RELATIVISTIC THERMODYNAMICS OF IRREVERSIBLE PROCESSES

G. A. KLUITENBERG

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PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE WIS- EN NATUURKUNDE AAN DE RIJKSUNIVERSITEIT TE LEIDEN, OP GEZAG VAN DE RECTOR MAGNIFICUS DR J. N. BAKHUIZEN VAN DEN BRINK, HOOGLERAAR IN DE FACULTEIT DER GODGELEERDHEID, PUBLIEK TE VERDEDIGEN OP WOENSDAG 15 DECEMBER 1954 TE 16 UUR

DOOR

GERRIT ALFRED KLUITENBERG

GEBOREN TE ROTTERDAM IN 1925

Promotor: PROF. DR S. R. DE GROOT

Aan mijn moeder

Aan de nagedachtenis van mijn vader

VOORWOORD

Op verzoek van de faculteit der Wis- en Natuurkunde volgt hier een beknopt levensoverzicht.

In 1925 werd ik te Rotterdam geboren, waar mijn vader architect was en leraar aan de Middelbare Technische School van de Academie van Beeldende Kunsten en Technische Wetenschappen. Na van 1937 tot 1942 de Chr. H.B.S. te Apeldoorn te hebben doorlopen, liet ik mij in laatstgenoemd jaar inschrijven aan de Technische Hogeschool te Delft voor de studie van natuurkundig ingenieur. Door de maatregelen van de Duitse bezetter was ik reeds na enkele maanden gedwongen met mijn studie op te houden. In 1944 bezocht ik enige tijd het Conservatorium voor muziek te Arnhem. Mijn studie aan de Technische Hogeschool hervatte ik in 1945. Het propaedeutisch examen legde ik af in 1947. Door de N.V. De Bataafsche Petroleum Maatschappij werd mij in 1948 een studieprijs toegekend. Gedurende vier jaren ontving ik een aan deze prijs verbonden stipendium. In 1949 legde ik het candidaatsexamen af en in 1951 het examen als natuurkundig ingenieur. Mijn afstudeerwerk stond onder leiding van Prof. Dr R. Kronig. In het studiejaar 1951–1952 studeerde ik aan de Universiteit te Leiden en deed enig onderzoekingswerk onder leiding van Prof. Dr J. Korringa en Prof. Dr H. A. Kramers. In 1952 werd ik benoemd tot wetenschappelijk medewerker van de Werkgemeenschap voor Molecuulphysica van de Stichting voor Fundamenteel Onderzoek der Materie. Als zodanig verrichtte ik onder leiding van Prof. Dr S. R. de Groot onderzoekingen aan de Instituten voor Theoretische Natuurkunde van de Universiteiten te Utrecht en Leiden op het gebied van de thermodynamica van de irreversibele processen.

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CONTENTS

NOTE Formulae are numbered separately in each chapter. Formula (5.6) of chapter II, say, is referred to as (5.6) in chapter II and as (II.5.6) in any other chapter.

> *"Truth is more likely to come out of error, if this is clear and definite, than out of confusion".*
>
> FRANCIS BACON

INTRODUCTION AND SUMMARY

The main purpose of this thesis is to develop the relativistic thermodynamics of irreversible processes in a continuous mixture consisting of an arbitrary number of chemical components. The second purpose is to investigate the energy-momentum tensor of the macroscopic electromagnetic field in ponderable matter.

We shall restrict ourselves to the special theory of relativity and we shall assume that atomic particles are neither created nor vanish. The validity of the thermodynamical theory is limited by the condition that, for an observer moving with the barycentric velocity, the variations in temperature, pressure etc. must be small over a distance comparable with, say, the mean free path of the molecules.

In chapter I we give the theory for systems which are influenced by forces which do not depend on the velocities of the chemical components. The theory is presented in four-dimensional tensor form. First, we introduce some useful notions such as densities, concentrations and flows of matter, the barycentric velocity and the barycentric Lorentz frame. The four-vectors which represent the relative flows of matter and the heat flow are defined in such a way that they are perpendicular to the four-vector which represents the barycentric velocity. The tensors which represent the stresses possess similar orthogonality properties. From the relativistic macroscopic fundamental laws (*i.e.*, the balance equation for rest mass, the momentum and energy laws and the second law of thermodynamics) the entropy balance is derived. The phenomenological equations are given for isotropic media and it is shown that the Onsager relations are Lorentz invariant. A new cross-effect is found between diffusion and heat conduction, arising from a relativistic term in the force conjugate to the heat flow. It appears that due to this cross-effect the diffusion phenomena are influenced by the barycentric motion.

As far as heat conduction, diffusion and entropy are concerned, the results of the theory given in chapter I are elaborated in chapter II. Moreover, this chapter contains considerations of heats of transfer and of almost Lorentz invariant quantities. The results of the theory given in chapter I concerning heat conduction and diffusion are reformulated in three-dimensional tensor form with the help of quantities which are used in the non-relativistic theory.

Formulae are given from which the difference between the results of the relativistic and the non-relativistic theory may easily be surveyed. The transformation properties of diverse quantities are examined. As a consequence of the developed formalism it is seen that the density of entropy is the fourth component of a four-vector and it appears that in general the entropy in a small element of volume is not a Lorentz invariant quantity. The connection between different sets of heats of transfer, occurring in the literature, is derived. Some of the quantities occurring in the theory appear to be almost Lorentz invariant. A formulation of the theory with the help of relative flows of matter which are defined with respect to a reference velocity other than the barycentric velocity is deduced from the formalism developed.

In chapter III we deal with systems, without polarization and magnetization, in an electromagnetic field. The entropy balance is derived from the relativistic macroscopic fundamental laws by means of a procedure which is slightly different from the one used in chapter I. The phenomenological equations for isotropic media are given in four-dimensional and three-dimensional tensor form. Also the Onsager relations are discussed. The relativistic law of Ohm appears to be a special case of the general equations which are obtained for diffusion phenomena. It appears that the electric current is a function not only of the electric and the magnetic field vectors and of the gradients of the temperature and of the partial specific Gibbs potentials of the chemical components, but is a function also of the local derivatives with respect to time of the two latter quantities and a function of the barycentric acceleration.

The thermodynamical theory for systems with polarization and magnetization is given in chapter IV. We restrict ourselves to systems which are isotropic as far as polarization and magnetization are concerned. In the case that the medium is polarized and magnetized terms occur in the non-relativistic second law of thermodynamics which are due to the polarization and magnetization of the matter. In this chapter we first derive the relativistic second law of thermodynamics for the case under consideration. If we wish to deduce a satisfactory form for the entropy balance from the fundamental equations, it appears that the explicit expression for the ponderomotive force must be closely connected to the form of the relativistic second law of thermodynamics. The phenomenological equations and the Onsager relations are given for media which are anisotropic with respect to irreversible processes.

In chapter V we consider the energy-momentum tensor of the macroscopic electromagnetic field. We also give further discussions of the first and second laws of thermodynamics and of the macroscopic forces which the electromagnetic field exerts on the matter. As in chapter IV we restrict ourselves to systems which are isotropic as far as polarization and magnetization are concerned. To have our considerations as general as possible we introduced in chapter IV several quantities for which we did not give further specification. It is seen that it is possible to make such choices for these quantities that an explicit expression can be deduced for a symmetric energy-momentum tensor of the macroscopic electromagnetic field. It appears that the non-diagonal elements of the tensor found in this way are equal to the corresponding elements of the tensor of Abraham. It is shown that Abraham's tensor leads to an equivalent formalism. It appears, however, that the form for the relativistic second law of thermodynamics which follows from Abraham's tensor corresponds to a rather unusual form for the non-relativistic second law of thermodynamics. Finally, it is shown that from the point of view of the developed formalism Abraham's tensor is preferable to Minkowski's tensor.

This work forms a part of the research programme of the „Stichting voor Fundamenteel Onderzoek der Materie" (F.O.M.). The latter foundation is financially supported by the „Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek" (Z.W.O.).

Parts of the contents of this thesis have been published (Physica, Amsterdam **19** (1953) 689; **19** (1953) 1079; **20** (1954) 199). The rest will appear shortly.

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Chapter I

SYSTEMS INFLUENCED BY FORCES WHICH DO NOT DEPEND ON THE VELOCITIES OF THE CHEMICAL COMPONENTS

§ 1. *Introduction*. The purpose of this chapter is to extend Eckart's theory [1]) of the relativistic thermodynamics of irreversible processes in a simple fluid to a mixture of an arbitrary number of chemical components and to derive physical results with the help of the Onsager relations. We shall assume that matter (rest mass) cannot change into other forms of energy and we shall limit ourselves to the special theory of relativity. Further, we shall make the restrictions that there are no external forces depending on the velocity of matter and that the medium is isotropic. We give the theory in four-dimensional tensor form, hence, the relativistic invariance is assured. We shall deal with the phenomena of diffusion, heat conduction, viscous flow and chemical reactions and with their cross-effects.

Having defined a barycentric velocity, the Lorentz frame in which this velocity vanishes will be called the barycentric Lorentz frame. As guiding principle we shall assume that in the barycentric Lorentz frame all equations have to correspond closely to the non-relativistic equations. Our method is analogous to Eckart's procedure [1]) (except in some points of interpretation) and is closely related to the non-relativistic one [2]).

The validity of the theory is limited by the condition that in the barycentric Lorentz frame the variations in temperature, pressure etc. must be small over a distance comparable with the mean free path of the molecules. This state of affairs is analogous to the non-relativistic case [3]).

In §§ 2 and 3 we discuss some preliminaries needed in the development of the theory. In § 4 we introduce four fundamental laws. In the first place we have the momentum law and the balance equation for the energy. Since we assume that matter cannot change into other forms of energy we can also introduce a conservation law for total rest mass. As fourth fundamental equation we introduce the second law of thermodynamics (Gibbs relation). In § 5 we derive the first law of thermodynamics for the internal energy of the system, measured by an observer in the barycentric Lorentz frame, from the first three fundamental equations mentioned before. In § 6 we deduce the relativistic analog of the entropy production, well-known from the non-relativistic theory. The phenomenological laws are formulated in § 7 and it is shown that the Onsager relations are invariant under Lorentz transformations. Eckart has found that acceleration of matter causes a heat flow and it will be shown that it also gives rise to a diffusion flow. This phenomenon resembles thermal diffusion because both are cross-effects of heat conduction and diffusion.

In the following chapter we shall formulate the theory with the aid of three-dimensional vectors, by means of which concepts of physical interest will be introduced.

§ 2. *Flows of matter and related notions.* Before stating the fundamental equations, which we need for the calculation of the entropy production, we shall first introduce some useful notions. In § 3 we shall consider the energy-momentum tensor and some quantities which may be derived from this tensor, while in this section we shall deal with such notions as densities, concentrations and flows of matter, the barycentric Lorentz frame, the substantial derivative with respect to time and an auxiliary tensor.

We assume a four-dimensional coordinate system (x_1, x_2, x_3, $x_4 = ict$), where x_1, x_2 and x_3 are the coordinates in ordinary space, c is the velocity of light and t is the time. By taking $x_4 = ict$, we have the metric tensor given by the Kronecker tensor and thus associated contravariant and covariant tensors become identical.

Furthermore, we shall assume that we have a mixture of n components. If $N^{(j)}$ is the number of atomic particles (electrons and atomic nuclei) of component j per unit volume (the volume being at rest with respect to the observer) and $M^{(j)}_{(0)(k)}$ is the rest mass of particle k if it is free (not bound in an atom, molecule or ion) we define as density of rest mass of component j the quantity $\varrho^{(j)} \equiv \Sigma_{k=1}^{N(j)} M^{(j)}_{(0)(k)}$. We can represent *the flow of matter* of component j, as is well-known, by a four-vector of which the components in the space-time continuum are defined by

$$m_1^{(j)} \equiv \varrho^{(j)} v_1^{(j)};\ m_2^{(j)} \equiv \varrho^{(j)} v_2^{(j)};\ m_3^{(j)} \equiv \varrho^{(j)} v_3^{(j)};\ m_4^{(j)} \equiv ic\varrho^{(j)}, (j=1,\ldots,n) \tag{2.1}$$

where $\mathbf{v}^{(j)}$ is the velocity of component j. Chemical components will always be denoted by a superscript and tensor components by a subscript.

The total density of rest mass is given by

$$\varrho \equiv \Sigma_{j=1}^{n} \varrho^{(j)}. \tag{2.2}$$

The specific volume is defined by

$$v \equiv \varrho^{-1}, \tag{2.3}$$

and the concentration of component j by

$$c^{(j)} \equiv \varrho^{(j)}/\varrho. \qquad (j = 1, \ldots, n) \tag{2.4}$$

From the preceding equation and (2.2) we have

$$\Sigma_{j=1}^{n} c^{(j)} = 1. \tag{2.5}$$

We shall define the barycentric velocity by

$$\mathbf{v} \equiv \Sigma_{j=1}^{n} c^{(j)} \mathbf{v}^{(j)}. \tag{2.6}$$

We now introduce a four-vector with components

$$m_a \equiv \Sigma_{j=1}^{n} m_a^{(j)}. \qquad (a = 1, \ldots, 4) \tag{2.7}$$

It is easily seen that

$$m_1 = \varrho v_1;\ m_2 = \varrho v_2;\ m_3 = \varrho v_3;\ m_4 = ic\varrho. \tag{2.8}$$

We see that this four-vector represents the total flow of rest mass. By m we shall indicate the scalar

$$m \equiv (-\Sigma_{a=1}^{4} m_a^2)^{\frac{1}{2}} = \varrho(c^2 - \mathbf{v}^2)^{\frac{1}{2}}. \tag{2.9}$$

For the following considerations it is useful to introduce the dimensionless four-vector

$$u_a \equiv m_a/m. \qquad (a = 1, \ldots, 4) \tag{2.10}$$

From this equation we have with the help of (2.8) and (2.9)

$$\begin{aligned} u_1 &= v_1(c^2 - \mathbf{v}^2)^{-\frac{1}{2}};\ u_2 = v_2(c^2 - \mathbf{v}^2)^{-\frac{1}{2}};\\ u_3 &= v_3(c^2 - \mathbf{v}^2)^{-\frac{1}{2}};\ u_4 = ic(c^2 - \mathbf{v}^2)^{-\frac{1}{2}}. \end{aligned} \tag{2.11}$$

We see that this four-vector can be interpreted as the four-dimensional analog of the barycentric velocity $\mathbf{v}$. Further, we see from the preceding equation that

$$\Sigma_{a=1}^{4} u_a^2 = -1, \tag{2.12}$$

hence, it follows that

$$\Sigma_{a=1}^{4} u_a(\partial u_a/\partial x_\beta) = 0. \qquad (\beta = 1, \ldots, 4) \tag{2.13}$$

At any particular time we can assign to every point of the system a Lorentz frame in which $\mathbf{v}$ vanishes. We shall call this frame the barycentric Lorentz frame belonging to the point in the space-time continuum under consideration. *All quantities at a point in the space-time continuum measured in the barycentric Lorentz frame belonging to this point will be distinguished by primes.* According to (2.9) we then have

$$\varrho' = m/c = \varrho(1 - \mathbf{v}^2/c^2)^{\frac{1}{2}}. \tag{2.14}$$

From this equation and (2.3) we have

$$v' = c/m, \tag{2.15}$$

and from (2.11) we have

$$u'_\alpha = i\delta(\alpha; 4), \qquad (\alpha = 1, .., 4) \tag{2.16}$$

where $\delta(\alpha; \beta)$ is the Kronecker symbol. Further, we have according to (2.4)

$$c'^{(j)} = \varrho'^{(j)}/\varrho'. \qquad (j = 1, \ldots, n) \tag{2.17}$$

We can show that $c'^{(j)}$ may also be expressed as

$$c'^{(j)} = -m^{-1} \Sigma_{\beta=1}^{4} u_\beta m_\beta^{(j)}. \qquad (j = 1, \ldots, n) \tag{2.18}$$

This can be done in the following way. In the first place we remark that the right hand side of this equation is a scalar. Hence, if we prove the validity of (2.18) in one Lorentz frame we may infer that the equation is valid in any Lorentz frame. Inserting (2.1), (2.9) and (2.11) into (2.18) gives with the help of (2.4)

$$c'^{(j)} = \frac{c^2 - \mathbf{v}^{(j)} \cdot \mathbf{v}}{c^2 - \mathbf{v}^2} c^{(j)}. \quad (j = 1, \ldots, n) \tag{2.19}$$

It is seen that this equation is identically fulfilled in the barycentric Lorentz frame. Thus, (2.18) is proved and therefore (2.19) too. Furthermore, we have

$$\Sigma_{j=1}^{n} c'^{(j)} = 1. \tag{2.20}$$

The most convenient way to represent by four-vectors the *relative flows of matter* of the components with respect to the barycentric motion is

$$I_\alpha^{(j)} \equiv m_\alpha^{(j)} - c'^{(j)} m_\alpha. \quad (\alpha = 1, .., 4; \ j = 1, \ldots, n) \tag{2.21}$$

Substitution of (2.1), (2.8) and (2.19) into this equation gives with the help of (2.4)

$$I_1^{(j)} = \varrho^{(j)}\left(v_1^{(j)} - \frac{c^2 - \mathbf{v}^{(j)} \cdot \mathbf{v}}{c^2 - \mathbf{v}^2} v_1\right); \quad I_2^{(j)} = \varrho^{(j)}\left(v_2^{(j)} - \frac{c^2 - \mathbf{v}^{(j)} \cdot \mathbf{v}}{c^2 - \mathbf{v}^2} v_2\right);$$

$$I_3^{(j)} = \varrho^{(j)}\left(v_3^{(j)} - \frac{c^2 - \mathbf{v}^{(j)} \cdot \mathbf{v}}{c^2 - \mathbf{v}^2} v_3\right); \quad I_4^{(j)} = ic\varrho^{(j)}\left(1 - \frac{c^2 - \mathbf{v}^{(j)} \cdot \mathbf{v}}{c^2 - \mathbf{v}^2}\right).$$

$$(j=1, \ldots, n) \tag{2.22}$$

These equations give the flows in terms of densities and velocities. The four-vectors $I_\alpha^{(j)}$ have been defined by the preceding equations in such a way as to have two important properties. First, it follows from (2.21) with the help of (2.7) and (2.20) that

$$\Sigma_{j=1}^{n} I_\alpha^{(j)} = 0, \qquad (\alpha = 1, .., 4). \tag{2.23}$$

which expresses that the sum of the relative flows of the components vanishes. Further, we deduce from (2.21) with the help of (2.10), (2.12) and (2.18)

$$\Sigma_{\alpha=1}^{4} u_\alpha I_\alpha^{(j)} = 0. \qquad (j = 1, \ldots, n) \tag{2.24}$$

From this equation we see that all the relative flows, $I_\alpha^{(j)}$, are perpendicular to the four-vector u_α representing the barycentric velocity.

We shall define the substantial derivative with respect to time as the Lorentz invariant operator

$$\mathrm{D} \equiv c\, \Sigma_{\alpha=1}^{4} u_\alpha\, (\partial/\partial x_\alpha). \tag{2.25}$$

With the help of (2.16) we see that

$$\mathrm{D} = \partial/\partial t', \tag{2.26}$$

where t' is the time measured by an observer in the barycentric Lorentz frame.

The density of rest mass, $\varrho_{(0)}^{(j)}$, of component j measured by an observer *moving with this component* is given by

$$\varrho_{(0)}^{(j)} = \varrho^{(j)}\, (1 - \mathbf{v}^{(j)2}/c^2)^{\frac{1}{2}}. \qquad (j = 1, \ldots, n) \tag{2.27}$$

In principle the quantities $\varrho_{(0)}^{(j)}$ and $\varrho'^{(j)}$ are different; however, in practical cases their difference in value is very small.

Finally, we introduce the tensor

$$\Delta_{\alpha\beta} \equiv \delta_{\alpha\beta} + u_\alpha u_\beta, \quad (\alpha, \beta = 1, \ldots, 4) \tag{2.28}$$

$\delta_{\alpha\beta}$ being the Kronecker tensor. We immediately see that

$$\Delta_{\alpha\beta} = \Delta_{\beta\alpha}, \qquad (\alpha, \beta = 1, \ldots, 4) \tag{2.29}$$

and with the help of (2.12) we deduce that

$$\Sigma_{\alpha=1}^{4} u_\alpha \Delta_{\alpha\beta} = \Sigma_{\alpha=1}^{4} \Delta_{\beta\alpha} u_\alpha = 0. \quad (\beta = 1, \ldots, 4) \tag{2.30}$$

Using (2.16) it follows from (2.28) that

$$\Delta'_{\alpha\beta} = \delta(\alpha; \beta) - \delta(\alpha; 4)\, \delta(\beta; 4), \quad (\alpha, \beta = 1, \ldots, 4) \tag{2.31}$$

or

$$\Delta'_{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.32}$$

The sum of the diagonal elements of a tensor is a scalar. Using (2.12) and (2.28) we get for the sum of the diagonal elements of $\Delta_{\alpha\beta}$

$$\Sigma_{\alpha=1}^{4} \Delta_{\alpha\alpha} = 3. \tag{2.33}$$

In the following it is seen that the tensor $\Delta_{\alpha\beta}$ $(\alpha, \beta = 1, .., 4)$ plays the role which $\delta_{\alpha\beta}(\alpha, \beta = 1, 2, 3)$ has in the non-relativistic theory. From (2.10), (2.18), (2.21) and (2.28) we have

$$I_\alpha^{(j)} = \Sigma_{\beta=1}^4 \Delta_{\alpha\beta} m_\beta^{(j)}. \quad (\alpha = 1, .., 4;\ j = 1, ..., n) \tag{2.34}$$

§ 3. *The energy-momentum tensor and some deduced quantities.* In this section we shall consider the energy-momentum tensor and some other quantities which may be defined with its help. We denote by $e_{(v)}$ the energy per unit volume and by $\mathbf{J}_{(e)}$ the energy flow. In principle both quantities differ from the corresponding non-relativistic quantities because the theory of relativity recognizes the fact that rest mass is a form of energy. Since the barycentric Lorentz frame is defined in such a way that the total flow of rest mass vanishes, $\mathbf{J}'_{(e)}$ corresponds closely to the non-relativistic energy flow in the barycentric Lorentz frame. According to the theory of relativity an energy flow is associated with a momentum density $\mathfrak{g}$ given by

$$\mathfrak{g} = c^{-2}\, \mathbf{J}_{(e)}. \tag{3.1}$$

We write the energy-momentum tensor in the form

$$W_{\alpha\beta} = t_{\alpha\beta} + g_\alpha v_\beta \ (\alpha, \beta = 1, 2, 3); \quad W_{\alpha 4} = icg_\alpha \ (\alpha = 1, 2, 3);$$
$$W_{4\alpha} = ic^{-1} J_{(e)\alpha} \quad (\alpha = 1, 2, 3); \ W_{44} = -e_{(v)}. \tag{3.2}$$

The components $W_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$ correspond to the momentum flows. These terms have been split up into a part $g_\alpha v_\beta$ corresponding to the transfer of momentum with the barycentric velocity (convective part) and a remaining part which defines the stress tensor. According to (3.1) we have $W_{\alpha 4} = W_{4\alpha}$ $(\alpha = 1, 2, 3)$ and we extend this by assuming $W_{\alpha\beta}$ to be a symmetric tensor. Thus,

$$W_{\alpha\beta} = W_{\beta\alpha}. \quad (\alpha, \beta = 1, .., 4) \tag{3.3}$$

It is easily shown that $e'_{(v)}$ is given by

$$e'_{(v)} = \Sigma_{\gamma,\zeta=1}^4 u_\gamma W_{\gamma\zeta} u_\zeta. \tag{3.4}$$

The right hand side of this equation is Lorentz invariant. If we calculate the right hand side in the barycentric Lorentz frame we get with (2.16) and (3.2) just the left hand side. Hence, the equation is proved.

We represent the heat flow by a four-vector defined by

$$I_\alpha^{(0)} \equiv -c\, \Sigma_{\gamma,\zeta=1}^4 \Delta_{\alpha\gamma} W_{\gamma\zeta} u_\zeta. \quad (\alpha = 1, .., 4) \tag{3.5}$$

With the help of (2.16), (2.31), (3.2) and (3.3) we find from the preceding definition

$$I_1'^{(0)} = J'_{(e)1};\ \ I_2'^{(0)} = J'_{(e)2};\ \ I_3'^{(0)} = J'_{(e)3};\ \ I_4'^{(0)} = 0. \tag{3.6}$$

From this equation we see that $I_\alpha'^{(0)}$ corresponds to $\mathbf{J}'_{(e)}$. Further, $\mathbf{J}'_{(e)}$ corresponds to the non-relativistic energy flow in the barycentric Lorentz frame as was stressed above. The heat flow in the non-relativistic thermodynamics of irreversible processes is usually defined in such a way that it equals the energy flow in the barycentric frame. (Discussion of various ways to define the heat flow in ref. 4.) From these considerations it follows that $I_a^{(0)}$ may represent the heat flow. With the help of (2.30) and (3.5) the following important property of the heat flow may be derived

$$\Sigma_{a=1}^{4} u_a I_a^{(0)} = 0, \tag{3.7}$$

showing that $I_a^{(0)}$ is perpendicular to the four-vector u_a representing the barycentric velocity (cf. (2.24)).

Further, we may represent the stresses by the tensor

$$w_{\alpha\beta} \equiv \Sigma_{\gamma,\zeta=1}^{4} \Delta_{\alpha\gamma} W_{\gamma\zeta} \Delta_{\zeta\beta}. \quad (\alpha, \beta = 1, \ldots, 4) \tag{3.8}$$

As a matter of fact we find from this definition with the help of (2.31) and (3.2)

$$w'_{\alpha\beta} = t'_{\alpha\beta} \ (\alpha, \beta = 1, 2, 3); \quad w'_{\alpha 4} = w'_{4\alpha} = 0 \quad (\alpha = 1, \ldots, 4), \tag{3.9}$$

showing that $w_{\alpha\beta}$ indeed may represent the stresses. From (2.29), (3.3) and (3.8) it follows that

$$w_{\alpha\beta} = w_{\beta\alpha}, \qquad (\alpha, \beta = 1, \ldots, 4) \tag{3.10}$$

and from (2.30) and (3.8) we have

$$\Sigma_{\alpha=1}^{4} u_\alpha w_{\alpha\beta} = \Sigma_{\alpha=1}^{4} w_{\beta\alpha} u_\alpha = 0. \quad (\beta = 1, \ldots, 4) \tag{3.11}$$

The equations (3.10) and (3.11) reduce the number of independent components of the tensor $w_{\alpha\beta}$ to six.

It may be readily verified that

$$W_{\alpha\beta} = u_\alpha u_\beta e'_{(v)} + c^{-1}(u_\beta I_\alpha^{(0)} + u_\alpha I_\beta^{(0)}) + w_{\alpha\beta} \quad (\alpha, \beta = 1, \ldots, 4) \tag{3.12}$$

by substituting (3.4), (3.5) and (3.8) into the right hand side of this equation and making use of (2.28) and (3.3).

We now define, analogous to the specific internal energy in the non-relativistic thermodynamics, the specific energy measured by an observer in the barycentric Lorentz frame by

$$e' \equiv v' e'_{(v)} - a, \tag{3.13}$$

where a is an arbitrary constant, fixing the zero point of e'. It will appear that a drops out of the final results.

By p' we denote the hydrostatic pressure measured in the barycentric Lorentz frame. We may define as viscous stress tensor

$$P_{\alpha\beta} \equiv - w_{\alpha\beta} + p' \Delta_{\alpha\beta}. \quad (\alpha, \beta = 1, \ldots, 4) \tag{3.14}$$

From the preceding definition we have with the help of (2.32) and (3.9)

$$P'_{\alpha\beta} = -t'_{\alpha\beta} + p'\delta_{\alpha\beta} \ (\alpha, \beta = 1,2,3); \ P'_{\alpha 4} = P'_{4\alpha} = 0 \ (\alpha = 1, .., 4), \quad (3.15)$$

showing that $P_{\alpha\beta}$ indeed may represent the viscous stress tensor. From (2.29), (3.10) and (3.14) it follows that

$$P_{\alpha\beta} = P_{\beta\alpha}, \qquad (\alpha, \beta = 1, .., 4) \qquad (3.16)$$

and from (2.30), (3.11) and (3.14) we have

$$\Sigma_{\alpha=1}^{4} u_\alpha P_{\alpha\beta} = \Sigma_{\alpha=1}^{4} P_{\beta\alpha} u_\alpha = 0. \quad (\beta = 1, .., 4) \qquad (3.17)$$

We shall not introduce the simplifying assumption $p' = \frac{1}{3} \Sigma_{\alpha=1}^{4} w_{\alpha\alpha}$ so that volume viscosity effects will not be neglected.

§ 4. *The fundamental laws.* We can now formulate four fundamental laws which are the starting point for the calculation of the entropy production.

I. The balance equation for rest mass. We assume one chemical reaction among the components of the system. We shall denote by $\nu^{(k)} J_{(c)}$ the chemical production of rest mass of component k per unit volume and per unit time. It is obvious that this quantity is Lorentz invariant. The quantity $\nu^{(k)}$ divided by the molecular mass of substance k is proportional to the stoechiometric number of this component in the chemical reaction. Thus, $\nu^{(k)}$ is Lorentz invariant too. Hence, it follows that $J_{(c)}$, called the chemical reaction rate in mass per unit volume and per unit time, is also Lorentz invariant. Now, we can write the balance equation for rest mass in the form

$$\partial \varrho^{(k)}/\partial t = -\operatorname{div} \varrho^{(k)} \mathbf{v}^{(k)} + \nu^{(k)} J_{(c)}. \qquad (k = 1, \ldots, n) \qquad (4.1)$$

(For several reactions the last term would be a sum of similar expressions for each reaction.) With the help of (2.1) we can write this law in the four-dimensional form

$$\Sigma_{\alpha=1}^{4} \partial m_\alpha^{(k)}/\partial x_\alpha = \nu^{(k)} J_{(c)}. \quad (k=1, \ldots, n) \qquad (4.2)$$

Hence, it follows that this law is Lorentz invariant. Summing the n equations (4.2) over all values of k, we get with the help of (2.7) and of $\Sigma_{k=1}^{n} \nu^{(k)} = 0$

$$\Sigma_{\alpha=1}^{4} \partial m_\alpha/\partial x_\alpha = 0. \qquad (4.3)$$

This means that the total rest mass is conserved.

II. The momentum and energy law. We shall assume that the external forces acting on the system do not depend on the velocities of the chemical components. If $\mathbf{F}^{(k)}$ is the force per unit of rest mass on component

k, we can define, as is well-known, a four-vector with components

$$K_1^{(k)} \equiv \varrho^{(k)} F_1^{(k)}/\varrho_{(0)}^{(k)};\ K_2^{(k)} \equiv \varrho^{(k)} F_2^{(k)}/\varrho_{(0)}^{(k)};\ K_3^{(k)} \equiv \varrho^{(k)} F_3^{(k)}/\varrho_{(0)}^{(k)};$$

$$K_4^{(k)} \equiv i\varrho^{(k)} (\mathbf{v}^{(k)} \cdot \mathbf{F}^{(k)})/(c\varrho_{(0)}^{(k)}). \quad (k = 1, \ldots, n) \tag{4.4}$$

From (2.1) and the preceding equation we have

$$\Sigma_{\alpha=1}^{4}\, m_\alpha^{(j)} K_\alpha^{(j)} = 0. \quad (j = 1, \ldots, n) \tag{4.5}$$

Thus, we see that the vectors $m_\alpha^{(j)}$ and $K_\alpha^{(j)}$ are perpendicular. The balance equation for momentum is expressed by

$$\partial g_\alpha/\partial t + \Sigma_{\beta=1}^{3}\, \partial(g_\alpha v_\beta)/\partial x_\beta = \Sigma_{j=1}^{n}\, \varrho^{(j)} F_\alpha^{(j)} - \Sigma_{\beta=1}^{3}\, \partial t_{\alpha\beta}/\partial x_\beta. \quad (\alpha = 1,2,3) \tag{4.6}$$

The energy balance reads

$$\partial e_{(v)}/\partial t = -\operatorname{div} \mathbf{J}_{(e)} + \Sigma_{j=1}^{n}\, \varrho^{(j)}\, \mathbf{v}^{(j)} \cdot \mathbf{F}^{(j)}. \tag{4.7}$$

With the help of (3.2) and (4.4) we can combine (4.6) and (4.7) into the four-dimensional equation

$$\Sigma_{\beta=1}^{4}\, \partial W_{\alpha\beta}/\partial x_\beta = \Sigma_{j=1}^{n}\, \varrho_{(0)}^{(j)}\, K_\alpha^{(j)}, \quad (\alpha = 1, \ldots, 4) \tag{4.8}$$

showing the relativistic invariance of the two laws.

III. The second law of thermodynamics (Gibbs relation). Just as in the non-relativistic thermodynamics of irreversible processes we assume that the second law,

$$T'(\partial s'/\partial t') = \partial e'/\partial t' + p'(\partial v'/\partial t') - \Sigma_{j=1}^{n}\, \mu'^{(j)}(\partial c'^{(j)}/\partial t'), \tag{4.9}$$

is valid in the barycentric frame. T is the temperature, s the specific entropy and $\mu^{(j)}$ the partial specific Gibbs function of component j (chemical potential). With the help of the Lorentz invariant operator D defined by (2.25) we can write

$$T'\mathrm{D}s' = \mathrm{D}e' + p'\mathrm{D}v' - \Sigma_{j=1}^{n}\, \mu'^{(j)}\mathrm{D}c'^{(j)}. \tag{4.10}$$

The quantities with primes, measured in the barycentric frame, are here expressed as functions of space coordinates and time in an arbitrary Lorentz frame.

§ 5. *The first law of thermodynamics.* In the non-relativistic theory the first law of thermodynamics is obtained by multiplying the momentum law by $\mathbf{v}$ and subtracting the result from the energy equation. By multiplying the equation (4.8) by u_α and summing the result over all values of α, it is obvious that we perform an analogous procedure. Therefore, we must study the equation

$$\Sigma_{\alpha,\beta=1}^{4}\, u_\alpha(\partial W_{\alpha\beta}/\partial x_\beta) = \Sigma_{j=1}^{n}\, \Sigma_{\alpha=1}^{4}\, \varrho_{(0)}^{(j)}\, u_\alpha\, K_\alpha^{(j)} \tag{5.1}$$

in more detail.

With the help of (2.10), (2.21) and (4.5) we can transform the right hand side of this equation into

$$\Sigma_{j=1}^{n} \Sigma_{\alpha=1}^{4} \varrho_{(0)}^{(j)} u_\alpha K_\alpha^{(j)} = \Sigma_{j=1}^{n} \Sigma_{\alpha=1}^{4} \varrho_{(0)}^{(j)} (mc'^{(j)})^{-1} c'^{(j)} m_\alpha K_\alpha^{(j)} =$$
$$= -\Sigma_{j=1}^{n} \Sigma_{\alpha=1}^{4} c^{-1} \omega^{(j)} I_\alpha^{(j)} K_\alpha^{(j)}, \qquad (5.2)$$

with the Lorentz invariant quantity $\omega^{(j)}$ being defined by

$$\omega^{(j)} \equiv \varrho_{(0)}^{(j)} c (mc'^{(j)})^{-1}. \qquad (j = 1, \ldots, n) \qquad (5.3)$$

Considering the left hand side of (5.1) we have with the help of (3.12)

$$\Sigma_{\alpha,\beta=1}^{4} u_\alpha (\partial W_{\alpha\beta}/\partial x_\beta) =$$
$$= \Sigma_{\beta=1}^{4} (\partial/\partial x_\beta) [\Sigma_{\alpha=1}^{4} u_\alpha \{u_\alpha u_\beta e'_{(v)} + c^{-1} (u_\beta I_\alpha^{(0)} + u_\alpha I_\beta^{(0)}) + w_{\alpha\beta}\}] -$$
$$- \Sigma_{\alpha,\beta=1}^{4} \{u_\alpha u_\beta e'_{(v)} + c^{-1}(u_\beta I_\alpha^{(0)} + u_\alpha I_\beta^{(0)}) + w_{\alpha\beta}\} (\partial u_\alpha/\partial x_\beta). \qquad (5.4)$$

Using the relations (2.12), (2.13), (3.7) and (3.11) we can simplify this expression

$$\Sigma_{\alpha,\beta=1}^{4} u_\alpha (\partial W_{\alpha\beta}/\partial x_\beta) =$$
$$= -\Sigma_{\beta=1}^{4} (\partial/\partial x_\beta) (u_\beta e'_{(v)} + c^{-1} I_\beta^{(0)}) - \Sigma_{\alpha,\beta=1}^{4} (c^{-1} u_\beta I_\alpha^{(0)} + w_{\alpha\beta}) (\partial u_\alpha/\partial x_\beta). \qquad (5.5)$$

We now transform two terms on the right hand side of (5.5). With the help of (2.10), (2.14), (2.15), (2.25), (3.13) and (4.3) we deduce

$$\Sigma_{\beta=1}^{4} (\partial/\partial x_\beta) (u_\beta e'_{(v)}) = c^{-1} \varrho' \mathrm{D} e'. \qquad (5.6)$$

Using (2.10), (2.13), (2.14), (2.15), (2.25), (2.28) and (4.3) we derive

$$\Sigma_{\alpha,\beta=1}^{4} \Delta_{\alpha\beta} (\partial u_\beta/\partial x_\alpha) = c^{-1} \varrho' \mathrm{D} v', \qquad (5.7)$$

and from this equation and (3.14) we have

$$\Sigma_{\alpha,\beta=1}^{4} w_{\alpha\beta} (\partial u_\beta/\partial x_\alpha) = c^{-1} \varrho' p' \mathrm{D} v' - \Sigma_{\alpha,\beta=1}^{4} P_{\alpha\beta} (\partial u_\beta/\partial x_\alpha). \qquad (5.8)$$

Substitution of (5.6) and (5.8) into (5.5) and the use of the definition (2.25) gives with the help of (3.10)

$$\Sigma_{\alpha,\beta=1}^{4} u_\alpha (\partial W_{\alpha\beta}/\partial x_\beta) = -c^{-1} \varrho' \mathrm{D} e' - c^{-1} \Sigma_{\beta=1}^{4} (\partial I_\beta^{(0)}/\partial x_\beta + c^{-1} I_\beta^{(0)} \mathrm{D} u_\beta) -$$
$$- c^{-1} \varrho' p' \mathrm{D} v' + \Sigma_{\alpha,\beta=1}^{4} P_{\alpha\beta} (\partial u_\beta/\partial x_\alpha). \qquad (5.9)$$

Substitution of the results (5.2) and (5.9) into (5.1) gives the equation

$$\varrho'(\mathrm{D}e' + p'\mathrm{D}v') = -\Sigma_{\beta=1}^{4} (\partial I_\beta^{(0)}/\partial x_\beta + c^{-1} I_\beta^{(0)} \mathrm{D} u_\beta) +$$
$$+ \Sigma_{j=1}^{n} \Sigma_{\alpha=1}^{4} \omega^{(j)} I_\alpha^{(j)} K_\alpha^{(j)} + c\, \Sigma_{\alpha,\beta=1}^{4} P_{\alpha\beta} (\partial u_\beta/\partial x_\alpha), \qquad (5.10)$$

which may be considered as the first law of thermodynamics for the energy e'. The left hand side of this equation is completely analogous to the left hand side of the corresponding equation of the non-relativistic theory. The first term on the right hand side of (5.10) corresponds to the divergence of the heat flow in the non-relativistic theory. It should be remarked, however, that in (5.10) the four-dimensional divergence of the heat flow occurs. The second term on the right hand side has no non-relativistic analog and was first found by E c k a r t [1]). The third and fourth terms are analogous to the corresponding non-relativistic ones, viz. energy dissipated by external forces and by viscous stresses.

