local U }(1))(\Lambda(x) = \Lambda_{U\,(1)}) \, , \end{aligned} \qquad (7.35)$$

where we have used the same notation as in eq. (3.38). In (7.35) (rigid U (1)) represents the U (1) subgroup of SU (1,1). Under this group Φ transforms according to

$$(U\,(\Phi))' = \exp\,(\tfrac{i}{2}\Lambda\tau_3)\,U\,(\Phi) \qquad (7.36)$$

or

$$\begin{aligned}\Phi'_1 &= \exp\,(\tfrac{i}{2}\Lambda)\,\Phi_1 \qquad , \qquad (\text{rigid U }(1)) \\ \Phi'_2 &= \exp\,(-\tfrac{i}{2}\Lambda)\,\Phi_2 \qquad , \end{aligned} \qquad (7.37)$$

with the matrix U (Φ) defined in eq. (7.14). Applying the decomposition rule (7.35) we find that the complex scalar field C transforms under the new chiral

U (1) transformations with a weight factor $c = -2$ (cf. table 1):

$$C' = \exp(-2\, i\Lambda_{U(1)})\, C \quad . \quad ((\text{rigid } U(1))') \tag{7.38}$$

In the context of $N = 4$ conformal supergravity the gauge condition (7.33) also leads to a decomposition rule for the Q transformations. After imposing this condition the new Q transformations are given by the previous Q transformations augmented with a field-dependent local U(1) transformation. This explains the U (1) component in the nonlinear transformation rules, which we mentioned at the end of the previous section. In the next section we give the explicit form of this decomposition rule and present the construction of the full $N = 4$ conformal supergravity theory with a manifest rigid SU (1,1) and local U (1) invariance.

4. Transformations of $N = 4$ conformal supergravity

Assuming that the field C in the linearized transformation rules (7.11) corresponds to a parametrization of the coset space SU (1,1)/U (1) some of the transformations of $N = 4$ conformal supergravity take a unique form. Modulo an overall factor the only supersymmetry variation of an SU (1,1) doublet ϕ_α that is consistent with SU (1,1), chiral U (1) x SU (4) and dilatational invariance is of the form

$$\delta\phi_\alpha = -\,\bar{\varepsilon}^i \Lambda_i\, \varepsilon_{\alpha\beta}\phi^\beta \quad , \quad \delta\phi^\alpha = \bar{\varepsilon}_i \Lambda^i\, \varepsilon^{\alpha\beta}\phi_\beta \quad . \tag{7.39}$$

Imposing the gauge condition (7.33) this result is indeed consistent with the linearized transformation of C given in (7.11), where C is defined by (7.34). To preserve the gauge condition the supersymmetry transformations are uniformly modified by the addition of a Λ-dependent local U(1) transformation. To determine this field-dependent parameter we use that before imposing (7.33) the Q and local U (1) transformations of $(\phi_1 - \phi_1^*)$ are given by

$$\delta\,(\Phi_1 - \Phi_1^*) = \bar{\varepsilon}^i \Lambda_i \Phi_2^* - \bar{\varepsilon}_i \Lambda^i \Phi_2 - i\Lambda\,(\Phi_1 + \Phi_1^*) \quad . \qquad (7.40)$$

After imposing the gauge condition $(\Phi_1 - \Phi_1^*) = 0$ the first two terms on the right-hand side of (7.40) cancel against the third term for a special value of the parameter Λ. This is specified by the following decomposition rule:

$$Q'\,(\varepsilon^i) = Q\,(\varepsilon^i) \otimes (\text{local U (1)})(\Lambda(x) = \tfrac{1}{2}\, i\,(\bar{\varepsilon}_i \Lambda^i)\, C + c.c.). \qquad (7.41)$$

Here $Q\,(\varepsilon^i)$ and $Q'\,(\varepsilon^i)$ denote a Q-supersymmetry transformation with parameter ε^i before and after imposing the gauge condition respectively. The second term on the right-hand side of (7.41) explains the previously mentioned U (1) component in the nonlinear transformation rules.

Besides the transformation of C itself the only place where this field occurs in the linearized transformation rules is the transformation of Λ_i. In the formulation with rigid SU (1,1) and local U (1) invariance this field transforms according to

$$\delta\Lambda_i = 2\,\varepsilon^{\alpha\beta}\Phi_\alpha \not{D}\Phi_\beta \varepsilon_i + E_{ij}\varepsilon^j + \varepsilon_{ijkl}\sigma\cdot T^{kl}\varepsilon^j \quad , \qquad (7.42)$$

where we have used the covariant bilinear expression

$$\varepsilon^{\alpha\beta}\Phi_\alpha D_a \Phi_\beta = \varepsilon^{\alpha\beta}\Phi_\alpha \partial_a \Phi_\beta - \tfrac{1}{2}\,\bar{\psi}_a^i \Lambda_i \quad . \qquad (7.43)$$

In lowest order the transformation (7.42) reduces to the form given in (7.11).

The gauge field for the local U (1) transformations is not an independent field. It is only defined modulo U (1) invariant terms. One definition is

$$a_\mu = \tfrac{1}{2}\, i\, \Phi^\alpha \overleftrightarrow{\partial}_\mu \Phi_\alpha + \tfrac{1}{2}\, i\, \bar{\Lambda}^i \gamma_\mu \Lambda_i \quad , \qquad (7.44)$$

which contains a term $C^* \overleftrightarrow{\partial}_\mu C$ in lowest order. This explains the $C^* \overleftrightarrow{\partial}_\mu C$ terms in the nonlinear transformation rules, which we mentioned at the end of section 2. Using eqs. (7.32), (7.39) and (7.42) one can verify that this gauge field has the following variations:

$$
\begin{aligned}
\delta_{U(1)} a_\mu &= \partial_\mu \Lambda \quad , \\
\delta_Q a_\mu &= -\tfrac{1}{2} i \, \bar{\epsilon}_i \gamma_\mu \epsilon^{\alpha\beta} \Phi_\alpha \not{D} \, \Phi_\beta \Lambda^i + \tfrac{1}{4} i \, \bar{\Lambda}^i \gamma_\mu \epsilon^j E_{ij} \\
&\quad + \tfrac{1}{4} i \, \varepsilon_{ijkl} \bar{\Lambda}^i \gamma_\mu \sigma \cdot T^{kl} \epsilon^j - \tfrac{1}{4} i \, (\bar{\Lambda}^i \gamma^a \Lambda_j - \delta^i_j \bar{\Lambda}^k \gamma^a \Lambda_k) \, \bar{\epsilon}_i \gamma_a \psi^j_\mu + \text{c.c.} , \\
\delta_S a_\mu &= 0 \quad .
\end{aligned}
\tag{7.45}
$$