§ 6. *The entropy balance.* In the preceding section we derived the first law of thermodynamics from the balance equation for the energy with the help of the momentum and mass laws. We shall now calculate the entropy balance from the first and second laws of thermodynamics and the balance equation for rest mass.

We first derive with the aid of (2.10), (2.14), (2.21), (2.25), (4.2) and (4.3)

$$\mathrm{D}c'^{(j)} = (1/\varrho')\, \Sigma_{\alpha=1}^{4}\, (\partial/\partial x_\alpha)\, (c'^{(j)} m_\alpha) = -(1/\varrho')\{\Sigma_{\alpha=1}^{4}\, (\partial I_\alpha^{(j)}/\partial x_\alpha) - \nu^{(j)} J_{(c)}\}. \qquad (j = 1, \ldots, n) \quad (6.1)$$

Substitution of this expression and (5.10) into (4.10) gives after some calculation

$$\begin{aligned} \varrho' \mathrm{D}s' = & -\Sigma_{\alpha=1}^{4}\, (\partial/\partial x_\alpha)\, \{(1/T')\, (I_\alpha^{(0)} - \Sigma_{j=1}^{n}\, \mu'^{(j)} I_\alpha^{(j)})\} - \\ & - (1/T')\, \Sigma_{\alpha=1}^{4}\, I_\alpha^{(0)}\, \{(1/T')\, (\partial T'/\partial x_\alpha) + c^{-1} \mathrm{D}u_\alpha\} + \\ & + (1/T')\, \Sigma_{j=1}^{n}\, \Sigma_{\alpha=1}^{4}\, I_\alpha^{(j)}\, \{\omega^{(j)} K_\alpha^{(j)} - T'(\partial/\partial x_\alpha)\, (\mu'^{(j)}/T')\} + \\ & + (c/T')\, \Sigma_{\alpha,\beta=1}^{4}\, P_{\alpha\beta}\, (\partial u_\beta/\partial x_\alpha) - (1/T')\, J_{(c)}\, \Sigma_{j=1}^{n}\, \nu^{(j)}\, \mu'^{(j)}. \end{aligned} \quad (6.2)$$

We now define the scalar quantity

$$\Pi \equiv \tfrac{1}{3}\, \Sigma_{\alpha=1}^{4}\, P_{\alpha\alpha}. \quad (6.3)$$

Substitution of (3.14) into the preceding equation gives with the help of (2.33)

$$\Pi = p' - \tfrac{1}{3}\, \Sigma_{\alpha=1}^{4}\, w_{\alpha\alpha}. \quad (6.4)$$

From this equation we see that Π is the difference between the hydrostatic pressure and $\frac{1}{3}$ of the sum of the diagonal elements of the stress tensor. Further, we introduce the tensor

$$\overline{P}_{\alpha\beta} \equiv P_{\alpha\beta} - \Pi \Delta_{\alpha\beta}\,. \quad (\alpha, \beta = 1, \ldots, 4) \quad (6.5)$$

Using (2.33) and (6.3) we have from the preceding equation

$$\Sigma_{\alpha=1}^{4}\, \overline{P}_{\alpha\alpha} = 0. \quad (6.6)$$

With the aid of (2.29) and (3.16) we immediately see from (6.5) that

$$\overline{P}_{\alpha\beta} = \overline{P}_{\beta\alpha}, \quad (\alpha, \beta = 1, \ldots, 4) \quad (6.7)$$

and using (2.30) and (3.17) we have from (6.5)

$$\Sigma_{a=1}^{4} u_a \overline{P}_{a\beta} = \Sigma_{a=1}^{4} \overline{P}_{\beta a} u_a = 0. \quad (\beta = 1, \ldots, 4) \tag{6.8}$$

We now define as "forces" (affinities) the four-vectors

$$Y_a^{(0)} \equiv -\{(1/T')\,(\partial T'/\partial x_a) + c^{-1}\,\mathrm{D}u_a\}, \qquad (a = 1, \ldots, 4) \tag{6.9}$$

$$Y_a^{(j)} \equiv \omega^{(j)} K_a^{(j)} - T'\,(\partial/\partial x_a)\,(\mu'^{(j)}/T'), \quad (a=1,\ldots,4;\, j=1,\ldots,n) \tag{6.10}$$

and the scalar

$$A \equiv -\Sigma_{j=1}^{n} \nu^{(j)} \mu'^{(j)}. \tag{6.11}$$

Substituting (6.5) and the three preceding equations into (6.2) gives with the aid of (5.7)

$$\begin{aligned} \varrho'\mathrm{D}s' = &-\Sigma_{a=1}^{4} (\partial/\partial x_a)\{(1/T')\,(I_a^{(0)} - \Sigma_{j=1}^{n} \mu'^{(j)} I_a^{(j)})\} + \\ &+(1/T')\,[\Sigma_{j=0}^{n} \Sigma_{a=1}^{4} I_a^{(j)} Y_a^{(j)} + c\,\Sigma_{a,\beta=1}^{4} \overline{P}_{a\beta}(\partial u_\beta/\partial x_a) + \Pi\varrho'\mathrm{D}v' + J_{(c)}A]. \end{aligned} \tag{6.12}$$

The first and second parts on the right hand side of this expression are analogous respectively to the divergence of the entropy flow and the entropy production of the non-relativistic theory. The first term in the second part contains the contribution of the heat conduction ($j = 0$) and the diffusion ($j \neq 0$), the second and third terms the contributions of ordinary and volume viscosity and the last term the contribution of the chemical reaction.

§ 7. *The phenomenological equations and the Onsager relations.* Taking into account Curie's law we introduce the phenomenological laws in such a way that a certain flux only depends on forces having the same tensorial character as this flux. On the other hand, this flux may depend on all the forces having its tensorial character.

Therefore, we introduce for the vectorial fluxes and forces $I_a^{(j)}$ and $Y_a^{(j)}$ the equations

$$I_a^{(j)} = \Sigma_{k=0}^{n} \Sigma_{\beta=1}^{4} L_{a\beta}^{(j)(k)} Y_\beta^{(k)}, \quad (a=1,\ldots,4;\, j=0,1,\ldots,n) \tag{7.1}$$

where the $L_{a\beta}^{(j)(k)}$ are $(n + 1)^2$ tensors $(j, k = 0, 1, \ldots, n)$ each having 4^2 components $(a, \beta = 1, \ldots, 4)$.

We shall now show that we can derive an explicit form for $L_{a\beta}^{(j)(k)}$ from the assumption that the medium is isotropic, using the postulate that all equations should correspond closely to the non-relativistic equations in the barycentric Lorentz frame. From (2.22) we have

$$\begin{aligned} I_a'^{(j)} &= \varrho'^{(j)} v_a'^{(j)} \quad (a = 1, 2, 3;\ j = 1, \ldots, n); \\ I_4'^{(j)} &= 0 \qquad (j = 1, \ldots, n). \end{aligned} \tag{7.2}$$

Using (2.14) and (2.17) we have from (5.3)

$$\omega^{(j)} = \varrho^{(j)}_{(0)}/\varrho'^{(j)}. \qquad (j = 1, \ldots, n) \qquad (7.3)$$

Substitution of the preceding equation and (4.4) into (6.10) gives

$$\begin{aligned} Y'^{(j)}_{\alpha} &= F'^{(j)}_{\alpha} - T'(\partial/\partial x'_{\alpha})\,(\mu'^{(j)}/T') \qquad (\alpha = 1, 2, 3;\ j = 1, \ldots, n); \\ Y'^{(j)}_{4} &= (i/c)\,\mathbf{v}'^{(j)} \cdot \mathbf{F}'^{(j)} + (iT'/c)\,(\partial/\partial t')\,(\mu'^{(j)}/T') \quad (j = 1, \ldots, n). \end{aligned} \qquad (7.4)$$

From (7.2) and (7.4) and from (3.6) and (6.9) we can conclude that (7.1) corresponds to the non-relativistic equations for an isotropic medium in the barycentric Lorentz frame if

$$L'^{(j)(k)}_{\alpha\beta} = L^{(j)(k)}\{\delta(\alpha;\beta) - \delta(\alpha;4)\,\delta(\beta;4)\},$$
$$(\alpha, \beta = 1, \ldots, 4;\ j, k = 0, 1, \ldots, n) \qquad (7.5)$$

where the $L^{(j)(k)}$ are the phenomenological coefficients of the non-relativistic theory. With the help of (2.31) we can write for (7.5)

$$L'^{(j)(k)}_{\alpha\beta} = L^{(j)(k)}\,\Delta'_{\alpha\beta}. \quad (\alpha, \beta = 1, \ldots, 4;\ j, k = 0, 1, \ldots, n) \qquad (7.6)$$

Since if two tensors are equal in one Lorentz frame they are equal in all Lorentz frames, we can conclude from (7.6)

$$L^{(j)(k)}_{\alpha\beta} = L^{(j)(k)}\,\Delta_{\alpha\beta}. \quad (\alpha, \beta = 1, \ldots, 4;\ j, k = 0, 1, \ldots, n) \qquad (7.7)$$

As the $L^{(j)(k)}$ are the phenomenological coefficients of the non-relativistic theory we have among them the Onsager relations

$$L^{(j)(k)} = L^{(k)(j)}. \qquad (j, k = 0, 1, \ldots, n) \qquad (7.8)$$

From (2.29) and the two preceding equations we get

$$L^{(j)(k)}_{\alpha\beta} = L^{(k)(j)}_{\alpha\beta} = L^{(j)(k)}_{\beta\alpha} = L^{(k)(j)}_{\beta\alpha}. \quad (\alpha, \beta = 1, \ldots, 4;\ j, k = 0, 1, \ldots, n) \qquad (7.9)$$

We see that the Onsager relations enter again in the relativistic theory and that they are invariant under Lorentz transformations.

In a mixture of n chemical components we may have $n - 1$ independent relative flows of matter and one heat flow. Together these flows have $3n$ components in ordinary space. Hence, we should expect $3n$ independent phenomenological equations; however, (7.1) gives $4(n+1)$ equations. Therefore, we must now prove that $n + 4$ of the equations are dependent on the others. From (2.30) and (7.7) it follows that

$$\Sigma^4_{\alpha=1}\, u_\alpha\, (\Sigma^n_{k=0}\, \Sigma^4_{\beta=1}\, L^{(j)(k)}_{\alpha\beta} Y^{(k)}_{\beta}) = 0. \qquad (j = 0, 1, \ldots, n) \qquad (7.10)$$

According to (2.11) we have $u_4 \neq 0$ in every Lorentz frame. Hence, from the preceding equation, (2.24) and (3.7) we can draw the conclusion that in (7.1) for each value of j the equation with $\alpha = 4$ depends on the equations with $\alpha = 1, 2, 3$ for the same value of j. This reduces the number of independent equations by $n + 1$. As is well-known from the non-relativistic theory we have

$$\Sigma_{j=1}^{n} L^{(j)(k)} = 0. \qquad (k = 0, 1, \ldots, n) \qquad (7.11)$$

Using the preceding equation and (7.7) we find

$$\Sigma_{j=1}^{n} (\Sigma_{k=0}^{n} \Sigma_{\beta=1}^{4} L_{\alpha\beta}^{(j)(k)} Y_{\beta}^{(k)}) = 0. \; (\alpha = 1, 2, 3) \qquad (7.12)$$

From this equation and (2.23) we see that in (7.1) for $\alpha = 1$, 2 or 3 the equation with $j = n$ depends on the equations with $j = 1, 2, \ldots, n - 1$ for the same value of α and this reduces the number of independent equations by 3. Thus, finally, we get the right number of $3n$ independent equations.

It should be emphasized that the term $c^{-1}\, \mathrm{D}u_{\alpha}$, occurring in (6.9), represents an effect which the non-relativistic theory does not predict. This term, discovered already by Eckart [1]), shows that acceleration of matter causes a heat flow. Moreover, as we now see, it also gives a cross-effect with diffusion.

For the tensor $\overline{P}_{\alpha\beta}$ we can introduce the phenomenological equations

$$\overline{P}_{\alpha\beta} = c\, \Sigma_{\gamma,\zeta=1}^{4} L_{\alpha\beta\gamma\zeta}\, (\partial u_{\gamma}/\partial x_{\zeta}), \qquad (\alpha, \beta = 1, \ldots, 4) \qquad (7.13)$$

where $L_{\alpha\beta\gamma\zeta}$ is a tensor of the fourth order. Taking into account the assumption that the medium is isotropic, the postulate that all equations have to correspond closely to the equations of the non-relativistic theory in the barycentric frame and the equations (6.6), (6.7) and (6.8), which equations express properties of the tensor $\overline{P}_{\alpha\beta}$, we can derive, along the same lines which gave the result (7.7), a form for $L_{\alpha\beta\gamma\zeta}$ which leads to the equation

$$\overline{P}_{\alpha\beta} = \eta c\, \Sigma_{\gamma,\zeta=1}^{4} [\Delta_{\alpha\gamma}\Delta_{\beta\zeta}\{(\partial u_{\gamma}/\partial x_{\zeta}) + (\partial u_{\zeta}/\partial x_{\gamma})\} - \tfrac{2}{3}\Delta_{\alpha\beta}\Delta_{\gamma\zeta}\, (\partial u_{\gamma}/\partial x_{\zeta})],$$
$$(\alpha, \beta = 1, \ldots, 4) \qquad (7.14)$$

where the scalar η is the ordinary viscosity. Again, we may show that among the sixteen equations given by (7.14) eleven equations are dependent on the others. This reduces the number of independent equations to five which would be expected from physical considerations.

For the scalar quantities Π and $J_{(c)}$ we can introduce phenomenological equations of the form

$$\Pi = \eta_{(v)}\varrho'\, \mathrm{D}v' + L_{(p)(c)}\, A, \qquad (7.15)$$

$$J_{(c)} = L_{(c)(p)}\, \varrho'\, \mathrm{D}v' + LA, \qquad (7.16)$$

where $\eta_{(v)}$ is called the volume viscosity. All quantities occurring in (7.15) and (7.16) are Lorentz invariant. The Onsager relations, in the Casimir form, read

$$L_{(c)(p)} = - L_{(p)(c)}. \qquad (7.17)$$

Substitution of (7.14) and (7.15) into (6.5) gives

$$P_{\alpha\beta} = \eta c \, \Sigma^4_{\gamma,\zeta=1} [\varDelta_{\alpha\gamma} \varDelta_{\beta\zeta} \{(\partial u_\gamma/\partial x_\zeta) + (\partial u_\zeta/\partial x_\gamma)\} - \tfrac{2}{3}\varDelta_{\alpha\beta}\varDelta_{\gamma\zeta} (\partial u_\gamma/\partial x_\zeta)] + + \eta_{(v)} \varDelta_{\alpha\beta} \varrho' \, \mathrm{D}v' + L_{(p)(c)} \varDelta_{\alpha\beta} A. \qquad (\alpha, \beta = 1, .., 4) \qquad (7.18)$$

The first term in(7.16) and the last terms in (7.15) and (7.18) represent cross-effects of volume viscosity and chemical reactions which one could call "visco-chemical" effects.

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CHAPTER II

FURTHER DEVELOPMENT OF THE THEORY

§ 1. *Introduction.* In the preceding chapter we developed the relativistic thermodynamics of irreversible processes in an isotropic mixture of an arbitrary number of chemical components. Heat conduction, diffusion, viscous flow, chemical reactions and the cross-effects of these phenomena were studied. The four-dimensional tensor form in which the theory was presented warranted relativistic invariance.

In this chapter the results of the theory concerning the entropy, heat conduction and diffusion will be studied in more detail. The theory will be presented in three-dimensional tensor form. It should be emphasized that *the relativistic invariance of the theory is maintained.* Further, we shall consider to what extent the relativistic theory deviates from the non-relativistic one. We shall also discuss the transformation properties of various quantities.

In § 2 we give the connection between the four-vectors introduced in the preceding chapter, which represented the relative flows of matter of the chemical components with respect to the barycentric velocity and the three-dimensional vectors, $\mathbf{J}^{(j)}$ $(j = 1, \ldots, n)$, which are used in the non-relativistic theory for this representation. We also introduce another heat flow, $\mathbf{J}^{(0)}$, in this section. We consider the phenomenological equations for the flows $\mathbf{J}^{(j)}$ $(j = 0, 1, \ldots, n)$ in § 3. It appears to be useful to introduce new three-dimensional forces $\mathbf{X}^{(j)}$ $(j = 0, 1, \ldots, n)$. We derive in § 4 the transformation properties of the flows $\mathbf{J}^{(j)}$ and the forces $\mathbf{X}^{(j)}$ for the transition from the barycentric Lorentz frame to an arbitrary Lorentz frame. The entropy and the entropy balance are discussed in § 5. Further, we draw some conclusions in this section concerning the phenomenological coefficients from the positive definite character of the entropy production. In § 6 the heats of transfer are introduced. Their transformation properties are examined and the connection is given between diverse definitions for these quantities occurring in the literature. Another form for the forces and the phenomenological equations, with the help of which the results of the relativistic and the non-relativistic theory may easily be compared, is derived in § 7. We discuss some almost Lorentz invariant quantities in § 8. Finally, in § 9 we formulate the theory with the help of relative flows of matter which are defined with respect to a different reference velocity.

§ 2. *Relative flows of matter and heat flow.* In chapter I we defined with the help of equation (I.2.21) *) a set of n four-vectors, $I_a^{(j)}$ $(a=1, \ldots, 4; j=1, \ldots, n)$, which represented in our theory the relative flows of matter with respect to the barycentric velocity. The first three components of every four-vector form a three-vector in ordinary space. Hence, according to (I.2.22) we used as relative flow of matter of component j the three-dimensional vector

$$\mathbf{I}^{(j)} \equiv \varrho^{(j)} \left(\mathbf{v}^{(j)} - \frac{c^2 - \mathbf{v}^{(j)} \cdot \mathbf{v}}{c^2 - \mathbf{v}^2} \mathbf{v} \right). \qquad (j = 1, \ldots, n) \qquad (2.1)$$

Thus, we see, that we do not take $\mathbf{v}$ as reference velocity, but $\mathbf{v}$ multiplied by the factor $(c^2 - \mathbf{v}^{(j)} \cdot \mathbf{v})\,(c^2 - \mathbf{v}^2)^{-1}$. This factor, however, still depends on $\mathbf{v}^{(j)}$ and therefore we have in fact a different reference velocity for each chemical component. Because of this it seems to be useful to reformulate our results, without giving up the relativistic invariance of the theory, with the help of the relative flows of matter as used in the non-relativistic theory and defined by

$$\mathbf{J}^{(j)} \equiv \varrho^{(j)} \,(\mathbf{v}^{(j)} - \mathbf{v}). \qquad (j = 1, \ldots, n) \qquad (2.2)$$

The physical picture with this description is simpler than with the description using the vectors $\mathbf{I}^{(j)}$ as we now take $\mathbf{v}$ as reference velocity for each chemical component. Moreover, we may now easily compare our results with those of the non-relativistic theory.

By eliminating the vector $\varrho^{(j)}\mathbf{v}^{(j)}$ with the help of (2.2) from the right hand side of (2.1) we find as relation between $\mathbf{I}^{(j)}$ and $\mathbf{J}^{(j)}$

$$\mathbf{I}^{(j)} = \mathbf{J}^{(j)} + (\mathbf{v} \cdot \mathbf{J}^{(j)})\, (c^2 - \mathbf{v}^2)^{-1}\, \mathbf{v}. \qquad (j = 1, \ldots, n) \qquad (2.3)$$

Written out in components these equations read

$$(c^2 - \mathbf{v}^2)\, I_a^{(j)} = \Sigma_{\beta=1}^3 \{(c^2 - \mathbf{v}^2)\, \delta_{a\beta} + v_a v_\beta\}\, J_\beta^{(j)}, \quad (a=1,2,3; j=1, \ldots, n) \qquad (2.4)$$

where $\delta_{a\beta}$ is the three-dimensional Kronecker tensor. Thus, for each value of j (2.4) gives three equations with the help of which we can express the three components of $\mathbf{J}^{(j)}$ in those of $\mathbf{I}^{(j)}$ and $\mathbf{v}$. This gives

$$J_a^{(j)} = \Sigma_{\beta=1}^3\, (\delta_{a\beta} - c^{-2}\, v_a v_\beta) I_\beta^{(j)}, \quad (a = 1, 2, 3; j = 1, \ldots, n) \qquad (2.5)$$

or

$$\mathbf{J}^{(j)} = \mathbf{I}^{(j)} - c^{-2}\, (\mathbf{v} \cdot \mathbf{I}^{(j)})\, \mathbf{v}. \qquad (j = 1, \ldots, n) \qquad (2.6)$$

*) Equation (2.21) of chapter I of this thesis will be indicated as (I.2.21) and in the same way we denote the other equations of chapter I.

With the aid of (I.2.2), (I.2.4) and (I.2.6) it follows from (2.2) that

$$\Sigma_{j=1}^{n} \mathbf{J}^{(j)} = 0. \tag{2.7}$$

Again, the first three components of the four-vector $I_\alpha^{(0)}$ $(\alpha = 1, .., 4)$, representing the heat flow in the theory and defined by (I.3.5), form a three-dimensional vector. Substitution of (I.2.28) and (I.3.2) into (I.3.5) gives with the help of (I.2.11) and (I.3.3)

$$\begin{aligned}\mathbf{I}^{(0)} = c(c^2 - \mathbf{v}^2)^{-\frac{1}{2}} [(1 - \mathbf{v}^2/c^2)\, \mathbf{J}_{(e)} + \\ + \{c^{-2} + (c^2 - \mathbf{v}^2)^{-1}\}(\mathbf{v}\cdot\mathbf{J}_{(e)})\,\mathbf{v} - c^2 (c^2 - \mathbf{v}^2)^{-1} e_{(v)}\,\mathbf{v} - \\ - \Sigma_{\alpha,\beta=1}^{3} \mathbf{i}_\alpha t_{\alpha\beta} v_\beta - (c^2 - \mathbf{v}^2)^{-1} (\Sigma_{\alpha,\beta=1}^{3} v_\alpha t_{\alpha\beta} v_\beta)\,\mathbf{v}],\end{aligned} \tag{2.8}$$

where $\mathbf{i}_\alpha$ is the unit vector in the direction of the positive α-axis in ordinary space. We now split up $\mathbf{J}_{(e)}$ into two parts, one being parallel and the other being perpendicular to $\mathbf{v}$. Thus,

$$\mathbf{J}_{(e)} = \mathbf{J}_{(e)\|} + \mathbf{J}_{(e)\perp}, \tag{2.9}$$

with

$$\mathbf{J}_{(e)\|} \equiv \mathbf{v}\,(\mathbf{v}\cdot\mathbf{J}_{(e)})/\mathbf{v}^2, \tag{2.10}$$

and

$$\mathbf{J}_{(e)\perp} \equiv \mathbf{J}_{(e)} - \mathbf{v}\,(\mathbf{v}\cdot\mathbf{J}_{(e)})/\mathbf{v}^2. \tag{2.11}$$

With the help of (2.9), (2.10) and (2.11) we find for (2.8)

$$\begin{aligned}\mathbf{I}^{(0)} = c(c^2-\mathbf{v}^2)^{-\frac{1}{2}}\{\mathbf{J}_{(e)}-(\mathbf{v}^2/c^2)\,\mathbf{J}_{(e)\perp} + \mathbf{v}^2(c^2-\mathbf{v}^2)^{-1}\mathbf{J}_{(e)\|}-c^2(c^2-\mathbf{v}^2)^{-1}e_{(v)}\mathbf{v}- \\ - \Sigma_{\alpha,\beta=1}^{3} \mathbf{i}_\alpha t_{\alpha\beta} v_\beta - (c^2 - \mathbf{v}^2)^{-1} (\Sigma_{\alpha,\beta=1}^{3} v_\alpha t_{\alpha\beta} v_\beta)\,\mathbf{v}\}.\end{aligned} \tag{2.12}$$

We now define a new heat flow, $\mathbf{J}^{(0)}$, by the equation

$$J_\alpha^{(0)} \equiv \Sigma_{\beta=1}^{3} (\delta_{\alpha\beta} - c^{-2} v_\alpha v_\beta)\, I_\beta^{(0)}, \qquad (\alpha = 1, 2, 3) \tag{2.13}$$

or

$$\mathbf{J}^{(0)} = \mathbf{I}^{(0)} - c^{-2}\,(\mathbf{v}\cdot\mathbf{I}^{(0)})\,\mathbf{v}. \tag{2.14}$$

Substitution of (2.12) into (2.14) gives with the help of (2.10) and (2.11)

$$\mathbf{J}^{(0)} = c(c^2 - \mathbf{v}^2)^{-\frac{1}{2}}\{\mathbf{J}_{(e)} - (\mathbf{v}^2/c^2)\,\mathbf{J}_{(e)\perp} - e_{(v)}\mathbf{v} - \Sigma_{\alpha,\beta=1}^{3} \mathbf{i}_\alpha t_{\alpha\beta} v_\beta\}. \tag{2.15}$$

Comparison of the expressions (2.12) for $\mathbf{I}^{(0)}$ and (2.15) for $\mathbf{J}^{(0)}$ with the definitions for the heat flow of the non-relativistic theory [1]) shows that $\mathbf{J}^{(0)}$ is more closely related to the heat flows introduced in the non-relativistic theory than $\mathbf{I}^{(0)}$ and in the following it is seen that for the three-dimensional formulation $\mathbf{J}^{(0)}$ is to be preferred over $\mathbf{I}^{(0)}$.

As is well-known from the non-relativistic theory, there is, however, a certain freedom in defining the heat flow. For instance, we may also introduce as heat flow the vector

$$\mathbf{J}_{(q)} \equiv c(c^2 - \mathbf{v}^2)^{-\frac{1}{2}} \{\mathbf{J}^{(0)} - c^{-2}(\mathbf{v}\cdot\mathbf{J}^{(0)})\,\mathbf{v}\}. \tag{2.16}$$

Inserting (2.15) into (2.16) gives with the aid of (2.9), (2.10) and (2.11)

$$\mathbf{J}_{(q)} = \mathbf{J}_{(e)} - e_{(v)}\mathbf{v} - c^2(c^2 - \mathbf{v}^2)^{-1} (\Sigma^3_{\alpha,\beta=1}\, \mathbf{i}_\alpha\, t_{\alpha\beta} v_\beta - \mathbf{v} c^{-2} \Sigma^3_{\alpha,\beta=1}\, v_\alpha t_{\alpha\beta} v_\beta). \tag{2.17}$$

From the preceding equation we have

$$\mathbf{J}_{(q)} = \mathbf{J}_{(e)} - (e_{(v)} + P)\,\mathbf{v} \quad \text{if} \quad t_{\alpha\beta} = P\,\delta_{\alpha\beta},\ (\alpha, \beta = 1, 2, 3) \tag{2.18}$$

which form is the same as a well-known definition in the non-relativistic theory for the heat flow in a non-viscous medium [1]).

§ 3. *Forces, phenomenological equations and Onsager relations.* To obtain the phenomenological equations for the flows $\mathbf{J}^{(j)}$ $(j = 0, 1, \ldots, n)$, we substitute (I.7.1) into (2.5) and (2.13). Using (I.7.7) we then find

$$J^{(j)}_\alpha = \Sigma^n_{k=0}\, L^{(j)(k)} \{\Sigma^3_{\beta=1}\, \Sigma^4_{\gamma=1}\, (\delta_{\alpha\beta} - c^{-2} v_\alpha v_\beta)\, \Delta_{\beta\gamma}\, Y^{(k)}_\gamma\}.$$
$$(\alpha = 1, 2, 3;\, j = 0, 1, \ldots, n) \tag{3.1}$$

We now define the three-dimensional "forces" (affinities) $\mathbf{X}^{(k)} (k = 0, 1, \ldots, n)$ by

$$X^{(k)}_\alpha \equiv \Sigma^3_{\beta=1}\, \Sigma^4_{\gamma=1}\, (\delta_{\alpha\beta} - c^{-2} v_\alpha v_\beta)\, \Delta_{\beta\gamma}\, Y^{(k)}_\gamma. \quad (\alpha = 1, 2, 3;\, k = 0, 1, \ldots, n) \tag{3.2}$$

With these forces (3.1) takes the simple form

$$\mathbf{J}^{(j)} = \Sigma^n_{k=0}\, L^{(j)(k)}\, \mathbf{X}^{(k)}. \qquad (j = 0, 1, \ldots, n) \tag{3.3}$$

The definitions of the relative flows of matter, $\mathbf{J}^{(j)}$ $(j = 1, \ldots, n)$, are analogous to those of the non-relativistic theory (cf. (2.2) and [2])). This is not the case with the definitions of the heat flow, $\mathbf{J}^{(0)}$ (cf. (2.15) and [1])), and the forces, $\mathbf{X}^{(k)}$ $(k = 0, 1, \ldots, n)$ (cf. (3.9), (3.12) and [2])). The phenomenological coefficients are the same as in the non-relativistic theory and satisfy the Onsager relations

$$L^{(j)(k)} = L^{(k)(j)} \qquad (j, k = 0, 1, \ldots, n) \tag{3.4}$$

according to (I.7.8). From (I.7.11) we have with the aid of the preceding equation

$$L^{(j)(n)} = -\Sigma^{n-1}_{k=1}\, L^{(j)(k)}. \qquad (j = 0, 1, \ldots, n) \tag{3.5}$$

By substituting (3.5) into (3.3) we obtain the equation

$$\mathbf{J}^{(j)} = \Sigma^{n-1}_{k=1}\, L^{(j)(k)}\, (\mathbf{X}^{(k)} - \mathbf{X}^{(n)}) + L^{(j)(0)}\, \mathbf{X}^{(0)}, \quad (j = 0, 1, \ldots, n-1) \tag{3.6}$$

which contains only independent quantities. The forms of the preceding equation and of (3.3) are well-known in the non-relativistic theory [2]). For $I_a^{(j)}$ $(a = 1, .., 4; j = 0, 1, \ldots, n-1)$ we can derive a similar expression

$$I_a^{(j)} = \Sigma_{\beta=1}^4 \{\Sigma_{k=1}^{n-1} L_{a\beta}^{(j)(k)}(Y_\beta^{(k)} - Y_\beta^{(n)}) + L_{a\beta}^{(j)(0)} Y_\beta^{(0)}\}.$$
$$(a = 1, .., 4; j = 0, 1, \ldots, n-1) \quad (3.7)$$

We shall now consider the explicit form of $\mathbf{X}^{(k)}$ in more detail. For that purpose we substitute (I.2.28) into (3.2). Using (I.2.11) we then find

$$X_a^{(k)} = Y_a^{(k)} + ic^{-1} v_a Y_4^{(k)}. \; (a = 1, 2, 3; k = 0, 1, \ldots, n) \quad (3.8)$$

Inserting (I.6.9) into (3.8) gives with the help of (I.2.11) and (I.2.25) for $k = 0$

$$\mathbf{X}^{(0)} = -\{(1/T') \text{ grad } T' + (c^2 - \mathbf{v}^2)^{-1} (\mathrm{d}\mathbf{v}/\mathrm{d}t) + (\mathbf{v}/T')c^{-2}(\partial T'/\partial t)\}, \quad (3.9)$$

where the operator $\mathrm{d}/\mathrm{d}t$ is the substantial derivative with respect to time defined by

$$\mathrm{d}/\mathrm{d}t \equiv \partial/\partial t + \Sigma_{\beta=1}^3 v_\beta (\partial/\partial x_\beta). \quad (3.10)$$

From (I.5.3) we have with the aid of (I.2.4), (I.2.14) and (I.2.19)

$$\omega^{(k)}\varrho^{(k)}/\varrho_{(0)}^{(k)} = c(c^2 - \mathbf{v}^2)^{\frac{1}{2}} (c^2 - \mathbf{v}^{(k)} \cdot \mathbf{v})^{-1}. \quad (k = 1, \ldots, n) \quad (3.11)$$

Substitution of (I.6.10) into (3.8) gives with the help of (I.4.4) and (3.11) for $k = 1, \ldots, n$

$$\mathbf{X}^{(k)} = c(c^2 - \mathbf{v}^2)^{\frac{1}{2}} (c^2 - \mathbf{v}^{(k)} \cdot \mathbf{v})^{-1} \{\mathbf{F}^{(k)} - c^{-2}(\mathbf{v}^{(k)} \cdot \mathbf{F}^{(k)}) \mathbf{v}\} -$$
$$- T' [\text{grad } (\mu'^{(k)}/T') + c^{-2}\{\partial(\mu'^{(k)}/T')/\partial t\} \mathbf{v}]. \quad (k = 1, \ldots, n). \quad (3.12)$$

From (3.9) it follows that in the barycentric Lorentz frame $\mathbf{X}^{(0)}$ has the form

$$\mathbf{X}'^{(0)} = -\{(1/T') \text{ grad}' \, T' + c^{-2} (\partial \mathbf{v}/\partial t)'\}, \quad (3.13)$$

while we find from (3.12) for the form of $\mathbf{X}^{(k)}$ $(k = 1, \ldots, n)$ in the barycentric Lorentz frame

$$\mathbf{X}'^{(k)} = \mathbf{F}'^{(k)} - T' \text{ grad}' \, (\mu'^{(k)}/T'). \quad (k = 1, \ldots, n) \quad (3.14)$$

In the two preceding formulae grad' means that the operation of the forming of the gradient must be performed with the help of x_1', x_2' and x_3'. The forms (3.13) and (3.14) for $\mathbf{X}'^{(k)}$ $(k = 0, 1, \ldots, n)$ correspond closely to the expressions for the forces in the non-relativistic theory [2]), the only difference being the second term on the right hand side of (3.13). It should be remarked that, though $\mathbf{v} = 0$ in the barycentric Lorentz frame, in general $(\partial \mathbf{v}/\partial t)'$ does not vanish. The general expressions for the forces (3.9) and (3.12) show that, in contradistinction to the non-relativistic theory, the forces depend on $\mathbf{v}^{(k)}$ and on derivatives with respect to time of several quantities. In § 7 we shall derive another form for the forces in which $\mathbf{v}^{(k)}$ and derivatives with respect to time, except the time derivative of the barycentric velocity, do not occur explicitly.

§ 4. *The transformation properties of flows and forces.* The phenomenological coefficients $L^{(j)(k)}$, occurring in (3.3), are Lorentz invariant. The flows and forces occurring in (3.3), however, do not transform as the components of a four-dimensional tensor and we shall now examine their transformation properties.