The formulation with explicit rigid SU (1,1) and local U (1) invariance offers important advantages because it restricts the nonlinearities that may occur in the full transformation rules. Clearly the rigid SU (1,1) invariance prevents non-polynomial modifications, since all invariants constructed from Φ are equal to constants. In terms of other fields such modifications were already excluded because of positive Weyl weights (some of the gauge fields have negative or zero Weyl weight but their presence is already restricted by corresponding gauge invariances). Of course, we should include derivatives on Φ as well, but $D_a \Phi$ has positive Weyl weight ($w = 1$). Hence the completion of the algebra and transformation rules will require only a few iterations. Indeed, application of the iterative procedure described in section (6.4) leads to the following Q-supersymmetry transformations:

$$
\begin{aligned}
\delta_Q \Phi_\alpha &= -\bar{\epsilon}^i \Lambda_i \varepsilon_{\alpha\beta} \Phi^\beta \quad , \\
\delta_Q \Lambda_i &= 2 \, \varepsilon^{\alpha\beta} \Phi_\alpha \not{D} \Phi_\beta \epsilon_i + E_{ij} \epsilon^j + \varepsilon_{ijkl} \, \sigma \cdot T^{kl} \epsilon^j \quad , \\
\delta_Q E_{ij} &= 2 \, \bar{\epsilon}_{(i} \not{D} \Lambda_{j)} - 2 \, \bar{\epsilon}^k \chi^{mn}{}_{(i} \varepsilon_{j)kmn} - \bar{\Lambda}_i \Lambda_j \bar{\epsilon}_k \Lambda^k + 2 \, \bar{\Lambda}_k \Lambda_{(i} \bar{\epsilon}_{j)} \Lambda^k \quad ,
\end{aligned}
$$

$$\delta_Q T_{ab}{}^{ij} = 2\,\bar\varepsilon^{[i}\hat R_{ab}{}^{j]}(Q) + \bar\varepsilon^k\sigma_{ab}\chi_k^{ij} + \frac{1}{2}\varepsilon^{ijkl}\,\bar\varepsilon_k\not{D}\sigma_{ab}\Lambda_l$$

$$- \frac{1}{3}E^{[ik}\bar\varepsilon^{j]}\sigma_{ab}\Lambda_k - \frac{2}{3}\varepsilon^{\alpha\beta}\bar\varepsilon^{[i}\sigma_{ab}\phi_\alpha\not{D}\phi_\beta\Lambda^{j]} \quad ,$$

$$\delta_Q\chi_k^{ij} = -\sigma\cdot T^{ij}\overleftrightarrow{\not{D}}\varepsilon_k - 2\,\sigma\cdot\hat R^{[i}{}_k(V)\varepsilon^{j]} - \frac{1}{2}\varepsilon^{ijlm}\not{D}E_{kl}\varepsilon_m$$

$$+ D^{ij}_{kl}\,\varepsilon^l - \frac{1}{3}\varepsilon_{klmn}E^{l[i}(\sigma\cdot T^{j]n}\varepsilon^m + \sigma\cdot T^{mn}\varepsilon^{j]})$$

$$+ \frac{1}{2}E_{kl}E^{l[i}\varepsilon^{j]} + \varepsilon^{ijlm}\varepsilon^{\alpha\beta}\phi_\alpha\not{D}\phi_\beta\sigma\cdot T_{kl}\varepsilon_m$$

$$+ \gamma^a\varepsilon_n\left(\frac{1}{2}\varepsilon^{ijln}\bar\chi^m_{lk} - \frac{1}{4}\varepsilon^{ijlm}\bar\chi^n_{lk}\right)\gamma_a\Lambda_m$$

$$+ \varepsilon^{[i}\left(\frac{1}{2}\bar\Lambda^{j]}\not{D}\Lambda_k + \frac{1}{4}\bar\Lambda_k\not{D}\Lambda^{j]}\right)$$

$$- \sigma_{ab}\varepsilon^{[i}\left(\bar\Lambda^{j]}\gamma_a D_b\Lambda_k - \frac{1}{2}\bar\Lambda_k\gamma_a D_b\Lambda^{j]}\right)$$

$$+ \frac{1}{12}\varepsilon^{ijlm}\Lambda_m\{5\,\bar\varepsilon_l(E_{kn}\Lambda^n + 2\,\varepsilon_{\alpha\beta}\phi^\alpha\not{D}\phi^\beta\Lambda_k) \qquad (7.46)$$

$$- \bar\varepsilon_k(E_{ln}\Lambda^n + 2\,\varepsilon_{\alpha\beta}\phi^\alpha\not{D}\phi^\beta\Lambda_l)\}$$

$$- \sigma\cdot T^{ij}\gamma_a\varepsilon_{[k}\bar\Lambda^l\gamma_a\Lambda_{l]} + \frac{1}{2}\sigma\cdot T^{[il}\gamma_a\varepsilon_{[k}\bar\Lambda^{j]}\gamma_a\Lambda_{l]}$$

$$+ \frac{1}{2}\varepsilon^{[i}\bar\Lambda^{j]}\Lambda^m\,\bar\Lambda_k\Lambda_m - (\text{traces}) \quad ,$$

$$\delta_Q D^{ij}_{kl} = -4\,\bar\varepsilon^{[i}\not{D}\chi^{j]}_{kl} + \varepsilon_{klmn}\,\bar\varepsilon^{[i}\{-2\,E^{j]p}\chi_p^{mn} + \sigma\cdot T^{mn}\overleftrightarrow{\not{D}}\Lambda^{j]}$$

$$+ \frac{1}{3}E^{j]m}E^{np}\Lambda_p + \frac{2}{3}\varepsilon^{\alpha\beta}\phi_\alpha\not{D}\phi_\beta\Lambda^m E^{j]n} + \sigma\cdot T^{mn}\Lambda_p\,\bar\Lambda^{j]}\Lambda^p\}$$

$$+ \bar\varepsilon^{[i}\{2\gamma_a\chi^m_{kl}\bar\Lambda^{j]}\gamma_a\Lambda_m - 4\,\varepsilon^{\alpha\beta}\phi_\alpha\not{D}\phi_\beta\sigma\cdot T_{kl}\Lambda^{j]}$$

$$+ \frac{1}{3}\Lambda_{[k}E_{l]m}\bar\Lambda^{j]}\Lambda^m + \frac{2}{3}\sigma_{ab}\varepsilon_{\alpha\beta}\phi^\alpha\not{D}\phi^\beta\Lambda^{j]}\bar\Lambda_k\sigma_{ab}\Lambda_l\}$$

$$+ \varepsilon^{ijmn}\bar\varepsilon^p T_{abkl}(2\,T_{abnp}\Lambda_m + T_{abmn}\Lambda_p) + (\text{h.c.; traceless}) \quad ,$$

$$\delta_Q e^a_\mu = \bar\varepsilon^i\gamma_a\psi_{\mu i} + \text{c.c.} \quad ,$$

$$\delta_Q\psi^i_\mu = 2\,\mathfrak{D}_\mu\varepsilon^i - \sigma\cdot T^{ij}\gamma_\mu\varepsilon_j - \varepsilon^{ijkl}\bar{\varepsilon}_j\psi_{\mu k}\Lambda_l \quad ,$$

$$\delta_Q b_\mu = \frac{1}{2}\,\bar{\varepsilon}^i\phi_{\mu i} + c\cdot c\cdot \quad ,$$

$$\delta_Q V^i_{\mu j} = \bar{\varepsilon}^i\phi_{\mu j} + \bar{\varepsilon}^k\gamma_\mu\chi^i_{kj} - \frac{1}{2}\,E^{ik}\,\varepsilon_{jkmn}\bar{\varepsilon}^m\psi^n_\mu - \frac{1}{6}\,E^{ik}\bar{\varepsilon}_j\gamma_\mu\Lambda_k$$

$$+ \frac{1}{2}\,\varepsilon^{iklm}\bar{\varepsilon}_k\,\sigma\cdot T_{lj}\gamma_\mu\Lambda_m + \frac{1}{3}\,\varepsilon_{\alpha\beta}\bar{\varepsilon}^i\gamma_\mu\phi^\alpha \not{D}\phi^\beta\Lambda_j$$

$$- \frac{1}{4}\,\varepsilon^{iklp}\varepsilon_{jmnp}\,\bar{\varepsilon}^m\gamma_a\psi_{\mu k}\bar{\Lambda}_l\gamma_a\Lambda^n - (\text{h.c.; traceless}) \quad .$$

Under S supersymmetry with parameter η^i the fields transform as follows:

$$\delta_S\phi_\alpha = 0 \quad ,$$

$$\delta_S\Lambda_i = 0 \quad ,$$

$$\delta_S E_{ij} = 2\,\bar{\eta}_{(i}\Lambda_{j)} \quad ,$$

$$\delta_S T^{ij}_{ab} = -\frac{1}{2}\,\varepsilon^{ijkl}\bar{\eta}_k\sigma_{ab}\Lambda_l \quad ,$$

$$\delta_S\chi^{ij}_k = \sigma\cdot T^{ij}\eta_k + \frac{2}{3}\,\delta^{[i}_k\sigma\cdot T^{j]l}\eta_l - \frac{1}{2}\,\varepsilon^{ijlm}E_{kl}\eta_m \qquad (7.47)$$

$$- \frac{1}{4}\,\bar{\Lambda}_k\gamma_a\Lambda^{[i}\gamma_a\eta^{j]} + \frac{1}{12}\,\delta^{[i}_k\left(\bar{\Lambda}_l\gamma_a\Lambda^l\gamma_a\eta^{j]} - \bar{\Lambda}_l\gamma_a\Lambda^{j]}\gamma_a\eta^l\right) \quad ,$$

$$\delta_S D^{ij}_{kl} = 0 \quad ,$$

$$\delta_S e_{\mu a} = 0 \quad ,$$

$$\delta_S\psi^i_\mu = -\gamma_\mu\eta^i \quad ,$$

$$\delta_S b_\mu = -\frac{1}{2}\,\bar{\psi}^i_\mu\eta_i + c\cdot c\cdot \quad ,$$

$$\delta_S V^i_{\mu j} = -\left(\psi^i_\mu\eta_j - \frac{1}{4}\,\delta^i_j\bar{\psi}^k_\mu\eta_k\right) - \text{h.c.} \quad .$$

In (7.46) and (7.47) the derivatives D_μ are covariant with respect to the superconformal and local U(1) symmetries. The curvatures $\hat{R}$ (Q) and $\hat{R}$ (V) are the fully covariant curvatures of ψ_μ^i and $V_{\mu j}^i$ respectively. Their explicit form is given in the original paper (see the references at the end of this chapter). In that paper one can also find the Q and S transformations of the dependent gauge fields $\omega_\mu^{ab}, f_\mu^a, \phi_\mu^i$ and of some covariant curvatures as well as many other details about N = 4 conformal supergravity.

In deriving the transformations (7.46) and (7.47) we have used the conventional constraints (5.46) for the fully covariant curvatures $\hat{R}$ (P), $\hat{R}$ (M) and $\hat{R}$ (Q). These curvatures are an extension of the SU (2,2|4) curvatures R (P), R (M), R (Q) (see the original paper and confer eq. (5.45)):

$$\begin{aligned} &\hat{R}_{\mu\nu}^a (P) - R_{\mu\nu}^a (P) = 0 , \\ &\hat{R}_{\mu\nu}^{ab} (M) - R_{\mu\nu}^{ab} (M) = - \bar{\psi}_{[\mu i}\gamma_{\nu]} \hat{R}_{ab}^i (Q) + \bar{\psi}_{\mu i} T_{ab}^{ij} \psi_{\nu j} + \text{c.c.} , \\ &\hat{R}_{\mu\nu}^{\ i} (Q) - R_{\mu\nu}^i (Q) = - \sigma \cdot T^{ij} \gamma_{[\mu} \psi_{\nu] j} + \frac{1}{2} \varepsilon^{ijkl} \bar{\psi}_{\mu j} \psi_{\nu k} \Lambda_l . \end{aligned} \tag{7.48}$$

The first term on the right-hand side of the second equation of (7.48) corresponds to the extra term in the transformation of ω_μ^{ab} in the presence of the conventional constraints. This term has been discussed in section (5.4) (cf. eq. (5.48)). The other terms correspond to the matter field-dependent modifications in the transformations of ω_μ^{ab} and ψ_μ^i.

We finally present the commutator of two Q transformations, and of a Q and an S transformation. These commutators have modifications in the form of field-dependent symmetry transformations. The results are given by

$$\begin{aligned} [\delta_Q (\varepsilon_1), \delta_Q (\varepsilon_2)] = \delta_{\text{g.c.t.}}^{\text{cov}} (\xi^\mu) + \delta_\Sigma (\varepsilon^{ab}) + \delta_Q (\varepsilon_3^i) + \delta_S (\eta^i) \\ + \delta_{SU\,(4)} (\Lambda_j^i) + \delta_{U\,(1)} (\Lambda) + \delta_K (\Lambda_K^a) , \end{aligned} \tag{7.49}$$

with $\delta_{\text{g.c.t.}}^{\text{cov}}$ a covariant translation (see section (4.6)),and with the parameters of the transformations on the right-hand side of (7.49) equal to

$$
\begin{aligned}
\xi^{\mu} &= 2\,\bar{\varepsilon}_1^i\,\gamma_\mu \varepsilon_{2i} + \text{c.c.} \quad , \\
\varepsilon^{ab} &= 4\,\bar{\varepsilon}_1^i\,\varepsilon_2^j\,T_{abij} + \text{c.c.} \quad , \\
\varepsilon_3^i &= \varepsilon^{ijkl}\,\bar{\varepsilon}_{1k}\varepsilon_{2l}\Lambda_j \quad , \\
\eta^i &= -2\,\bar{\varepsilon}_1^k \varepsilon_2^l \chi^i_{kl} - \frac{1}{2}\,(\bar{\varepsilon}_2^k \gamma_a \varepsilon_{1j} + \text{h.c.})\,(\gamma_a \chi_k^{ij} + \varepsilon^{ijlm}\sigma\cdot T_{km}\gamma_a \Lambda_l) \\
&\quad - \frac{1}{12}\,(\bar{\varepsilon}_2^i \gamma_a \varepsilon_{1j} - \delta_j^i \bar{\varepsilon}_2^l \gamma_a \varepsilon_{1l} + \text{h.c.}) \times \\
&\qquad \times \gamma_a\,(E^{jk}\Lambda_k - 2\,\varepsilon^{\alpha\beta}\phi_\alpha \not{D}\phi_\beta \Lambda^j) + \frac{1}{2}\,\varepsilon^{ijkl}\bar{\varepsilon}_{1k}\varepsilon_{2l}\not{D}\Lambda_j \\
&\quad + \frac{2}{3}\,\bar{\varepsilon}_2^{[i}\varepsilon_1^{j]}\,(E_{jk}\Lambda^k + 2\,\varepsilon_{\alpha\beta}\phi^\alpha \not{D}\phi^\beta \Lambda_j) \quad , \\
\Lambda_j^i &= E^{ik}\,\varepsilon_{klmj}\bar{\varepsilon}_2^l \varepsilon_1^m + \frac{1}{2}\,(\bar{\varepsilon}_2^k \gamma_a \varepsilon_{1j} + \text{h.c.})\,\bar{\Lambda}^i \gamma_a \Lambda_k \\
&\quad - \frac{1}{4}\,(\bar{\varepsilon}_2^k \gamma_a \varepsilon_{1k} + \text{c.c.})\,\bar{\Lambda}^i \gamma_a \Lambda_j \qquad\qquad (7.50) \\
&\quad - \frac{1}{4}\,(\bar{\varepsilon}_2^i \gamma_a \varepsilon_{1j} + \text{h.c.})\,\bar{\Lambda}^k \gamma_a \Lambda_k - (\text{h.c.; traceless}) \quad , \\
\Lambda &= -\frac{i}{2}\,(\bar{\varepsilon}_2^i \gamma_a \varepsilon_{1j} + \text{h.c.})\,(\bar{\Lambda}^j \gamma_a \Lambda_i - \delta_i^j \bar{\Lambda}^k \gamma_a \Lambda_k) \quad , \\
\Lambda_K^a &= \frac{2}{3}\,\bar{\varepsilon}_{2i}\gamma_b \varepsilon_1^j R_{abj}^{\ \ \ i}(V) + \frac{8}{3}\,\bar{\varepsilon}_{2i}\varepsilon_{1j}\,D_b T_{ab}^{ij} \\
&\quad - \frac{1}{3}\,\bar{\varepsilon}_{2i}\gamma_b \varepsilon_1^i \varepsilon^{abcd}\,(D_c \phi^\alpha D_d \phi_\alpha - \frac{1}{4}\,(\bar{\Lambda}^j \gamma_c D_d \Lambda_j - \text{c.c.})) \\
&\qquad + \bar{\varepsilon}_{2i}\sigma\cdot T_{jk}\gamma^a \sigma\cdot T^{ij}\varepsilon_1^k + \text{c.c.} \quad .
\end{aligned}
$$