We have from (I.3.6), (I.7.2), (2.6) and (2.14)

$$I_1'^{(j)} = J_1'^{(j)};\ I_2'^{(j)} = J_2'^{(j)};\ I_3'^{(j)} = J_3'^{(j)};\ I_4'^{(j)} = 0. \quad (j = 0, 1, \ldots, n) \tag{4.1}$$

We now consider two Lorentz frames. Quantities measured in one of the frames we denote by double primes. We then have according to the theory of the Lorentz transformations [3] [4]

$$I_\alpha^{(j)} = \Sigma_{\beta=1}^4 a_{\alpha\beta} I_\beta''^{(j)}. \qquad (\alpha = 1, \ldots, 4;\ j = 0, 1, \ldots, n) \tag{4.2}$$

Excluding rotations of the three-dimensional axis-frame, we have for the coefficients $a_{\alpha\beta}$ [3] [4]

$$\begin{aligned} a_{\alpha\beta} &= \delta(\alpha;\beta) + v_{(r)\alpha} v_{(r)\beta}\, \varkappa/\mathbf{v}_{(r)}^2 \qquad (\alpha, \beta = 1, 2, 3); \\ a_{\alpha 4} = -a_{4\alpha} &= i v_{(r)\alpha}\, c^{-1}(1-\mathbf{v}_{(r)}^2/c^2)^{-\frac{1}{2}} \quad (\alpha = 1,2,3);\ a_{44} = (1-\mathbf{v}_{(r)}^2/c^2)^{-\frac{1}{2}}, \end{aligned} \tag{4.3}$$

where $\mathbf{v}_{(r)}$ is the velocity of the Lorentz frame "without primes" with respect to the Lorentz frame "with double primes" and $\varkappa$ is given by

$$\varkappa \equiv (1 - \mathbf{v}_{(r)}^2/c^2)^{-\frac{1}{2}} - 1. \tag{4.4}$$

We now take for the Lorentz frame "with double primes" the barycentric Lorentz frame. We then have $\mathbf{v}_{(r)} = -\mathbf{v}$ and (4.2) becomes with the help of (4.1) and (4.3)

$$I_\alpha^{(j)} = \Sigma_{\beta=1}^3 (\delta_{\alpha\beta} + v_\alpha v_\beta\, \varkappa/\mathbf{v}^2)\, J_\beta'^{(j)}, \qquad (\alpha=1, 2, 3; j=0, 1, \ldots, n) \tag{4.5}$$

or

$$\mathbf{I}^{(j)} = \mathbf{J}'^{(j)} + \mathbf{v}(\mathbf{v}\cdot\mathbf{J}'^{(j)})\, \varkappa/\mathbf{v}^2. \quad (j = 0, 1, \ldots, n) \tag{4.6}$$

Substitution of (4.6) into (2.6) and (2.14) gives with the aid of (4.4)

$$\mathbf{J}^{(j)} = \mathbf{J}'^{(j)} + \mathbf{v}\,(\mathbf{v}\cdot\mathbf{J}'^{(j)})\,\{(1-\mathbf{v}^2/c^2)^{\frac{1}{2}} - 1\}/\mathbf{v}^2. \quad (j = 0,1, \ldots, n) \tag{4.7}$$

We now split up $\mathbf{J}'^{(j)}$ into two parts; one being perpendicular to $\mathbf{v}$ and the other parallel to $\mathbf{v}$. Thus,

$$\mathbf{J}'^{(j)}_{\parallel} \equiv \mathbf{v}\,(\mathbf{v}\cdot\mathbf{J}'^{(j)})/\mathbf{v}^2, \qquad (j = 0, 1, \ldots, n) \tag{4.8}$$

and

$$\mathbf{J}'^{(j)}_{\perp} \equiv \mathbf{J}'^{(j)} - \mathbf{v}(\mathbf{v}\cdot\mathbf{J}'^{(j)})/\mathbf{v}^2. \quad (j = 0, 1, \ldots, n) \tag{4.9}$$

With the aid of (4.8) and (4.9) we get for (4.7)

$$\mathbf{J}^{(j)} = \mathbf{J}'^{(j)}_{\perp} + \mathbf{J}'^{(j)}_{\parallel}(1 - \mathbf{v}^2/c^2)^{\frac{1}{2}}. \quad (j = 0, 1, \ldots, n) \tag{4.10}$$

Hence, we see that the component of $\mathbf{J}^{(j)}$ perpendicular to $\mathbf{v}$ is the same as the component of $\mathbf{J}'^{(j)}$ perpendicular to $\mathbf{v}$, however, the component of $\mathbf{J}^{(j)}$ parallel to $\mathbf{v}$ becomes smaller if one goes to a Lorentz frame in which $|\mathbf{v}|$ is larger and differs a factor $(1 - \mathbf{v}^2/c^2)^{\frac{1}{2}}$ from the component of $\mathbf{J}'^{(j)}$ parallel to $\mathbf{v}$.

From (I.2.31) and (3.2) we have

$$\begin{aligned} \Sigma_{\beta=1}^{4} \Delta'_{\alpha\beta} Y'^{(k)}_{\beta} = Y'^{(k)}_{\alpha} = X'^{(k)}_{\alpha} \quad (\alpha = 1, 2, 3; k = 0, 1, \ldots, n); \\ \Sigma_{\beta=1}^{4} \Delta_{4\beta} Y'^{(k)}_{\beta} = 0 \qquad (k = 0, 1, \ldots, n). \end{aligned} \tag{4.11}$$

With the help of the preceding equation we can deduce in the same way as above that

$$\mathbf{X}^{(k)} = \mathbf{X}'^{(k)}_{\perp} + \mathbf{X}'^{(k)}_{\parallel} (1 - \mathbf{v}^2/c^2)^{\frac{1}{2}}, \quad (k = 0, 1, \ldots, n) \tag{4.12}$$

where $\mathbf{X}'^{(k)}_{\parallel}$ and $\mathbf{X}'^{(k)}_{\perp}$ are defined by equations analogous to (4.8) and (4.9) respectively.

In the same way as we derived the transformation properties of $\mathbf{J}^{(k)}$ $(k = 0, 1, \ldots, n)$ we can find those of $\mathbf{J}_{(q)}$.

§ 5. *The entropy and the entropy balance.* In § 6 of chapter I we derived the entropy balance. We shall now examine this balance further. According to (I.6.12) we have

$$\varrho' \mathrm{D}s' = - \Sigma_{\alpha=1}^{4} (\partial I_{(s)\alpha}/\partial x_{\alpha}) + \sigma, \tag{5.1}$$

where $I_{(s)\alpha}$ and σ are given by

$$I_{(s)\alpha} \equiv (1/T') (I^{(0)}_{\alpha} - \Sigma_{j=1}^{n} \mu'^{(j)} I^{(j)}_{\alpha}), \quad (\alpha = 1, \ldots, 4) \tag{5.2}$$

and

$$\begin{aligned} \sigma \equiv (1/T') \{ \Sigma_{j=0}^{n} \Sigma_{\alpha=1}^{4} I^{(j)}_{\alpha} Y^{(j)}_{\alpha} + \\ + c \, \Sigma_{\alpha,\beta=1}^{4} \bar{P}_{\alpha\beta} (\partial u_{\beta}/\partial x_{\alpha}) + \Pi \varrho' \mathrm{D}v' + J_{(c)} A \} \end{aligned} \tag{5.3}$$

respectively.

We shall now transform the expression (5.1). For that purpose we first derive with the help of (I.2.10), (I.2.14), (I.2.25) and (I.4.3)

$$\varrho' \mathrm{D}s' = \Sigma_{\alpha=1}^{4} \partial(m_{\alpha} s')/\partial x_{\alpha}. \tag{5.4}$$

Inserting (5.4) into (5.1) gives

$$\Sigma_{\alpha=1}^{4} \partial S_{\alpha}/\partial x_{\alpha} = \sigma, \tag{5.5}$$

where the four-vector S_{α} is given by

$$S_{\alpha} \equiv m_{\alpha} s' + I_{(s)\alpha}. \quad (\alpha = 1, \ldots, 4) \tag{5.6}$$

We now introduce the quantities

$$s_{(v)} \equiv - ic^{-1} S_4, \tag{5.7}$$

and

$$J_{(s)\alpha} \equiv S_\alpha - s_{(v)} v_\alpha. \qquad (\alpha = 1, 2, 3) \tag{5.8}$$

Substitution of the two preceding equations into (5.5) gives

$$\partial s_{(v)}/\partial t = - \operatorname{div} (\mathbf{J}_{(s)} + s_{(v)} \mathbf{v}) + \sigma. \tag{5.9}$$

Introducing

$$s \equiv s_{(v)}/\varrho, \tag{5.10}$$

we find from (5.9) with the help of (I.2.8), (I.4.3) and (3.10)

$$\varrho (\mathrm{d}s/\mathrm{d}t) = - \operatorname{div} \mathbf{J}_{(s)} + \sigma. \tag{5.11}$$

The preceding equation has exactly the form of the entropy balance as it is usually given in the non-relativistic theory [2]). Hence, we can interpret s as the specific entropy, $s_{(v)}$ as the density of entropy and $\mathbf{J}_{(s)}$ as the density of the conductive flow of entropy.

Inserting (5.6) into (5.7) gives with the help of (I.2.8), (I.2.14) and (5.10)

$$s_{(v)} = s'_{(v)} (1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}} - ic^{-1} I_{(s)4}. \tag{5.12}$$

According to the Lorentz contraction we have

$$\mathrm{d}V = \mathrm{d}V' (1 - \mathbf{v}^2/c^2)^{\frac{1}{2}}, \tag{5.13}$$

where $\mathrm{d}V$ is an infinitesimal volume element in ordinary space. Multiplying (5.12) by $\mathrm{d}V$ and using (5.13) gives for the entropy in the volume element $\mathrm{d}V$

$$s_{(v)}\, \mathrm{d}V = s'_{(v)}\, \mathrm{d}V' - ic^{-1} I_{(s)4}\, \mathrm{d}V. \tag{5.14}$$

From (5.2) and the preceding equation we see that the entropy in the volume element is only a Lorentz invariant quantity if there is no diffusion and heat conduction ($I_\alpha^{(j)} = 0$ ($\alpha = 1, \ldots, 4; j = 0, 1, \ldots, n$)). We shall show, however, in § 8 that in all practical cases the second term on the right hand side of (5.14) is very small with respect to the first term, so that the entropy in a small volume element has almost the same value in all Lorentz frames. Hence, we can conclude that according to (5.7) *the density of entropy* multiplied by ic appears to be the fourth component of a four-vector, while the entropy in a small volume element is not a Lorentz invariant quantity because the transport of entropy is due not only to convection (the term $s_{(v)}\, \mathbf{v}$ in (5.9)), but also to conduction (the term $\mathbf{J}_{(s)}$ in (5.9)). Thus, Planck's point of view [5]) [6]), which is also adopted by Einstein [7]), and according to which the entropy is Lorentz invariant, is according to our formalism only correct if there is no diffusion and heat conduction. Eckart [8]) interprets the quantity s' as the entropy, whereas we think it is more correct to call this quantity the specific entropy measured by an observer in the barycentric Lorentz frame.

To avoid confusion we make the following remark. If in a Lorentz frame at the time t and at the position $\mathbf{r}$ we have $\mathbf{v}(\mathbf{r}, t) = 0$, then $s_{(v)}(\mathbf{r}, t) = s'_{(v)}(\mathbf{r}, t)$. In general, however, at the time $t + \mathrm{d}t$ we shall have $\mathbf{v}(\mathbf{r}, t + \mathrm{d}t) \neq 0$ and then $s_{(v)}(\mathbf{r}, t + \mathrm{d}t) \neq s'_{(v)}(\mathbf{r}, t + \mathrm{d}t)$. It appears that $\lim_{\mathrm{d}t=0} \{s_{(v)}(\mathbf{r}, t + \mathrm{d}t) - s'_{(v)}(\mathbf{r}, t)\}/\mathrm{d}t$ differs from $\lim_{\mathrm{d}t=0} \{s'_{(v)}(\mathbf{r}, t + \mathrm{d}t) - s'_{(v)}(\mathbf{r}, t)\}/\mathrm{d}t$. Applying (5.5) or (5.9) in the barycentric Lorentz frame, we must take for $\partial s_{(v)}/\partial t$ the first limit mentioned above. Also by applying (5.11) in the barycentric Lorentz frame we must take a limit of this kind for $\partial s/\partial t$. Similar considerations also hold for the derivatives with respect to space coordinates of the entropy and the entropy flow. In general if Ξ is some arbitrary quantity (for instance a tensor component) depending on x_1, x_2, x_3 and x_4, we denote by $(\partial \Xi/\partial x_a)'$ $(a = 1, \ldots, 4)$ a limit of the first kind (Cf. (3.13)) and by $\partial \Xi'/\partial x'_a$ $(a = 1, \ldots, 4)$ a limit of the second kind (Cf. (I.4.9)).

According to (5.3) the contribution, $\sigma_{(h)(d)}$, of heat conduction and diffusion to σ is given by

$$T' \sigma_{(h)(d)} = \Sigma_{j=0}^{n} \Sigma_{a=1}^{4} I_a^{(j)} Y_a^{(j)} = \Sigma_{j=0}^{n} \mathbf{J}'^{(j)} \cdot \mathbf{X}'^{(j)}, \tag{5.15}$$

where we have used (4.1) and (4.11). From (4.10) and (4.12) we have

$$\mathbf{J}'^{(j)} = \mathbf{J}_{\perp}^{(j)} + (1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}} \mathbf{J}_{\parallel}^{(j)}, \quad (j = 0, 1, \ldots, n) \tag{5.16}$$

$$\mathbf{X}'^{(j)} = \mathbf{X}_{\perp}^{(j)} + (1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}} \mathbf{X}_{\parallel}^{(j)}, \quad (j = 0, 1, \ldots, n) \tag{5.17}$$

where $\mathbf{J}_{\parallel}^{(j)}$ and $\mathbf{X}_{\parallel}^{(j)}$ are defined analogous to (2.10) and $\mathbf{J}_{\perp}^{(j)}$ and $\mathbf{X}_{\perp}^{(j)}$ are defined analogous to (2.11). Inserting the two preceding equations into (5.15) gives

$$T' \sigma_{(h)(d)} = \Sigma_{j=0}^{n} \{\mathbf{J}_{\perp}^{(j)} \cdot \mathbf{X}_{\perp}^{(j)} + (1 - \mathbf{v}^2/c^2)^{-1} \mathbf{J}_{\parallel}^{(j)} \cdot \mathbf{X}_{\parallel}^{(j)}\}, \tag{5.18}$$

or

$$T' \sigma_{(h)(d)} = \Sigma_{j=0}^{n} \mathbf{J}^{(j)} \cdot \overline{\mathbf{X}}^{(j)}, \tag{5.19}$$

where $\overline{\mathbf{X}}^{(j)}$ is given by

$$\overline{\mathbf{X}}^{(j)} \equiv \mathbf{X}_{\perp}^{(j)} + (1 - \mathbf{v}^2/c^2)^{-1} \mathbf{X}_{\parallel}^{(j)}. \quad (j = 0, 1, \ldots, n) \tag{5.20}$$

The form (5.19) is analogous to the form which is usually given in the non-relativistic theory for $\sigma_{(h)(d)}$.

According to the second law of thermodynamics σ must be a positive definite expression. Analogous to the non-relativistic theory [2]) we can draw some conclusions concerning the phenomenological coefficients from this positive definite character of σ. Substitution of (3.3) into (5.15) gives

$$T' \sigma_{(h)(d)} = \Sigma_{j,k=0}^{n} L^{(j)(k)} \mathbf{X}'^{(j)} \cdot \mathbf{X}'^{(k)}. \tag{5.21}$$

As $\sigma_{(h)(d)}$ is a part of σ we must have $\sigma_{(h)(d)} \geqslant 0$. Hence, from (5.21) we find, analogous to the non-relativistic theory [2]), that the $L^{(j)(k)}$ must satisfy several inequalities as, for instance

$$L^{(j)(j)} \geqslant 0, \qquad (j = 0, 1, \ldots, n) \tag{5.22}$$

and

$$L^{(j)(j)}L^{(k)(k)} - L^{(j)(k)}L^{(k)(j)} \geqslant 0. \qquad (j, k = 0, 1, \ldots, n) \tag{5.23}$$

From (I.7.15), (I.7.16), (I.7.17) and (5.3) we find for the contribution, $\sigma_{(c)(v)}$, of the chemical reaction and the volume viscosity to σ

$$T'\,\sigma_{(c)(v)} = LA^2 + \eta_{(v)}\,(\varrho'\,\mathrm{D}v')^2. \tag{5.24}$$

Again, $\sigma_{(c)(v)} \geqslant 0$ and thus,

$$L \geqslant 0, \tag{5.25}$$

and

$$\eta_{(v)} \geqslant 0. \tag{5.26}$$

It is interesting to note that there are no cross-terms in the expression (5.24); therefore, $L_{(p)(c)}$ and $L_{(c)(p)}$ need not satisfy an inequality of the type given above. Along arguments analogous to those which lead to (5.22) we can derive

$$\eta \geqslant 0, \tag{5.27}$$

which assures that the contribution of the viscous flow of the medium to σ is positive definite. As σ, $\sigma_{(h)(d)}$, etc. are Lorentz invariant quantities, the given inequalities assure the positive definite character of σ in all Lorentz frames.

§ 6. *The heats of transfer.* In the literature the heats of transfer are introduced in different ways. A set of $n - 1$ independent quantities, $Q^{*(1)}$, $Q^{*(2)}, \ldots, Q^{*(n-1)}$, to which one gives the name heats of transfer are defined by

$$L^{(j)(0)} = \Sigma_{k=1}^{n-1} L^{(j)(k)} Q^{*(k)}. \qquad (j = 1, \ldots, n - 1) \tag{6.1}$$

With the help of the $n - 1$ equations (6.1) we can express $Q^{*(1)}, Q^{*(2)}, \ldots, Q^{*(n-1)}$ in terms of the Lorentz invariant phenomenological coefficients. Hence, these heats of transfer are also Lorentz invariant. Substitution of (6.1) into (3.6) gives for $j = 1, \ldots, n - 1$ the equations

$$\mathbf{J}^{(j)} = \Sigma_{k=1}^{n-1} L^{(j)(k)}(\mathbf{X}^{(k)} - \mathbf{X}^{(n)} + Q^{*(k)}\mathbf{X}^{(0)}), \quad (j = 1, \ldots, n - 1) \tag{6.2}$$

while we find from (3.6) with the help of the two preceding equations and (3.4) for $j = 0$ the equation

$$\mathbf{J}^{(0)} = \Sigma_{j=1}^{n-1} Q^{*(j)}\mathbf{J}^{(j)} + \lambda T'\,\mathbf{X}^{(0)}, \tag{6.3}$$

where the Lorentz invariant quantity λ is defined by

$$\lambda \equiv (1/T')\,(L^{(0)(0)} - \Sigma_{k=1}^{n-1} L^{(0)(k)} Q^{*(k)}). \tag{6.4}$$

This quantity is the coefficient of heat conduction in the stationary state. The four preceding equations are analogous to those of the non-relativistic theory. According to (3.13) and (4.12) we have $\mathbf{X}^{(0)} = 0$, if in the barycentric Lorentz frame T' is uniform and at the same time $(\partial \mathbf{v}/\partial t)' = 0$; therefore, one calls the quantities $Q^{*(j)}$ $(j = 1, \ldots, n-1)$ heats of transfer. We remark that if for an observer at a certain point in the space-time continuum grad $T' = 0$ (uniformity of the temperature T') for an observer in a different Lorentz frame in general grad $T' \neq 0$.

From (6.3) we can derive another form for $\mathbf{J}^{(0)}$ which is also used in the literature [9]. For that purpose we substitute (2.2) into (6.3) and we then get with the aid of (I.2.4) and (I.2.6)

$$\mathbf{J}^{(0)} = \Sigma_{j=1}^{n-1} (Q^{*(j)} - \Sigma_{k=1}^{n-1} c^{(k)} Q^{*(k)})\, \varrho^{(j)} \mathbf{v}^{(j)} - (\Sigma_{k=1}^{n-1} c^{(k)} Q^{*(k)}) \varrho^{(n)} \mathbf{v}^{(n)} + \lambda T' \mathbf{X}^{(0)}. \tag{6.5}$$

The n quantities $\bar{Q}^{(1)}, \bar{Q}^{(2)}, \ldots, \bar{Q}^{(n)}$, which are related to $Q^{*(1)}, Q^{*(2)}, \ldots, Q^{*(n-1)}$ by the equations

$$\bar{Q}^{(j)} \equiv Q^{*(j)} - \Sigma_{k=1}^{n-1} c^{(k)} Q^{*(k)}, \qquad (j = 1, \ldots, n-1) \tag{6.6}$$

and

$$\bar{Q}^{(n)} \equiv - \Sigma_{k=1}^{n-1} c^{(k)} Q^{*(k)}, \tag{6.7}$$

are also denoted as heats of transfer, since (6.5) can be written with the help of the two preceding definitions in the form

$$\mathbf{J}^{(0)} = \Sigma_{j=1}^{n} \bar{Q}^{(j)} \varrho^{(j)} \mathbf{v}^{(j)} + \lambda T' \mathbf{X}^{(0)}. \tag{6.8}$$

In this equation the absolute flows of matter $\varrho^{(j)}\mathbf{v}^{(j)}$ are used instead of the relative flows of matter $\mathbf{J}^{(j)}$, defined by (2.2), which occur in (6.3). From (I.2.5), (6.6) and (6.7) we see that the quantities $\bar{Q}^{(j)}$ $(j = 1, \ldots, n)$ satisfy the relation

$$\Sigma_{j=1}^{n} c^{(j)} \bar{Q}^{(j)} = 0. \tag{6.9}$$

According to (I.2.19) the concentrations are not Lorentz invariant. Hence, we see from (6.6) and (6.7) that also the quantities $\bar{Q}^{(j)}$ are not Lorentz invariant. In § 8 we shall show, however, that in all practical cases these quantities have almost the same value in all Lorentz frames.

If we use the heat flow $\mathbf{I}^{(0)}$ we can obtain a set of n heats of transfer which are exactly Lorentz invariant. Analogous to (6.2) we find for $I_\alpha^{(j)}$ $(j \neq 0)$

$$I_\alpha^{(j)} = \Sigma_{k=1}^{n-1} \Sigma_{\beta=1}^{4} L_{\alpha\beta}^{(j)(k)} (Y_\beta^{(k)} - Y_\beta^{(n)} + Q^{*(k)} Y_\beta^{(0)}),$$
$$(\alpha = 1, \ldots, 4;\ j = 1, \ldots, n-1) \tag{6.10}$$

whereas, analogous to (6.3), we find for $I_a^{(0)}$

$$I_a^{(0)} = \Sigma_{j=1}^{n-1} Q^{*(j)} I_a^{(j)} + \lambda T' \Sigma_{\beta=1}^{4} \Delta_{a\beta} Y_\beta^{(0)}. \quad (a = 1, \ldots, 4) \tag{6.11}$$

Inserting (I.2.21) into this equation gives with the aid of (I.2.7)

$$I_a^{(0)} = \Sigma_{j=1}^{n} Q^{(j)} m_a^{(j)} + \lambda T' \Sigma_{\beta=1}^{4} \Delta_{a\beta} Y_\beta^{(0)}, \qquad (a = 1, \ldots, 4) \tag{6.12}$$

where the quantities $Q^{(j)}$ $(j = 1, \ldots, n)$ are defined by

$$Q^{(j)} \equiv Q^{*(j)} - \Sigma_{k=1}^{n-1} c'^{(k)} Q^{*(k)}, \qquad (j = 1, \ldots, n-1) \tag{6.13}$$

and

$$Q^{(n)} \equiv - \Sigma_{k=1}^{n-1} c'^{(k)} Q^{*(k)}. \tag{6.14}$$

We now see that in contradistinction to the $\bar{Q}^{(j)}$ $(j = 1, \ldots, n)$ the quantities $Q^{(j)}$ $(j = 1, \ldots, n)$ are exactly Lorentz invariant because the right hand sides of the two preceding equations contain only Lorentz invariant quantities. It is easily seen that

$$\Sigma_{j=1}^{n} c'^{(j)} Q^{(j)} = 0. \tag{6.15}$$

From the positive definite character of $\sigma_{(h)(d)}$ it follows that

$$\lambda \geqslant 0, \tag{6.16}$$

showing that the coefficient of heat conduction in the stationary state, λ, must be positive or zero (cf. [2]).

§ 7. *Other forms for the forces and the phenomenological equations.* We shall now give other expressions for the forces in which the velocities of the components and derivatives with respect to time, except the time-derivative of the barycentric velocity, do not occur explicitly. The desired expressions are readily obtained by inserting (3.13) and (3.14) into (4.12) which gives

$$\begin{aligned} \mathbf{X}^{(0)} = - [(1/T') \{(\text{grad}' \, T')_\perp + (\text{grad}' \, T')_{||} (1 - \mathbf{v}^2/c^2)^{\frac{1}{2}}\} \\ + c^{-2} \{(\partial \mathbf{v}/\partial t)'_\perp + (\partial \mathbf{v}/\partial t)'_{||} (1 - \mathbf{v}^2/c^2)^{\frac{1}{2}}\}], \end{aligned} \tag{7.1}$$

and

$$\begin{aligned} \mathbf{X}^{(k)} = \mathbf{F}'^{(k)}_\perp + \mathbf{F}'^{(k)}_{||} (1 - \mathbf{v}^2/c^2)^{\frac{1}{2}} - \\ - T'[\{\text{grad}'(\mu'^{(k)}/T')\}_\perp + \{\text{grad}'(\mu'^{(k)}/T')\}_{||} (1 - \mathbf{v}^2/c^2)^{\frac{1}{2}}] \quad (k = 1, \ldots, n) \end{aligned} \tag{7.2}$$

respectively. In these expressions the components of grad$'$ T', grad$'$ $(\mu'^{(k)}/T')$, $(\partial \mathbf{v}/\partial t)'$ and $\mathbf{F}'^{(k)}$, parallel and perpendicular to $\mathbf{v}$, are defined analogous to (4.8) and (4.9). The partial specific enthalpy of component k measured by an observer in the barycentric Lorentz frame is given by

$$h'^{(k)} = \mu'^{(k)} + T' s'^{(k)}, \qquad (k = 1, \ldots, n) \tag{7.3}$$

where $s^{(k)}$ is the partial specific entropy of chemical component k. We now consider $\mu'^{(k)}$ as a function of $c'^{(1)}, c'^{(2)}, \ldots, c'^{(n-1)}$, p' and T'. We then have the relation [2])

$$\operatorname{grad}' \mu'^{(k)} = - s'^{(k)} \operatorname{grad}' T' + v'^{(k)} \operatorname{grad}' p' + \Sigma_{j=1}^{n-1} (\partial \mu'^{(k)} / \partial c'^{(j)}) \operatorname{grad}' c'^{(j)}, \quad (k = 1, \ldots, n) \tag{7.4}$$

where $v^{(k)}$ is the partial specific volume of the chemical component k. Substitution of (7.4) into (7.2) gives with the help of (7.3)

$$\begin{aligned}\mathbf{X}^{(k)} &= \mathbf{F}'^{(k)}_{\perp} + \mathbf{F}'^{(k)}_{\parallel} (1 - \mathbf{v}^2/c^2)^{\frac{1}{2}} + \\ &+ (h'^{(k)}/T') \{(\operatorname{grad}' T')_{\perp} + (\operatorname{grad}' T')_{\parallel} (1 - \mathbf{v}^2/c^2)^{\frac{1}{2}}\} - \\ &- v'^{(k)} \{(\operatorname{grad}' p')_{\perp} + (\operatorname{grad}' p')_{\parallel} (1 - \mathbf{v}^2/c^2)^{\frac{1}{2}}\} - \\ &- \Sigma_{j=1}^{n-1} (\partial \mu'^{(k)}/\partial c'^{(j)}) \{(\operatorname{grad}' c'^{(j)})_{\perp} + (\operatorname{grad}' c'^{(j)})_{\parallel} (1 - \mathbf{v}^2/c^2)^{\frac{1}{2}}\}. \quad (k = 1, \ldots, n)\end{aligned} \tag{7.5}$$

Inserting (7.1) and (7.5) into (6.2) gives for the phenomenological equations for the relative flows of matter $\mathbf{J}^{(j)}$

$$\begin{aligned}\mathbf{J}^{(j)} &= \Sigma_{k=1}^{n-1} L^{(j)(k)} [(\mathbf{F}'^{(k)}_{\perp} - \mathbf{F}'^{(n)}_{\perp}) + (\mathbf{F}'^{(k)}_{\parallel} - \mathbf{F}'^{(n)}_{\parallel}) (1 - \mathbf{v}^2/c^2)^{\frac{1}{2}} + \\ &+ \{(h'^{(k)} - h'^{(n)} - Q^{*(k)})/T'\} \{(\operatorname{grad}' T')_{\perp} + (\operatorname{grad}' T')_{\parallel} (1 - \mathbf{v}^2/c^2)^{\frac{1}{2}}\} - \\ &- (v'^{(k)} - v'^{(n)}) \{(\operatorname{grad}' p')_{\perp} + (\operatorname{grad}' p')_{\parallel} (1 - \mathbf{v}^2/c^2)^{\frac{1}{2}}\} - \\ &- \Sigma_{l=1}^{n-1} \{\partial(\mu'^{(k)} - \mu'^{(n)})/\partial c'^{(l)}\} \{(\operatorname{grad}' c'^{(l)})_{\perp} + (\operatorname{grad}' c'^{(l)})_{\parallel} (1 - \mathbf{v}^2/c^2)^{\frac{1}{2}}\} - \\ &- (Q^{*(k)}/c^2) \{(\partial \mathbf{v}/\partial t)'_{\perp} + (\partial \mathbf{v}/\partial t)'_{\parallel} (1 - \mathbf{v}^2/c^2)^{\frac{1}{2}}\}]. \quad (j = 1, \ldots, n-1)\end{aligned} \tag{7.6}$$

If c tends to infinity (7.6) goes over into the well-known phenomenological equations for $\mathbf{J}^{(j)} (j = 1, \ldots, n-1)$ of the non-relativistic theory [2]).

The phenomenological equations may also be written in the form

$$J^{(j)}_{\alpha} = \Sigma_{k=0}^{n} \Sigma_{\beta=1}^{3} \bar{L}^{(j)(k)}_{\alpha\beta} \bar{X}^{(k)}_{\beta}, \quad (\alpha = 1, 2, 3; j = 0, 1, \ldots, n) \tag{7.7}$$

where

$$\bar{L}^{(j)(k)}_{\alpha\beta} \equiv L^{(j)(k)} (\delta_{\alpha\beta} - c^{-2} v_{\alpha} v_{\beta}). \quad (\alpha, \beta = 1, 2, 3; j, k = 0, 1, \ldots, n) \tag{7.8}$$

This is readily verified, for by inserting (5.20) and (7.8) into (7.7) we get the phenomenological equations (3.3) back again if we take into account that $\mathbf{X}^{(k)}_{\parallel}$ and $\mathbf{X}^{(k)}_{\perp}$ are defined analogous to (2.10) and (2.11) respectively.

§ 8. *Some almost Lorentz invariant quantities.* As emphasized in the introduction of chapter I, the validity of the theory is limited by the condition that the state of the system is not too far from the state of thermodynamical equilibrium; therefore, we must have

$$|\mathbf{v}'^{(j)}| \ll c, \quad (j = 1, \ldots, n) \tag{8.1}$$

and

$$|\mathbf{J}'_{(s)}| \ll c\, s'_{(v)}. \tag{8.2}$$

We shall now show that it follows from (8.1) that

$$(1 - \mathbf{v}^{(j)2}/c^2)^{\frac{1}{2}} (1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}} \cong 1 \qquad (j = 1, \ldots, n) \tag{8.3}$$

in any Lorentz frame. According to the well-known Einstein addition theorem we have [3] [4]

$$\mathbf{v}^{(j)2} = \frac{\mathbf{v}'^{(j)2} + \mathbf{v}^2 + 2|\mathbf{v}'^{(j)}|\,|\mathbf{v}| \cos\vartheta' - \mathbf{v}'^{(j)2}\mathbf{v}^2 c^{-2} \sin^2\vartheta'}{(1 + |\mathbf{v}'^{(j)}|\,|\mathbf{v}|\, c^{-2} \cos\vartheta')^2}, \quad (j=1,\ldots,n) \tag{8.4}$$

where ϑ' is the angle between $\mathbf{v}'^{(j)}$ and $\mathbf{v}$ measured by an observer in the barycentric Lorentz frame. After a short calculation we find from the preceding equation

$$(1 - \mathbf{v}^{(j)2}/c^2)^{\frac{1}{2}} (1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}} = (1 - \mathbf{v}'^{(j)2}/c^2)^{\frac{1}{2}} (1 + |\mathbf{v}'^{(j)}|\,|\mathbf{v}|\, c^{-2} \cos\vartheta')^{-1}, \quad (j = 1, \ldots, n) \tag{8.5}$$

and from this equation and (8.1) we immediately see that (8.3) is correct in any Lorentz frame.

It is now easily shown that the concentrations $c^{(j)}$ $(j = 1, \ldots, n)$ are almost Lorentz invariant quantities. Using (I.2.4), (I.2.14), (I.2.17) and (I.2.27) we have

$$c^{(j)} = (1 - \mathbf{v}^{(j)2}/c^2)^{-\frac{1}{2}} (1 - \mathbf{v}^2/c^2)^{\frac{1}{2}} (1 - \mathbf{v}'^{(j)2}/c^2)^{\frac{1}{2}} c'^{(j)}. \quad (j=1,\ldots,n) \tag{8.6}$$

From the preceding equation, (8.1) and (8.3) it follows that

$$c^{(j)} \cong c'^{(j)}. \qquad (j = 1, \ldots, n) \tag{8.7}$$

Comparing (6.6) and (6.7) with (6.13) and (6.14) gives with the help of (8.7)

$$\bar{Q}^{(j)} \cong Q^{(j)}, \qquad (j = 1, \ldots, n) \tag{8.8}$$

showing that the heats of transfer $\bar{Q}^{(j)}$ $(j = 1, \ldots, n)$ are also almost Lorentz invariant quantities.

We now consider the transformation properties of the entropy in a small volume element. From (4.1) and (5.2) we have

$$I'_{(s)4} = 0. \tag{8.9}$$

Using the Lorentz transformation (4.3), we have with the aid of the preceding equation

$$I_{(s)4} = ic^{-1} (1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}} \Sigma_{\beta=1}^{3} v_\beta I'_{(s)\beta}, \tag{8.10}$$

because in this case $\mathbf{v}_{(r)} = -\mathbf{v}$. Substitution of (8.10) into (5.12) gives with the help of (I.2.8), (5.6) and (5.8)

$$s_{(v)} = (1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}} (s'_{(v)} + c^{-2}\, \mathbf{v} \cdot \mathbf{J}'_{(s)}). \tag{8.11}$$

From the preceding equation, (5.13) and (8.2) we have

$$s_{(v)}\, \mathrm{d}V \cong s'_{(v)}\, \mathrm{d}V', \tag{8.12}$$

showing that the entropy in a small volume element is an almost Lorentz invariant quantity.

Finally, we remark that we have from (I.2.27) and (I.7.3)

$$\omega^{(j)} = (1 - \mathbf{v}'^{(j)2}/c^2)^{\frac{1}{2}}. \qquad (j = 1, \ldots, n) \tag{8.13}$$

Hence, we see that according to (8.1)

$$\omega^{(j)} \simeq 1, \qquad (j = 1, \ldots, n) \tag{8.14}$$

which may be used in practical calculations.

§ 9. *Formulation with other relative flows of matter*. Sometimes, for practical applications, it is useful to reformulate the theory with the help of other relative flows of matter. These relative flows of matter have the form $\mathbf{J}^{*(j)} \equiv \varrho^{(j)}(\mathbf{v}^{(j)} - \mathbf{v}^*)$ $(j = 1, \ldots, n)$, where $\mathbf{v}^*$ is a linear function of $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(n)}$ and differs from the barycentric velocity. For this reformulation we shall give a method which can be applied in the relativistic theory as well as in the non-relativistic theory. Hence, we introduce as relative flows of matter

$$\mathbf{J}^{*(j)} \equiv \varrho^{(j)} (\mathbf{v}^{(j)} - \mathbf{v}^*), \quad (j = 1, \ldots, n) \tag{9.1}$$

where

$$\mathbf{v}^* \equiv \Sigma_{k=1}^{n} \xi^{(k)} \mathbf{v}^{(k)}. \tag{9.2}$$

We shall assume that the quantities $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(n)}$ satisfy the relation

$$\Sigma_{k=1}^{n} \xi^{(k)} = 1. \tag{9.3}$$

From the three preceding equations and (I.2.4) we have

$$\mathbf{J}^{*(n)} = - \Sigma_{j=1}^{n-1} \zeta^{(j)} \mathbf{J}^{*(j)}, \tag{9.4}$$

where

$$\zeta^{(j)} \equiv \xi^{(j)} c^{(n)} (\xi^{(n)} c^{(j)})^{-1}. \qquad (j = 1, \ldots, n-1) \tag{9.5}$$

We now introduce the matrix $A^{(j)(k)}$ $(j, k = 1, \ldots, n-1)$ defined by

$$A^{(j)(k)} \equiv \delta(j; k) + c^{(j)} (\zeta^{(k)} - 1). \quad (j, k = 1, \ldots, n-1) \tag{9.6}$$

The matrix $A^{-1(j)(k)}$, given by

$$A^{-1(j)(k)} \equiv \delta(j; k) - (\xi^{(j)}/\zeta^{(j)}) (\zeta^{(k)} - 1), \quad (j, k = 1, \ldots, n-1) \tag{9.7}$$

has the property

$$\Sigma_{l=1}^{n-1} A^{(j)(l)} A^{-1(l)(k)} = \Sigma_{l=1}^{n-1} A^{-1(j)(l)} A^{(l)(k)} = \delta(j; k), \quad (j, k=1, \ldots, n-1) \tag{9.8}$$

as is easily derived with the help of (I.2.5), (9.3), (9.5), (9.6) and (9.7). Hence, $A^{-1(j)(k)}$ is the inverse matrix of $A^{(j)(k)}$. Using (I.2.4), (I.2.5), (I.2.6), (2.2), (9.1), (9.4) and (9.6) we can derive

$$\mathbf{J}^{(j)} = \Sigma_{k=1}^{n-1} A^{(j)(k)} \mathbf{J}^{*(k)}. \qquad (j = 1, \ldots, n-1) \tag{9.9}$$

From (5.19) we have with the help of (2.7) and the preceding equation

$$T'\sigma_{(h)(d)} = \Sigma_{j=0}^{n-1} \mathbf{J}^{*(j)} \cdot \mathbf{X}^{*(j)}, \tag{9.10}$$

where

$$\mathbf{J}^{*(0)} = \mathbf{J}^{(0)}, \tag{9.11}$$

$$\mathbf{X}^{*(0)} = \overline{\mathbf{X}}^{(0)}, \tag{9.12}$$

$$\mathbf{X}^{*(j)} = \Sigma_{k=1}^{n-1} A^{(k)(j)} (\overline{\mathbf{X}}^{(k)} - \overline{\mathbf{X}}^{(n)}). \quad (j = 1, \ldots, n-1) \tag{9.13}$$

Using (3.5) and (7.8) we have from (7.7)

$$J_\alpha^{(j)} = \Sigma_{\beta=1}^{3} \overline{L}_{\alpha\beta}^{(j)(0)} \overline{X}_\beta^{(0)} + \Sigma_{k=1}^{n-1} \Sigma_{\beta=1}^{3} \overline{L}_{\alpha\beta}^{(j)(k)} (\overline{X}_\beta^{(k)} - \overline{X}_\beta^{(n)}). \tag{9.14}$$

$$(\alpha = 1, 2, 3; \quad j = 0, 1, \ldots, n-1)$$

From this equation we have with the help of (9.8), (9.9), (9.11), (9.12) and (9.13)

$$J_\alpha^{*(j)} = \Sigma_{k=0}^{n-1} \Sigma_{\beta=1}^{3} L_{\alpha\beta}^{*(j)(k)} X_\beta^{*(k)}, \quad (\alpha = 1, 2, 3; \ j = 0, 1, \ldots, n-1) \tag{9.15}$$

where the new phenomenological coefficients, $L_{\alpha\beta}^{*(j)(k)}$, are given by

$$L_{\alpha\beta}^{*(0)(0)} = \overline{L}_{\alpha\beta}^{(0)(0)}, \quad (\alpha, \beta = 1, 2, 3) \tag{9.16}$$

$$L_{\alpha\beta}^{*(j)(0)} = \Sigma_{l=1}^{n-1} A^{-1(j)(l)} \overline{L}_{\alpha\beta}^{(l)(0)}, \quad (\alpha, \beta = 1, 2, 3; \ j = 1, \ldots, n-1) \tag{9.17}$$

$$L_{\alpha\beta}^{*(0)(k)} = \Sigma_{l=1}^{n-1} A^{-1(k)(l)} \overline{L}_{\alpha\beta}^{(0)(l)}, \quad (\alpha, \beta = 1, 2, 3; \ k = 1, \ldots, n-1) \tag{9.18}$$

$$L_{\alpha\beta}^{*(j)(k)} = \Sigma_{i,l=1}^{n-1} A^{-1(j)(i)} \overline{L}_{\alpha\beta}^{(i)(l)} A^{-1(k)(l)}. \quad (\alpha, \beta = 1, 2, 3; \ j, k = 1, \ldots, n-1) \tag{9.19}$$

From (3.4), (7.8) and the four preceding equations it follows that

$$L_{\alpha\beta}^{*(j)(k)} = L_{\beta\alpha}^{*(j)(k)} = L_{\alpha\beta}^{*(k)(j)} = L_{\beta\alpha}^{*(k)(j)}. \tag{9.20}$$

$$(\alpha, \beta = 1, 2, 3; j, k = 0, 1, \ldots, n-1)$$

We see that the Onsager relations are maintained. It is easily seen that we can derive analogous formulae for the transition from a formulation using relative flows of matter given by $\varrho^{(j)}(\mathbf{v}^{(j)} - \mathbf{v}^*)$ to a formulation using relative flows of matter given by $\varrho^{(j)}(\mathbf{v}^{(j)} - \mathbf{v}^{**})$, where neither $\mathbf{v}^*$ nor $\mathbf{v}^{**}$ are equal to the barycentric velocity.