Furthermore we have

$$
[\delta_Q(\varepsilon)\ ,\ \delta_S(\eta)] = \delta_D(\Lambda_D) + \delta_\Sigma(\varepsilon^{ab}) + \delta_S(\eta_2^i) + \delta_{SU(4)}(\Lambda_j^i) + \delta_\kappa(\Lambda_K^a)\ , \qquad (7.51)
$$

with the following transformation parameters:

$$
\begin{aligned}
\Lambda_D &= - \bar{\eta}_i \varepsilon^i + \text{c.c.} \,, \\
\varepsilon^{ab} &= - 2\, \bar{\eta}_i \sigma^{ab} \varepsilon^i + \text{c.c.} \,, \\
\Lambda^i_j &= - 2\, \bar{\varepsilon}^i \eta_j + \frac{1}{2} \delta^i_j \bar{\varepsilon}^k \eta_k - \text{h.c.} \,, \qquad (7.52) \\
\eta^i_2 &= - \frac{1}{4} \varepsilon^{ijkl} \bar{\eta}_k \gamma_a \varepsilon_j \gamma_a \Lambda_l \,, \\
\Lambda^a_K &= \frac{1}{3} \bar{\eta}_i \sigma \cdot T^{ij} \gamma_a \varepsilon_j + \text{c.c.} \,.
\end{aligned}
$$

5. Outlook

In this chapter we have presented conformal supergravity theories for $N \leq 4$. The complete nonlinear transformation rules and the corresponding commutator algebra of the superconformal gauge transformations were constructed for $N = 4$. This was done in a formulation with rigid SU (1,1) and local U (1) invariance.

The next step in this program is the construction of the corresponding Poincaré supergravity theories. To carry out this program requires knowledge of a variety of superconformal multiplets, which can be used to provide the necessary compensating fields when coupled to conformal supergravity. For $N = 1$ and $N = 2$ such a procedure has been applied successfully and in that context the $N = 1$ and $N = 2$ Weyl multiplets have been very useful in clarifying the off-shell structure of $N = 1$ and $N = 2$ Poincaré supergravity.

For $N = 4$ not much is known about off-shell representations of rigid supersymmetry and therefore the compensating mechanism has not been applied in this case. For instance, there exists no off-shell version of the $N = 4$ supersymmetric Yang-Mills theory which we have considered in section 2 to construct the $N = 4$, $d = 4$ supercurrent. W. Siegel and M. Roček have given the following counting argument that for the $N = 4$ super Yang-Mills theory the auxiliary field problem cannot have a solution within any previously known framework. On the one side the number n_F of fermionic field components of an off-shell Yang-Mills multiplet without central charge must be an integral multiple of the number of fermionic field components of the smallest off-shell supersymmetry representation, which has dimensionality 128 + 128 (see the table in section (5.4)). Thus, we have $n_F = n \times 128$ (n integer). On the other hand the physical Bose fields A_μ and ϕ^{ij} of the Yang-Mills multiplet have an even number of SU (4) indices. Since the supersymmetry generators Q^i_α have one spinor and one SU (4) index, all Fermi fields are spinors with an odd number of SU (4) indices. All SU (4) tensors with an odd number of SU (4) indices contain an integral multiple of four components. Hence each Fermi spinor field has an integral multiple of 4 x 4 = 16 components. Since all spinor auxiliary fields occur in pairs (one as the Lagrange multiplier for the other), the total Fermi dimensionality n_F of the off-shell representation is thus determined modulo 2 x 16 = 32 by the total dimensionality d = 16 of the physical Fermi fields ψ^i of the Yang-Mills multiplet, i.e. $n_F = 16 \bmod 32$. The compatibility of the first and second condition on the total dimensionality n_F of off-shell Fermi components thus gives the restriction 128 = 16 mod 32, which is clearly not consistent.

One way to circumvent this counting argument is by allowing the introduction of central charges. These are bosonic operators Z^{ij} (antisymmetric in the indices i and j) which occur in a modification of the super-Poincaré algebra (2.66). This modification has the form

$$\{Q^i_\alpha, Q^j_\beta\} = Z^{ij} C_{\alpha\beta} \quad , \qquad (7.53)$$

while the Z^{ij} commute with all other elements of the algebra. In (7.53) we have used the same (chiral) notations as in eq. (5.14). However, the presence of the central charge operators restricts the structure of the internal symmetry group. For instance, there exists an off-shell formulation of the super Yang-Mills theory with central charge, but it has only an invariance with respect to the Sp (4) subgroup of SU (4). In addition, one cannot extend the super-Poincaré algebra with central charge to include conformal transformations as well. Therefore multiplets with central charge do not fit in with the compensating mechanism that we want to apply.

One can give the following arguments (see the references at the end of this chapter) which suggest that a possible set of compensating field multiplets for

$N = 4$ Poincaré supergravity is six $N = 4$ (abelian) Yang-Mills multiplets. The field content of six (on-shell) Yang-Mills multiplets is given in the table below, where we have taken the six to be a 6 of SU (4). In this table we have also listed the field content of $N = 4$ (on-shell) Poincaré supergravity and the Weyl multiplet. We see that the Yang-Mills vectors A_μ^{ij} are the only physical fields of Poincaré supergravity not already contained in the Weyl multiplet. The Yang-Mills scalar ϕ and spinors ψ^i may be used to compensate for dilatations and S supersymmetry, while ϕ^i_j may be used to compensate for SU (4). The remaining fields ψ_k^{ij} and ϕ_{kl}^{ij} are auxiliary and of the right structure to act as Lagrange multipliers for

spin	$N = 4$ Poincaré	$N = 4$ Weyl	$6 \times N = 4$ Maxwell
2	e_μ^a (1)	e_μ^a (1)	
3/2	ψ_μ^i (4)	ψ_μ^i (4)	
1	A_μ^{ij} (6)	$V_{\mu j}^i$ (15), T_{ab}^{ij} (6^c)	A_μ^{ij} (6)
1/2	ψ^i (4)	Λ_i (4), χ_k^{ij} (20)	ψ^i (4), ψ_k^{ij} (20)
0	ϕ (1)	C (1^c), D_{kl}^{ij} (20), E_{ij}(10)	ϕ (1), ϕ^i_j (15), ϕ_{kl}^{ij} (20)

table 2. Field content of the $N = 4$ (on-shell) Poincaré supergravity, Weyl and six (on-shell) Maxwell multiplets. The numbers between brackets denote the SU (4) representation assignments of the fields.

the high dimension auxiliary fields χ_k^{ij} and D_{kl}^{ij} in the Weyl multiplet. Hence, coupling the $N = 4$ Weyl multiplet to six Maxwell multiplets and fixing the superconformal gauges $\phi = 1$, $\psi^i = \phi^i_j = b_\mu = 0$, should give $N = 4$ Poincaré supergravity with the missing auxiliary fields being those of the compensating multiplets. Giving that this is so, the full off-shell structure of $N = 4$ Poincaré supergravity will have to await the resolution of the auxiliary field problem for the $N = 4$ Yang-Mills theory.