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Chapter III

SYSTEMS WITHOUT POLARIZATION AND MAGNETIZATION IN AN ELECTROMAGNETIC FIELD

§ 1. *Introduction.* In this chapter we shall develop the relativistic thermodynamics of the irreversible processes in a continuous system which is influenced by an electromagnetic field. We shall limit ourselves to systems which are neither polarizable nor magnetizable. Further, we shall assume that the system is an isotropic mixture of an arbitrary number of chemical components. As in the two preceding chapters of this thesis we shall limit ourselves to the special theory of relativity.

As is well-known, the force exerted on each of the chemical components by the electromagnetic field is given by the formula of Lorentz. According to this formula, the force acting on a certain chemical component depends among other things on the velocity of the component under consideration. If we should adopt the formalism of chapter I without alterations, the consequence would be that also in the barycentric Lorentz frame the thermodynamical "force" (affinity), conjugate to the relative flow of matter of a certain chemical component, would depend on the velocity of this component. In the Appendix we shall show that this is not allowed. Therefore, in this chapter we shall follow a method which differs in some respects from the one given in chapter I.

In § 2 we give the equations of the electromagnetic field. The fundamental laws which form the starting point for the thermodynamical considerations are given in § 3. In § 4 we discuss the first law of thermodynamics, in § 5 the entropy balance and in § 6 the phenomenological equations and the Onsager relations. To compare our results with those of Mazur and Prigogine[1]) we formulate the phenomenological equations in three-dimensional tensor form in § 7. The relativistic law of Ohm is discussed in § 8.

§ 2. *The electromagnetic field.* The macroscopic electromagnetic field in ponderable matter is described by the electric field vectors **E** and **D** and the magnetic field vectors **H** and **B**. Throughout this chapter we shall assume that the medium is neither polarizable nor magnetizable *i.e.*,

$$\mathbf{D} = \mathbf{E}, \tag{2.1}$$

and

$$\mathbf{H} = \mathbf{B}. \tag{2.2}$$

Again, we shall consider a mixture of n chemical components. Ions and free electrons will be considered as separate chemical components. If $e^{(k)}$ is the charge per unit of rest mass of component k, the density of the electric charge is given by

$$\varrho_{(el)} = \Sigma_{k=1}^{n}\, e^{(k)}\, \varrho^{(k)}. \tag{2.3}$$

In the same way the density of the electric current, $\mathbf{j}$, is given by

$$\mathbf{j} = \Sigma_{k=1}^{n}\, e^{(k)}\, \varrho^{(k)}\, \mathbf{v}^{(k)}. \tag{2.4}$$

If we take into account (2.1) and (2.2), the Maxwell equations read

$$\text{rot}\, \mathbf{B} - c^{-1}\,(\partial \mathbf{E}/\partial t) = c^{-1}\,\mathbf{j}, \tag{2.5}$$

$$\text{div}\, \mathbf{E} = \varrho_{(el)}, \tag{2.6}$$

$$\text{rot}\, \mathbf{E} + c^{-1}(\partial \mathbf{B}/\partial t) = 0, \tag{2.7}$$

$$\text{div}\, \mathbf{B} = 0. \tag{2.8}$$

The quantities $e^{(k)}$ satisfy the relation

$$\Sigma_{k=1}^{n}\, e^{(k)}\, \nu^{(k)} = 0. \tag{2.9}$$

By multiplying (I.4.1) *) with $e^{(k)}$ and summing over k we obtain with the help of (2.3), (2.4) and the preceding equation

$$\partial \varrho_{(el)}/\partial t = -\,\text{div}\, \mathbf{j}. \tag{2.10}$$

In this chapter we shall assume that the forces acting on the matter are only of an electromagnetic nature. Hence, the force, $\mathbf{F}^{(k)}$, per unit mass on component k is given by the formula of Lorentz

$$\mathbf{F}^{(k)} = e^{(k)}\,\{\mathbf{E} + c^{-1}(\mathbf{v}^{(k)} \wedge \mathbf{B})\}\,.\ (k = 1, \ldots, n) \tag{2.11}$$

As is well-known E_1, E_2, E_3, B_1, B_2 and B_3 are the components of a tensor, $B_{\alpha\beta}$, given by

$$B_{\alpha\beta} \equiv \begin{bmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{bmatrix}. \tag{2.12}$$

This tensor is antisymmetric, *i.e.*, it possesses the property

$$B_{\alpha\beta} = -\,B_{\beta\alpha}.\quad (\alpha, \beta = 1, \ldots, 4) \tag{2.13}$$

*) Equation (4.1) of chapter I of this thesis will be indicated as (I.4.1) and in the same way we denote the other equations of chapter I and chapter II.

From (I.2.1) and (2.3) it follows that

$$ic\varrho_{(el)} = \Sigma_{k=1}^{n} e^{(k)} m_4^{(k)}. \tag{2.14}$$

According to (I.2.1) and (2.4) we also have for the components of **j**

$$j_\alpha = \Sigma_{k=1}^{n} e^{(k)} m_\alpha^{(k)}. \quad (\alpha = 1, 2, 3). \tag{2.15}$$

As the quantities $e^{(k)}$ $(k = 1, \ldots, n)$ are Lorentz invariant it follows from (2.14) and (2.15) that j_1, j_2, j_3 and $ic\varrho_{(el)}$ form a four-vector. Using (2.12), (2.14) and (2.15) we can combine the Maxwell equations (2.5) and (2.6) into the form

$$\Sigma_{\beta=1}^{4} \partial B_{\alpha\beta}/\partial x_\beta = c^{-1} \Sigma_{k=1}^{n} e^{(k)} m_\alpha^{(k)}. \quad (\alpha = 1, .., 4) \tag{2.16}$$

In the same way we find for the Maxwell equations (2.7) and (2.8) with the help of (2.12)

$$\partial B_{\alpha\beta}/\partial x_\gamma + \partial B_{\beta\gamma}/\partial x_\alpha + \partial B_{\gamma\alpha}/\partial x_\beta = 0. \quad (\alpha, \beta, \gamma = 1, .., 4) \tag{2.17}$$

Using (2.14) and (2.15) we can rewrite the equation of continuity (2.10) in the form

$$\Sigma_{\alpha=1}^{4} \partial(\Sigma_{k=1}^{n} e^{(k)} m_\alpha^{(k)})/\partial x_\alpha = 0. \tag{2.18}$$

Since the quantities $e^{(k)}$ $(k = 1, \ldots, n)$ and $v^{(k)}$ $(k = 1, \ldots, n)$ are Lorentz invariant, (2.9) is valid in any Lorentz frame. Using (I.2.1), (2.11) and (2.12) we find for $K_\alpha^{(k)}$ $(\alpha = 1, .., 4;\ k = 1, \ldots, n)$, defined by (I.4.4),

$$K_\alpha^{(k)} = e^{(k)} (c\varrho_{(0)}^{(k)})^{-1} \Sigma_{\beta=1}^{4} B_{\alpha\beta} m_\beta^{(k)}. \quad (\alpha = 1, .., 4;\ k = 1, \ldots, n) \tag{2.19}$$

From this equation and (I.2.1) we see that the quantities $\varrho_{(0)}^{(k)} K_\alpha^{(k)}$ $(\alpha = 1, .., 4;\ k = 1, \ldots, n)$ depend explicitly on $\mathbf{v}^{(k)}$.

§ 3. *The fundamental laws.* Analogous to (I.4.8) the equations of motion and the balance equation for the energy read

$$\Sigma_{\beta=1}^{4} \partial W_{\alpha\beta}/\partial x_\beta = \Sigma_{j=1}^{n} \varrho_{(0)}^{(j)} K_\alpha^{(j)}. \quad (\alpha = 1, .., 4) \tag{3.1}$$

Using (2.16) and (2.19) we find for the right-hand side of this equation

$$\Sigma_{j=1}^{n} \varrho_{(0)}^{(j)} K_\alpha^{(j)} = \Sigma_{\beta,\gamma=1}^{4} B_{\alpha\beta} (\partial B_{\beta\gamma}/\partial x_\gamma). \quad (\alpha = 1, .., 4). \tag{3.2}$$

We can also assign an energy-momentum tensor to the electromagnetic field. This tensor, $W_{(f)\alpha\beta}$, satisfies the relation

$$\Sigma_{j=1}^{n} \varrho_{(0)}^{(j)} K_\alpha^{(j)} = -\Sigma_{\beta=1}^{4} \partial W_{(f)\alpha\beta}/\partial x_\beta. \quad (\alpha = 1, .., 4). \tag{3.3}$$

From the preceding equation we can derive with the help of (2.13), (2.17) and (3.2)

$$W_{(f)\alpha\beta} = -\Sigma_{\gamma=1}^{4} B_{\alpha\gamma} B_{\gamma\beta} - \tfrac{1}{4}\delta_{\alpha\beta} \Sigma_{\gamma,\zeta=1}^{4} (B_{\gamma\zeta})^2. \quad (\alpha, \beta = 1, .., 4) \tag{3.4}$$

It follows from (3.1) and (3.3) that we can write for the equations of motion and the energy balance

$$\Sigma_{\beta=1}^{4} \partial W_{(t)\alpha\beta}/\partial x_\beta = 0, \quad (\alpha = 1, \ldots, 4) \tag{3.5}$$

where $W_{(t)\alpha\beta}$ is the energy-momentum tensor of the system (matter and electromagnetic field together) given by

$$W_{(t)\alpha\beta} = W_{\alpha\beta} + W_{(f)\alpha\beta}. \quad (\alpha, \beta = 1, \ldots, 4) \tag{3.6}$$

We have from (3.4)

$$W_{(f)\alpha\beta} = W_{(f)\beta\alpha}, \quad (\alpha, \beta = 1, \ldots, 4) \tag{3.7}$$

i.e., the energy-momentum tensor of the electromagnetic field is symmetric. As in chapter I we shall also assume that the energy-momentum tensor of the matter is symmetric. Hence,

$$W_{\alpha\beta} = W_{\beta\alpha}. \quad (\alpha, \beta = 1, \ldots, 4) \tag{3.8}$$

We have from the three preceding equations

$$W_{(t)\alpha\beta} = W_{(t)\beta\alpha}, \quad (\alpha, \beta = 1, \ldots, 4) \tag{3.9}$$

i.e., the energy-momentum tensor of the system is symmetric. From this symmetry of $W_{(t)\alpha\beta}$ we can derive [2]) that the total macroscopic angular momentum of the whole system (field and matter together) is constant, *i.e.*,

$$(\mathrm{d}/\mathrm{d}t) \int \{\mathbf{r} \wedge \mathbf{g}_{(t)}(\mathbf{r}, t)\} \, \mathrm{d}V = 0. \tag{3.10}$$

In this equation $\mathbf{g}_{(t)}(\mathbf{r}, t)$ is the density of momentum of the system at time t and position $\mathbf{r}$, while $\mathrm{d}V$ is an element of volume in the ordinary three-dimensional space. The integration must be extended over the whole of the system. Because the energy-momentum tensors of the electromagnetic field and of the matter are also symmetric separately, equations of type (3.10) are also valid for the electromagnetic field and the matter separately.

Since we assume that the medium is neither polarizable nor magnetizable the second law of thermodynamics (Gibbs relation) is again given by (I.4.10). The balance equation for rest mass is given by (I.4.1).

§ 4. *The first law of thermodynamics.* As stated already in the introduction to this chapter we cannot follow the same procedure as in the two preceding chapters of this thesis. For, according to (2.11) and (II.3.14) the consequence would be that $\mathbf{X}'^{(k)}$ would depend on $\mathbf{v}'^{(k)}$, *i.e.*, in the barycentric Lorentz frame the thermodynamical "force" (affinity) conjugate to the relative flow of matter of a certain chemical component would depend on the velocity of this component. In the Appendix we shall show that this is not allowed. Therefore, in this section we shall deduce an expression for the first law of thermodynamics which leads to a form for the entropy production such that $\mathbf{X}'^{(k)}$ does not depend on $\mathbf{v}'^{(k)}$.

Using (I.2.10), (I.2.21), (2.13) and (2.19) we can derive

$$\Sigma_{a=1}^{4} \varrho_{(0)}^{(k)} u_a K_a^{(k)} = e^{(k)} c^{-1} \Sigma_{a,\beta=1}^{4} u_a B_{a\beta}(m_\beta^{(k)} - c'^{(k)} m u_\beta) = \\ = -\Sigma_{a=1}^{4} I_a^{(k)}(c^{-1} e^{(k)} \Sigma_{\beta=1}^{4} B_{a\beta} u_\beta). \quad (k = 1, \ldots, n) \qquad (4.1)$$

It is now easily seen that we can follow the same procedure as in § 5 of chapter I, except that according to (I.5.2) and the preceding equation, we must replace $\omega^{(k)} K_a^{(k)}$ by $e^{(k)} \Sigma_{\beta=1}^{4} B_{a\beta} u_\beta$. We then get instead of (I.5.10)

$$\varrho'(\mathrm{D}e' + p'\mathrm{D}v') = -\Sigma_{\beta=1}^{4} (\partial I_\beta^{(0)}/\partial x_\beta + c^{-1} I_\beta^{(0)} \mathrm{D}u_\beta) + \\ + \Sigma_{j=1}^{n} \Sigma_{a,\beta=1}^{4} e^{(j)} I_a^{(j)} B_{a\beta} u_\beta + c\, \Sigma_{a,\beta=1}^{4} P_{a\beta} (\partial u_\beta/\partial x_a) \qquad (4.2)$$

as first law of thermodynamics.

§ 5. *The entropy balance*. In the same way as in § 6 of chapter I we can derive the entropy balance. This gives

$$\varrho'\mathrm{D}s' = -\Sigma_{a=1}^{4} (\partial/\partial x_a) \{(1/T')(I_a^{(0)} - \Sigma_{j=1}^{n} \mu'^{(j)} I_a^{(j)})\} + \\ + (1/T') [\Sigma_{j=0}^{n} \Sigma_{a=1}^{4} I_a^{(j)} Y_a^{(j)} + c\, \Sigma_{a,\beta=1}^{4} \overline{P}_{a\beta}(\partial u_\beta/\partial x_a) + \Pi\, \varrho'\mathrm{D}v' + J_{(c)} A], \quad (5.1)$$

where the "forces" (affinities), $Y_a^{(j)} (a = 1, .., 4; j = 0, 1, \ldots, n)$, are given by

$$Y_a^{(0)} \equiv -\{(1/T')(\partial T'/\partial x_a) + c^{-1} \mathrm{D}u_a\}, \qquad (a = 1, .., 4) \qquad (5.2)$$

$$Y_a^{(j)} \equiv e^{(j)} \Sigma_{\beta=1}^{4} B_{a\beta} u_\beta - T'(\partial/\partial x_a)(\mu'^{(j)}/T'). \quad (a = 1, .., 4; j = 1, \ldots, n) \quad (5.3)$$

Hence, we see from (I.2.11) and (2.12) that $Y_a^{(j)}$ $(a = 1, .., 4;\ j = 0, 1, \ldots, n)$, given by the preceding equation, does not depend explicitly on $\mathbf{v}^{(k)}$ in the barycentric Lorentz frame where $\mathbf{v} = 0$.

§ 6. *The phenomenological equations and the Onsager relations*. For the phenomenological equations, describing the diffusion phenomena and the heat conduction, we can write

$$I_a^{(j)} = \Sigma_{k=0}^{n} \Sigma_{\beta=1}^{4} L_{a\beta}^{(j)(k)} Y_\beta^{(k)}. \qquad (a = 1, .., 4;\ j = 0, 1, \ldots, n) \qquad (6.1)$$

As shown in § 7 of chapter I we have for the $(n + 1)^2$ tensors $L_{a\beta}^{(j)(k)}$ $(a, \beta = 1, .., 4;\ j, k = 0, 1, \ldots, n)$ in the case of an isotropic mixture

$$L_{a\beta}^{(j)(k)} = L^{(j)(k)} \Delta_{a\beta}. \quad (a, \beta = 1, .., 4;\ j, k = 0, 1, \ldots, n) \qquad (6.2)$$

The $(n + 1)^2$ quantities $L^{(j)(k)}$ are the phenomenological coefficients of the non-relativistic thermodynamics of the irreversible processes in an isotropic mixture of n chemical components. From the preceding equation we see that these coefficients enter in the relativistic theory as Lorentz invariant quantities. We may consider these coefficients as functions of p', T', $c'^{(1)}$,

$c'^{(2)}, \ldots, c'^{(n-1)}$, $\mathbf{E}'$ and $\mathbf{B}'$, *i.e.*, as functions of quantities measured in the barycentric Lorentz frame. As is well-known, the Onsager relations read [3-7])

$$L^{(j)(k)}(\mathbf{B}') = L^{(k)(j)}(-\mathbf{B}'). \quad (j, k = 0, 1, \ldots, n) \tag{6.3}$$

In the case $j = k$ the preceding equation expresses that the $L^{(j)(j)}$ ($j = 0, 1, \ldots, n$) are even functions of $\mathbf{B}'$. According to (I.7.11) we also have the relation

$$\Sigma_{j=1}^{n} L^{(j)(k)}(\mathbf{B}') = 0. \quad (k = 0, 1, \ldots, n) \tag{6.4}$$

Substituting (5.2), (5.3) and (6.2) into (6.1) gives with the help of (I.2.28) and (2.13)

$$I_{\alpha}^{(j)} = (\Sigma_{k=1}^{n} L^{(j)(k)} e^{(k)}) \, \Sigma_{\beta=1}^{4} B_{\alpha\beta} u_{\beta} -$$
$$-\Sigma_{\beta=1}^{4} \Delta_{\alpha\beta} [T' \Sigma_{k=1}^{n} L^{(j)(k)} \{\partial(\mu'^{(k)}/T')/\partial x_{\beta}\} + L^{(j)(0)} \{(1/T')(\partial T'/\partial x_{\beta}) + c^{-1} \mathrm{D} u_{\beta}\}]. \tag{6.5}$$
$$(\alpha = 1, \ldots, 4;\ j = 0, 1, \ldots, n)$$

The phenomenological equations for $\bar{P}_{\alpha\beta}$, Π and $J_{(c)}$ have the same forms as in chapter I. It is easily seen from symmetry considerations that in an isotropic medium $L_{(p)(c)}$ and $L_{(c)(p)}$ must be even functions of $\mathbf{B}'$. Hence, the Onsager relations for these phenomenological coefficients are again given by (I.7.17).

§ 7. *Three-dimensional tensor form for the phenomenological equations.* To be able to compare our results with those obtained by Mazur and Prigogine[1]), we shall reformulate the phenomenological equations for heat conduction and diffusion in three-dimensional tensor form. We shall use the procedure which was given in chapter II. It should be emphasized that *the relativistic invariance of the theory will be maintained.* As in chapter II we shall use relative flows of matter defined by

$$\mathbf{J}^{(j)} \equiv \varrho^{(j)}(\mathbf{v}^{(j)} - \mathbf{v}). \quad (j = 1, \ldots, n) \tag{7.1}$$

As shown in § 3 of chapter II we can write the phenomenological equations in the form

$$\mathbf{J}^{(j)} = \Sigma_{k=0}^{n} L^{(j)(k)} \mathbf{X}^{(k)}, \quad (j = 0, 1, \ldots, n) \tag{7.2}$$

where $\mathbf{J}^{(0)}$ is given by (II.2.13) and $\mathbf{X}^{(k)}$ ($k = 0, 1, \ldots, n$) by (II.3.8). According to (II.3.9) we have

$$\mathbf{X}^{(0)} = -\{(1/T') \operatorname{grad} T' + (c^2 - \mathbf{v}^2)^{-1}(\mathrm{d}\mathbf{v}/\mathrm{d}t) + (\mathbf{v}/T')c^{-2}(\partial T'/\partial t)\}. \tag{7.3}$$

Substitution of (5.3) into (II.3.8) gives with the help of (I.2.11) and (2.12) for $k = 1, \ldots, n$

$$\mathbf{X}^{(k)} = (1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}} e^{(k)} \{\mathbf{E} + c^{-1}(\mathbf{v} \wedge \mathbf{B}) - c^{-2}(\mathbf{v} \cdot \mathbf{E})\mathbf{v}\} -$$
$$- T'[\operatorname{grad}(\mu'^{(k)}/T') + c^{-2} \{\partial(\mu'^{(k)}/T')/\partial t\} \mathbf{v}]. \quad (k = 1, \ldots, n) \tag{7.4}$$

Substituting (7.3) and (7.4) into (7.2) gives

$$\mathbf{J}^{(j)} = (\Sigma_{k=1}^{n} L^{(j)(k)} e^{(k)}) (1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}} \{\mathbf{E} + c^{-1}(\mathbf{v} \wedge \mathbf{B}) - c^{-2} (\mathbf{v} \cdot \mathbf{E})\mathbf{v}\} -$$
$$- \Sigma_{k=1}^{n} L^{(j)(k)} [T' \operatorname{grad} (\mu'^{(k)}/T') + c^{-2} T' \{\partial(\mu'^{(k)}/T')/\partial t\} \mathbf{v}] -$$
$$- L^{(j)(0)} \{(1/T') \operatorname{grad} T' + (c^2 - \mathbf{v}^2)^{-1} (\mathrm{d}\mathbf{v}/\mathrm{d}t) + (\mathbf{v}/T') c^{-2} (\partial T'/\partial t)\}. \quad (7.5)$$
$$(j = 0, 1, \ldots, n)$$

If we compare the result which (7.5) gives for the diffusion flows $\mathbf{J}^{(j)}$ $(j = 1, \ldots, n)$ with that obtained by Mazur and Prigogine[1]) we see that besides the term $\mathbf{E} + c^{-1}(\mathbf{v} \wedge \mathbf{B})$, there occurs the term $- c^{-2}(\mathbf{v} \cdot \mathbf{E})\mathbf{v}$. This term is of the order $\mathbf{v}^2/c^2$. Moreover, these two terms are multiplied by the factor $(1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}}$. Besides terms containing the gradients of $\mu'^{(k)}/T'$ and T' we also get terms with local derivatives with respect to time of these quantities. Finally, there is the term with $\mathrm{d}\mathbf{v}/\mathrm{d}t$ which expresses that heat conduction and diffusion are influenced by the barycentric acceleration.

§ 8. *The relativistic law of Ohm.* We now consider a mixture of two chemical components. We then have from (6.3) and (6.4)

$$L^{(1)(2)}(\mathbf{B}') = L^{(2)(1)}(-\mathbf{B}') = -L^{(1)(1)}(-\mathbf{B}') = -L^{(1)(1)}(\mathbf{B}'). \qquad (8.1)$$

From the preceding equation and (6.3) it follows that

$$L^{(1)(2)}(\mathbf{B}') = -L^{(1)(1)}(-\mathbf{B}') = L^{(1)(2)}(-\mathbf{B}') = L^{(2)(1)}(\mathbf{B}'). \qquad (8.2)$$

Hence, we see that $L^{(1)(2)}(\mathbf{B}')$ and $L^{(2)(1)}(\mathbf{B}')$ are even functions of $\mathbf{B}'$ in the case of a binary mixture. With the help of (I.2.13), (I.2.25), (I.2.28) and (8.1) we now find for (6.5)

$$I_\alpha^{(1)} = (\chi/e^{(1)}) \Sigma_{\beta=1}^{4} B_{\alpha\beta} u_\beta - L^{(1)(1)} T'(\partial/\partial x_\alpha) \{(\mu'^{(1)} - \mu'^{(2)})/T'\} -$$
$$- L^{(1)(0)} (1/T')(\partial T'/\partial x_\alpha) - L^{(1)(1)} u_\alpha c^{-1} T' \mathrm{D} \{(\mu'^{(1)} - \mu'^{(2)})/T'\} -$$
$$- L^{(1)(0)} u_\alpha (cT')^{-1} \mathrm{D} T' - L^{(1)(0)} c^{-1} \mathrm{D} u_\alpha, \qquad (\alpha = 1, \ldots, 4) \qquad (8.3)$$

where the Lorentz invariant quantity χ, the electric conductivity, is given by

$$\chi = e^{(1)} L^{(1)(1)} (e^{(1)} - e^{(2)}). \qquad (8.4)$$

The first term on the right hand side of (8.3) gives the influence of the electromagnetic field on diffusion phenomena. The second and third terms are found in the same form in the non-relativistic theory. The second term is proportional to the gradient of $(\mu'^{(1)} - \mu'^{(2)})/T'$ ("eingeprägte Kraft") and the third term gives the cross-effect with the heat conduction (thermal diffusion). The remaining terms on the right hand side of (8.3), all containing substantial derivatives with respect to time, do not occur in the non-relativistic theory.

We now consider the case that one of the two chemical components consists of electrons. Taking the electrons as component 1, then in literature the relativistic law of Ohm is given by [2] [8–12])

$$e^{(1)} I^{(1)}_{\alpha} = \chi \, \Sigma^{4}_{\beta=1} B_{\alpha\beta} u_{\beta}. \quad (\alpha = 1, \ldots, 4) \tag{8.5}$$

It is seen that the right hand side of (8.5) is equal to the first term on the right hand side of (8.3). The latter equation gives the general expression for the diffusion flow in a mixture of two chemical components. The four-vector u_{α}, which is used in the definition of the four-vector $I^{(1)}_{\alpha}$ (Cf. (I.2.28) and (I.2.34)) and which also occurs on the right hand side of (8.5), is often not sharply defined in the literature. According to our formalism u_{α} is given by (I.2.6) and (I.2.11).

As the electrons have a very small rest mass we have $\varrho'^{(1)} \ll \varrho'$. Hence, according to (I.7.3) and (II.8.14) we also have $\varrho^{(1)}_{(0)} \ll \varrho'$. With the help of (I.2.2), (I.2.14), (I.2.27) and (II.8.3) it follows that in any Lorentz frame

$$\varrho^{(1)} \ll \varrho^{(2)}. \tag{8.6}$$

Using (I.2.7), (I.2.9), (I.2.10) and (I.2.28) we have from (I.2.34)

$$I^{(1)}_{\alpha} = \frac{\Sigma^{4}_{\beta=1} (m^{(1)}_{\alpha} m^{(2)}_{\beta} - m^{(2)}_{\alpha} m^{(1)}_{\beta}) m_{\beta}}{\Sigma^{4}_{\beta=1} (m_{\beta})^{2}}. \quad (\alpha = 1, \ldots, 4) \tag{8.7}$$

If $|\mathbf{v}|$ is sufficiently great, $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ and $\mathbf{v}$ have nearly the same direction while the length of these vectors is of the same order of magnitude. According to (I.2.1), (I.2.7) and (8.6) we then have in general $m^{(2)}_{\beta} \simeq m_{\beta}$ $(\beta = 1, \ldots, 4)$. If $|\mathbf{v}^{(1)}|$, $|\mathbf{v}^{(2)}|$ and $|\mathbf{v}|$ are not of the same order of magnitude we are in the non-relativistic region and it appears that we may replace m_{β} by $m^{(2)}_{\beta}$ $(\beta = 1, \ldots, 4)$ in the numerator of the right hand side of (8.7) for $\alpha = 1, 2, 3$ (Cf. (I.2.22)). It is also easily seen that we may replace $\Sigma^{4}_{\beta=1}(m_{\beta})^{2}$ by $\Sigma^{4}_{\beta=1}(m^{(2)}_{\beta})^{2}$ in the denominator of the right hand side of (8.7). Hence, from the preceding considerations we can infer that in any Lorentz frame

$$I^{(1)}_{\alpha} \simeq m^{(1)}_{\alpha} - m^{(2)}_{\alpha} \frac{\Sigma^{4}_{\beta=1} m^{(1)}_{\beta} m^{(2)}_{\beta}}{\Sigma^{4}_{\beta=1} (m^{(2)}_{\beta})^{2}}. \quad (\alpha = 1, 2, 3) \tag{8.8}$$

By analogous considerations we can find that we may replace u_{β} by $m^{(2)}_{\beta} \{-\Sigma^{4}_{\gamma=1} (m^{(2)}_{\gamma})^{2}\}^{-\frac{1}{2}}$ in the right hand side of (8.5).

Appendix

As already stated in the introduction and in § 4, the procedure given in the chapters I and II of this thesis would be wrong in case the medium is influenced by an electromagnetic field. We shall now show that even in the barycentric Lorentz frame this procedure would lead to wrong results.

According to (7.5) we have in the barycentric Lorentz frame

$$\mathbf{J}'^{(j)} = (\Sigma_{k=1}^{n} L^{(j)(k)} e^{(k)})\mathbf{E}' - T'\Sigma_{k=1}^{n} L^{(j)(k)} \operatorname{grad}' (\mu'^{(k)}/T') - \\ - L^{(j)(0)} \{(1/T') \operatorname{grad}' T' + c^{-2} (\partial \mathbf{v}/\partial t)'\}, \quad (j = 0, 1, \ldots, n) \qquad \text{(A.1)}$$

where we have taken into account the definition (II.3.10) for $\mathrm{d}/\mathrm{d}t$. From (II.3.3), (II.3.13), (II.3.14) and (2.11) we should have according to the procedure given in the chapters I and II

$$\mathbf{J}'^{(j)} = (\Sigma_{k=1}^{n} L^{(j)(k)} e^{(k)})\{\mathbf{E}' + c^{-1}(\mathbf{v}'^{(k)} \wedge \mathbf{B}')\} - \\ - T' \Sigma_{k=1}^{n} L^{(j)(k)} \operatorname{grad}' (\mu'^{(k)}/T') - L^{(j)(0)}\{(1/T') \operatorname{grad}' T' + c^{-2} (\partial \mathbf{v}/\partial t)'\}. \text{(A.2)} \\ (j = 0, 1, \ldots, n)$$

According to Fieschi, de Groot, Mazur and Vlieger [7]) the result (A.1) is correct. From (7.1) we have $\mathbf{v}'^{(k)} = (1/\varrho'^{(k)})\mathbf{J}'^{(k)}$ ($k = 1, \ldots, n$). Using this expression for $\mathbf{v}'^{(k)}$ and comparing (A.1) and (A.2), one can easily see that (A.2) gives a result which is different from the one given by (A.1). Hence, (A.2) is wrong in the barycentric Lorentz frame and then, of course, the procedure given in the chapters I and II gives wrong results in any Lorentz frame.

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Chapter IV

SYSTEMS WITH POLARIZATION AND MAGNETIZATION IN AN ELECTROMAGNETIC FIELD

§ 1. *Introduction*. The purpose of this chapter is to extend the considerations given in chapter III to systems which are polarizable and magnetizable. We shall deal with the case where the medium is isotropic as far as polarization and magnetization are concerned. As in chapter III we shall assume that the forces acting on the matter are only of an electromagnetic nature.

In the case that the medium is polarized and magnetized terms occur in the non-relativistic second law of thermodynamics which are connected with the polarization and magnetization of the matter. In this chapter our first aim will be to derive the appropriate relativistic second law of thermodynamics.

If there is no polarization and magnetization, the electromagnetic field only influences the medium by exerting a force on each chemical component which is electrically charged. As a consequence of the polarization and magnetization, however, the electromagnetic field exerts a force which we cannot regard as acting on each chemical component separately. This force will be called the ponderomotive force. It will appear that the explicit expression for the ponderomotive force is closely connected with the form of the relativistic second law of thermodynamics.

Again, we consider a continuous mixture of an arbitrary number of chemical components. The barycentric velocity, defined by (I.2.6) *) and measured by an observer in a Lorentz frame A at the position $\mathbf{r}$ and at the time t, will be denoted by $\mathbf{v}(\mathbf{r}, t)$. As in the preceding chapters we shall assign to each point of the space-time continuum a special Lorentz frame (the barycentric Lorentz frame) such, that in the point under consideration, for an observer in this special Lorentz frame, the barycentric velocity vanishes. It should be remarked that by this condition the barycentric frame assigned to a point of the space-time continuum is not uniquely determined. For, if for an observer in some Lorentz frame at a certain point of the space-time continuum the barycentric velocity vanishes, the

*) Equation (2.6) of chapter I of this thesis will be indicated as (I.2.6) and in the same way we denote the other equations of the chapters I, II and III.

barycentric velocity also vanishes at the same point of the space-time continuum for an observer in a Lorentz frame which is at rest with respect to the Lorentz frame first mentioned but for which the ordinary three-dimensional axis-frame has been rotated with respect to the ordinary three-dimensional axis-frame of the first Lorentz frame. This indefiniteness did not play a role in the considerations in the preceding chapters. For example, in the relativistic second law of thermodynamics, given by (I. 4.10), quantities occurred which were measured in the barycentric Lorentz frame, but which were invariant with respect to rotations of the ordinary three-dimensional axis-frame. It will appear that also in the considerations in this chapter this indefiniteness of the orientations of the ordinary three-dimensional axis-frames of the barycentric Lorentz frames does not play a role.

We shall denote by $\mathrm{B}_{\mathbf{r},t}$ the barycentric Lorentz frame assigned to that point of the space-time continuum which is described by position $\mathbf{r}$ and time t by an observer in Lorentz frame A. We shall denote by $\mathbf{r}'$ and t' the position and time respectively which an observer in the Lorentz frame $\mathrm{B}_{\mathbf{r},t}$ assigns to the point of the space-time continuum which is described by an observer in Lorentz frame A by position $\mathbf{r}$ and time t.

Again, we shall distinguish by primes all quantities at a point in the space-time continuum measured in the barycentric Lorentz frame belonging to this point.

The second law of thermodynamics is discussed in § 2. In § 3 we deal with two four-dimensional tensors introduced in § 2. In § 4 further considerations are given on the second law of thermodynamics. We then discuss in § 5 the balance equations for momentum and energy and in § 6 the balance equation for rest mass. The first law of thermodynamics and the entropy balance are derived in § 7 and § 8 respectively. It is also shown in § 8 that the explicit expression for the ponderomotive force is closely connected with the form of the second law of thermodynamics. The phenomenological equations and the Onsager relations for vectorial, tensorial and scalar fluxes are given for anisotropic media in § 9, § 10 and § 11 respectively. Finally, in § 12 we deal with the phenomenological equations for isotropic media.

§ 2. *The second law of thermodynamics.* We first remark that the components of the electric field vector $\mathbf{E}$ and the magnetic field vector $\mathbf{B}$ form the components of a four-dimensional tensor, $B_{\alpha\beta}$, defined by

$$B_{\alpha\beta} \equiv \begin{bmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{bmatrix}. \tag{2.1}$$

It is seen from this definition that this tensor is antisymmetric *i.e.*,

$$B_{\alpha\beta} = -B_{\beta\alpha}. \quad (\alpha, \beta = 1, \ldots, 4) \tag{2.2}$$

The components of the polarization vector **P** and the magnetization vector **M** also form the components of a four-dimensional tensor, $M_{\alpha\beta}$. This tensor is defined by

$$M_{\alpha\beta} \equiv \begin{bmatrix} 0 & M_3 & -M_2 & iP_1 \\ -M_3 & 0 & M_1 & iP_2 \\ M_2 & -M_1 & 0 & iP_3 \\ -iP_1 & -iP_2 & -iP_3 & 0 \end{bmatrix}, \tag{2.3}$$

and is also antisymmetric. Thus,

$$M_{\alpha\beta} = -M_{\beta\alpha}. \quad (\alpha, \beta = 1, \ldots, 4) \tag{2.4}$$

Before dealing with the relativistic second law of thermodynamics we shall first consider the non-relativistic second law of thermodynamics. In the case that the medium is polarized and magnetized, terms occur in the non-relativistic second law of thermodynamics which are connected with the polarization and magnetization of the matter. As said already in the introduction to this chapter we shall only deal with systems which are isotropic as far as polarization and magnetization are concerned. For such systems M a z u r and P r i g o g i n e [1]) take as the form for these special terms $-\mathbf{E}\cdot\{\mathrm{d}(v\mathbf{P})/\mathrm{d}t\}$, where v is the specific volume and $\mathrm{d}/\mathrm{d}t$ is the operator defined by (II.3.10). (The authors quoted leave magnetization out of consideration.) By definition we have $\mathrm{d}\mathbf{P}/\mathrm{d}t \equiv \lim_{\mathrm{d}t=0}\{\mathbf{P}(\mathbf{r}+\mathbf{v}\mathrm{d}t, t+\mathrm{d}t) - \mathbf{P}(\mathbf{r}, t)\}/\mathrm{d}t$. Usually, $\mathbf{P}(\mathbf{r}+\mathbf{v}\mathrm{d}t, t+\mathrm{d}t)$ and $\mathbf{P}(\mathbf{r}, t)$ are measured in the same three-dimensional axis-frame. It should be remarked, however, that $\mathbf{P}(\mathbf{r}+\mathbf{v}\mathrm{d}t, t+\mathrm{d}t)$ and $\mathbf{P}(\mathbf{r}, t)$ may be measured in three-dimensional axis-frames which are rotated with respect to each other over an arbitrary angle (of which, however, the (mathematical) order of magnitude is not greater than the order of magnitude of $\mathrm{d}t$). To make this clear we remark that $\mathrm{d}\mathbf{P} \equiv \mathbf{P}(\mathbf{r}+\mathbf{v}\mathrm{d}t, t+\mathrm{d}t) - \mathbf{P}(\mathbf{r}, t)$ may be split up into two parts. The first part, $\mathrm{d}_r\mathbf{P}$, is due to the rotation of the axis-frame and the second part, $\mathrm{d}_s\mathbf{P}$, equals $\mathrm{d}\mathbf{P}$ if $\mathbf{P}(\mathbf{r}+\mathbf{v}\mathrm{d}t, t+\mathrm{d}t)$ and $\mathbf{P}(\mathbf{r}, t)$ are measured in the same axis-frame. It is obvious that $\mathrm{d}_r\mathbf{P} \perp \mathbf{P}$. Since the medium is assumed to be isotropic as far as polarization and magnetization are concerned we have $\mathbf{P} \,/\!/\, \mathbf{E}$. Hence, we have $\mathrm{d}_r\mathbf{P} \perp \mathbf{E}$ and therefore $\mathbf{E}\cdot\mathrm{d}\mathbf{P} = \mathbf{E}\cdot\mathrm{d}_s\mathbf{P}$. Thus, we see that it is not necessary to measure $\mathbf{P}(\mathbf{r}+\mathbf{v}\mathrm{d}t, t+\mathrm{d}t)$ and $\mathbf{P}(\mathbf{r}, t)$ with respect to the same axis-frame. It is easily seen that if the theory of M a z u r and P r i g o g i n e [1]) is extended to the case where we have polarization as well as magnetization the special terms in the non-relativistic second law of thermodynamics will have the form $-\mathbf{E}\cdot\{\mathrm{d}(v\mathbf{P})/\mathrm{d}t\} - \mathbf{B}\cdot\{\mathrm{d}(v\mathbf{M})/\mathrm{d}t\}$.

We shall now discuss the relativistic second law of thermodynamics. As is well-known, the non-relativistic second law of thermodynamics for systems without polarization and magnetization reads $T(\mathrm{d}s/\mathrm{d}t) = (\mathrm{d}e/\mathrm{d}t) + p(\mathrm{d}v/\mathrm{d}t) - \Sigma_{j=1}^{n}\mu^{(j)}(\mathrm{d}c^{(j)}/\mathrm{d}t)$ [2]. According to (I.4.10) the corresponding relativistic second law of thermodynamics reads $T'\mathrm{D}s' = \mathrm{D}e' + p'\,\mathrm{D}v' - \Sigma_{j=1}^{n}\mu'^{(j)}\mathrm{D}c'^{(j)}$. Thus, we see that the operator $\mathrm{d}/\mathrm{d}t$, defined by (II.3.10), is replaced by the operator D, defined by (I.2.25), and the quantities $T, s, e, p, v, \mu^{(j)}$ and $c^{(j)}$ are replaced by $T', s', e', p', v', \mu'^{(j)}$ and $c'^{(j)}$. On taking the forms mentioned above for the special terms, the non-relativistic second law of thermodynamics for systems with polarization and magnetization reads $T(\mathrm{d}s/\mathrm{d}t) = (\mathrm{d}e/\mathrm{d}t) + p(\mathrm{d}v/\mathrm{d}t) - \mathbf{E}\cdot\{\mathrm{d}(v\mathbf{P})/\mathrm{d}t\} - \mathbf{B}\cdot\{\mathrm{d}(v\mathbf{M})/\mathrm{d}t\} - \Sigma_{j=1}^{n}\mu^{(j)}(\mathrm{d}c^{(j)}/\mathrm{d}t)$. Hence, the most natural assumption for the relativistic second law of thermodynamics seems to be $T'\mathrm{D}s' = \mathrm{D}e' + p'\mathrm{D}v' - \mathbf{E}'\cdot\mathrm{D}(v'\mathbf{P}') - \mathbf{B}'\cdot\mathrm{D}(v'\mathbf{M}') - \Sigma_{j=1}^{n}\mu'^{(j)}\mathrm{D}c'^{(j)}$ or, using (2.1) and (2.3), $T'\mathrm{D}s' = \mathrm{D}e' + p'\mathrm{D}v' - \frac{1}{2}\Sigma_{\alpha,\beta=1}^{4} B'_{\alpha\beta}\mathrm{D}(v'M'_{\alpha\beta}) - \Sigma_{j=1}^{n}\mu'^{(j)}\mathrm{D}c'^{(j)}$. It should be remarked, however, that the special terms in the non-relativistic second law of thermodynamics which are connected with polarization and magnetization need not necessarily have the form given above. Examples of different forms will be discussed in the following chapter. (Cf. also formula (2.67) of reference 1.) To have our discussion as general as possible, we introduce two four-dimensional tensors, $G_{\alpha\beta}$ and $Z_{\alpha\beta}(\alpha, \beta, = 1, \ldots, 4)$, which will further be specified in § 3, and we take as the relativistic second law of thermodynamics

$$T'\mathrm{D}s' = \mathrm{D}e' + p'\mathrm{D}v' + \tfrac{1}{2}\Sigma_{\alpha,\beta=1}^{4} G'_{\alpha\beta}\,\mathrm{D}Z'_{\alpha\beta} - \Sigma_{j=1}^{n}\mu'^{(j)}\,\mathrm{D}c'^{(j)}. \tag{2.5}$$

(By taking $G_{\alpha\beta} = -B_{\alpha\beta}$ and $Z_{\alpha\beta} = vM_{\alpha\beta}$ we obtain from (2.5) the form for the relativistic second law of thermodynamics which corresponds to the form for the non-relativistic second law of thermodynamics assumed above.) It should be remarked that the quantities $\mu'^{(j)}(j = 1, \ldots, n)$ may be considered as functions of $T', p', G'_{\alpha\beta}(\alpha, \beta = 1, \ldots, 4)$ and $c'^{(j)}\,(j = 1, \ldots, n-1)$.

From (I.2.11), (I.2.25) and (II.3.10) we have $\mathrm{D} = (1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}}(\mathrm{d}/\mathrm{d}t)$ and hence, $\mathrm{D}Z'_{\alpha\beta} = (1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}}[\lim_{\mathrm{d}t=0}\{Z'_{\alpha\beta}(\mathbf{r} + \mathbf{v}\,\mathrm{d}t, t + \mathrm{d}t) - Z'_{\alpha\beta}(\mathbf{r}, t)\}/\mathrm{d}t]$. It should be noted that $Z'_{\alpha\beta}(\mathbf{r} + \mathbf{v}\mathrm{d}t, t + \mathrm{d}t)$ is measured in $\mathrm{B}_{\mathbf{r}+\mathbf{v}\mathrm{d}t,\, t+\mathrm{d}t}$ and $Z'_{\alpha\beta}(\mathbf{r}, t)$ in $\mathrm{B}_{\mathbf{r},t}$. As is well-known, the pure Lorentz transformations (*i.e.*, Lorentz transformations without rotation of the ordinary three-dimensional axis-frame, for which transformations the coefficients are given by (II.4.3)) do not form a group. Thus, it is impossible to choose the barycentric Lorentz frames such that they all transform into each other by means of pure Lorentz transformations. Since, as we have said already above, $Z'_{\alpha\beta}(\mathbf{r} + \mathbf{v}\mathrm{d}t, t + \mathrm{d}t)$ is measured in $\mathrm{B}_{\mathbf{r}+\mathbf{v}\mathrm{d}t,\, t+\mathrm{d}t}$ and $Z'_{\alpha\beta}(\mathbf{r}, t)$ in $\mathrm{B}_{\mathbf{r},t}$ the quantity $\mathrm{D}Z'_{\alpha\beta}$ will depend on the choice for the orientations of the ordinary three-dimensional axis-frames of $\mathrm{B}_{\mathbf{r}+\mathbf{v}dt,\,t+dt}$ and $\mathrm{B}_{\mathbf{r},t}$. In § 4 we shall show that, though $\mathrm{D}Z'_{\alpha\beta}$ depends on this choice, the second law of thermodynamics, given by (2.5), does not depend on it. This is the same situation as the non-relativistic one which we have discussed above.

§ 3. *Discussion of the tensors $G_{\alpha\beta}$ and $Z_{\alpha\beta}$.* We shall now further specify the tensors $G_{\alpha\beta}$ and $Z_{\alpha\beta}$ $(\alpha, \beta = 1, \ldots, 4)$. For that purpose we introduce the four-dimensional tensors $B^*_{\alpha\beta}$ and $M^*_{\alpha\beta}$ $(\alpha, \beta = 1, \ldots, 4)$ defined by

$$B^*_{\alpha\beta} \equiv \Sigma^4_{\gamma,\zeta=1}\, \Delta_{\alpha\gamma}\, B_{\gamma\zeta}\, \Delta_{\zeta\beta}, \qquad (\alpha, \beta = 1, \ldots, 4) \tag{3.1}$$

$$M^*_{\alpha\beta} \equiv \Sigma^4_{\gamma,\zeta=1}\, \Delta_{\alpha\gamma}\, M_{\gamma\zeta}\, \Delta_{\zeta\beta}, \qquad (\alpha, \beta = 1, \ldots, 4) \tag{3.2}$$

and the four-vectors B^*_α and $M^*_\alpha (\alpha = 1, \ldots, 4)$ defined by

$$B^*_\alpha \equiv \Sigma^4_{\beta=1}\, B_{\alpha\beta}\, u_\beta, \qquad (\alpha = 1, \ldots, 4) \tag{3.3}$$

$$M^*_\alpha \equiv \Sigma^4_{\beta=1}\, M_{\alpha\beta}\, u_\beta. \qquad (\alpha = 1, \ldots, 4) \tag{3.4}$$

With the help of (I.2.29), (2.2) and (2.4) we have from (3.1) and (3.2)

$$B^*_{\alpha\beta} = -\, B^*_{\beta\alpha}, \qquad (\alpha, \beta = 1, \ldots, 4) \tag{3.5}$$

$$M^*_{\alpha\beta} = -\, M^*_{\beta\alpha}. \qquad (\alpha, \beta = 1, \ldots, 4) \tag{3.6}$$

Using (I.2.30) we also find from (3.1) and (3.2)

$$\Sigma^4_{\beta=1}\, B^*_{\alpha\beta}\, u_\beta = \Sigma^4_{\beta=1}\, u_\beta\, B^*_{\beta\alpha} = 0, \qquad (\alpha = 1, \ldots, 4) \tag{3.7}$$

$$\Sigma^4_{\beta=1}\, M^*_{\alpha\beta}\, u_\beta = \Sigma^4_{\beta=1}\, u_\beta\, M^*_{\beta\alpha} = 0. \qquad (\alpha = 1, \ldots, 4) \tag{3.8}$$

From (3.3) and (3.4) we obtain with the help of (2.2) and (2.4)

$$\Sigma^4_{\alpha=1}\, u_\alpha\, B^*_\alpha = 0, \tag{3.9}$$

$$\Sigma^4_{\alpha=1}\, u_\alpha\, M^*_\alpha = 0. \tag{3.10}$$

Using (I.2.32), (2.1) and (2.3) it follows from (3.1) and (3.2) that

$$B'^*_{\alpha\beta} = \begin{bmatrix} 0 & B'_3 & -\,B'_2 & 0 \\ -\,B'_3 & 0 & B'_1 & 0 \\ B'_2 & -\,B'_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.11}$$

$$M'^*_{\alpha\beta} = \begin{bmatrix} 0 & M'_3 & -\,M'_2 & 0 \\ -\,M'_3 & 0 & M'_1 & 0 \\ M'_2 & -\,M'_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.12}$$

and from (I.2.16), (2.1), (2.3), (3.3) and (3.4) we get

$$B_1'^* = E_1'; \quad B_2'^* = E_2'; \quad B_3'^* = E_3'; \quad B_4'^* = 0, \tag{3.13}$$

$$M_1'^* = -P_1'; \; M_2'^* = -P_2'; \; M_3'^* = -P_3'; \; M_4'^* = 0. \tag{3.14}$$

With the help of (I.2.28), (2.2), (2.4), (3.1), (3.2), (3.3) and (3.4) we derive

$$B_{\alpha\beta} = B^*_{\alpha\beta} - B^*_\alpha u_\beta + u_\alpha B^*_\beta, \qquad (\alpha, \beta = 1, .., 4) \tag{3.15}$$

$$M_{\alpha\beta} = M^*_{\alpha\beta} - M^*_\alpha u_\beta + u_\alpha M^*_\beta. \qquad (\alpha, \beta = 1, .., 4) \tag{3.16}$$

We now define the tensors $G^*_{\alpha\beta}$ and $Z^*_{\alpha\beta}$ $(\alpha, \beta = 1, .., 4)$ by

$$G^*_{\alpha\beta} \equiv \lambda'_{(1)} B^*_{\alpha\beta} + \lambda'_{(2)} M^*_{\alpha\beta}, \qquad (\alpha, \beta = 1, .., 4) \tag{3.17}$$

$$Z^*_{\alpha\beta} \equiv \lambda'_{(3)} B^*_{\alpha\beta} + \lambda'_{(4)} M^*_{\alpha\beta}, \qquad (\alpha, \beta = 1, .., 4) \tag{3.18}$$

and the four-vectors G^*_α and Z^*_α $(\alpha = 1, .., 4)$ by

$$G^*_\alpha \equiv \lambda'_{(5)} B^*_\alpha + \lambda'_{(6)} M^*_\alpha, \qquad (\alpha = 1, .., 4) \tag{3.19}$$

$$Z^*_\alpha \equiv \lambda'_{(7)} B^*_\alpha + \lambda'_{(8)} M^*_\alpha. \qquad (\alpha = 1, .., 4) \tag{3.20}$$

In these equations the coefficients $\lambda'_{(k)} (k = 1, \ldots, 8)$ are Lorentz invariant quantities which will be specified in the following chapter. It will appear that they do not occur explicitly in the final results (*i.e.*, the entropy balance and the phenomenological equations) obtained in this chapter. We now take for the tensors $G_{\alpha\beta}$ and $Z_{\alpha\beta}$ analogous to (3.15) and (3.16)

$$G_{\alpha\beta} = G^*_{\alpha\beta} - G^*_\alpha u_\beta + u_\alpha G^*_\beta, \qquad (\alpha, \beta = 1, .., 4) \tag{3.21}$$

$$Z_{\alpha\beta} = Z^*_{\alpha\beta} - Z^*_\alpha u_\beta + u_\alpha Z^*_\beta. \qquad (\alpha, \beta = 1, .., 4) \tag{3.22}$$

Substituting (3.17), (3.18), (3.19) and (3.20) into the two preceding equations gives

$$G_{\alpha\beta} = \lambda'_{(1)} B^*_{\alpha\beta} + \lambda'_{(2)} M^*_{\alpha\beta} + \lambda'_{(5)} (u_\alpha B^*_\beta - B^*_\alpha u_\beta) + \\ + \lambda'_{(6)} (u_\alpha M^*_\beta - M^*_\alpha u_\beta), \qquad (\alpha, \beta = 1, .., 4) \tag{3.23}$$

$$Z_{\alpha\beta} = \lambda'_{(3)} B^*_{\alpha\beta} + \lambda'_{(4)} M^*_{\alpha\beta} + \lambda'_{(7)} (u_\alpha B^*_\beta - B^*_\alpha u_\beta) + \\ + \lambda'_{(8)} (u_\alpha M^*_\beta - M^*_\alpha u_\beta). \qquad (\alpha, \beta = 1, .., 4) \tag{3.24}$$

From the two above equations it is seen with the help of (3.5) and (3.6) that

$$G_{\alpha\beta} = -G_{\beta\alpha}, \qquad (\alpha, \beta = 1, .., 4) \tag{3.25}$$

$$Z_{\alpha\beta} = -Z_{\beta\alpha}. \qquad (\alpha, \beta = 1, .., 4) \tag{3.26}$$

We shall now derive some properties of the tensors $G^*_{\alpha\beta}$ and $Z^*_{\alpha\beta}$ and of the four-vectors G^*_α and Z^*_α. For that purpose we first consider the constitutive equations. As is well-known these equations read for media which are isotropic with respect to polarization and magnetization

$$\mathbf{M}' = (1 - \mu^{-1})\mathbf{B}', \tag{3.27}$$

$$\mathbf{P}' = (\varepsilon - 1)\mathbf{E}', \tag{3.28}$$

where the Lorentz invariant quantities μ and ε are the magnetic permeability and the dielectric constant respectively. From (3.11), (3.12), (3.13) and (3.14) it is seen that the two preceding equations can also be written in the form

$$M'^*_{\alpha\beta} = (1 - \mu^{-1})B'^*_{\alpha\beta}, \qquad (\alpha, \beta = 1, \ldots, 4) \tag{3.29}$$

$$M'^*_\alpha = (1 - \varepsilon)B'^*_\alpha. \qquad (\alpha = 1, \ldots, 4) \tag{3.30}$$

Since these tensor-equations hold at position $\mathbf{r}'$ and at time t' for an observer in $\mathrm{B}_{\mathbf{r},t}$ they also hold for an observer in A at position $\mathbf{r}$ and at time t. Thus,

$$M^*_{\alpha\beta} = (1 - \mu^{-1})B^*_{\alpha\beta}, \qquad (\alpha, \beta = 1, \ldots, 4) \tag{3.31}$$

$$M^*_\alpha = (1 - \varepsilon)B^*_\alpha. \qquad (\alpha = 1, \ldots, 4) \tag{3.32}$$

It follows from (3.17) and (3.18) with the help of (3.5) and (3.6) that

$$G^*_{\alpha\beta} = - G^*_{\beta\alpha}, \qquad (\alpha, \beta = 1, \ldots, 4) \tag{3.33}$$

$$Z^*_{\alpha\beta} = - Z^*_{\beta\alpha}, \qquad (\alpha, \beta = 1, \ldots, 4) \tag{3.34}$$

and with the help of (3.7) and (3.8) that

$$\Sigma^4_{\beta=1}\, G^*_{\alpha\beta}\, u_\beta = \Sigma^4_{\beta=1}\, u_\beta\, G^*_{\beta\alpha} = 0, \qquad (\alpha = 1, \ldots, 4) \tag{3.35}$$

$$\Sigma^4_{\beta=1}\, Z^*_{\alpha\beta}\, u_\beta = \Sigma^4_{\beta=1}\, u_\beta\, Z^*_{\beta\alpha} = 0. \qquad (\alpha = 1, \ldots, 4) \tag{3.36}$$

Using (3.9) and (3.10) we have from (3.19) and (3.20)

$$\Sigma^4_{\alpha=1}\, u_\alpha\, G^*_\alpha = 0, \tag{3.37}$$

$$\Sigma^4_{\alpha=1}\, u_\alpha\, Z^*_\alpha = 0. \tag{3.38}$$

From the four preceding equations we get with the help of (I.2.16)

$$G'^*_{\alpha4} = G'^*_{4\alpha} = 0, \qquad (\alpha = 1, \ldots, 4) \tag{3.39}$$

$$Z'^*_{\alpha4} = Z'^*_{4\alpha} = 0, \qquad (\alpha = 1, \ldots, 4) \tag{3.40}$$

$$G'^*_4 = 0, \tag{3.41}$$

$$Z'^*_4 = 0. \tag{3.42}$$

Inserting (3.31) and (3.32) into (3.17), (3.18), (3.19) and (3.20) gives

$$G^*_{\alpha\beta} = \{\lambda'_{(1)} + \lambda'_{(2)}\,(1 - \mu^{-1})\}\,B^*_{\alpha\beta}, \qquad (\alpha, \beta = 1, \ldots, 4) \qquad (3.43)$$

$$Z^*_{\alpha\beta} = \{\lambda'_{(3)} + \lambda'_{(4)}\,(1 - \mu^{-1})\}\,B^*_{\alpha\beta}, \qquad (\alpha, \beta = 1, \ldots, 4) \qquad (3.44)$$

$$G^*_{\alpha} = \{\lambda'_{(5)} + \lambda'_{(6)}\,(1 - \varepsilon)\}\,B^*_{\alpha}, \qquad (\alpha = 1, \ldots, 4) \qquad (3.45)$$

$$Z^*_{\alpha} = \{\lambda'_{(7)} + \lambda'_{(8)}\,(1 - \varepsilon)\}\,B^*_{\alpha}. \qquad (\alpha = 1, \ldots, 4) \qquad (3.46)$$

From these equations we find

$$Z^*_{\alpha\beta} = \Gamma G^*_{\alpha\beta}, \qquad (\alpha, \beta = 1, \ldots, 4) \qquad (3.47)$$

$$Z^*_{\alpha} = \Omega G^*_{\alpha}, \qquad (\alpha = 1, \ldots, 4) \qquad (3.48)$$

where

$$\Gamma \equiv \frac{\lambda'_{(3)} + \lambda'_{(4)}\,(1 - \mu^{-1})}{\lambda'_{(1)} + \lambda'_{(2)}\,(1 - \mu^{-1})}, \qquad (3.49)$$

$$\Omega \equiv \frac{\lambda'_{(7)} + \lambda'_{(8)}\,(1 - \varepsilon)}{\lambda'_{(5)} + \lambda'_{(6)}\,(1 - \varepsilon)}. \qquad (3.50)$$

Finally, we remark that we have from (3.21) and (3.22) with the help of (I.2.28), (I.2.30), (3.35) and (3.36)

$$G^*_{\alpha\beta} = \Sigma^4_{\gamma,\zeta=1}\, \Delta_{\alpha\gamma}\, G_{\gamma\zeta}\, \Delta_{\zeta\beta}, \qquad (\alpha, \beta = 1, \ldots, 4) \qquad (3.51)$$

$$Z^*_{\alpha\beta} = \Sigma^4_{\gamma,\zeta=1}\, \Delta_{\alpha\gamma}\, Z_{\gamma\zeta}\, \Delta_{\zeta\beta}. \qquad (\alpha, \beta = 1, \ldots, 4) \qquad (3.52)$$

We also have

$$G^*_{\alpha} = \Sigma^4_{\beta=1}\, G_{\alpha\beta}\, u_{\beta}, \qquad (\alpha = 1, \ldots, 4) \qquad (3.53)$$

$$Z^*_{\alpha} = \Sigma^4_{\beta=1}\, Z_{\alpha\beta}\, u_{\beta}. \qquad (\alpha = 1, \ldots, 4) \qquad (3.54)$$

These two equations are readily verified with the help of (I.2.12), (3.21), (3.22), (3.35), (3.36), (3.37) and (3.38).

§ 4. *Further discussion of the second law of thermodynamics.* We first remark that, according to (I.2.16) u'_α ($\alpha = 1, \ldots, 4$) does not depend on x_1, x_2, x_3 or x_4 and hence, using (I.2.25), we have

$$\mathrm{D}u'_\alpha = 0. \qquad (\alpha = 1, \ldots, 4) \qquad (4.1)$$

In the same way we find from (3.39), (3.40), (3.41) and (3.42)

$$\mathrm{D}G'^*_{\alpha 4} = \mathrm{D}G'^*_{4\alpha} = 0, \qquad (\alpha = 1, \ldots, 4) \qquad (4.2)$$

$$\mathrm{D}Z'^*_{\alpha 4} = \mathrm{D}Z'^*_{4\alpha} = 0, \qquad (\alpha = 1, \ldots, 4) \qquad (4.3)$$

$$\mathrm{D}G'^*_4 = 0, \qquad (4.4)$$

$$\mathrm{D}Z'^*_4 = 0. \qquad (4.5)$$

We now substitute (3.21) and (3.22) into (2.5). We then get with the help of (4.1)

$$T'\mathrm{D}s' = \mathrm{D}e' + p'\mathrm{D}v' - \Sigma_{j=1}^{n} \mu'^{(j)} \mathrm{D}c'^{(j)} +$$
$$+ \tfrac{1}{2} \Sigma_{\alpha,\beta=1}^{4} (G_{\alpha\beta}'^{*} - G_{\alpha}'^{*} u_{\beta}' + u_{\alpha}' G_{\beta}'^{*}) (\mathrm{D}Z_{\alpha\beta}'^{*} - u_{\beta}' \mathrm{D}Z_{\alpha}'^{*} + u_{\alpha}' \mathrm{D}Z_{\beta}'^{*}), \quad (4.6)$$

or, using (I.2.16), (3.39), (3.41) and (4.3)

$$T'\mathrm{D}s' = \mathrm{D}e' + p'\mathrm{D}v' + \tfrac{1}{2} \Sigma_{\alpha,\beta=1}^{3} G_{\alpha\beta}'^{*} \mathrm{D}Z_{\alpha\beta}'^{*} - \Sigma_{\alpha=1}^{3} G_{\alpha}'^{*} \mathrm{D}Z_{\alpha}'^{*} - \Sigma_{j=1}^{n} \mu'^{(j)} \mathrm{D}c'^{(j)}. \quad (4.7)$$

Inserting (3.17), (3.18), (3.19) and (3.20) into the preceding equation gives

$$T'\mathrm{D}s' = \mathrm{D}e' + p'\mathrm{D}v' - \Sigma_{j=1}^{n} \mu'^{(j)} \mathrm{D}c'^{(j)} +$$
$$+ \tfrac{1}{2} \Sigma_{\alpha,\beta=1}^{3} (\lambda_{(1)}' B_{\alpha\beta}'^{*} + \lambda_{(2)}' M_{\alpha\beta}'^{*}) \mathrm{D}(\lambda_{(3)}' B_{\alpha\beta}'^{*} + \lambda_{(4)}' M_{\alpha\beta}'^{*}) -$$
$$- \Sigma_{\alpha=1}^{3} (\lambda_{(5)}' B_{\alpha}'^{*} + \lambda_{(6)}' M_{\alpha}'^{*}) \mathrm{D}(\lambda_{(7)}' B_{\alpha}'^{*} + \lambda_{(8)}' M_{\alpha}'^{*}), \quad (4.8)$$

or, with the help of (3.11), (3.12), (3.13) and (3.14)

$$T'\mathrm{D}s' = \mathrm{D}e' + p'\mathrm{D}v' - \Sigma_{j=1}^{n} \mu'^{(j)} \mathrm{D}c'^{(j)} +$$
$$+ (\lambda_{(1)}' \mathbf{B}' + \lambda_{(2)}' \mathbf{M}') \cdot \mathrm{D}(\lambda_{(3)}' \mathbf{B}' + \lambda_{(4)}' \mathbf{M}') -$$
$$- (\lambda_{(5)}' \mathbf{E}' - \lambda_{(6)}' \mathbf{P}') \cdot \mathrm{D}(\lambda_{(7)}' \mathbf{E}' - \lambda_{(8)}' \mathbf{P}'). \quad (4.9)$$

We obtain the non-relativistic second law of thermodynamics corresponding to (4.9) by dropping the primes and replacing D by d/dt in (4.9). (Cf. § 2). Hence, we consider all those forms for the relativistic second law of thermodynamics of which the non-relativistic analogs can be written in the form of the non-relativistic analog of (4.9). It should be remarked that not all sets of values for $\lambda_{(k)}'$ ($k = 1, \ldots, 8$) give correct forms for the second law of thermodynamics.

We shall now derive another form for the relativistic second law of thermodynamics which will be useful for the considerations in the following sections. Using (3.39) and (3.41) we can write for (4.7)

$$T'\mathrm{D}s' = \mathrm{D}e' + p'\mathrm{D}v' + \tfrac{1}{2} \Sigma_{\alpha,\beta=1}^{4} G_{\alpha\beta}'^{*} \mathrm{D}Z_{\alpha\beta}'^{*} - \Sigma_{\alpha=1}^{4} G_{\alpha}'^{*} \mathrm{D}Z_{\alpha}'^{*} - \Sigma_{j=1}^{n} \mu'^{(j)} \mathrm{D}c'^{(j)}. \quad (4.10)$$

Substituting (3.47) and (3.48) into the preceding equation gives

$$T'\mathrm{D}s' = \mathrm{D}e' + p'\mathrm{D}v' - \Sigma_{j=1}^{n} \mu'^{(j)} \mathrm{D}c'^{(j)} +$$
$$+ \tfrac{1}{2} \Sigma_{\alpha,\beta=1}^{4} \{(G_{\alpha\beta}'^{*})^2 \mathrm{D}\Gamma + \tfrac{1}{2}\Gamma \mathrm{D}(G_{\alpha\beta}'^{*})^2\} -$$
$$- \Sigma_{\alpha=1}^{4} \{(G_{\alpha}'^{*})^2 \mathrm{D}\Omega + \tfrac{1}{2}\Omega \mathrm{D}(G_{\alpha}'^{*})^2\}. \quad (4.11)$$

We have

$$\Sigma_{\alpha,\beta=1}^4 (G'^*_{\alpha\beta})^2 = \Sigma_{\alpha,\beta=1}^4 (G^*_{\alpha\beta})^2, \tag{4.12}$$

$$\Sigma_{\alpha=1}^4 (G'^*_\alpha)^2 = \Sigma_{\alpha=1}^4 (G^*_\alpha)^2, \tag{4.13}$$

since the right hand sides of these equations have the same values in all Lorentz frames. Inserting the two preceding equations into (4.11) gives with the help of (3.47) and (3.48)

$$T'\mathrm{D}s' = \mathrm{D}e' + p'\mathrm{D}v' + \tfrac{1}{2}\,\Sigma_{\alpha,\beta=1}^4 G^*_{\alpha\beta}\,\mathrm{D}Z^*_{\alpha\beta} - \Sigma_{\alpha=1}^4 G^*_\alpha\,\mathrm{D}Z^*_\alpha - $$
$$- \Sigma_{j=1}^n \mu'^{(j)}\mathrm{D}\, c'^{(j)}. \tag{4.14}$$

Using (3.21) and (3.22) we can write for (4.14)

$$T'\mathrm{D}s' = \mathrm{D}e' + p'\mathrm{D}v' - \Sigma_{j=1}^n \mu'^{(j)}\,\mathrm{D}c'^{(j)} - \Sigma_{\alpha=1}^4 G^*_\alpha\,\mathrm{D}Z^*_\alpha +$$
$$+ \tfrac{1}{2}\,\Sigma_{\alpha,\beta=1}^4 (G_{\alpha\beta} + G^*_\alpha u_\beta - u_\alpha G^*_\beta)\mathrm{D}(Z_{\alpha\beta} + Z^*_\alpha u_\beta - u_\alpha Z^*_\beta). \tag{4.15}$$

With the help of (I.2.12), (I.2.13), (I.2.25), (3.25), (3.26), (3.37) and (3.38) we find for the above equation

$$T'\mathrm{D}s' = \mathrm{D}e' + p'\mathrm{D}v' - \Sigma_{j=1}^n \mu'^{(j)}\,\mathrm{D}c'^{(j)} + \tfrac{1}{2}\,\Sigma_{\alpha,\beta=1}^4 G_{\alpha\beta}\,\mathrm{D}Z_{\alpha\beta} +$$
$$+ \Sigma_{\alpha,\beta=1}^4 Z^*_\alpha\, G_{\alpha\beta}\,\mathrm{D}u_\beta - \Sigma_{\alpha,\beta=1}^4 u_\alpha\, G_{\alpha\beta}\,\mathrm{D}Z^*_\beta + \Sigma_{\alpha,\beta=1}^4 G^*_\alpha\, u_\beta\,\mathrm{D}Z_{\alpha\beta} -$$
$$- 2\Sigma_{\alpha=1}^4 G^*_\alpha\,\mathrm{D}Z^*_\alpha. \tag{4.16}$$

Using (3.54) we have

$$\Sigma_{\alpha,\beta=1}^4 G^*_\alpha\, u_\beta\,\mathrm{D}Z_{\alpha\beta} = \Sigma_{\alpha=1}^4 G^*_\alpha\,\mathrm{D}Z^*_\alpha - \Sigma_{\alpha,\beta=1}^4 G^*_\alpha\, Z_{\alpha\beta}\,\mathrm{D}u_\beta. \tag{4.17}$$

Substituting the preceding equation into (4.16) gives with the help of (3.25) and (3.53)

$$T'\mathrm{D}s' = \mathrm{D}e' + p'\mathrm{D}v' - \Sigma_{j=1}^n \mu'^{(j)}\,\mathrm{D}c'^{(j)} +$$
$$+ \Sigma_{\alpha,\beta=1}^4 (\tfrac{1}{2}G_{\alpha\beta}\,\mathrm{D}Z_{\alpha\beta} + Z^*_\alpha\, G_{\alpha\beta}\,\mathrm{D}u_\beta - G^*_\alpha\, Z_{\alpha\beta}\,\mathrm{D}u_\beta), \tag{4.18}$$

or, using (3.25), (3.26), (3.53) and (3.54)

$$T'\mathrm{D}s' = \mathrm{D}e' + p'\mathrm{D}v' - \Sigma_{j=1}^n \mu'^{(j)}\,\mathrm{D}c'^{(j)} + \tfrac{1}{2}\Sigma_{\alpha,\beta=1}^4 G_{\alpha\beta}\,\mathrm{D}Z_{\alpha\beta} +$$
$$+ \Sigma_{\alpha,\beta,\gamma=1}^4 u_\alpha\,(G_{\alpha\gamma}\, Z_{\gamma\beta} - Z_{\alpha\gamma}\, G_{\gamma\beta})\,\mathrm{D}u_\beta. \tag{4.19}$$

From this final form for the second law of thermodynamics it is seen that, as is required, our result is independent of the choice for the orientations of the three-dimensional axis-frames of the barycentric Lorentz frames. This may also be seen from (4.11) since the quantities $\Sigma_{\alpha,\beta=1}^4 (G'^*_{\alpha\beta})^2$ and

$\Sigma_{a=1}^{4}(G_a'^*)^2$ are invariant with respect to rotations of the ordinary three-dimensional axis-frames of the barycentric frames. (In our derivations, however, we have assumed that these orientations are chosen such that derivatives with respect to x_1, x_2, x_3 and x_4 of such quantities as $Z'_{\alpha\beta}$, $Z'^*_{\alpha\beta}$ and Z'^*_{α} exist.)

§ 5. *The balance equations for momentum and energy.* Again, we shall assume that the energy-momentum tensor of the matter is symmetric. Hence,

$$W_{\alpha\beta} = W_{\beta\alpha}. \qquad (\alpha, \beta = 1, \ldots, 4) \tag{5.1}$$

We shall assume that the forces acting on the matter are only of an electromagnetic nature. The balance equations for momentum and energy read

$$\Sigma_{\beta=1}^{4}\, \partial W_{\alpha\beta}/\partial x_\beta = k_\alpha. \qquad (\alpha = 1, \ldots, 4) \tag{5.2}$$

The first three components of the four-vector k_α $(\alpha = 1, \ldots, 4)$ are the components of the force per unit volume which the electromagnetic field exerts on the matter. The quantity $(c/i)k_4$ is the energy which the electromagnetic field contributes to the matter per unit volume and per unit time.

In two ways the electromagnetic field exerts a force on the medium. In the first place the Lorentz force acts on each chemical component which is electrically charged. In the second place, in consequence of the polarization and magnetization of the matter, the electromagnetic field exerts a force on the medium which cannot be interpreted as acting on the chemical components separately. This force will be called the ponderomotive force. Hence, we have

$$k_\alpha = \Sigma_{j=1}^{n}\, \varrho_{(0)}^{(j)}\, K_\alpha^{(j)} + k_{(P)\alpha}, \qquad (\alpha = 1, \ldots, 4) \tag{5.3}$$

where $\varrho_{(0)}^{(j)} K_\alpha^{(j)}$ $(\alpha = 1, \ldots, 4)$ is the four-vector representing the Lorentz force per unit volume acting on component j and $k_{(P)\alpha}$ $(\alpha = 1, \ldots, 4)$ is the four-vector representing the ponderomotive force per unit volume.

In chapter III $K_\alpha^{(j)}$ was given by (III.2.19). We shall now assume that

$$K_\alpha^{(j)} = e^{(j)}\, (c\varrho_{(0)}^{(j)})^{-1}\, \Sigma_{\beta=1}^{4}\, F_{\alpha\beta}^{(j)}\, m_\beta^{(j)}, \qquad (\alpha = 1, \ldots, 4;\ j = 1, \ldots, n) \tag{5.4}$$

where $F_{\alpha\beta}^{(j)}$ $(j = 1, \ldots, n)$ is a four-dimensional tensor. This tensor represents the "local" electric and magnetic fields to which the ions of the chemical component j $(j = 1, \ldots, n)$ are subjected. To have our discussions as general as possible we shall not yet completely specify this tensor but we shall assume that it is given by an equation analogous to (3.23) and (3.24). Hence,

$$F_{\alpha\beta}^{(j)} = \lambda_{(1)}'^{(j)}\, B_{\alpha\beta}^* + \lambda_{(2)}'^{(j)}\, M_{\alpha\beta}^* + \lambda_{(3)}'^{(j)}\, (u_\alpha B_\beta^* - B_\alpha^* u_\beta) +$$
$$+ \lambda_{(4)}'^{(j)}\, (u_\alpha M_\beta^* - M_\alpha^* u_\beta). \qquad (\alpha, \beta = 1, \ldots, 4;\ j = 1, \ldots, n) \tag{5.5}$$

From the above equation it is seen with the help of (3.5) and (3.6) that

$$F^{(j)}_{\alpha\beta} = - F^{(j)}_{\beta\alpha}. \qquad (\alpha, \beta = 1, .., 4; j = 1, \ldots, n) \qquad (5.6)$$

In § 8 we shall investigate the explicit form of $k_{(P)\alpha}$ $(\alpha = 1, .., 4)$.