An alternative way to study the auxiliary field problem is by leaving $d = 4$ dimensions and by considering supergravity models in as high a dimension as possible. The motivation is that by studying supergravity in another context than the familiar four-dimensional one we may learn something about the way in which Poincaré supergravity can be realized in four dimensions. Recently we have considered an implementation of the superconformal ideas presented in this thesis in the context of supergravity in ten dimensions (this corresponds to $N = 4$ in four dimensions).

The linearized transformation rules of the $N = 1$, $d = 10$ Weyl multiplet can be found from an analysis of the $d = 10$ Maxwell supercurrent. This supercurrent is reducible; it contains a submultiplet of 128 + 128 components, whereas the remaining degrees of freedom form a constrained scalar (chiral) superfield. In the nonabelian case the scalar superfield part of the current is unconstrained. The 128 + 128 current submultiplet is associated with the fields of conformal supergravity, because it is the smallest off-shell multiplet that contains the energy-momentum tensor. However, a nontrivial aspect is that the decomposition of the $d = 10$ supercurrent into its two submultiplets is realized in a nonlocal way. As a consequence the linear transformation rules of the $d = 10$ Weyl multiplet contain nonlocal terms. One can avoid the nonlocal character of the transformations by introducing new fields which are subject to differential constraints. Hence these fields do not represent new degrees of freedom. However, the presence of the differential constraints presents an obstacle for a straightforward application of the compensating mechanism.

In a recent paper (see the references at the end of this chapter) we have shown that it is in principle straightforward to avoid the differential constraints by introducing new degrees of freedom. After ignoring the constraints one adds new fields in the transformation laws of the superconformal fields whose variations are then required to reestablish the closure of the superconformal algebra. Subsequently the results may be completed by iteration. It is not obvious that such a program will be successful for the full nonlinear theory, although there are no conceivable problems at the linearized level. A crucial point is, that the original commutation relations of the superconformal algebra will be modified by terms that contain the new fields.

In order to avoid the differential constraints one must at least add a scalar multiplet to the Weyl multiplet. This is called the minimal field representation. We have investigated some nonlinear aspects of this minimal field representation. We find a number of nonlinear modifications associated with the scalar submultipet, and at this stage a completion along these lines seems perfectly possible. These results are in fact relevant for the off-shell formulation of linearized d = 10 Poincaré supergravity, which has been obtained recently by P. Howe, H. Nicolai and A. Van Proeyen (H, N. V.P.).

Using a scalar multiplet of Lagrange multipliers it is possible to construct a superconformally invariant action for the minimal field representation. After imposing the appropriate superconformal gauge conditions one obtains a Poincaré supergravity action with its auxiliary fields. We expect the results to coincide with those of (H, N, V.P.). This is confirmed by a calculation of some of the new terms in the action, which has many of the same ingredients although the superconformal scheme leads to a different arrangement of terms.

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APPENDIX A. NOTATIONS AND CONVENTIONS

In this appendix we collect the notations and conventions used throughout the text. When we consider global Lorentz invariance, we denote vectors by the indices μ, ν,... or a, b,... and spinors by the indices α, β,.... Both run from 1 to 4 in four-dimensional spacetime. We use the Pauli metric

$$\delta_{\mu\nu} = \text{diag.}\ (+,+,+,+) \quad , \qquad (A.1)$$

with imaginary time components of four-vectors:

$$k_\mu = (\vec{k},\ k_4) = (\vec{k},\ i\ k_0) \quad . \qquad (A.2)$$

Hence there is no need for distinguishing upper and lower indices. In all cases repeated indices imply a summation, unless explicitly stated otherwise. The four-dimensional Levi-Civita tensor is defined by

$$\varepsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & , (\mu\nu\rho\sigma) = \text{even permutation of } (1234) \\ -1 & , (\mu\nu\rho\sigma) = \text{odd permutation of } (1234) \\ 0 & . \text{ otherwise} \end{cases} \qquad (A.3)$$

This tensor satisfies the relations

$$\begin{aligned} \varepsilon^{\mu'\nu'\rho'\sigma'}\ \varepsilon_{\mu\nu\rho\sigma} &= \delta^{\mu'\nu'\rho'\sigma'}_{\mu\nu\rho\sigma} \quad , \\ \varepsilon^{\mu\nu'\rho'\sigma'}\ \varepsilon_{\mu\nu\rho\sigma} &= \delta^{\nu'\rho'\sigma'}_{\nu\rho\sigma} \quad , \\ \varepsilon^{\mu\nu\rho'\sigma'}\ \varepsilon_{\mu\nu\rho\sigma} &= 2!\ \delta^{\rho'\sigma'}_{\rho\sigma} \quad , \qquad (A.4) \\ \varepsilon^{\mu\nu\rho\sigma'}\ \varepsilon_{\mu\nu\rho\sigma} &= 3!\ \delta^{\sigma'}_{\sigma} \quad , \\ \varepsilon^{\mu\nu\rho\sigma}\ \varepsilon_{\mu\nu\rho\sigma} &= 4! \quad , \end{aligned}$$

where $\delta^{\nu'\rho'..}_{\nu\rho..}$ are the standard permutation symbols:

$$\delta^{\rho'\sigma'}_{\rho\sigma} = \delta^{\rho'}_{\rho}\,\delta^{\sigma'}_{\sigma} - \delta^{\sigma'}_{\rho}\,\delta^{\rho'}_{\sigma} \quad \text{etc.} \qquad . \tag{A.5}$$

When we discuss local Lorentz invariance, we use the indices $\mu,\nu,..$ to denote world indices, whereas a, b, .. denote local Lorentz indices. Vierbeins e^a_μ and inverse vierbeins e^μ_a convert world indices into local Lorentz indices and vice versa. World tensors with upper and lower indices are related by a contraction with the metric tensor

$$g_{\mu\nu} = e^a_\mu e^a_\nu \tag{A.6}$$

in the following way:

$$\xi_\mu = g_{\mu\nu}\xi^\nu \qquad . \tag{A.7}$$

The summation convention for world indices implies a contraction over $g_{\mu\nu}$.

We now discuss gamma matrices in four dimensions. We will do this in the context of global Lorentz invariance. When considering the local case the Dirac algebra remains unchanged when one defines all elements with local Lorentz indices. The four-dimensional Dirac algebra is defined by

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\,\delta_{\mu\nu} \qquad . \tag{A.8}$$

Here the γ_μ are four 4 x 4 hermitean matrices:

$$\gamma^\dagger_\mu = \gamma_\mu \qquad . \tag{A.9}$$

One can construct explicit representations by taking tensor products of the standard Pauli matrices τ_i (i = 1,2,3):

$$\gamma_1 = \tau_1 \otimes \mathbb{1} \quad ,$$
$$\gamma_2 = \tau_3 \otimes \tau_3 \quad ,$$
$$\gamma_3 = \tau_3 \otimes \tau_1 \quad ,$$
$$\gamma_4 = \tau_2 \otimes \mathbb{1} \quad , \tag{A.10}$$

with

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{A.11}$$

This representationis called the Majorana representation. From the gamma matrices we define the following quantities:

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 \quad , \quad \gamma_5^\dagger = \gamma_5 \quad ,$$
$$\sigma_{\mu\nu} = \frac{1}{4} (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \quad , \quad \sigma_{\mu\nu}^\dagger = - \sigma_{\mu\nu} \quad . \tag{A.12}$$

In the Majorana representation γ_5 is given by

$$\gamma_5 = - \tau_3 \otimes \tau_2 \quad . \tag{A.13}$$

The set of sixteen 4 x 4 matrices

$$\Gamma_A = (\mathbb{1}, \gamma_\mu, 2\, i\, \sigma_{\mu\nu}, i\, \gamma_5\gamma_\mu, \gamma_5) \tag{A.14}$$

is complete and satisfies the relations

$$\Gamma_A^\dagger = \Gamma_A \quad ,$$
$$\Gamma_A^2 = \mathbb{1} \quad ,$$
$$\mathrm{Tr}\, (\Gamma_A\Gamma_B) = 4\, \delta_{AB} \quad ,$$
$$(\Gamma_A)_{\alpha\beta}\, (\Gamma_A)_{\gamma\delta} = 4\, \delta_{\alpha\delta}\delta_{\beta\gamma} \quad . \tag{A.15}$$

Using these relations one easily verifies that an arbitrary 4 x 4 matrix X can be expanded in terms of the Γ_A according to

$$X = \frac{1}{4} \sum_A \mathrm{Tr}\,(X\Gamma_A)\,\Gamma_A \quad . \qquad \text{(A.16)}$$

We define a charge conjugation matrix C by

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T \quad . \qquad \text{(A.17)}$$

Here the superscript T denotes transposition. The following symmetry relations hold:

$$\left.\begin{array}{lll} C, \gamma_5 C, \gamma_5\gamma_\mu C & \text{are antisymmetric} & , \\ \gamma_\mu C, \sigma_{\mu\nu} C & \text{are symmetric} & . \end{array}\right\} \qquad \text{(A.18)}$$

The matrix C has the following form in the Majorana representation (A-10):

$$C = -\tau_2 \otimes \mathbb{1} \quad . \qquad \text{(A.19)}$$

The four-dimensional representation space of the Dirac algebra is called spinor space. The elements of this space, the spinors, are denoted by ψ_α $(\alpha = 1..4)$. In quantum field theory ψ_α (x) represents a field with spin 1/2. For consistency such fields have to be anticommuting, i.e.:

$$\psi_\alpha\psi_\beta = -\psi_\beta\psi_\alpha \quad . \qquad \text{(A.20)}$$

The Pauli conjugate $\bar\psi$ of a spinor is defined by

$$\bar\psi = \psi^\dagger\gamma_4 \quad . \qquad \text{(A.21)}$$

A Majorana spinor is defined by the relation

$$\psi = C\,\bar\psi^T \quad . \qquad \text{(A.22)}$$

In the Majorana representation this is just a reality condition:

$$\psi = (\tau_2 \otimes \mathbb{1})\,(\tau_2 \otimes \mathbb{1})\,\psi^* = \psi^* \quad . \qquad \text{(A.23)}$$

The completeness relation (A.16) may be used to expand the product of two spinors:

$$\psi_1\bar{\psi}_2 = -\frac{1}{4}\{\bar{\psi}_2\psi_1 + \bar{\psi}_2\gamma_\mu\psi_1\gamma_\mu - 2\,\bar{\psi}_2\sigma_{\mu\nu}\psi_1\sigma_{\mu\nu} - \bar{\psi}_2\gamma_5\gamma_\mu\psi_1\gamma_5\gamma_\mu + \bar{\psi}_2\gamma_5\psi_1\gamma_5\} \quad . \tag{A.24}$$

This is called the Fierz rearrangement formula. Using a chiral decomposition of the spinors

$$\begin{aligned} \psi &= \frac{1}{2}(1+\gamma_5)\psi + \frac{1}{2}(1-\gamma_5)\psi \equiv \psi^{\cdot} + \psi_{.} \quad , \\ \bar{\psi} &= \bar{\psi}\,\frac{1}{2}(1+\gamma_5) + \bar{\psi}\,\frac{1}{2}(1-\gamma_5) \equiv \bar{\psi}^{\cdot} + \bar{\psi}_{.} \quad , \end{aligned} \tag{A.25}$$

one can derive an alternative form of the Fierz rearrangement formula:

$$\begin{aligned} \psi_{1.}\bar{\psi}_2^{\cdot} &= \{-\frac{1}{2}\bar{\psi}_2^{\cdot}\gamma_\mu\psi_{1.}\gamma_\mu\}\,\frac{1}{2}(1+\gamma_5) \quad , \\ \psi_1^{\cdot}\bar{\psi}_2^{\cdot} &= \{-\frac{1}{2}\bar{\psi}_2^{\cdot}\psi_1^{\cdot} + \frac{1}{2}\bar{\psi}_2^{\cdot}\sigma_{\mu\nu}\psi_1^{\cdot}\sigma_{\mu\nu}\}\,\frac{1}{2}(1+\gamma_5) \quad . \end{aligned} \tag{A.26}$$

We denote commutators and anticommutators by

$$\begin{aligned} [\Gamma_A, \Gamma_B] &\equiv \Gamma_A\Gamma_B - \Gamma_B\Gamma_A \quad , \\ \{\Gamma_A, \Gamma_B\} &\equiv \Gamma_A\Gamma_B + \Gamma_B\Gamma_A \quad , \end{aligned} \tag{A.27}$$

and have the following conventions on symmetrization:

$$\Gamma_{[A}\Gamma_{B]} = \frac{1}{2}(\Gamma_A\Gamma_B - \Gamma_B\Gamma_A) \quad ,$$

$$\Gamma_{(A}\Gamma_{B)} = \frac{1}{2}(\Gamma_A\Gamma_B + \Gamma_B\Gamma_A) \quad , \qquad (A.28)$$

$$\Gamma_{[A}\Gamma_B\Gamma_{C]} = \frac{1}{3!}(\Gamma_A\Gamma_B\Gamma_C + \Gamma_C\Gamma_A\Gamma_B + \Gamma_B\Gamma_C\Gamma_A - \Gamma_B\Gamma_A\Gamma_C - \Gamma_A\Gamma_C\Gamma_B - \Gamma_C\Gamma_B\Gamma_A) \text{ etc..}$$

We finally give some useful identities:

$$\{\gamma_\mu,\sigma_{\nu\rho}\} = \varepsilon_{\mu\nu\rho\sigma}\gamma_5\gamma_\sigma \quad ,$$

$$[\gamma_\mu,\sigma_{\nu\rho}] = \gamma_\rho\delta_{\nu\mu} - \gamma_\nu\delta_{\rho\mu} \quad ,$$

$$\{\sigma_{\mu\nu},\sigma_{\rho\sigma}\} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\gamma_5 - \frac{1}{2}\delta^{\rho\sigma}_{\mu\nu} \quad ,$$

$$[\sigma_{\mu\nu},\sigma_{\rho\sigma}] = 4\,\sigma_{[\nu}{}^{[\rho}\delta_{\mu]}{}^{\sigma]} \quad , \qquad (A.29)$$

$$\gamma_\mu\gamma_\nu\gamma_\mu = -2\,\gamma_\nu \quad ,$$

$$\gamma_\mu\sigma_{\nu\rho}\gamma_\mu = 0 \quad ,$$

$$\sigma_{\mu\nu}\sigma_{\rho\sigma}\sigma_{\mu\nu} = \sigma_{\rho\sigma} \quad ,$$

$$\sigma_{\mu\nu} = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\sigma_{\rho\sigma}\gamma_5 \quad .$$

APPENDIX B. THE CHIRAL NOTATION

In this appendix we explain the chiral notation, which we use throughout this thesis for spinors which combine into representations of (S)U(N) (we consider SU(N) in this appendix).