§ 6. *The balance equation for rest mass.* Analogous to (I.4.1) the balance equation for rest mass reads

$$\partial \varrho^{(k)}/\partial t = - \operatorname{div} \varrho^{(k)} \mathbf{v}^{(k)} + \nu^{(k)} J_{(c)}. \qquad (k = 1, \ldots, n) \qquad (6.1)$$

Using the four-vectors $m^{(k)}_\alpha$ $(\alpha = 1, .., 4; k = 1, \ldots, n)$, defined by (I.2.1), we can write for the preceding equation

$$\Sigma^4_{\alpha=1} \partial m^{(k)}_\alpha/\partial x_\alpha = \nu^{(k)} J_{(c)}. \qquad (k = 1, \ldots, n) \qquad (6.2)$$

Analogous to (I.4.3) we have the conservation law

$$\Sigma^4_{\alpha=1} \partial m_\alpha/\partial x_\alpha = 0. \qquad (6.3)$$

The equation (6.1) can also be written in the form

$$\varrho' \mathrm{D} c'^{(j)} = - \Sigma^4_{\alpha=1} \partial I^{(j)}_\alpha/\partial x_\alpha + \nu^{(j)} J_{(c)}, \qquad (j = 1, \ldots, n) \qquad (6.4)$$

which equation is identical with (I.6.1).

§ 7. *The first law of thermodynamics.* To deduce the first law of thermodynamics, we use the same method as in § 5 of chapter I; *i.e.*, we study the equation which we obtain by multiplying (5.2) by u_α and summing over α. Hence, we must consider the equation

$$\Sigma^4_{\alpha,\beta=1} u_\alpha(\partial W_{\alpha\beta}/\partial x_\beta) = \Sigma^4_{\alpha=1} u_\alpha k_\alpha. \qquad (7.1)$$

Using (I.2.10), (I.2.21), (5.3), (5.4) and (5.6) we derive

$$\Sigma^4_{\alpha=1} u_\alpha k_\alpha = - \Sigma^n_{k=1} \Sigma^4_{\alpha=1} I^{(k)}_\alpha (c^{-1} e^{(k)} \Sigma^4_{\beta=1} F^{(k)}_{\alpha\beta} u_\beta) + \Sigma^4_{\alpha=1} u_\alpha k_{(P)\alpha}. \quad (7.2)$$

Substitution of (I.5.9) and the above equation into (7.1) gives

$$\varrho'(\mathrm{D}e' + p'\mathrm{D}v') = - \Sigma^4_{\beta=1}(\partial I^{(0)}_\beta/\partial x_\beta + c^{-1} I^{(0)}_\beta \mathrm{D}u_\beta) +$$
$$+ c\,\Sigma^4_{\alpha,\beta=1} P_{\alpha\beta}(\partial u_\beta/\partial x_\alpha) + \Sigma^n_{k=1} \Sigma^4_{\alpha=1} I^{(k)}_\alpha (e^{(k)} \Sigma^4_{\beta=1} F^{(k)}_{\alpha\beta} u_\beta) - c\,\Sigma^4_{\alpha=1} u_\alpha k_{(P)\alpha} \quad (7.3)$$

as first law of thermodynamics.

§ 8. *The entropy balance and the ponderomotive forces.* We have not yet discussed the form of the four-vector $k_{(P)\alpha}$ $(\alpha = 1, .., 4)$ which represents the ponderomotive force. In this section it will appear that certain conditions must be imposed on $k_{(P)\alpha}$ if we want to obtain a satisfactory expression for the entropy balance. These conditions, however, are not such that $k_{(P)\alpha}$ is uniquely determined.

We first substitute (6.4) and (7.3) into (4.19). Using (I. 2.25) we then find

$$\begin{aligned}\varrho' \mathrm{D}s' = & - \Sigma_{\alpha=1}^{4} (\partial/\partial x_\alpha) \{(1/T')(I_\alpha^{(0)} - \Sigma_{j=1}^{n} \mu'^{(j)} I_\alpha^{(j)})\} + \\ & + (1/T') [- \Sigma_{\alpha=1}^{4} I_\alpha^{(0)} \{(1/T')(\partial T'/\partial x_\alpha) + c^{-1} \mathrm{D}u_\alpha\} + \\ & + \Sigma_{j=1}^{n} \Sigma_{\alpha=1}^{4} I_\alpha^{(j)} \{e^{(j)} \Sigma_{\beta=1}^{4} F_{\alpha\beta}^{(j)} u_\beta - T'(\partial/\partial x_\alpha)(\mu'^{(j)}/T')\} + \\ & + c\, \Sigma_{\alpha,\beta=1}^{4} P_{\alpha\beta}(\partial u_\beta/\partial x_\alpha) - J_{(c)} \Sigma_{j=1}^{n} \nu^{(j)} \mu'^{(j)} - \\ & - c\, \Sigma_{\alpha=1}^{4} u_\alpha \{k_{(P)\alpha} - \tfrac{1}{2} \varrho' \Sigma_{\beta,\gamma=1}^{4} G_{\beta\gamma}(\partial Z_{\beta\gamma}/\partial x_\alpha) - \\ & - \varrho' \Sigma_{\beta,\gamma,\zeta=1}^{4} u_\beta (G_{\beta\gamma} Z_{\gamma\zeta} - Z_{\beta\gamma} G_{\gamma\zeta}) (\partial u_\zeta/\partial x_\alpha)\}]. \qquad (8.1)\end{aligned}$$

The last two lines of the preceding equation do not contain quantities which may be interpreted as fluxes in the sense of thermodynamics, because $k_{(P)\alpha}$ represents the ponderomotive force, the tensors $G_{\alpha\beta}$ and $Z_{\alpha\beta}$ describe the electromagnetic field and the polarization and magnetization of the matter, the four-vector u_α represents the barycentric velocity and ϱ is the total density of rest mass. The entropy production always consists of a sum of products of fluxes and "forces" (affinities), while the entropy flow equals a sum of terms where each term contains a flux as factor. Hence, if we want to obtain a satisfactory form for the entropy balance the last two lines of the preceding equation must vanish. This means that $k_{(P)\alpha}$ must have the form

$$\begin{aligned}k_{(P)\alpha} = \varrho' \{ & \tfrac{1}{2} \Sigma_{\beta,\gamma=1}^{4} G_{\beta\gamma}(\partial Z_{\beta\gamma}/\partial x_\alpha) + \Sigma_{\beta=1}^{4} \Delta_{\alpha\beta}\, \Phi_\beta + \\ & + \Sigma_{\beta,\gamma,\zeta=1}^{4} u_\beta (G_{\beta\gamma} Z_{\gamma\zeta} - Z_{\beta\gamma} G_{\gamma\zeta}) (\partial u_\zeta/\partial x_\alpha)\}. \qquad (\alpha = 1, \ldots, 4) \qquad (8.2)\end{aligned}$$

The four-vector $\Phi_\beta (\beta = 1, \ldots, 4)$ which occurs on the right hand side of this equation cannot be determined with the help of pure thermodynamical considerations; for, if we insert the preceding expression for $k_{(P)\alpha}$ into (7.3) or (8.1) the term containing Φ_β vanishes according to (I.2.30).

We now define

$$Y_\alpha^{(0)} \equiv - \{(1/T')\, (\partial T'/\partial x_\alpha) + c^{-1}\, \mathrm{D}u_\alpha\}, \qquad (\alpha = 1, \ldots, 4) \qquad (8.3)$$

$$Y_\alpha^{(j)} \equiv e^{(j)} \Sigma_{\beta=1}^{4} F_{\alpha\beta}^{(j)} u_\beta - T'(\partial/\partial x_\alpha)\, (\mu'^{(j)}/T'). \quad (\alpha = 1, \ldots, 4;\ j = 1, \ldots, n) \qquad (8.4)$$

Substituting (I.6.5), (I.6.11) and the three preceding equations into (8.1) gives with the help of (I.2.30) and (I.5.7)

$$\begin{aligned}\varrho' \mathrm{D}s' = & - \Sigma_{\alpha=1}^{4} (\partial/\partial x_\alpha) \{(1/T')\, (I_\alpha^{(0)} - \Sigma_{j=1}^{n} \mu'^{(j)} I_\alpha^{(j)})\} + \\ & + (1/T') \{\Sigma_{j=0}^{n} \Sigma_{\alpha=1}^{4} I_\alpha^{(j)} Y_\alpha^{(j)} + c\, \Sigma_{\alpha,\beta=1}^{4} \overline{P}_{\alpha\beta}(\partial u_\beta/\partial x_\alpha) + \Pi \varrho' \mathrm{D}v' + J_{(c)} A\}. \qquad (8.5)\end{aligned}$$

If we define

$$I_{(s)\alpha} \equiv (1/T')\, (I_\alpha^{(0)} - \Sigma_{j=1}^{n} \mu'^{(j)} I_\alpha^{(j)}), \qquad (\alpha = 1, \ldots, 4) \qquad (8.6)$$

$$\sigma \equiv (1/T') \{\Sigma_{j=0}^{n} \Sigma_{\alpha=1}^{4} I_\alpha^{(j)} Y_\alpha^{(j)} + c\, \Sigma_{\alpha,\beta=1}^{4} \overline{P}_{\alpha\beta}(\partial u_\beta/\partial x_\alpha) + \Pi \varrho' \mathrm{D}v' + J_{(c)} A\}, \qquad (8.7)$$

we have for (8.5)

$$\varrho' \mathrm{D} s' = - \Sigma_{a=1}^{4} \, \partial I_{(s)a} / \partial x_a + \sigma. \tag{8.8}$$

This is the final form for the entropy balance. We can interpret σ as the entropy production per unit time and per unit volume. The four-vector $I_{(s)a}$ represents the conductive flow of entropy per unit surface.

From (8.7) it follows that σ is a Lorentz invariant quantity. It is seen that the entropy balance (8.5) is formally equal to the entropy balance (III.5.1) for the case without polarization and magnetization. Mazur and Prigogine also find this formal analogy as one of the results of their non-relativistic theory [1]).

§ 9. *The phenomenological equations for vectorial fluxes in anisotropic media and the Onsager relations.* In the preceding chapters we have given the phenomenological equations for isotropic media. We shall now treat the general case of media which are anisotropic as far as the irreversible processes are concerned. It should be remarked that the medium can become anisotropic (as far as the irreversible processes are concerned) owing to the polarization and magnetization of the matter. In this section we shall deal with vectorial fluxes. These fluxes represent the heat flow and the relative flows of matter.

According to (8.7) the contribution, $\sigma_{(h)(d)}$, of the vectorial fluxes to the entropy production, σ, is given by

$$\sigma_{(h)(d)} = (1/T') \, \Sigma_{j=0}^{n} \, \Sigma_{a=1}^{4} \, I_a^{(j)} \, Y_a^{(j)}. \tag{9.1}$$

We now introduce as new vectorial "forces" (affinities) the four-vectors $\tilde{Y}_a^{(j)}$ $(a = 1, \ldots, 4;\ j = 0, 1, \ldots, n)$ defined by

$$\tilde{Y}_a^{(j)} \equiv \Sigma_{\beta=1}^{4} \, \Delta_{a\beta} \, Y_\beta^{(j)}. \qquad (a = 1, \ldots, 4;\ j = 0, 1, \ldots, n) \tag{9.2}$$

Using (I.2.30) it follows from this definition that

$$\Sigma_{a=1}^{4} \, u_a \, \tilde{Y}_a^{(j)} = 0. \qquad (j = 0, 1, \ldots, n) \tag{9.3}$$

With the help of (I.2.24), (I.2.28), (I.3.7) and (9.2) we can write for (9.1)

$$\sigma_{(h)(d)} = (1/T') \, \Sigma_{j=0}^{n} \, \Sigma_{a=1}^{4} \, I_a^{(j)} \, \tilde{Y}_a^{(j)}. \tag{9.4}$$

We shall leave out of consideration cross-effects between quantities of different tensorial character. (It should be remarked that such cross-effects might exist in anisotropic media.) We shall assume, however, that a flux depends on all "forces" (affinities) having the same tensorial character as this flux. Therefore, taking into account the above form for $\sigma_{(h)(d)}$, we have for the four-vectors $I_a^{(j)}$ $(a = 1, \ldots, 4;\ j = 0, 1, \ldots, n)$ the phenomenological equations

$$I_a^{(j)} = \Sigma_{k=0}^{n} \, \Sigma_{\beta=1}^{4} \, \tilde{L}_{a\beta}^{*(j)(k)} \, \tilde{Y}_\beta^{(k)}. \qquad (a = 1, \ldots, 4;\ j = 0, 1, \ldots, n) \tag{9.5}$$

From (I.2.24), (I.3.7) and the preceding equation it follows that

$$\Sigma_{k=0}^{n} \Sigma_{\alpha,\beta=1}^{4} u_\alpha \tilde{L}_{\alpha\beta}^{*(j)(k)} \tilde{Y}_\beta^{(k)} = 0. \qquad (j = 0, 1, \ldots, n) \tag{9.6}$$

Using (I.2.28), (9.3) and the above equation we write for (9.5)

$$I_\alpha^{(j)} = \Sigma_{k=0}^{n} \Sigma_{\beta=1}^{4} \tilde{L}_{\alpha\beta}^{(j)(k)} \tilde{Y}_\beta^{(k)}, \qquad (\alpha = 1, \ldots, 4;\ j = 0, 1, \ldots, n) \tag{9.7}$$

where

$$\tilde{L}_{\alpha\beta}^{(j)(k)} \equiv \Sigma_{\gamma,\zeta=1}^{4} \Delta_{\alpha\gamma} \tilde{L}_{\gamma\zeta}^{*(j)(k)} \Delta_{\zeta\beta}. \qquad (\alpha, \beta = 1, \ldots, 4;\ j, k = 0, 1, \ldots, n) \tag{9.8}$$

We shall now deduce some properties of the phenomenological tensors $\tilde{L}_{\alpha\beta}^{(j)(k)}$ $(\alpha, \beta = 1, \ldots, 4;\ j, k = 0, 1, \ldots, n)$. With the help of (I.2.30) we have from (9.8)

$$\Sigma_{\alpha=1}^{4} u_\alpha \tilde{L}_{\alpha\beta}^{(j)(k)} = 0, \qquad (\beta = 1, \ldots, 4;\ j, k = 0, 1, \ldots, n) \tag{9.9}$$

and

$$\Sigma_{\beta=1}^{4} \tilde{L}_{\alpha\beta}^{(j)(k)} u_\beta = 0. \qquad (\alpha = 1, \ldots, 4;\ j, k = 0, 1, \ldots, n) \tag{9.10}$$

Using (I.2.16) it follows from the two preceding equations that

$$\tilde{L}'^{(j)(k)}_{4\alpha} = \tilde{L}'^{(j)(k)}_{\alpha 4} = 0. \qquad (\alpha = 1, \ldots, 4;\ j, k = 0, 1, \ldots, n) \tag{9.11}$$

From (9.7) and the above equation we conclude

$$I'^{(j)}_\alpha = \Sigma_{k=0}^{n} \Sigma_{\beta=1}^{3} \tilde{L}'^{(j)(k)}_{\alpha\beta} \tilde{Y}'^{(k)}_\beta. \qquad (\alpha = 1, \ldots, 4;\ j = 0, 1, \ldots, n) \tag{9.12}$$

Since the $\tilde{Y}'^{(k)}_\beta (\beta = 1, 2, 3;\ k = 0, 1, \ldots, n)$ are independent from each other we have from (I.2.23) and the preceding equation

$$\Sigma_{j=1}^{n} \tilde{L}'^{(j)(k)}_{\alpha\beta} = 0, \qquad (\alpha = 1, \ldots, 4;\ \beta = 1, 2, 3;\ k = 0, 1, \ldots, n) \tag{9.13}$$

or with the help of (9.11)

$$\Sigma_{j=1}^{n} \tilde{L}'^{(j)(k)}_{\alpha\beta} = 0. \qquad (\alpha, \beta = 1, \ldots, 4;\ k = 0, 1, \ldots, n) \tag{9.14}$$

As this tensor relation holds at position $\mathbf{r}'$ and at time t' for an observer in $\mathrm{B}_{\mathbf{r},t}$ it also holds at position $\mathbf{r}$ and at time t for an observer in A. Hence,

$$\Sigma_{j=1}^{n} \tilde{L}_{\alpha\beta}^{(j)(k)} = 0. \qquad (\alpha, \beta = 1, \ldots, 4;\ k = 0, 1, \ldots, n) \tag{9.15}$$

Using (I.3.6) and (I.7.2) we have from (9.4)

$$T'\sigma_{(h)(d)} = \Sigma_{j=0}^{n} \Sigma_{\alpha=1}^{3} I'^{(j)}_\alpha \tilde{Y}'^{(j)}_\alpha. \tag{9.16}$$

With the help of (I.2.23) we find from the above equation

$$T'\sigma_{(h)(d)} = \Sigma_{\alpha=1}^{3} I'^{(0)}_\alpha \tilde{Y}'^{(0)}_\alpha + \Sigma_{j=1}^{n-1} \Sigma_{\alpha=1}^{3} I'^{(j)}_\alpha (\tilde{Y}'^{(j)}_\alpha - \tilde{Y}'^{(n)}_\alpha). \tag{9.17}$$

Inserting (9.12) into the preceding equation gives

$$\begin{aligned} T'\sigma_{(h)(d)} = {} & \Sigma_{k=0}^{n} \Sigma_{\alpha,\beta=1}^{3} \tilde{L}'^{(0)(k)}_{\alpha\beta} \tilde{Y}'^{(k)}_\beta \tilde{Y}'^{(0)}_\alpha + \\ & + \Sigma_{k=0}^{n} \Sigma_{j=1}^{n-1} \Sigma_{\alpha,\beta=1}^{3} \tilde{L}'^{(j)(k)}_{\alpha\beta} \tilde{Y}'^{(k)}_\beta (\tilde{Y}'^{(j)}_\alpha - \tilde{Y}'^{(n)}_\alpha). \end{aligned} \tag{9.18}$$

From this equation it follows that

$$T'\sigma_{(h)(d)} = \tilde{L}'^{(0)(0)}_{\xi\xi} (\tilde{Y}'^{(0)}_{\xi})^2 + (\Sigma^n_{k=1} \tilde{L}'^{(0)(k)}_{\xi\zeta}) \tilde{Y}'^{(n)}_{\zeta} \tilde{Y}'^{(0)}_{\xi} \text{ if}$$

$$\tilde{Y}'^{(j)}_{a} = \tilde{Y}'^{(n)}_{a} (j=1, \ldots, n-1), \tilde{Y}'^{(0)}_{a} = 0 \text{ for } a \neq \xi, \tilde{Y}'^{(n)}_{a} = 0 \text{ for } a \neq \zeta, \quad (9.19)$$

where ξ and ζ are two numbers, each of them having one of the values 1, 2 or 3. Since $T'\sigma_{(h)(d)}$ must be a positive definite quantity, it follows from the preceding equation that

$$\tilde{L}'^{(0)(0)}_{\xi\xi} > 0, \qquad (\xi = 1, 2, 3) \qquad (9.20)$$

and

$$\Sigma^n_{k=1} \tilde{L}'^{(0)(k)}_{\xi\zeta} = 0, \qquad (\xi, \zeta = 1, 2, 3) \qquad (9.21)$$

because $\tilde{Y}'^{(0)}_{\xi}$ and $\tilde{Y}'^{(n)}_{\zeta}$ may be chosen arbitrarily. From (9.18) we also have

$$T'\sigma_{(h)(d)} = \tilde{L}'^{(l)(l)}_{\xi\xi} (\tilde{Y}'^{(l)}_{\xi} - \tilde{Y}'^{(n)}_{\xi})^2 + (\Sigma^n_{k=1} \tilde{L}'^{(l)(k)}_{\xi\zeta}) \tilde{Y}'^{(n)}_{\zeta} (\tilde{Y}'^{(l)}_{\xi} - \tilde{Y}'^{(n)}_{\xi}) \text{ if}$$

$$\tilde{Y}'^{(j)}_{a} = \tilde{Y}'^{(n)}_{a} (j=1, \ldots, n-1) \text{ for } j \neq l, \ \tilde{Y}'^{(0)}_{a} = 0, \ \tilde{Y}'^{(l)}_{a} = \tilde{Y}'^{(n)}_{a} \text{ for } a \neq \xi,$$

$$\tilde{Y}'^{(n)}_{a} = 0 \text{ for } a \neq \zeta, \qquad (9.22)$$

where ξ and ζ are two numbers, each of them having one of the values 1, 2, or 3 and l is a number which may have one of the values $1, 2, \ldots, n-1$. Since $\tilde{Y}'^{(l)}_{\xi} - \tilde{Y}'^{(n)}_{\xi}$ and $\tilde{Y}'^{(n)}_{\zeta}$ may be chosen arbitrarily and since $T'\sigma_{(h)(d)}$ must be a positive definite quantity it follows from the above equation that

$$\tilde{L}'^{(l)(l)}_{\xi\xi} \geqslant 0, \qquad (\xi = 1, 2, 3; \ l = 1, \ldots, n-1) \qquad (9.23)$$

and

$$\Sigma^n_{k=1} \tilde{L}'^{(l)(k)}_{\xi\zeta} = 0. \qquad (\xi, \zeta = 1, 2, 3; \ l = 1, \ldots, n-1) \qquad (9.24)$$

It is easily seen that the two preceding equations are also valid for $l = n$. Moreover, on account of (9.11) ξ and ζ may run from 1 to 4 in the equations (9.20), (9.21), (9.23) and (9.24). Thus, we get

$$\tilde{L}'^{(j)(j)}_{aa} \geqslant 0, \qquad (a = 1, \ldots, 4; \ j = 0, 1, \ldots, n) \qquad (9.25)$$

and

$$\Sigma^n_{k=1} \tilde{L}'^{(j)(k)}_{a\beta} = 0. \qquad (a, \beta = 1, \ldots, 4; \ j = 0, 1, \ldots, n) \qquad (9.26)$$

Since the last tensor equation holds for an observer in $B_{\mathbf{r},t}$ at position $\mathbf{r}'$ and at time t' this equation is also valid for an observer in A at position $\mathbf{r}$ and time t. Hence,

$$\Sigma^n_{k=1} \tilde{L}^{(j)(k)}_{a\beta} = 0. \qquad (a, \beta = 1, \ldots, 4; \ j = 0, 1, \ldots, n) \qquad (9.27)$$

It may easily be verified that (9.12) is identical with the non-relativistic phenomenological equations for heat conduction and diffusion in anisotropic media for the case that $\mathbf{v}$ and $d\mathbf{v}/dt$ vanish [2]). Thus, the coefficients $\tilde{L}'^{(j)(k)}_{a\beta}$ $(a, \beta = 1, 2, 3; \ j, k = 0, 1, \ldots, n)$ are the phenomenological coefficients, of the non-relativistic theory, among which we have the Onsager relations

$$\tilde{L}'^{(j)(k)}_{\alpha\beta}(\mathbf{B}') = \tilde{L}'^{(k)(j)}_{\beta\alpha}(-\mathbf{B}'). \qquad (\alpha, \beta = 1, 2, 3;\ j, k = 0, 1, \ldots, n) \quad (9.28)$$

From (9.11) and the preceding equation we find

$$\tilde{L}'^{(j)(k)}_{\alpha\beta}(\mathbf{B}') = \tilde{L}'^{(k)(j)}_{\beta\alpha}(-\mathbf{B}'). \qquad (\alpha, \beta = 1, \ldots, 4;\ j, k = 0, 1, \ldots, n) \quad (9.29)$$

Since these tensor relations hold for an observer in $B_{\mathbf{r},t}$ at position $\mathbf{r}'$ and at time t' they also hold for an observer in A at position $\mathbf{r}$ and at time t. Therefore, we get

$$\tilde{L}^{(j)(k)}_{\alpha\beta}(\mathbf{B}') = \tilde{L}^{(k)(j)}_{\beta\alpha}(-\mathbf{B}') \qquad (\alpha, \beta = 1, \ldots, 4;\ j, k = 0, 1, \ldots, n) \quad (9.30)$$

as relativistic Onsager relations for heat conduction and diffusion in anisotropic media.

It should be noted that it is also possible to derive (9.27) from (9.15) and (9.30). As we have seen, however, it is not necessary to use the Onsager relations for the derivation of (9.27) since this equation may also be deduced from the positive definite character of $T'\sigma_{(h)(d)}$. The method used above to deduce (9.27) is an extension of a procedure used by de Groot [2]) to derive this equation for isotropic media. Finally, we remark that the tensors $\tilde{L}^{*(j)(k)}_{\alpha\beta}$ occurring in (9.5) need not have any of the properties which we have derived for the tensors $\tilde{L}^{(j)(k)}_{\alpha\beta}$.

§ 10. *The phenomenological equations for tensorial fluxes in anisotropic media and the Onsager relations.* The only tensorial flux occurring in the expression (8.7) for σ is the ordinary viscous pressure tensor $\overline{P}_{\alpha\beta}$ ($\alpha, \beta = 1, \ldots, 4$). It is seen from (8.7) that the contribution, $\sigma_{(v)}$, of the viscous flow to σ is given by

$$\sigma_{(v)} = (c/T')\, \Sigma^4_{\alpha,\beta=1}\, \overline{P}_{\alpha\beta}\, (\partial u_\beta/\partial x_\alpha). \quad (10.1)$$

We now introduce the four-dimensional tensor $Y_{\alpha\beta}$ ($\alpha, \beta = 1, \ldots, 4$)

$$Y_{\alpha\beta} \equiv c\, \Sigma^4_{\gamma,\zeta=1}\, [\tfrac{1}{2}\Delta_{\alpha\gamma}\Delta_{\beta\zeta}\{(\partial u_\gamma/\partial x_\zeta) + (\partial u_\zeta/\partial x_\gamma)\} - \tfrac{1}{3}\Delta_{\alpha\beta}\,\Delta_{\gamma\zeta}\,(\partial u_\gamma/\partial x_\zeta)]. \quad (10.2)$$

$$(\alpha, \beta = 1, \ldots, 4)$$

From (I.2.30) and the preceding equation we have

$$\Sigma^4_{\alpha=1}\, u_\alpha\, Y_{\alpha\beta} = 0, \qquad (\beta = 1, \ldots, 4) \quad (10.3)$$

$$\Sigma^4_{\beta=1}\, Y_{\alpha\beta}\, u_\beta = 0. \qquad (\alpha = 1, \ldots, 4) \quad (10.4)$$

With the help of (I.2.29) we find from (10.2)

$$Y_{\alpha\beta} = Y_{\beta\alpha}. \qquad (\alpha, \beta = 1, \ldots, 4) \quad (10.5)$$

Using (I.2.28), (I.2.29), (I.2.30) and (I.2.33) we get from (10.2)

$$\Sigma_{\alpha=1}^{4} Y_{\alpha\alpha} = 0. \tag{10.6}$$

With the help of (I.2.28), (I.6.6), (I.6.7), (I.6.8) and (10.2) we can write for (10.1)

$$\sigma_{(v)} = (1/T') \, \Sigma_{\alpha,\beta=1}^{4} \overline{P}_{\alpha\beta} \, Y_{\alpha\beta}. \tag{10.7}$$

As noted already in the preceding section we shall leave out of consideration cross-effects among quantities of different tensorial character. Therefore, taking into account the preceding form for $\sigma_{(v)}$, we have for the tensor $\overline{P}_{\alpha\beta}$ $(\alpha, \beta = 1, .., 4)$ the phenomenological equations

$$\overline{P}_{\alpha\beta} = \Sigma_{\gamma,\zeta=1}^{4} L_{\alpha\beta\gamma\zeta}^{***} \, Y_{\gamma\zeta}, \qquad (\alpha, \beta = 1, .., 4) \tag{10.8}$$

where $L_{\alpha\beta\gamma\zeta}^{***}$ $(\alpha, \beta, \gamma, \zeta = 1, .., 4)$ is a phenomenological tensor. We now introduce the four-dimensional tensor $L_{\alpha\beta\gamma\zeta}^{**}$ $(\alpha, \beta, \gamma, \zeta = 1, .., 4)$ defined by

$$L_{\alpha\beta\gamma\zeta}^{**} \equiv \Sigma_{\eta,\lambda,\xi,\nu=1}^{4} \Delta_{\alpha\eta} \, \Delta_{\beta\lambda} \, \Delta_{\gamma\xi} \, \Delta_{\zeta\nu} \, L_{\eta\lambda\xi\nu}^{***}. \qquad (\alpha, \beta, \gamma, \zeta = 1, .., 4) \tag{10.9}$$

Using (I.2.28), (I.6.8), (10.3), (10.4) and (10.9) we can write for (10.8)

$$\overline{P}_{\alpha\beta} = \Sigma_{\gamma,\zeta=1}^{4} L_{\alpha\beta\gamma\zeta}^{**} \, Y_{\gamma\zeta}. \qquad (\alpha, \beta = 1, .., 4) \tag{10.10}$$

Finally, we introduce the four-dimensional tensor $L_{\alpha\beta\gamma\zeta}^{*}$ $(\alpha, \beta, \gamma, \zeta = 1, .., 4)$ defined by

$$\begin{aligned} L_{\alpha\beta\gamma\zeta}^{*} \equiv {} & \tfrac{1}{4}(L_{\alpha\beta\gamma\zeta}^{**} + L_{\beta\alpha\gamma\zeta}^{**} + L_{\alpha\beta\zeta\gamma}^{**} + L_{\beta\alpha\zeta\gamma}^{**}) - \\ & - \tfrac{1}{6}\Delta_{\alpha\beta} \, \Sigma_{\xi=1}^{4} (L_{\xi\xi\gamma\zeta}^{**} + L_{\xi\xi\zeta\gamma}^{**}) - \tfrac{1}{6}\Delta_{\gamma\zeta} \, \Sigma_{\xi=1}^{4} (L_{\alpha\beta\xi\xi}^{**} + L_{\beta\alpha\xi\xi}^{**}) + \\ & + \tfrac{1}{9}\Delta_{\alpha\beta} \, \Delta_{\gamma\zeta} \, \Sigma_{\lambda,\xi=1}^{4} L_{\lambda\lambda\xi\xi}^{**}. \qquad (\alpha, \beta, \gamma, \zeta = 1, .., 4) \end{aligned} \tag{10.11}$$

With the help of (I.2.28), (I.6.6), (I.6.7), (10.3), (10.4), (10.5), (10.6) and the preceding equation we find for (10.10)

$$\overline{P}_{\alpha\beta} = \Sigma_{\gamma,\zeta=1}^{4} L_{\alpha\beta\gamma\zeta}^{*} \, Y_{\gamma\zeta}. \qquad (\alpha, \beta = 1, .., 4) \tag{10.12}$$

We shall now deduce some properties of the tensor $L_{\alpha\beta\gamma\zeta}^{*}$. Using (I.2.30) and (10.9) it is seen from (10.11) that

$$\Sigma_{\alpha=1}^{4} u_\alpha \, L_{\alpha\beta\gamma\zeta}^{*} = 0, \qquad (\beta, \gamma, \zeta = 1, .., 4) \tag{10.13}$$

$$\Sigma_{\beta=1}^{4} u_\beta \, L_{\alpha\beta\gamma\zeta}^{*} = 0, \qquad (\alpha, \gamma, \zeta = 1, .., 4) \tag{10.14}$$

$$\Sigma_{\gamma=1}^{4} u_\gamma \, L_{\alpha\beta\gamma\zeta}^{*} = 0, \qquad (\alpha, \beta, \zeta = 1, .., 4) \tag{10.15}$$

$$\Sigma_{\zeta=1}^{4} u_\zeta \, L_{\alpha\beta\gamma\zeta}^{*} = 0. \qquad (\alpha, \beta, \gamma = 1, .., 4) \tag{10.16}$$

With the help of (I.2.33) we obtain from (10.11)

$$\Sigma_{\gamma=1}^{4} L_{\alpha\beta\gamma\gamma}^{*} = 0, \qquad (\alpha, \beta = 1, .., 4) \tag{10.17}$$

$$\Sigma_{\alpha=1}^{4} L_{\alpha\alpha\gamma\zeta}^{*} = 0. \qquad (\gamma, \zeta = 1, .., 4) \tag{10.18}$$

Further, we get from (I.2.29) and (10.11)

$$L^*_{\alpha\beta\gamma\zeta} = L^*_{\beta\alpha\gamma\zeta}, \qquad (\alpha, \beta, \gamma, \zeta = 1, \ldots, 4) \tag{10.19}$$

$$L^*_{\alpha\beta\gamma\zeta} = L^*_{\alpha\beta\zeta\gamma}. \qquad (\alpha, \beta, \gamma, \zeta = 1, \ldots, 4) \tag{10.20}$$

We shall now derive the Onsager relations. Using (I.2.16) we find from (I.6.8), (10.3), (10.4), (10.13), (10.14), (10.15) and (10.16)

$$\overline{P}'_{\alpha 4} = \overline{P}'_{4\alpha} = 0, \qquad (\alpha = 1, \ldots, 4) \tag{10.21}$$

$$Y'_{\alpha 4} = Y'_{4\alpha} = 0, \qquad (\alpha = 1, \ldots, 4) \tag{10.22}$$

$$L'^*_{4\alpha\beta\gamma} = L'^*_{\alpha 4\beta\gamma} = L'^*_{\alpha\beta 4\gamma} = L'^*_{\alpha\beta\gamma 4} = 0. \qquad (\alpha, \beta, \gamma = 1, \ldots, 4) \tag{10.23}$$

Using the three preceding equations it follows from (10.12) that

$$\overline{P}'_{\alpha\beta} = \Sigma^3_{\gamma,\zeta=1} L'^*_{\alpha\beta\gamma\zeta} Y'_{\gamma\zeta}. \qquad (\alpha, \beta = 1, 2, 3) \tag{10.24}$$

It may be easily verified that (10.24) is identical with the non-relativistic phenomenological equations for the ordinary viscous pressure tensor in anisotropic media and that the coefficients $L'^*_{\alpha\beta\gamma\zeta}$ $(\alpha, \beta, \gamma, \zeta = 1, 2, 3)$ are the same coefficients as the phenomenological coefficients used by de Groot and Mazur [3]). Therefore, we have the Onsager relations

$$L'^*_{\alpha\beta\gamma\zeta} (\mathbf{B}') = L'^*_{\gamma\zeta\alpha\beta} (-\mathbf{B}'). \qquad (\alpha, \beta, \gamma, \zeta = 1, 2, 3) \tag{10.25}$$

Using (10.23) we may extend the preceding equation to

$$L'^*_{\alpha\beta\gamma\zeta} (\mathbf{B}') = L'^*_{\gamma\zeta\alpha\beta} (-\mathbf{B}'). \qquad (\alpha, \beta, \gamma, \zeta = 1, \ldots, 4) \tag{10.26}$$

Since this tensor relation holds at position $\mathbf{r}'$ and at time t' for an observer in $\mathrm{B}_{\mathbf{r},t}$ it also holds at position $\mathbf{r}$ and at time t for an observer in A. Hence,

$$L^*_{\alpha\beta\gamma\zeta} (\mathbf{B}') = L^*_{\gamma\zeta\alpha\beta} (-\mathbf{B}'). \qquad (\alpha, \beta, \gamma, \zeta = 1, \ldots, 4) \tag{10.27}$$

This equation is the relativistic form for the Onsager relations for viscous flow in anisotropic media.

Finally, it should be remarked that the tensor $L^{***}_{\alpha\beta\gamma\zeta}$ $(\alpha, \beta, \gamma, \zeta = 1, \ldots, 4)$ need not have any of the properties which we have derived for the tensor $L^*_{\alpha\beta\gamma\zeta}$ $(\alpha, \beta, \gamma, \zeta = 1, \ldots, 4)$.

§ 11. *The phenomenological equations for scalar fluxes in anisotropic media and the Onsager relations.* For the scalar quantities Π and $J_{(c)}$ we have the phenomenological equations

$$\Pi = \eta_{(v)} \varrho' \mathrm{D} v' + L_{(p)(c)} A, \tag{11.1}$$

and

$$J_{(c)} = L_{(c)(p)} \varrho' \mathrm{D} v' + LA. \tag{11.2}$$

The coefficients $L_{(p)(c)}$ and $L_{(c)(p)}$ are identical with those of the non-relativistic theory. Between them we have

$$L_{(p)(c)}(\mathbf{B}') = -L_{(c)(p)}(-\mathbf{B}'), \tag{11.3}$$

which is the Onsager relation for visco-chemical effects in anisotropic media.