By definition SU(N) is the group of complex N x N matrices with unit determinant that leave the metric δ = diag (+,+,..,+) invariant. There are (N^2-1) independent matrices that satisfy this requirement. The Lie algebra su(n) of SU(N) consists of all N x N antihermitean traceless matrices. For a N-component vector we use the following notation:

$$V^i \equiv (V_i)^* \qquad . \; (i = 1..N) \tag{B.1}$$

In this section V^i and V_i transform according to

$$\begin{aligned} \delta V^i &= \Lambda^i{}_j V^j \qquad , \quad (N \text{ representation}) \\ \delta V_i &= \Lambda_i{}^j V_j \qquad , \quad (\bar{N} \text{ representation}) \end{aligned} \tag{B.2}$$

where Λ is a N x N antihermitean traceless matrix parameter that characterizes the infinitesimal SU(N) transformation:

$$(\Lambda^i{}_j)^* \equiv \Lambda_i{}^j = -\Lambda^j{}_i , \; \Lambda^i{}_i = 0 \qquad . \tag{B.3}$$

Once can verify the invariance of the metric δ^i_j (i,j = i..N) under such transformations:

$$\delta(\delta^i_j) = \Lambda^i{}_k \delta^k_j + \Lambda_j{}^k \delta^i_k = \Lambda^i{}_j + \Lambda_j{}^i = 0 \qquad . \tag{B.4}$$

In addition, one can verify that the bilinear $V^i W_i$ (i = 1..N) is a scalar:

$$\delta(V^i W_i) = \Lambda^i{}_k V^k W_i + \Lambda_i{}^k V^i W_k = 0 \qquad . \tag{B.5}$$

For N Majorana spinors ψ_M^i, with the property that the chiral projections $\frac{1}{2}(1+\gamma_5)\,\psi_M^i$ transform under SU(N) as V^i in (B.2), we use the following notations:

$$\psi^i \equiv \frac{1}{2}(1+\gamma_5)\,\psi_M^i \quad ,$$
$$\psi_i \equiv C\,\overline{(\psi^i)}^T = \frac{1}{2}(1-\gamma_5)\,\psi_M^i \quad . \tag{B.6}$$

In the Majorana representation (A.10) the second definition reduces to $\psi_i \equiv (\psi^i)^*$. For the Pauli conjugate spinors we use the notations.

$$\bar\psi^i \equiv \bar\psi_M^i\,\frac{1}{2}(1+\gamma_5) \quad ,$$
$$\bar\psi_i \equiv \bar\psi_M^i\,\frac{1}{2}(1-\gamma_5) \quad . \tag{B.7}$$

This notation is consistent with (B.6) in the sense that

$$\overline{(\psi^i)} = \bar\psi_i \quad . \tag{B.8}$$

If the chiral projections $\frac{1}{2}(1+\gamma_5)\,\psi_M^i$ transform under SU(N) as V_i in (B.2) we use instead of (B.6) and (B.7) the definitions:

$$\psi^i \equiv \frac{1}{2}(1-\gamma_5)\,\psi_M^i \quad , \tag{B.9}$$
$$\psi_i \equiv C\,\overline{(\psi^i)}^T = \frac{1}{2}(1+\gamma_5)\,\psi_M^i \quad ,$$

and

$$\bar\psi^i \equiv \bar\psi_M^i\,\frac{1}{2}(1-\gamma_5) \quad , \tag{B.10}$$
$$\bar\psi_i \equiv \bar\psi_M^i\,\frac{1}{2}(1+\gamma_5) \quad .$$

If no confusion is possible we often drop the index M of a Majorana spinor.

For N = 1 the above still applies, but with SU(N) replaced by U(1). In that case we denote the index i (i=1) by a dot. If no confusion is possible we sometimes omit this dot in the text, but not in the formulae.

SAMENVATTING

Alle tot nu toe bekende deeltjes in de natuur laten zich onderverdelen in twee klassen: deeltjes met heeltallige en deeltjes met halftallige spin. De eerste heten bosonen, terwijl de laatste fermionen genoemd worden. Supersymmetrie is de enige symmetrie die in staat is deze opsplitsing in bosonen en fermionen te doorbreken. In de aanwezigheid van deze symmetrie komen bosonen en fermionen voor als gelijkwaardige partners van een gemeenschappelijk multiplet en dienen in hun onderlinge samenhang bestudeerd te worden.

Een belangrijk gevolg van supersymmetrie is dat de oneindige resultaten, die optreden bij de berekening van quantum mechanische correcties, vaak afwezig zijn in theorieën met supersymmetrie. Supersymmetrie heeft tevens tot gevolg dat de betreffende theorie invariant is onder translaties. Daarom moet een theorie die invariant is onder ruimte en tijd afhankelijke supersymmetrie transformaties, ook invariant zijn onder ruimte en tijd afhankelijke translaties, d.w.z. algemene coördinaten transformaties. Dit betekent dat een dergelijke theorie een beschrijving van de zwaartekracht inhoudt. We noemen dit soort theorieën supergravitatie. De bijzondere eigenschappen van supersymmetrie blijven in supergravitatie behouden. Men hoopt dat deze eigenschappen zullen leiden tot een consistente beschrijving van de quantumtheorie van de gravitatie. Een dergelijke beschrijving is tot nu toe niet mogelijk gebleken vanwege de eerder genoemde oneindige resultaten.

Naast de zwaartekracht komen nog drie andere fundamentele wisselwerkingen in de natuur voor: de elektromagnetische, de zwakke en de sterke wisselwerking. Het blijkt dat deze drie wisselwerkingen bijzonder goed beschreven kunnen worden met behulp van zogeheten ijktheorieën. Dit zijn theorieën, die als uitgangspunt de aanwezigheid van een bepaalde interne symmetrie vooronderstellen. Met intern bedoelen we hier dat deze symmetrieën geen betrekking hebben op de ruimte en tijd. Men kan deze interne symmetrieën opvatten als afkomstig zijnde van één grote interne symmetrie. De ijktheorie van deze interne symmetrie geeft een <u>geünificeerde</u> beschrijving van bovengenoemde drie wisselwerkingen.

Het blijkt dat de aanwezigheid van meerdere onafhankelijke supersymmetrieën op unieke wijze de invariantie onder een bepaalde interne symmetrie tot gevolg heeft. Een ijktheorie van dergelijke onafhankelijke supersymmetrieën heet uitgebreide supergravitatie. De multipletten van uitgebreide supergravitatie bevatten zowel ijkvelden (van ruimte en tijd en interne symmetrieën) als materie velden. Op deze manier zou uitgebreide supergravitatie een geünificeerde beschrijving kunnen geven van de elementaire deeltjes en hun onderlinge fundamentele wisselwerkingen.

De structuur van uitgebreide supergravitatie theorieën ligt in principe vast. Zij is echter vrij ingewikkeld. In het bijzonder hebben de tot nu toe bestaande formuleringen het nadeel dat zij alleen consistent zijn onder gebruikmaking van de bewegingsvergelijkingen voor de velden. Deze beperking staat een aantal praktische toepassingen in de weg. In dit proefschrift worden technieken ontwikkeld die de structuur kunnen verhelderen van formuleringen die deze beperking niet hebben. Dergelijke formuleringen heten "off-shell". De onderliggende gedachte is om door het invoeren van extra symmetrieën de theorie op te delen in een aantal onderdelen om deze vervolgens afzonderlijk te bestuderen. Deze extra symmetrieën zijn de conforme (super)symmetrieën die in hoofdstuk IV en V van dit proefschrift besproken worden.

Het construeren van een "off-shell" onderdeel van de theorie blijkt in sommige gevallen al zeer moeilijk te zijn. In het tweede gedeelte van dit proefschrift wordt een methode ontwikkeld, waarmede men voor een aantal theorieën het "off-shell" stuk dat het graviton bevat, kan construeren. Dit is het deeltje dat de zwaartekrachts wisselwerking overbrengt. In het laatste hoofdstuk wordt deze methode toegepast om het graviton gedeelte te construeren van een theorie die invariant is onder vier onafhankelijke supersymmetrieën. Tot nu toe is het niet mogelijk gebleken om een formulering van de overige onderdelen van deze theorie te geven zonder bewegingsvergelijkingen te gebruiken. Recentelijk heeft men wel vooruitgang geboekt bij het construeren van "off-shell" formuleringen van supergravitatie in hogere dimensies.

Een deel van het in dit proefschrift beschreven onderzoek is gepubliceerd in Nucl.Phys.B. Mevr.A.v.d.Werf-v.d.Vlist verleende assistentie bij de verzorging van het manuscript.

CURRICULUM VITAE

Eric Arnold Bergshoeff, geboren op 5 augustus 1955 te Alphen a/d Rijn, behaalde in 1973 het einddiploma Gymnasium B aan het Christelijk Lyceum te Alphen a/d Rijn. Daarna studeerde hij natuurkunde aan de Rijksuniversiteit te Leiden, hetgeen resulteerde in het kandidaatsexamen natuurkunde met bijvak sterrenkunde in 1976, het kandidaatsexamen wiskunde in 1977 en het doctoraal examen natuurkunde met bijvak wiskunde in 1979. Tijdens zijn studie verrichtte hij experimenteel onderzoek in de groep van Prof.Dr.R.de Bruyn Ouboter. Vanaf augustus 1979 werkt hij als medewerker aan het Instituut-Lorentz voor theoretische natuurkunde in Leiden onder leiding van Prof.Dr.F.A.Berends en Prof.Dr.B.de Wit op het gebied van de veldentheorie en hoge-energiefysica. Sinds 1980 doet hij dit binnen de werkgroep H-th-L van de Stichting voor Fundamenteel Onderzoek der Materie. Het onderzoek heeft met name betrekking op de theorie van supergravitatie en geschiedt in samenwerking met Prof.Dr.B.de Wit en Dr.M.de Roo. Daarnaast verrichtte hij ook een aantal onderwijstaken, zoals het geven van werkcolleges en studentenseminaria. Ter ondersteuning van het onderzoek bezocht hij enkele zomerscholen en conferenties, deels met financiële steun van de Stichting F.O.M. en de Nederlandse Organisatie voor Zuiver-Wetenschappelijk Onderzoek (Z.W.O.).

LIJST VAN PUBLIKATIES

1. E.Bergshoeff, M.de Roo, J.W.van Holten, B.de Wit and A.Van Proeyen:
Extended conformal supergravity and its applications.
Proc.Nuffield Supergravity Workshop, Cambridge, ed. S.W.Hawking and M.Roček (Cambridge University Press, 1981), pp. 237-256.

2. E.Bergshoeff, M.de Roo and B.de Wit:
Extended conformal supergravity.
Nucl.Phys. B182 (1981), pp. 173-204.

3. E.Bergshoeff, M.de Roo, B.de Wit and P.van Nieuwenhuizen:
Ten-dimensional Maxwell-Einstein supergravity, its currents, and the issue of its auxiliary fields.
Nucl.Phys. B195 (1982), pp. 97-136.

4. E.Bergshoeff and M.de Roo:
The supercurrent in ten dimensions.
Phys.Lett. 112B (1982), pp. 53-58.

5. E.Bergshoeff, M.de Roo and B.de Wit:
Conformal supergravity in ten dimensions.
Nucl.Phys. B217 (1983), pp. 489-530.

STELLINGEN

1. De voor het eerst door Bauer opgemerkte dimensionele gelijkheid tussen spinor representaties van orthogonale groepen en representaties van symplectische groepen kan niet zonder meer gebruikt worden bij het bepalen van het Kronecker produkt van tensor en spinor representaties van speciale-orthogonale groepen in even dimensies.

F.L. Bauer, Math.Ann., Bd. 128 (1954) 228.

2. De kinematica van de verdamping uit een vast natrium oppervlak kan niet beschreven worden door een model dat gebaseerd is op een directe overgang tussen de vaste en de gasvormige phase.

3. Een antisymmetrisch tensor-ijkveld $A_{\mu\nu}$ kan op consistente wijze gekoppeld worden aan een foton A_μ, indien men de Maxwell transformatie $\delta A_\mu = \partial_\mu \Lambda$, $\delta A_{\mu\nu} = 0$ uitbreidt tot $\delta A_\mu = \partial_\mu \Lambda$, $\delta A_{\mu\nu} = g\Lambda\,(\partial_\mu A_\nu - \partial_\nu A_\mu)$, waarbij g een koppelingsconstante is die de dimensie van een inverse massa heeft.

E. Bergshoeff, M. de Roo, B. de Wit en P. van Nieuwenhuizen, Nucl.Phys. B195 (1982) 97.
H. Nicolai en P.K. Townsend, Phys.Lett. 98B (1981) 257.

4. Het multiplet van stromen dat behoort bij de koppeling van een supersymmetrisch Maxwell systeem aan supergravitatie in tien dimensies, is reducibel. Het bevat een irreducibel submultiplet van 128 + 128 componenten, terwijl de overige vrijheidsgraden een chiraal scalair superveld vormen dat aan een beperkende voorwaarde voldoet.

E. Bergshoeff en M. de Roo, Phys.Lett. 112B (1982) 53.

5. Een "off-shell" formulering van Poincaré supergravitatie in tien dimensies in termen waarvan men een invariante actie kan construeren, moet gebaseerd zijn op een multiplet dat naast de superconforme velden op zijn minst een scalair submultiplet bevat.

E. Bergshoeff, M. de Roo en B. de Wit, Nucl.Phys. B217 (1983) 489.

6. De bewering van Marvin en Toigo dat het capillaire golfmodel voor de grenslaag vloeistof-gas niet geschikt is om lichtexperimenten hieraan te verklaren, is ongegrond.

A.M. Marvin en F. Toigo, Phys.Rev. A26 (1982) 2927.

7. Bij de berekeningen van de demping van het vierde geluid in vloeibaar helium door Kaganov et al. wordt een aantal veronderstellingen gemaakt die niet alle noodzakelijk zijn en niet alle verantwoord kunnen worden.

B.N. Esel'son, M.I. Kaganov, É.Ya. Rudavskii en I.A. Serbin, Sov.Phys. Usp. 17, 2 (1974) 215.
Zie ook: A. Hartoog, proefschrift Leiden 1979.

8. Het is aan twijfel onderhevig dat de lange relaxatietijden die optreden bij soortelijke-warmte metingen aan TTF-$AuS_4C_4(CF_3)_4$ beneden 3K uitsluitend veroorzaakt worden door de intrinsieke eigenschappen van deze verbinding.

J.A. Northby, F.J.A.M. Greidanus, W.J. Huiskamp, L.J. de Jongh, I.S. Jacobs en L.V. Interrante, J.Appl.Phys. 53 (1982) 8032.

9. De conclusie van Rosenberg et al. dat de door adrenaline geïnduceerde aggregatie van trombocyten bij maligne hyperthermie gevoelige patiënten normaal zou zijn, is onjuist.

H. Rosenberg, C. Fisher, S. Reed en P. Addonizio, Anesthesiology 55 (1981) 621.

E.A. Bergshoeff

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