§ 12. *The case of isotropy.* The three-dimensional tensor $\overline{P}'_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$ is the ordinary three-dimensional viscous stress tensor of the non-relativistic theory. The three-dimensional tensor $Y'_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$ is the "force" (affinity) conjugate to $P'_{\alpha\beta}$ in the non-relativistic theory. As may be seen from (I.6.7) and (10.5), the two tensors are symmetrical. Moreover, the trace of these two three-dimensional tensors vanishes according to (I.6.6), (10.6), (10.21) and (10.22). As is well-known, isotropy means that the two tensors are proportional to each other. Hence, we can write

$$\overline{P}'_{\alpha\beta} = 2\eta Y'_{\alpha\beta}. \qquad (\alpha, \beta = 1, 2, 3) \tag{12.1}$$

From (10.21), (10.22) and the preceding equation we have

$$\overline{P}'_{\alpha\beta} = 2\eta Y'_{\alpha\beta}. \qquad (\alpha, \beta = 1, \ldots, 4) \tag{12.2}$$

In the same way we can derive

$$I'^{(j)}_{\alpha} = \Sigma^{n}_{k=0} L^{(j)(k)} \tilde{Y}'^{(k)}_{\alpha} \qquad (\alpha = 1, \ldots, 4;\ j = 0, 1, \ldots, n) \tag{12.3}$$

for isotropic media. Since the two preceding tensor relations hold for an observer in $B_{\mathbf{r},t}$ at position $\mathbf{r}'$ and at time t' these relations also hold for an observer in A at position $\mathbf{r}$ and at time t. Hence,

$$\overline{P}_{\alpha\beta} = 2\eta Y_{\alpha\beta}, \qquad (\alpha, \beta = 1, \ldots, 4) \tag{12.4}$$

and

$$I^{(j)}_{\alpha} = \Sigma^{n}_{k=0} L^{(j)(k)} \tilde{Y}^{(k)}_{\alpha}. \qquad (\alpha = 1, \ldots, 4;\ j = 0, 1, \ldots, n) \tag{12.5}$$

Inserting (10.2) into (12.4) shows that (12.4) is identical with (I.7.14). From (I.7.7) and (9.2) we see that also (12.5) is identical with (I.7.1).

Comparing (10.8) and (12.4) we see that in the case of isotropy

$$L^{***}_{\alpha\beta\gamma\zeta} = 2\eta\delta_{\alpha\gamma}\delta_{\beta\zeta}. \qquad (\alpha, \beta, \gamma, \zeta = 1, \ldots, 4) \tag{12.6}$$

Similarly we see from (9.5) and (12.5) that in the case of isotropy

$$\tilde{L}^{*(j)(k)}_{\alpha\beta} = L^{(j)(k)}\delta_{\alpha\beta}. \qquad (\alpha, \beta = 1, \ldots, 4;\ j,k = 0, 1, \ldots, n) \tag{12.7}$$

Substitution of (12.6) into (10.9) gives with the help of (I.2.28) and (I.2.30)

$$L^{**}_{\alpha\beta\gamma\zeta} = 2\eta\Delta_{\alpha\gamma}\Delta_{\beta\zeta}. \qquad (\alpha, \beta, \gamma, \zeta = 1, \ldots, 4) \tag{12.8}$$

Inserting the preceding equation into (10.11) gives with the aid of (I.2.28), (I.2.30) and (I.2.33)

$$L^{*}_{\alpha\beta\gamma\zeta} = \eta(\Delta_{\alpha\gamma}\,\Delta_{\beta\zeta} + \Delta_{\alpha\zeta}\,\Delta_{\beta\gamma} - \tfrac{2}{3}\Delta_{\alpha\beta}\,\Delta_{\gamma\zeta}). \qquad (12.9)$$

$$(\alpha, \beta, \gamma, \zeta = 1, \ldots, 4)$$

Substitution of (12.7) into (9.8) gives with the help of (I.2.28) and (I.2.30)

$$\tilde{L}^{(j)(k)}_{\alpha\beta} = L^{(j)(k)}\,\Delta_{\alpha\beta}. \qquad (\alpha, \beta = 1, \ldots, 4;\ j, k = 0, 1, \ldots, n) \qquad (12.10)$$

If we take $\alpha = \beta = \gamma = \zeta$ we have from (10.27) and (12.9)

$$\eta(\mathbf{B}') = \eta(-\,\mathbf{B}'), \qquad (12.11)$$

i.e., η is an even function of $\mathbf{B}'$. The Onsager relations for heat conduction and diffusion read

$$L^{(j)(k)}\,(\mathbf{B}') = L^{(k)(j)}\,(-\,\mathbf{B}'), \qquad (j, k = 0, 1, \ldots, n) \qquad (12.12)$$

which follows from (I.2.29), (9.30) and (12.10).

In isotropic media the phenomenological equations for the scalar fluxes Π and $J_{(c)}$ are again given by (11.1) and (11.2). It is easily seen from symmetry considerations that in an isotropic medium $L_{(p)(c)}$ and $L_{(c)(p)}$ must be even functions of $\mathbf{B}'$. Hence, we have from (11.3)

$$L_{(p)(c)}\,(\mathbf{B}') = -\,L_{(c)(p)}\,(\mathbf{B}') \qquad (12.13)$$

for the Onsager relation for visco-chemical effects in an isotropic medium.

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CHAPTER V

THE ENERGY-MOMENTUM TENSOR OF THE MACROSCOPIC ELECTROMAGNETIC FIELD, THE MACROSCOPIC FORCES ACTING ON THE MATTER AND THE FIRST AND SECOND LAWS OF THERMODYNAMICS

§ 1. *Introduction.* In chapter IV of this thesis quantities occurred, in the expressions for the Lorentz force and the ponderomotive force and in the first and second laws of thermodynamics, which were not completely specified. In this chapter we shall give such explicit expressions for these quantities that the four-vector which represents the total force exerted by the electromagnetic field on the matter may be taken as the divergence of a four-dimensional tensor (the energy-momentum tensor of the electromagnetic field).

We shall deal with media which are isotropic as far as polarization and magnetization are concerned.

The equations for the electromagnetic field are given in § 2. In § 3 we give explicit expressions for the quantities mentioned above and we derive an expression for the energy-momentum tensor of the electromagnetic field. In § 4 we discuss the conservation laws for energy, momentum and angular momentum. The energy-momentum tensor of the electromagnetic field, derived in § 3, and the forces exerted by the electromagnetic field on the matter are discussed in § 5 and § 6 respectively. In § 7 we compare the energy-momentum tensor of the electromagnetic field, found in § 3, with the tensors of Abraham and Minkowski. The first and second laws of thermodynamics are discussed in § 8. In § 9 we discuss the indefiniteness of the energy-momentum tensors of the matter and of the electromagnetic field. Finally, in § 10 it is shown that Abraham's tensor leads to an equivalent formalism and that from the point of view of the developed theory this tensor is preferable to Minkowski's tensor.

§ 2. *The electromagnetic field.* The electromagnetic field in ponderable matter is described by the electric field vectors **E** and **D** and the magnetic field vectors **H** and **B**. The polarization vector, **P**, is defined by

$$\mathbf{P} \equiv \mathbf{D} - \mathbf{E}, \tag{2.1}$$

and the magnetization vector, **M**, by

$$\mathbf{M} \equiv \mathbf{B} - \mathbf{H}. \tag{2.2}$$

The Maxwell equations read

$$\operatorname{rot} \mathbf{H} - c^{-1} (\partial \mathbf{D}/\partial t) = c^{-1} \mathbf{j}, \tag{2.3}$$

$$\operatorname{div} \mathbf{D} = \varrho_{(el)}, \tag{2.4}$$

$$\operatorname{rot} \mathbf{E} + c^{-1} (\partial \mathbf{B}/\partial t) = 0, \tag{2.5}$$

$$\operatorname{div} \mathbf{B} = 0. \tag{2.6}$$

As is well-known E_1, E_2, E_3, B_1, B_2 and B_3 are the components of a tensor, $B_{\alpha\beta}$ $(\alpha, \beta = 1, \ldots, 4)$, defined by

$$B_{\alpha\beta} \equiv \begin{bmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{bmatrix}. \tag{2.7}$$

This tensor is antisymmetric, *i.e.*, it possesses the property

$$B_{\alpha\beta} = -B_{\beta\alpha}. \qquad (\alpha, \beta = 1, \ldots, 4) \tag{2.8}$$

Also D_1, D_2, D_3, H_1, H_2 and H_3 are the components of a tensor, $H_{\alpha\beta}$ $(\alpha, \beta = 1, \ldots, 4)$, given by

$$H_{\alpha\beta} \equiv \begin{bmatrix} 0 & H_3 & -H_2 & -iD_1 \\ -H_3 & 0 & H_1 & -iD_2 \\ H_2 & -H_1 & 0 & -iD_3 \\ iD_1 & iD_2 & iD_3 & 0 \end{bmatrix}. \tag{2.9}$$

This tensor is also antisymmetric. Hence,

$$H_{\alpha\beta} = -H_{\beta\alpha}. \qquad (\alpha, \beta = 1, \ldots, 4) \tag{2.10}$$

From (2.1), (2.2), (2.7) and (2.9) it follows that P_1, P_2, P_3, M_1, M_2 and M_3 are the components of a tensor, $M_{\alpha\beta}$ $(\alpha, \beta = 1, \ldots, 4)$, given by

$$M_{\alpha\beta} \equiv \begin{bmatrix} 0 & M_3 & -M_2 & iP_1 \\ -M_3 & 0 & M_1 & iP_2 \\ M_2 & -M_1 & 0 & iP_3 \\ -iP_1 & -iP_2 & -iP_3 & 0 \end{bmatrix}, \tag{2.11}$$

where also

$$M_{\alpha\beta} = -M_{\beta\alpha}. \qquad (\alpha, \beta = 1, \ldots, 4) \tag{2.12}$$

Using (2.7), (2.9) and (2.11) we can combine the equations (2.1) and (2.2) into the tensor equation

$$M_{\alpha\beta} = B_{\alpha\beta} - H_{\alpha\beta}. \qquad (\alpha, \beta = 1, .., 4) \qquad (2.13)$$

With the help of (III.2.14) *), (III.2.15) and (2.9) we can combine the Maxwell equations (2.3) and (2.4) into the form

$$\Sigma_{\beta=1}^{4} \partial H_{\alpha\beta}/\partial x_\beta = c^{-1} \Sigma_{k=1}^{n} e^{(k)} m_\alpha^{(k)}. \qquad (\alpha = 1, .., 4) \qquad (2.14)$$

In the same way we can write for the Maxwell equations (2.5) and (2.6)

$$\partial B_{\alpha\beta}/\partial x_\gamma + \partial B_{\beta\gamma}/\partial x_\alpha + \partial B_{\gamma\alpha}/\partial x_\beta = 0, \qquad (\alpha, \beta, \gamma = 1, .., 4) \qquad (2.15)$$

where we have used (2.7).

We now introduce some four-dimensional vectors and tensors which are useful for the discussions in this chapter. We define the tensors $B_{\alpha\beta}^*$ and $H_{\alpha\beta}^*$ $(\alpha, \beta = 1, .., 4)$ by the equations

$$B_{\alpha\beta}^* \equiv \Sigma_{\zeta,\xi=1}^{4} \Delta_{\alpha\zeta} B_{\zeta\xi} \Delta_{\xi\beta}, \qquad (\alpha, \beta = 1, .., 4) \qquad (2.16)$$

$$H_{\alpha\beta}^* \equiv \Sigma_{\zeta,\xi=1}^{4} \Delta_{\alpha\zeta} H_{\zeta\xi} \Delta_{\xi\beta}, \qquad (\alpha, \beta = 1, .., 4) \qquad (2.17)$$

and the four-vectors B_α^* and H_α^* $(\alpha = 1, .., 4)$ by the equations

$$B_\alpha^* \equiv \Sigma_{\beta=1}^{4} B_{\alpha\beta} u_\beta, \qquad (\alpha = 1, .., 4) \qquad (2.18)$$

$$H_\alpha^* \equiv \Sigma_{\beta=1}^{4} H_{\alpha\beta} u_\beta. \qquad (\alpha = 1, .., 4) \qquad (2.19)$$

Using (I.2.28), (2.8), (2.10) and the four preceding equations we derive

$$B_{\alpha\beta} = B_{\alpha\beta}^* - B_\alpha^* u_\beta + u_\alpha B_\beta^*, \qquad (\alpha, \beta = 1, .., 4) \qquad (2.20)$$

$$H_{\alpha\beta} = H_{\alpha\beta}^* - H_\alpha^* u_\beta + u_\alpha H_\beta^*. \qquad (\alpha, \beta = 1, .., 4) \qquad (2.21)$$

From (2.16) and (2.18) it follows with the help of (I.2.30) and (2.8) that

$$\Sigma_{\beta=1}^{4} B_{\alpha\beta}^* u_\beta = \Sigma_{\beta=1}^{4} u_\beta B_{\beta\alpha}^* = 0, \qquad (\alpha = 1, .., 4) \qquad (2.22)$$

$$\Sigma_{\alpha=1}^{4} u_\alpha B_\alpha^* = 0. \qquad (2.23)$$

In the same way we deduce

$$\Sigma_{\beta=1}^{4} H_{\alpha\beta}^* u_\beta = \Sigma_{\beta=1}^{4} u_\beta H_{\beta\alpha}^* = 0, \qquad (\alpha = 1, .., 4) \qquad (2.24)$$

$$\Sigma_{\alpha=1}^{4} u_\alpha H_\alpha^* = 0. \qquad (2.25)$$

Using (I.2.29), (2.8) and (2.10) it follows from (2.16) and (2.17) that

$$B_{\alpha\beta}^* = - B_{\beta\alpha}^*, \qquad (\alpha, \beta = 1, .., 4) \qquad (2.26)$$

$$H_{\alpha\beta}^* = - H_{\beta\alpha}^*. \qquad (\alpha, \beta = 1, .., 4) \qquad (2.27)$$

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*) Equation (2.14) of chapter III of this thesis will be indicated as (III.2.14) and in the same way we denote the other equations of the chapters I, II, III and IV.

From (2.16) we derive with the help of (I.2.32) and (2.7)

$$B'^{*}_{\alpha\beta} = \begin{bmatrix} 0 & B'_3 & -B'_2 & 0 \\ -B'_3 & 0 & B'_1 & 0 \\ B'_2 & -B'_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.28}$$

and using (I.2.32) and (2.9) we find from (2.17)

$$H'^{*}_{\alpha\beta} = \begin{bmatrix} 0 & H'_3 & -H'_2 & 0 \\ -H'_3 & 0 & H'_1 & 0 \\ H'_2 & -H'_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.29}$$

Using (I.2.16), (2.7) and (2.9) it follows from (2.18) and (2.19) that

$$B'^{*}_{\alpha} = E'_{\alpha} \qquad (\alpha = 1, 2, 3); \quad B'^{*}_4 = 0, \tag{2.30}$$

$$H'^{*}_{\alpha} = D'_{\alpha} \qquad (\alpha = 1, 2, 3); \quad H'^{*}_4 = 0. \tag{2.31}$$

For media which are isotropic, as far as polarization and magnetization are concerned, we have the constitutive equations

$$\mathbf{H}' = \mu^{-1}\,\mathbf{B}', \tag{2.32}$$

$$\mathbf{D}' = \varepsilon\mathbf{E}', \tag{2.33}$$

where the Lorentz invariant quantities μ and ε are the magnetic permeability and the dielectric constant respectively. From the six preceding equations we find

$$H'^{*}_{\alpha\beta} = \mu^{-1}\,B'^{*}_{\alpha\beta}, \qquad (\alpha, \beta = 1, \ldots, 4) \tag{2.34}$$

$$H'^{*}_{\alpha} = \varepsilon B'^{*}_{\alpha}, \qquad (\alpha = 1, \ldots, 4) \tag{2.35}$$

and since these tensor equations hold for an observer in $\mathrm{B}_{\mathbf{r},t}$ at position $\mathbf{r}'$ and at time t' they also hold at position $\mathbf{r}$ and at time t for an observer in A. Hence,

$$H^{*}_{\alpha\beta} = \mu^{-1}\,B^{*}_{\alpha\beta}, \qquad (\alpha, \beta = 1, \ldots, 4) \tag{2.36}$$

$$H^{*}_{\alpha} = \varepsilon B^{*}_{\alpha}. \qquad (\alpha = 1, \ldots, 4) \tag{2.37}$$

A well-known different form for (2.36) reads

$$H_{\alpha\beta}u_{\gamma} + H_{\gamma\alpha}u_{\beta} + H_{\beta\gamma}u_{\alpha} = \mu^{-1}\,(B_{\alpha\beta}\,u_{\gamma} + B_{\gamma\alpha}\,u_{\beta} + B_{\beta\gamma}\,u_{\alpha}). \tag{2.38}$$

$$(\alpha, \beta, \gamma = 1, \ldots, 4)$$

With the help of (I.2.16), (2.7) and (2.9) one easily verifies that this relation holds for an observer in $\mathrm{B}_{\mathbf{r},t}$ at position $\mathbf{r}'$ and at time t'. Hence, this equation also holds at position $\mathbf{r}$ and at time t for an observer in A. From (2.37) and

(2.38) we have with the help of (I.2.11), (2.7), (2.9), (2.18) and (2.19)

$$\mathbf{H} - c^{-1} (\mathbf{v} \wedge \mathbf{D}) = \mu^{-1} \{\mathbf{B} - c^{-1} (\mathbf{v} \wedge \mathbf{E})\}, \tag{2.39}$$

$$\mathbf{D} + c^{-1} (\mathbf{v} \wedge \mathbf{H}) = \varepsilon\{\mathbf{E} + c^{-1} (\mathbf{v} \wedge \mathbf{B})\}. \tag{2.40}$$

By solving these equations for the vectors **H** and **D** we obtain

$$\mathbf{H} = (1 - \mathbf{v}^2/c^2)^{-1} \mu^{-1} [(1 - \varepsilon\mu\mathbf{v}^2/c^2)\mathbf{B} + \\ + (\varepsilon\mu - 1)\{c^{-1} (\mathbf{v} \wedge \mathbf{E}) + c^{-2} (\mathbf{v}\cdot\mathbf{B})\mathbf{v}\}], \tag{2.41}$$

$$\mathbf{D} = (1 - \mathbf{v}^2/c^2)^{-1} \varepsilon[(1 - \varepsilon^{-1} \mu^{-1} \mathbf{v}^2/c^2)\mathbf{E} + \\ + (1 - \varepsilon^{-1} \mu^{-1})\{c^{-1} (\mathbf{v} \wedge \mathbf{B}) - c^{-2} (\mathbf{v}\cdot\mathbf{E})\mathbf{v}\}], \tag{2.42}$$

which are the three-dimensional tensor forms for the relativistic constitutive equations.

§ 3. *Derivation of the energy-momentum tensor of the electromagnetic field.* As in chapter III we shall assign an energy-momentum tensor, $W_{(f)\alpha\beta}(\alpha, \beta = 1, \ldots, 4)$, to the electromagnetic field such that

$$k_\alpha = - \Sigma_{\beta=1}^4 \, \partial W_{(f)\alpha\beta}/\partial x_\beta, \qquad (\alpha = 1, \ldots, 4) \tag{3.1}$$

where $k_\alpha(\alpha = 1, \ldots, 4)$ is the four-vector given by (IV.5.3). In this section we shall derive an expression for $W_{(f)\alpha\beta}$.

In chapter IV we did not specify the tensors $Z_{\alpha\beta}$ and $G_{\alpha\beta}(\alpha, \beta = 1, \ldots, 4)$ and the four-vector $\Phi_\alpha(\alpha = 1, \ldots, 4)$ which occurred in the expression (IV.8.2) for the ponderomotive force and the tensor $F_{\alpha\beta}^{(j)}(\alpha, \beta = 1, \ldots, 4; j = 1, \ldots, n)$ which occurred in the expression (IV.5.4) for the Lorentz force. We shall now choose these quantities such that an explicit expression may be obtained for the tensor $W_{(f)\alpha\beta}$ which is closely related to the expressions of Abraham and Minkowski for the energy-momentum tensor of the electromagnetic field (Cf. § 7). We shall make the choices (Cf. (IV.3.16) and (IV.3.23), (IV.3.15) and (IV.3.24), (IV.3.15) and (IV.5.5))

$$G_{\alpha\beta} = v' M_{\alpha\beta}, \qquad (\alpha, \beta = 1, \ldots, 4) \tag{3.2}$$

$$Z_{\alpha\beta} = B_{\alpha\beta}, \qquad (\alpha, \beta = 1, \ldots, 4) \tag{3.3}$$

$$F_{\alpha\beta}^{(j)} = B_{\alpha\beta}, \qquad (\alpha, \beta = 1, \ldots, 4; j = 1, \ldots, n) \tag{3.4}$$

and

$$\Phi_\alpha = v' \, \Sigma_{\beta,\gamma,\zeta=1}^4 \, u_\beta (B_{\beta\gamma} M_{\gamma\zeta} - M_{\beta\gamma} B_{\gamma\zeta}) \, (\partial u_\zeta/\partial x_\alpha) + \\ + c^{-1} \, \mathrm{D} \{v' \, \Sigma_{\beta,\gamma=1}^4 \, u_\beta (B_{\beta\gamma} \, M_{\gamma\alpha} - M_{\beta\gamma} \, B_{\gamma\alpha})\}. \qquad (\alpha = 1, \ldots, 4) \tag{3.5}$$

Using (I.2.25), (2.8) and (2.12) we have from (3.5)

$$\Sigma_{\alpha=1}^4 \, u_\alpha \, \Phi_\alpha = 0, \tag{3.6}$$

and from this equation we have with the help of (I.2.28)

$$\Sigma_{\beta=1}^{4} \Delta_{\alpha\beta}\, \Phi_\beta = \Phi_\alpha. \qquad (\alpha = 1, \ldots, 4) \tag{3.7}$$

Inserting (3.4) into (IV.5.4) gives for the four-vector representing the Lorentz force

$$K_\alpha^{(j)} = e^{(j)}\, (c\varrho_{(0)}^{(j)})^{-1}\, \Sigma_{\beta=1}^{4} B_{\alpha\beta}\, m_\beta^{(j)}. \qquad (\alpha = 1, \ldots, 4;\ j = 1, \ldots, n) \tag{3.8}$$

Substitution of (3.2), (3.3) and (3.5) into (IV.8.2) gives with the help of (I.2.3) and (3.7) for the ponderomotive force

$$\begin{aligned} k_{(P)\alpha} = {} & \tfrac{1}{2}\, \Sigma_{\beta,\gamma=1}^{4} M_{\beta\gamma}(\partial B_{\beta\gamma}/\partial x_\alpha) + \\ & + c^{-1}\, \varrho'\, \mathrm{D}\, \{v'\, \Sigma_{\beta,\gamma=1}^{4} u_\beta (B_{\beta\gamma} M_{\gamma\alpha} - M_{\beta\gamma} B_{\gamma\alpha})\}. \qquad (\alpha = 1, \ldots, 4) \end{aligned} \tag{3.9}$$

From (IV.5.3) and the two preceding equations we have

$$\begin{aligned} k_\alpha = {} & c^{-1}\, \Sigma_{j=1}^{n} e^{(j)}\, \Sigma_{\gamma=1}^{4} B_{\alpha\gamma}\, m_\gamma^{(j)} + \tfrac{1}{2}\, \Sigma_{\beta,\gamma=1}^{4} M_{\beta\gamma}(\partial B_{\beta\gamma}/\partial x_\alpha) + \\ & + c^{-1}\, \varrho' \mathrm{D}\, \{v'\, \Sigma_{\beta,\gamma=1}^{4} u_\beta (B_{\beta\gamma} M_{\gamma\alpha} - M_{\beta\gamma} B_{\gamma\alpha})\}. \qquad (\alpha = 1, \ldots, 4) \end{aligned} \tag{3.10}$$

Using (I.2.10), (I.2.14), (I.2.25), (2.13) and (2.14) we can write for the above equation.

$$\begin{aligned} k_\alpha = {} & \Sigma_{\beta,\gamma=1}^{4} B_{\alpha\gamma}(\partial H_{\gamma\beta}/\partial x_\beta) - \tfrac{1}{2}\, \Sigma_{\beta,\gamma=1}^{4} H_{\beta\gamma}(\partial B_{\beta\gamma}/\partial x_\alpha) - \\ & - c^{-1}\, \Sigma_{\beta=1}^{4} m_\beta (\partial/\partial x_\beta)\, \{v'\, \Sigma_{\gamma,\zeta=1}^{4} u_\zeta (B_{\zeta\gamma} H_{\gamma\alpha} - H_{\zeta\gamma} B_{\gamma\alpha})\} + \\ & + \tfrac{1}{4}\, \Sigma_{\beta,\gamma=1}^{4} \{\partial (B_{\beta\gamma})^2/\partial x_\alpha\}, \qquad (\alpha = 1, \ldots, 4) \end{aligned} \tag{3.11}$$

and from this equation it follows with the help of (IV.6.3) and (2.15) that

$$\begin{aligned} k_\alpha = {} & \Sigma_{\beta,\gamma=1}^{4} B_{\alpha\gamma}(\partial H_{\gamma\beta}/\partial x_\beta) + \tfrac{1}{2}\, \Sigma_{\beta,\gamma=1}^{4} H_{\beta\gamma}\, \{(\partial B_{\alpha\beta}/\partial x_\gamma) + (\partial B_{\gamma\alpha}/\partial x_\beta)\} - \\ & - c^{-1}\, \Sigma_{\beta=1}^{4} (\partial/\partial x_\beta)\, \{v' m_\beta\, \Sigma_{\gamma,\zeta=1}^{4} u_\zeta (B_{\zeta\gamma} H_{\gamma\alpha} - H_{\zeta\gamma} B_{\gamma\alpha})\} + \\ & + \tfrac{1}{4}\, \Sigma_{\beta,\gamma=1}^{4} \{\partial (B_{\beta\gamma})^2/\partial x_\alpha\}. \qquad (\alpha = 1, \ldots, 4) \end{aligned} \tag{3.12}$$

Finally, we find from the last equation with the help of (I.2.10), (I.2.15), (2.8) and (2.10)

$$\begin{aligned} k_\alpha = {} & \Sigma_{\beta=1}^{4} (\partial/\partial x_\beta)\, \{\Sigma_{\gamma=1}^{4} B_{\alpha\gamma} H_{\gamma\beta} + \tfrac{1}{4}\, \delta_{\alpha\beta}\, \Sigma_{\gamma,\zeta=1}^{4} (B_{\gamma\zeta})^2\} - \\ & - \Sigma_{\beta=1}^{4} (\partial/\partial x_\beta)\, \{u_\beta\, \Sigma_{\gamma,\zeta=1}^{4} u_\zeta (B_{\zeta\gamma} H_{\gamma\alpha} - H_{\zeta\gamma} B_{\gamma\alpha})\}. \qquad (\alpha = 1, \ldots, 4) \end{aligned} \tag{3.13}$$

From (3.1) and (3.13) it follows that we may take

$$\begin{aligned} W_{(f)\alpha\beta} = {} & -\, \Sigma_{\gamma=1}^{4} B_{\alpha\gamma} H_{\gamma\beta} - \tfrac{1}{4}\, \delta_{\alpha\beta}\, \Sigma_{\gamma,\zeta=1}^{4} (B_{\gamma\zeta})^2 + \\ & + \{\Sigma_{\gamma,\zeta=1}^{4} u_\zeta (B_{\zeta\gamma} H_{\gamma\alpha} - H_{\zeta\gamma} B_{\gamma\alpha})\}\, u_\beta \qquad (\alpha, \beta = 1, \ldots, 4) \end{aligned} \tag{3.14}$$

as an explicit expression for the energy-momentum tensor of the electromagnetic field.

§ 4. *The conservation laws for energy, momentum and angular momentum.* We define the tensor $W_{(t)\alpha\beta}$ by

$$W_{(t)\alpha\beta} \equiv W_{\alpha\beta} + W_{(f)\alpha\beta}. \qquad (\alpha, \beta = 1, \ldots, 4) \qquad (4.1)$$

We can consider this tensor as the energy-momentum tensor of the system (electromagnetic field and matter together).

From (IV.5.2), (3.1) and the preceding equation we have

$$\Sigma_{\beta=1}^{4} \, \partial W_{(t)\alpha\beta}/\partial x_\beta = 0. \qquad (\alpha = 1, \ldots, 4) \qquad (4.2)$$

For $\alpha = 1, 2$ and 3 it follows from this equation that the total momentum of the whole system (*i.e.*, the density of momentum of the matter and the field together, integrated over the whole volume of the system) is conserved. For $\alpha = 4$ it follows from (4.2) that the total energy of the whole system (*i.e.*, the density of energy of the matter and the field together, integrated over the whole volume of the system) is conserved.

To be able to discuss the law of conservation of angular momentum we shall first show that the tensor $W_{(f)\alpha\beta}$ $(\alpha, \beta = 1, \ldots, 4)$ is symmetric for media which are isotropic as far as polarization and magnetization are concerned. Using (I.2.28), (2.8) and (2.10) we have from (3.14)

$$W_{(f)\alpha\beta} - W_{(f)\beta\alpha} = - \Sigma_{\gamma,\zeta,\xi=1}^{4} \Delta_{\alpha\zeta}(B_{\zeta\gamma} H_{\gamma\xi} - H_{\zeta\gamma} B_{\gamma\xi}) \Delta_{\xi\beta}. \quad (\alpha, \beta = 1, \ldots, 4) \quad (4.3)$$

With the help of (I.2.12), (2.20), (2.21), (2.22), (2.23), (2.24) and (2.25) we can derive that

$$\begin{aligned} \Sigma_{\gamma=1}^{4}(B_{\zeta\gamma} H_{\gamma\xi} - H_{\zeta\gamma} B_{\gamma\xi}) &= \Sigma_{\gamma=1}^{4}(B^*_{\zeta\gamma} H^*_{\gamma\xi} - H^*_{\zeta\gamma} B^*_{\gamma\xi}) - \\ &- \{\Sigma_{\gamma=1}^{4} (B^*_{\zeta\gamma} H^*_{\gamma} - H^*_{\zeta\gamma} B^*_{\gamma})\} u_\xi + B^*_\zeta H^*_\xi - H^*_\zeta B^*_\xi + \\ &+ u_\zeta \{\Sigma_{\gamma=1}^{4} (B^*_\gamma H^*_{\gamma\xi} - H^*_\gamma B^*_{\gamma\xi})\}. \qquad (\zeta, \xi = 1, \ldots, 4) \end{aligned} \qquad (4.4)$$

Inserting (4.4) into (4.3) gives with the help of (I.2.28), (I.2.30), (2.22), (2.23), (2.24) and (2.25)

$$\begin{aligned} W_{(f)\alpha\beta} - W_{(f)\beta\alpha} &= \Sigma_{\gamma=1}^{4} (B^*_{\alpha\gamma} H^*_{\gamma\beta} - H^*_{\alpha\gamma} B^*_{\gamma\beta}) + \\ &+ B^*_\alpha H^*_\beta - H^*_\alpha B^*_\beta. \qquad (\alpha, \beta = 1, \ldots, 4) \end{aligned} \qquad (4.5)$$

We now introduce into (4.5) the relations (2.36) and (2.37) which hold for media which are isotropic as far as polarization and magnetization are concerned. This gives

$$W_{(f)\alpha\beta} = W_{(f)\beta\alpha}. \qquad (\alpha, \beta = 1, \ldots, 4) \qquad (4.6)$$

In our considerations we assumed that the energy-momentum tensor of the matter, $W_{\alpha\beta}$ $(\alpha, \beta = 1, \ldots, 4)$, is symmetric (Cf. § 5 of chapter IV). Hence,

$$W_{\alpha\beta} = W_{\beta\alpha}. \qquad (\alpha, \beta = 1, \ldots, 4) \qquad (4.7)$$

From (4.1) and the two preceding equations we have

$$W_{(t)\alpha\beta} = W_{(t)\beta\alpha}, \qquad (\alpha, \beta = 1, \ldots, 4) \tag{4.8}$$

i.e., the energy-momentum tensor of the system is symmetric.

We introduce the three-dimensional vector $\mathfrak{g}_{(f)}$ of which the components are given by

$$g_{(f)\alpha} = (ic)^{-1} W_{(f)\alpha 4}. \qquad (\alpha = 1, 2, 3) \tag{4.9}$$

This vector gives the density of momentum of the electromagnetic field. The density of momentum of the system, $\mathfrak{g}_{(t)}$, is given by

$$\mathfrak{g}_{(t)} = \mathfrak{g} + \mathfrak{g}_{(f)}, \tag{4.10}$$

where $\mathfrak{g}$ is the density of momentum of the matter, given by (I.3.2).

We now consider a finite system. Using (4.7) we then can derive [1])

$$(\mathrm{d}/\mathrm{d}t) \int \{\mathbf{r} \wedge \mathfrak{g}(\mathbf{r}, t)\} \mathrm{d}V = 0, \tag{4.11}$$

i.e., the total macroscopic angular momentum of all the matter within the whole system is conserved. In the same way we have from (4.6)

$$(\mathrm{d}/\mathrm{d}t) \int \{\mathbf{r} \wedge \mathfrak{g}_{(f)} (\mathbf{r}, t)\} \mathrm{d}V = 0, \tag{4.12}$$

i.e., also the total macroscopic angular momentum of the entire electromagnetic field within the whole system is conserved. From the three preceding equations we find

$$(\mathrm{d}/\mathrm{d}t) \int \{\mathbf{r} \wedge \mathfrak{g}_{(t)} (\mathbf{r}, t)\} \mathrm{d}V = 0, \tag{4.13}$$

which shows that the total macroscopic angular momentum of the whole system is conserved.

§ 5. *Further discussion of the energy-momentum tensor.* In this section we shall express the components of the energy-momentum tensor of the electromagnetic field with the help of three-dimensional vectors. For that purpose we first deduce from (2.7) and (2.9)

$$\Sigma_{\gamma=1}^{4} B_{\alpha\gamma} H_{\gamma\beta} = -(\mathbf{B}\cdot\mathbf{H})\, \delta_{\alpha\beta} + H_\alpha B_\beta + E_\alpha D_\beta, \qquad (\alpha, \beta = 1, 2, 3) \tag{5.1}$$

$$\Sigma_{\gamma=1}^{4} B_{\alpha\gamma} H_{\gamma 4} = i(\mathbf{B} \wedge \mathbf{D})_\alpha, \qquad (\alpha = 1, 2, 3) \tag{5.2}$$

$$\Sigma_{\gamma=1}^{4} B_{4\gamma} H_{\gamma\beta} = -i(\mathbf{E} \wedge \mathbf{H})_\beta, \qquad (\beta = 1, 2, 3) \tag{5.3}$$

$$\Sigma_{\gamma=1}^{4} B_{4\gamma} H_{\gamma 4} = \mathbf{E}\cdot\mathbf{D}. \tag{5.4}$$

Using (I.2.11), (2.8), (2.10), (5.1), (5.2) and (5.3) we derive

$$\Sigma_{\gamma,\zeta=1}^{4} u_\zeta (B_{\zeta\gamma} H_{\gamma\alpha} - H_{\zeta\gamma} B_{\gamma\alpha}) =$$

$$(c^2 - \mathbf{v}^2)^{-\frac{1}{2}} [\{\mathbf{v} \wedge (\mathbf{B} \wedge \mathbf{H})\} - \{\mathbf{v} \wedge (\mathbf{E} \wedge \mathbf{D})\} + c(\mathbf{E} \wedge \mathbf{H}) + c(\mathbf{B} \wedge \mathbf{D})]_\alpha, \tag{5.5}$$

$$(\alpha = 1, 2, 3)$$

and using (I.2.11), (2.8), (2.10), (5.2) and (5.3) we deduce

$$\Sigma_{\gamma,\zeta=1}^{4} u_\zeta (B_{\zeta\gamma} H_{\gamma 4} - H_{\zeta\gamma} B_{\gamma 4}) = i(c^2 - \mathbf{v}^2)^{-\frac{1}{2}} \{\mathbf{v}\cdot(\mathbf{B} \wedge \mathbf{D}) + \mathbf{v}\cdot(\mathbf{E} \wedge \mathbf{H})\}. \tag{5.6}$$

Inserting (I.2.11), (2.7), (5.1) and (5.5) into (3.14) gives for $\alpha, \beta = 1, 2, 3$

$$W_{(f)\alpha\beta} = \{(\mathbf{B}\cdot\mathbf{H}) + \tfrac{1}{2}\mathbf{E}^2 - \tfrac{1}{2}\mathbf{B}^2\}\,\delta_{\alpha\beta} - H_\alpha B_\beta - E_\alpha D_\beta + $$
$$+ (c^2-\mathbf{v}^2)^{-1}[\{\mathbf{v} \wedge (\mathbf{B} \wedge \mathbf{H})\} - \{\mathbf{v} \wedge (\mathbf{E} \wedge \mathbf{D})\} + c(\mathbf{E} \wedge \mathbf{H}) + c(\mathbf{B} \wedge \mathbf{D})]_\alpha v_\beta. \tag{5.7}$$
$$(\alpha, \beta = 1, 2, 3)$$

Substitution of (I.2.11), (5.3) and (5.6) into (3.14) gives for $\alpha = 4$ and $\beta = 1, 2, 3$,

$$W_{(f)4\beta} = i(\mathbf{E} \wedge \mathbf{H})_\beta + i(c^2 - \mathbf{v}^2)^{-1} \{\mathbf{v}\cdot(\mathbf{B} \wedge \mathbf{D}) + \mathbf{v}\cdot(\mathbf{E} \wedge \mathbf{H})\} v_\beta. \tag{5.8}$$
$$(\beta = 1, 2, 3)$$

Inserting (I.2.11), (2.7), (5.4) and (5.6) into (3.14) gives with the help of (2.1) for $\alpha = \beta = 4$,

$$W_{(f)44} = - \{\tfrac{1}{2}\mathbf{E}^2 + \tfrac{1}{2}\mathbf{B}^2 + (\mathbf{E}\cdot\mathbf{P})\} + $$
$$+ c(c^2 - \mathbf{v}^2)^{-1} \{\mathbf{v}\cdot(\mathbf{D} \wedge \mathbf{B}) + \mathbf{v}\cdot(\mathbf{H} \wedge \mathbf{E})\}. \tag{5.9}$$

The Maxwell stress tensor is given by (5.7). On account of (4.6) this three-dimensional tensor is symmetric. The components of the Poynting vector $\mathbf{J}_{(P)}$, which gives the density of the energy flow of the electromagnetic field, are given by

$$J_{(P)\beta} = (c/i) W_{(f)4\beta}. \qquad (\beta = 1, 2, 3) \tag{5.10}$$

From (5.8) and the preceding equation we have

$$\mathbf{J}_{(P)} = c(\mathbf{E} \wedge \mathbf{H}) + c(c^2 - \mathbf{v}^2)^{-1} \{\mathbf{v}\cdot(\mathbf{B} \wedge \mathbf{D}) + \mathbf{v}\cdot(\mathbf{E} \wedge \mathbf{H})\}\mathbf{v}. \tag{5.11}$$

It follows from (4.6), (4.9) and (5.10) that

$$\mathbf{g}_{(f)} = c^{-2}\, \mathbf{J}_{(P)}. \tag{5.12}$$

From the two preceding equations we get

$$\mathbf{g}_{(f)} = c^{-1}(\mathbf{E} \wedge \mathbf{H}) + c^{-1} (c^2 - \mathbf{v}^2)^{-1} \{\mathbf{v}\cdot(\mathbf{B} \wedge \mathbf{D}) + \mathbf{v}\cdot(\mathbf{E} \wedge \mathbf{H})\}\mathbf{v}. \tag{5.13}$$

The density of energy, $e_{(f)(v)}$, of the electromagnetic field is given by

$$e_{(f)(v)} = - W_{(f)44}. \tag{5.14}$$

From (5.9) and the preceding equation we find

$$e_{(f)(v)} = \tfrac{1}{2}\mathbf{E}^2 + \tfrac{1}{2}\mathbf{B}^2 + (\mathbf{E}\cdot\mathbf{P}) + $$
$$+ c(c^2 - \mathbf{v}^2)^{-1} \{\mathbf{v}\cdot(\mathbf{B} \wedge \mathbf{D}) + \mathbf{v}\cdot(\mathbf{E} \wedge \mathbf{H})\}. \tag{5.15}$$

Using (2.8), (2.10) and (2.13) it follows from (3.14) that

$$\Sigma_{\alpha=1}^{4} W_{(f)\alpha\alpha} = - \Sigma_{\beta,\gamma=1}^{4} B_{\beta\gamma} M_{\beta\gamma}. \tag{5.16}$$

Substitution of (2.7) and (2.11) into this equation gives

$$\Sigma_{\alpha=1}^{4} W_{(f)\alpha\alpha} = - 2\{(\mathbf{B}\cdot\mathbf{M}) + (\mathbf{E}\cdot\mathbf{P})\} \tag{5.17}$$

for the trace of the energy-momentum tensor of the electromagnetic field.

§ 6. *Discussion of the forces acting on the matter.* We shall also express the forces acting on the matter with the help of three-dimensional vectors.

Using (I.2.1), (2.7) and (3.8) we find from (I.4.4) for the Lorentz force, $\mathbf{F}^{(k)}$, per unit mass, acting on component k

$$\mathbf{F}^{(k)} = e^{(k)} \{\mathbf{E} + c^{-1} (\mathbf{v}^{(k)} \wedge \mathbf{B})\}. \qquad (k = 1, \ldots, n) \tag{6.1}$$

We have from (I.2.1), (I.2.27), (2.7) and (3.8)

$$K_4^{(k)} = ie^{(k)} \varrho^{(k)} (c\varrho_{(0)}^{(k)})^{-1} (\mathbf{v}^{(k)}\cdot\mathbf{E}) =$$
$$= ic^{-1} e^{(k)} \{1 - (\mathbf{v}^{(k)})^2/c^2\} (\mathbf{v}^{(k)}\cdot\mathbf{E}). \tag{6.2}$$
$$(k = 1, \ldots, n)$$

It follows from the two preceding equations that the quantity $(c/i)\varrho_{(0)}^{(k)} K_4^{(k)}$ can be interpreted as the work done per unit time and per unit volume on component k by the Lorentz force.

We shall now express the components of the four-vector, $k_{(P)\alpha}$, representing the ponderomotive force, in terms of three-dimensional vectors. For that purpose we first remark that we have from (2.7) and (2.11)

$$\Sigma_{\gamma=1}^{4} B_{\alpha\gamma} M_{\gamma\beta} = - (\mathbf{B}\cdot\mathbf{M})\delta_{\alpha\beta} + M_\alpha B_\beta - E_\alpha P_\beta, \quad (\alpha, \beta = 1, 2, 3) \tag{6.3}$$

$$\Sigma_{\gamma=1}^{4} B_{\alpha\gamma} M_{\gamma 4} = - i(\mathbf{B} \wedge \mathbf{P})_\alpha, \quad (\alpha = 1, 2, 3) \tag{6.4}$$

$$\Sigma_{\gamma=1}^{4} B_{4\gamma} M_{\gamma\beta} = - i(\mathbf{E} \wedge \mathbf{M})_\beta, \quad (\beta = 1, 2, 3) \tag{6.5}$$

$$\Sigma_{\gamma=1}^{4} B_{4\gamma} M_{\gamma 4} = - (\mathbf{E}\cdot\mathbf{P}). \tag{6.6}$$

With the help of (I.2.11), (2.8), (2.12), (6.3), (6.4) and (6.5) we find

$$\Sigma_{\beta,\gamma=1}^{4} u_\beta (B_{\beta\gamma} M_{\gamma\alpha} - M_{\beta\gamma} B_{\gamma\alpha}) =$$
$$= (c^2 - \mathbf{v}^2)^{-\frac{1}{2}} [\{\mathbf{v} \wedge (\mathbf{B} \wedge \mathbf{M})\} + \{\mathbf{v} \wedge (\mathbf{E} \wedge \mathbf{P})\} + c(\mathbf{E} \wedge \mathbf{M}) - c(\mathbf{B} \wedge \mathbf{P})]_\alpha, \tag{6.7}$$
$$(\alpha = 1, 2, 3)$$

and

$$\Sigma_{\beta,\gamma=1}^{4} u_\beta (B_{\beta\gamma} M_{\gamma 4} - M_{\beta\gamma} B_{\gamma 4}) =$$
$$i(c^2 - \mathbf{v}^2)^{-\frac{1}{2}} \{\mathbf{v}\cdot(\mathbf{E} \wedge \mathbf{M}) - \mathbf{v}\cdot(\mathbf{B} \wedge \mathbf{P})\}. \tag{6.8}$$

The first three components of the four-vector $k_{(P)\alpha}$ are the components of a three-dimensional vector $\mathbf{k}_{(P)}$. Inserting (I.2.25), (2.7), (2.11) and (6.7) into (3.9) gives with the help of (I.2.11), (I.2.14) and (II.3.10) for $\mathbf{k}_{(P)}$

$$\mathbf{k}_{(P)} = (\text{Grad}\,\mathbf{E})\cdot\mathbf{P} + (\text{Grad}\,\mathbf{B})\cdot\mathbf{M} + \\ + c^{-1}\varrho(\mathrm{d}/\mathrm{d}t)\,\{v'(c^2 - \mathbf{v}^2)^{-\frac{1}{2}}\,[\{\mathbf{v}\wedge(\mathbf{B}\wedge\mathbf{M})\} + \{\mathbf{v}\wedge(\mathbf{E}\wedge\mathbf{P})\} + \\ + c(\mathbf{E}\wedge\mathbf{M}) - c(\mathbf{B}\wedge\mathbf{P})]\}, \tag{6.9}$$

where

$$(\text{Grad}\,\mathbf{E})\cdot\mathbf{P} \equiv \Sigma_{\alpha=1}^{3}\,\{\partial\mathbf{E}/\partial x_\alpha)\cdot\mathbf{P}\}\mathbf{i}_\alpha, \tag{6.10}$$

and

$$(\text{Grad}\,\mathbf{B})\cdot\mathbf{M} \equiv \Sigma_{\alpha=1}^{3}\,\{(\partial\mathbf{B}/\partial x_\alpha)\cdot\mathbf{M}\}\mathbf{i}_\alpha. \tag{6.11}$$

We can interpret $\mathbf{k}_{(P)}$ as the ponderomotive force per unit volume.

We now define

$$\mathbf{k}_{(\mathbf{EP})} \equiv (\text{Grad}\,\mathbf{E})\cdot\mathbf{P} + c^{-1}\varrho(\mathrm{d}/\mathrm{d}t)[v'(c^2 - \mathbf{v}^2)^{-\frac{1}{2}}\,\{\mathbf{v}\wedge(\mathbf{E}\wedge\mathbf{P})\}], \tag{6.12}$$

$$\mathbf{k}_{(\mathbf{BM})} \equiv (\text{Grad}\,\mathbf{B})\cdot\mathbf{M} + c^{-1}\varrho(\mathrm{d}/\mathrm{d}t)\,[v'(c^2 - \mathbf{v}^2)^{-\frac{1}{2}}\{\mathbf{v}\wedge(\mathbf{B}\wedge\mathbf{M})\}], \tag{6.13}$$

$$\mathbf{k}_{(\mathbf{EM})} \equiv c^{-1}\varrho(\mathrm{d}/\mathrm{d}t)\,\{v'(1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}}\,(\mathbf{E}\wedge\mathbf{M})\}, \tag{6.14}$$

$$\mathbf{k}_{(\mathbf{BP})} \equiv -\,c^{-1}\varrho(\mathrm{d}/\mathrm{d}t)\,\{v'(1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}}\,(\mathbf{B}\wedge\mathbf{P})\}. \tag{6.15}$$

From (6.9) and the four preceding equations we have

$$\mathbf{k}_{(P)} = \mathbf{k}_{(\mathbf{EP})} + \mathbf{k}_{(\mathbf{BM})} + \mathbf{k}_{(\mathbf{EM})} + \mathbf{k}_{(\mathbf{BP})}. \tag{6.16}$$

We can interpret $\mathbf{k}_{(\mathbf{EP})}$ and $\mathbf{k}_{(\mathbf{EM})}$ as the forces which the electric field exerts on the medium in consequence of the polarization of the matter and the magnetization of the matter respectively. In the same way we can interpret $\mathbf{k}_{(\mathbf{BM})}$ and $\mathbf{k}_{(\mathbf{BP})}$ as the forces which the magnetic field exerts on the medium as a consequence of the magnetization and the polarization of the matter respectively. The term $(\text{Grad}\,\mathbf{E})\cdot\mathbf{P}$ reduces, in case the magnetic field is constant (*i.e.*, $\partial\mathbf{B}/\partial t = 0$), to the Kelvin form for the force which the electric field exerts on polarized matter [2]). The term $(\text{Grad}\,\mathbf{B})\cdot\mathbf{M}$ is the magnetic analog of the term $(\text{Grad}\,\mathbf{E})\cdot\mathbf{P}$. Only terms of this kind are taken into consideration by Smith-White [3]) and by Mazur and Prigogine [2]).

Substitution of (I.2.25), (2.7), (2.11) and (6.8) into (3.9) gives with the help of (I.2.11), (I.2.14) and (II.3.10)

$$k_{(P)4} = -\,(i/c)\,\{\mathbf{P}\cdot(\partial\mathbf{E}/\partial t) + \mathbf{M}\cdot(\partial\mathbf{B}/\partial t)\} + \\ + (i/c)\varrho(\mathrm{d}/\mathrm{d}t)[v'(c^2 - \mathbf{v}^2)^{-\frac{1}{2}}\,\{\mathbf{v}\cdot(\mathbf{E}\wedge\mathbf{M}) - \mathbf{v}\cdot(\mathbf{B}\wedge\mathbf{P})\}]. \tag{6.17}$$

The first three components of k_α $(\alpha = 1, \ldots, 4)$ are the components of a three-dimensional vector $\mathbf{k}$. From (I.4.4) and (IV.5.3) we have

$$\mathbf{k} = \Sigma_{j=1}^{n}\,\varrho^{(j)}\,\mathbf{F}^{(j)} + \mathbf{k}_{(P)}, \tag{6.18}$$

where **k** is the total macroscopic density of force which is exerted by the electromagnetic field on the matter.

Further, it follows from (IV.5.3) that

$$k_4 = \Sigma_{j=1}^{n} \varrho_{(0)}^{(j)} K_4^{(j)} + k_{(P)4}. \tag{6.19}$$

The quantity $(c/i)k_4$ is the total work done on the matter by the electromagnetic field per unit volume and per unit time. As we have seen at the beginning of this section $(c/i)\varrho_{(0)}^{(j)} K_4^{(j)}$ is the work done on component j by the Lorentz force. Hence, $(c/i)\Sigma_{j=1}^{n} \varrho_{(0)}^{(j)} K_4^{(j)}$ is the work done by the electromagnetic field on the medium because the matter is bearing electric charges. From the preceding equation and the given interpretation of $(c/i)k_4$ and $(c/i)\Sigma_{j=1}^{n} \varrho_{(0)}^{(j)} K_4^{(j)}$ it follows that we can interpret $(c/i)k_{(P)4}$ as the work which the electromagnetic field does on the matter because the matter is polarized and magnetized.

§ 7. *Comparison with the tensors of Abraham and Minkowski.* To be able to compare the energy-momentum tensor derived in § 3 with the energy-momentum tensors which Abraham and Minkowski assign to the macroscopic electromagnetic field in media which are isotropic as far as polarization and magnetization are concerned, we shall change the expression (3.14) for $W_{(f)\alpha\beta}$.

For that purpose we first introduce the Minkowski vector, $\Psi_\alpha (\alpha = 1,.., 4)$, ("Ruhstrahlvector") which is given by

$$\Psi_\alpha \equiv \Sigma_{\gamma,\zeta,\xi=1}^{4} u_\xi B_{\gamma\zeta} u_\zeta (H_{\gamma\alpha} u_\xi + H_{\xi\gamma} u_\alpha + H_{\alpha\xi} u_\gamma), \quad (\alpha = 1, .., 4) \tag{7.1}$$

and the four-vector $\Psi_\alpha^* \ (\alpha = 1, .., 4)$ given by

$$\Psi_\alpha^* \equiv \Sigma_{\gamma,\zeta,\xi=1}^{4} u_\xi H_{\gamma\zeta} u_\zeta (B_{\gamma\alpha} u_\xi + B_{\xi\gamma} u_\alpha + B_{\alpha\xi} u_\gamma). \quad (\alpha = 1, .., 4) \tag{7.2}$$

Using (I.2.12), (2.8) and (2.10) we can also write for these two equations

$$\Psi_\alpha = \Sigma_{\gamma,\zeta=1}^{4} u_\zeta B_{\zeta\gamma} H_{\gamma\alpha} + u_\alpha \Sigma_{\gamma,\zeta,\xi=1}^{4} u_\zeta B_{\zeta\gamma} H_{\gamma\xi} u_\xi, \quad (\alpha = 1, .., 4) \tag{7.3}$$

$$\Psi_\alpha^* = \Sigma_{\gamma,\zeta=1}^{4} u_\zeta H_{\zeta\gamma} B_{\gamma\alpha} + u_\alpha \Sigma_{\gamma,\zeta,\xi=1}^{4} u_\zeta H_{\zeta\gamma} B_{\gamma\xi} u_\xi, \quad (\alpha = 1, .., 4) \tag{7.4}$$

or with the help of (I.2.28)

$$\Psi_\alpha = \Sigma_{\gamma,\zeta,\xi=1}^{4} u_\zeta B_{\zeta\gamma} H_{\gamma\xi} \Delta_{\xi\alpha}, \quad (\alpha = 1, .., 4) \tag{7.5}$$

$$\Psi_\alpha^* = \Sigma_{\gamma,\zeta,\xi=1}^{4} u_\zeta H_{\zeta\gamma} B_{\gamma\xi} \Delta_{\xi\alpha}. \quad (\alpha = 1, .., 4) \tag{7.6}$$

From the two preceding equations we find with the help of (I.2.30)

$$\Sigma_{\alpha=1}^{4} u_\alpha \Psi_\alpha = 0, \tag{7.7}$$

$$\Sigma_{\alpha=1}^{4} u_\alpha \Psi_\alpha^* = 0. \tag{7.8}$$

Using (I.2.28), (2.8) and (2.10) we have from (7.5) and (7.6)

$$\Psi_\alpha - \Psi_\alpha^* = \Sigma_{\gamma,\zeta=1}^{4} u_\zeta (B_{\zeta\gamma} H_{\gamma\alpha} - H_{\zeta\gamma} B_{\gamma\alpha}). \tag{7.9}$$
$$(\alpha = 1, .., 4)$$

Inserting (7.9) into (3.14) gives

$$W_{(f)\alpha\beta} = -\Sigma_{\gamma=1}^{4} B_{\alpha\gamma} H_{\gamma\beta} - \tfrac{1}{4}\,\delta_{\alpha\beta}\,\Sigma_{\gamma,\zeta=1}^{4} (B_{\gamma\zeta})^2 + (\Psi_\alpha - \Psi_\alpha^*)u_\beta. \qquad (\alpha, \beta = 1, .., 4) \tag{7.10}$$

In the case of media which are isotropic as far as polarization and magnetization are concerned we have from (7.1) and (7.2) with the help of (2.18), (2.19), (2.37) and (2.38)

$$\Psi_\alpha^* = \varepsilon\mu\Psi_\alpha. \qquad (\alpha = 1, .., 4) \tag{7.11}$$

By substituting the preceding equation into (7.10) we find

$$W_{(f)\alpha\beta} = -\Sigma_{\gamma=1}^{4} B_{\alpha\gamma} H_{\gamma\beta} - \tfrac{1}{4}\,\delta_{\alpha\beta}\,\Sigma_{\gamma,\zeta=1}^{4} (B_{\gamma\zeta})^2 - (\varepsilon\mu - 1)\Psi_\alpha u_\beta. \qquad (\alpha, \beta = 1, .., 4) \tag{7.12}$$

For media which are isotropic as far as polarization and magnetization are concerned the symmetric tensor of A b r a h a m, $W_{(f)\alpha\beta}^{A}$ $(\alpha, \beta = 1, .., 4)$, has the form [4] [5] [6]

$$W_{(f)\alpha\beta}^{A} = -\Sigma_{\gamma=1}^{4} B_{\alpha\gamma} H_{\gamma\beta} - \tfrac{1}{4}\,\delta_{\alpha\beta}\,\Sigma_{\gamma,\zeta=1}^{4} B_{\gamma\zeta} H_{\gamma\zeta} - (\varepsilon\mu - 1)\Psi_\alpha u_\beta. \qquad (\alpha, \beta = 1, .., 4) \tag{7.13}$$

From the two preceding equations we see that the first and the third terms in $W_{(f)\alpha\beta}^{A}$ and $W_{(f)\alpha\beta}$ are equal; the second terms, however, differ. Hence, we see that $W_{(f)\alpha\beta}^{A} = W_{(f)\alpha\beta}$ if $\alpha \neq \beta$, *i.e.*, the non-diagonal elements of the tensor found in § 3 equal the corresponding elements of the tensor of A b r a h a m. The asymmetric tensor of M i n k o w s k i, $W_{(f)\alpha\beta}^{M}$ $(\alpha, \beta = 1, .., 4)$, has the form [6]

$$W_{(f)\alpha\beta}^{M} = -\Sigma_{\gamma=1}^{4} B_{\alpha\gamma} H_{\gamma\beta} - \tfrac{1}{4}\,\delta_{\alpha\beta}\,\Sigma_{\gamma,\zeta=1}^{4} B_{\gamma\zeta} H_{\gamma\zeta}. \qquad (\alpha, \beta = 1, .., 4) \tag{7.14}$$

Only the first terms occurring in (7.12) and (7.14) are equal. We conclude that our form for the energy-momentum tensor of the electromagnetic field is thus essentially (apart from a difference in the diagonal terms) the same as A b r a h a m's symmetric tensor. In § 10 we shall show that A b r a h a m's tensor leads to an equivalent formalism and that from the point of view of our theory $W_{(f)\alpha\beta}^{A}$ is to be preferred over $W_{(f)\alpha\beta}^{M}$.

§ 8. *The first and second laws of thermodynamics.* We shall now discuss the first and second laws of thermodynamics. For that purpose we introduce the quantity φ defined by

$$\varphi \equiv -(c^2 - \mathbf{v}^2)^{\frac{1}{2}}\,\Sigma_{\alpha=1}^{4} u_\alpha k_{(P)\alpha}. \tag{8.1}$$

Inserting (I.2.11) into this equation gives

$$\varphi = (c/i)k_{(P)4} - (\mathbf{v}\cdot\mathbf{k}_{(P)}). \tag{8.2}$$

Using (3.4) and (8.1) we can write for the first law of thermodynamics given by (IV.7.3)

$$\varrho'(\mathrm{D}e' + p'\mathrm{D}v') = -\Sigma_{\beta=1}^{4} (\partial I_\beta^{(0)}/\partial x_\beta + c^{-1} I_\beta^{(0)} \mathrm{D}u_\beta) +$$
$$+ c\,\Sigma_{\alpha,\beta=1}^{4} P_{\alpha\beta}(\partial u_\beta/\partial x_\alpha) + \Sigma_{k=1}^{n} \Sigma_{\alpha=1}^{4} I_\alpha^{(k)} (e^{(k)} \Sigma_{\beta=1}^{4} B_{\alpha\beta} u_\beta) + \varphi(1-\mathbf{v}^2/c^2)^{-\frac{1}{2}}. \quad (8.3)$$

From this equation it follows that we can interpret φ as that part of the work done by the electromagnetic field on the medium per unit volume and per unit time which is used to change the internal energy e' of the matter. We can draw this conclusion also from (8.2), because, as we have seen at the end of § 6, we can interpret $(c/i)k_{(P)4}$ as the work done by the electromagnetic field on the medium due to the polarization and magnetization of the matter. Hence, we can also say that φ is the work done by the electromagnetic field to change the state of polarization and magnetization of the matter. With the help of (I.2.3), (I.2.25), (I.2.30), (IV.8.2) and (8.1) we can write the second law of thermodynamics, given by (IV.4.19), in the form

$$T'\mathrm{D}s' = \mathrm{D}e' + p'\mathrm{D}v' - \varphi v'(1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}} - \Sigma_{j=1}^{n} \mu'^{(j)} \mathrm{D}c'^{(j)}. \quad (8.4)$$

We shall now give an explicit expression for φ. For that purpose we first substitute (3.2) and (3.3) into (IV.8.2). We then obtain with the help of (I.2.3)

$$k_{(P)\alpha} = \tfrac{1}{2}\Sigma_{\beta,\gamma=1}^{4} M_{\beta\gamma} (\partial B_{\beta\gamma}/\partial x_\alpha) + \varrho'\Sigma_{\beta=1}^{4} \Delta_{\alpha\beta}\, \Phi_\beta -$$
$$- \Sigma_{\beta,\gamma,\zeta=1}^{4} u_\beta (B_{\beta\gamma} M_{\gamma\zeta} - M_{\beta\gamma} B_{\gamma\zeta}) (\partial u_\zeta/\partial x_\alpha). \quad (\alpha = 1, \ldots, 4) \quad (8.5)$$

Substitution of (8.5) into (8.1) gives with the help of (I.2.25) and (I.2.30)

$$\varphi = -\tfrac{1}{2}(1 - \mathbf{v}^2/c^2)^{\frac{1}{2}} \Sigma_{\alpha,\beta=1}^{4} M_{\alpha\beta} \mathrm{D}B_{\alpha\beta} +$$
$$+ (1 - \mathbf{v}^2/c^2)^{\frac{1}{2}} \Sigma_{\alpha,\beta,\gamma=1}^{4} u_\alpha (B_{\alpha\gamma} M_{\gamma\beta} - M_{\alpha\gamma} B_{\gamma\beta}) \mathrm{D}u_\beta. \quad (8.6)$$

Inserting this equation into (8.3) gives for the first law of thermodynamics

$$\varrho'(\mathrm{D}e' + p'\mathrm{D}v') = -\Sigma_{\beta=1}^{4} (\partial I_\beta^{(0)}/\partial x_\beta + c^{-1} I_\beta^{(0)} \mathrm{D}u_\beta) +$$
$$+ c\Sigma_{\alpha,\beta=1}^{4} P_{\alpha\beta} (\partial u_\beta/\partial x_\alpha) + \Sigma_{k=1}^{n} \Sigma_{\alpha=1}^{4} I_\alpha^{(k)} (e^{(k)} \Sigma_{\beta=1}^{4} B_{\alpha\beta} u_\beta) -$$
$$- \tfrac{1}{2}\Sigma_{\alpha,\beta=1}^{4} M_{\alpha\beta} \mathrm{D}B_{\alpha\beta} + \Sigma_{\alpha,\beta,\gamma=1}^{4} u_\alpha (B_{\alpha\gamma} M_{\gamma\beta} - M_{\alpha\gamma} B_{\gamma\beta}) \mathrm{D}u_\beta. \quad (8.7)$$

For the second law of thermodynamics we obtain by substituting (8.6) into (8.4)

$$T'\mathrm{D}s' = \mathrm{D}e' + p'\mathrm{D}v' + \tfrac{1}{2}v' \Sigma_{\alpha,\beta=1}^{4} M_{\alpha\beta} \mathrm{D}B_{\alpha\beta} -$$
$$- v' \Sigma_{\alpha,\beta,\gamma=1}^{4} u_\alpha(B_{\alpha\gamma} M_{\gamma\beta} - M_{\alpha\gamma} B_{\gamma\beta}) \mathrm{D}u_\beta - \Sigma_{j=1}^{n} \mu'^{(j)} \mathrm{D}c'^{(j)}. \quad (8.8)$$

To be able to compare our results with those of non-relativistic theories, we shall still formulate the second law of thermodynamics in another way.

Inserting (I.2.11), (2.7), (2.11), (6.7) and (6.8) into (8.6) gives with the help of (I.2.25) and (II.3.10)

$$\begin{aligned}\varphi = &- \mathbf{P}\cdot(\mathrm{d}\mathbf{E}/\mathrm{d}t) - \mathbf{M}\cdot(\mathrm{d}\mathbf{B}/\mathrm{d}t) + \\ &+ (c^2 - \mathbf{v}^2)^{-\frac{1}{2}}\{\mathbf{v}\cdot(\mathbf{B}\wedge\mathbf{P}) - \mathbf{v}\cdot(\mathbf{E}\wedge\mathbf{M})\}\{(\mathrm{d}/\mathrm{d}t)\,(1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}}\} + \\ &+ (c^2 - \mathbf{v}^2)^{-\frac{1}{2}}\,[\{\mathbf{v}\wedge(\mathbf{B}\wedge\mathbf{M})\} + \{\mathbf{v}\wedge(\mathbf{E}\wedge\mathbf{P})\} - c(\mathbf{B}\wedge\mathbf{P}) + \\ &+ c(\mathbf{E}\wedge\mathbf{M})]\cdot[(\mathrm{d}/\mathrm{d}t)\,\{\mathbf{v}(c^2 - \mathbf{v}^2)^{-\frac{1}{2}}\}]. \qquad (8.9)\end{aligned}$$

Substituting (8.9) into (8.4) gives with the aid of (I.2.11), (I.2.25) and (II.3.10)

$$\begin{aligned}T'(\mathrm{d}s'/\mathrm{d}t) = &(\mathrm{d}e'/\mathrm{d}t) + p'(\mathrm{d}v'/\mathrm{d}t) - \Sigma_{j=1}^{n}\,\mu'^{(j)}\,(\mathrm{d}c'^{(j)}/\mathrm{d}t) + \\ &+ v'\mathbf{P}\cdot(\mathrm{d}\mathbf{E}/\mathrm{d}t) + v'\mathbf{M}\cdot(\mathrm{d}\mathbf{B}/\mathrm{d}t) - \\ &- v'(c^2 - \mathbf{v}^2)^{-\frac{1}{2}}\{\mathbf{v}\cdot(\mathbf{B}\wedge\mathbf{P}) - \mathbf{v}\cdot(\mathbf{E}\wedge\mathbf{M})\}\{(\mathrm{d}/\mathrm{d}t)\,(1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}}\} - \\ &- v'(c^2 - \mathbf{v}^2)^{-\frac{1}{2}}\,[\{\mathbf{v}\wedge(\mathbf{B}\wedge\mathbf{M})\} + \{\mathbf{v}\wedge(\mathbf{E}\wedge\mathbf{P})\} - c(\mathbf{B}\wedge\mathbf{P}) + \\ &+ c(\mathbf{E}\wedge\mathbf{M})]\cdot[(\mathrm{d}/\mathrm{d}t)\,\{\mathbf{v}(c^2 - \mathbf{v}^2)^{-\frac{1}{2}}\}]. \qquad (8.10)\end{aligned}$$

The first three terms on the right hand side of this equation are analogous to terms which also occur in the non-relativistic second law of thermodynamics. Instead of the fourth and fifth terms several authors give different forms. By introducing, for example, a different definition for the specific energy of the matter measured by an observer moving with the barycentric velocity, other forms may be obtained.

For instance we may define the Lorentz invariant quantity $\bar{e}'$ by

$$\bar{e}' \equiv e' + \tfrac{1}{2}\,v'\,\Sigma_{\alpha,\beta=1}^{4}\,M_{\alpha\beta}\,B_{\alpha\beta}. \qquad (8.11)$$

Using (2.7) and (2.11) we can write for this equation

$$\bar{e}' = e' + v'(\mathbf{M}\cdot\mathbf{B}) + v'(\mathbf{P}\cdot\mathbf{E}). \qquad (8.12)$$

Hence, with the help of (I.2.3) and (8.11) we can also write for (8.7)

$$\begin{aligned}\varrho'(\mathrm{D}\bar{e}' + p'\mathrm{D}v') = &- \Sigma_{\beta=1}^{4}(\partial I_\beta^{(0)}/\partial x_\beta + c^{-1}\,I_\beta^{(0)}\,\mathrm{D}u_\beta) + \\ &+ c\,\Sigma_{\alpha,\beta=1}^{4}\,P_{\alpha\beta}(\partial u_\beta/\partial x_\alpha) + \Sigma_{k=1}^{n}\,\Sigma_{\alpha=1}^{4}\,I_\alpha^{(k)}\,(e^{(k)}\,\Sigma_{\beta=1}^{4}\,B_{\alpha\beta}\,u_\beta) + \\ &+ \tfrac{1}{2}\,\varrho'\,\Sigma_{\alpha,\beta=1}^{4}\,B_{\alpha\beta}\,\mathrm{D}(v'M_{\alpha\beta}) + \Sigma_{\alpha,\beta,\gamma=1}^{4}\,u_\alpha(B_{\alpha\gamma}\,M_{\gamma\beta} - M_{\alpha\gamma}\,B_{\gamma\beta})\,\mathrm{D}u_\beta. \qquad (8.13)\end{aligned}$$

Inserting (8.12) into (8.10) gives for the second law of thermodynamics

$$\begin{aligned}T'(\mathrm{d}s'/\mathrm{d}t) = &(\mathrm{d}\bar{e}'/\mathrm{d}t) + p'(\mathrm{d}v'/\mathrm{d}t) - \Sigma_{j=1}^{n}\,\mu'^{(j)}\,(\mathrm{d}c'^{(j)}/\mathrm{d}t) - \\ &- \mathbf{E}\cdot\{\mathrm{d}(v'\mathbf{P})/\mathrm{d}t\} - \mathbf{B}\cdot\{\mathrm{d}(v'\mathbf{M})/\mathrm{d}t\} - \\ &- v'(c^2 - \mathbf{v}^2)^{-\frac{1}{2}}\{\mathbf{v}\cdot(\mathbf{B}\wedge\mathbf{P}) - \mathbf{v}\cdot(\mathbf{E}\wedge\mathbf{M})\}\{(\mathrm{d}/\mathrm{d}t)(1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}}\} - \\ &- v'(c^2 - \mathbf{v}^2)^{-\frac{1}{2}}\,[\{\mathbf{v}\wedge(\mathbf{B}\wedge\mathbf{M})\} + \{\mathbf{v}\wedge(\mathbf{E}\wedge\mathbf{P})\} - c(\mathbf{B}\wedge\mathbf{P}) + \\ &+ c(\mathbf{E}\wedge\mathbf{M})]\cdot[(\mathrm{d}/\mathrm{d}t)\,\{\mathbf{v}(c^2 - \mathbf{v}^2)^{-\frac{1}{2}}\}]. \qquad (8.14)\end{aligned}$$

The first five terms on the right hand side of this equation are analogous to terms which occur in the non-relativistic second law of thermodynamics used by Mazur and Prigogine[2]). The other terms are of a special relativistic nature.

By introducing still other definitions, analogous to (8.11), for the specific internal energy of the matter, we may obtain other forms for the fourth and fifth terms occurring in (8.10). It is seen, however, that all these forms for the second law of thermodynamics are equivalent. In the following section we shall discuss this question from a more general point of view.

§ 9. *On the indefiniteness of the energy-momentum tensors of the matter and of the field.* By means of two examples we shall show that there remains a certain indefiniteness in the energy-momentum tensors of the field and of the matter.

We introduce a new energy-momentum tensor of the matter, $W^*_{\alpha\beta}(\alpha, \beta = 1, .., 4)$, defined by

$$W^*_{\alpha\beta} \equiv W_{\alpha\beta} - \bar{p}\Delta_{\alpha\beta}, \qquad (\alpha, \beta = 1, .., 4) \qquad (9.1)$$

where $\bar{p}$ is a Lorentz invariant quantity having the dimension of a pressure. We also introduce a new energy-momentum tensor, $W^*_{(f)\alpha\beta}(\alpha, \beta = 1, .., 4)$, for the electromagnetic field which is defined by

$$W^*_{(f)\alpha\beta} \equiv W_{(f)\alpha\beta} + \bar{p}\Delta_{\alpha\beta}. \qquad (\alpha, \beta = 1, .., 4) \qquad (9.2)$$

Further, we introduce a new hydrostatic pressure p'^* defined by

$$p'^* \equiv p' - \bar{p}, \qquad (9.3)$$

and a new stress tensor, $w^*_{\alpha\beta}(\alpha, \beta = 1, .., 4)$, defined by

$$w^*_{\alpha\beta} \equiv w_{\alpha\beta} - \bar{p}\Delta_{\alpha\beta}. \qquad (\alpha, \beta = 1, .., 4) \qquad (9.4)$$

Finally, we introduce a new four-vector, $k^*_\alpha(\alpha = 1, .., 4)$, representing the total force exerted on the matter by the electromagnetic field, defined by

$$k^*_\alpha \equiv -\Sigma^4_{\beta=1}\, \partial W^*_{(f)\alpha\beta}/\partial x_\beta = k_\alpha - \Sigma^4_{\beta=1}\, \partial(\bar{p}\Delta_{\alpha\beta})/\partial x_\beta, \qquad (\alpha = 1, .., 4) \qquad (9.5)$$

where the last form has been obtained with the help of (3.1) and (9.2), and a new four-vector, $k^*_{(P)\alpha}(\alpha = 1, .., 4)$, representing the ponderomotive force, defined by

$$k^*_{(P)\alpha} \equiv k_{(P)\alpha} - \Sigma^4_{\beta=1}\, \partial(\bar{p}\Delta_{\alpha\beta})/\partial x_\beta. \qquad (\alpha = 1, .., 4) \qquad (9.6)$$

We shall now deduce some relations which are useful for the following considerations. From (9.1) and (9.2) we have with the help of (I.2.29), (IV.5.1) and (4.6)

$$W^*_{\alpha\beta} = W^*_{\beta\alpha}, \qquad (\alpha, \beta = 1, .., 4) \qquad (9.7)$$

and

$$W^*_{(f)\alpha\beta} = W^*_{(f)\beta\alpha}. \qquad (\alpha, \beta = 1, .., 4) \qquad (9.8)$$

Using (I.2.29), (I.2.30), (I.3.10) and (I.3.11) we get from (9.4)

$$w^*_{\alpha\beta} = w^*_{\beta\alpha}, \qquad (\alpha, \beta = 1, .., 4) \qquad (9.9)$$

and

$$\Sigma^4_{\beta=1}\, u_\beta\, w^*_{\beta\alpha} = \Sigma^4_{\beta=1}\, w^*_{\alpha\beta}\, u_\beta = 0. \qquad (\alpha = 1, .., 4) \qquad (9.10)$$

With the help of (I.2.3), (I.2.29) and (I.5.7) we deduce from (9.3) and (9.6)

$$p'^*\mathrm{D}v' + v'c\, \Sigma^4_{\alpha=1}\, u_\alpha\, k^*_{(P)\alpha} = p'\mathrm{D}v' + v'c\, \Sigma^4_{\alpha=1}\, u_\alpha\, k_{(P)\alpha}. \qquad (9.11)$$

Using (I.3.12) and (9.4) we find from (9.1)

$$W^*_{\alpha\beta} = u_\alpha u_\beta e'_{(v)} + c^{-1}(u_\beta I^{(0)}_\alpha + u_\alpha I^{(0)}_\beta) + w^*_{\alpha\beta}. \qquad (\alpha, \beta = 1, .., 4) \qquad (9.12)$$

It follows from (I.3.14), (9.3) and (9.4) that

$$P_{\alpha\beta} = -\, w^*_{\alpha\beta} + p'^* \Delta_{\alpha\beta}. \qquad (\alpha, \beta = 1, .., 4) \qquad (9.13)$$

We have from (4.1), (9.1) and (9.2)

$$W_{(t)\alpha\beta} = W^*_{\alpha\beta} + W^*_{(f)\alpha\beta}. \qquad (\alpha, \beta = 1, .., 4) \qquad (9.14)$$

If we consider $W^*_{\alpha\beta}$ and $W^*_{(f)\alpha\beta}$ as the energy-momentum tensors of the matter and of the electromagnetic field respectively, we see from the preceding equation that the energy-momentum tensor of the system remains unchanged. Hence, also the laws of conservation of momentum and energy, given by (4.2), and the law of conservation of angular momentum, given by (4.13), remain unchanged. Since, according to (9.7) and (9.8), the new energy-momentum tensors of the matter and of the field are symmetric, we also have relations which are analogous to (4.11) and (4.12).

We shall now discuss the first law of thermodynamics. For that purpose we remark that according to (IV.5.2), (9.1) and (9.5) we can write for the balance equations for energy and momentum of the matter

$$\Sigma^4_{\beta=1}\, \partial W^*_{\alpha\beta}/\partial x_\beta = k^*_\alpha. \qquad (\alpha = 1, .., 4) \qquad (9.15)$$

We now multiply (9.15) by u_α and we sum over α. Analogous to (IV.7.1) we then get

$$\Sigma^4_{\alpha,\beta=1}\, u_\alpha(\partial W^*_{\alpha\beta}/\partial x_\beta) = \Sigma^4_{\alpha=1}\, u_\alpha k^*_\alpha. \qquad (9.16)$$

For the derivation of (I.5.9) we used, among other things, the relations (I.3.11), (I.3.12) and (I.3.14). These relations correspond with (9.10), (9.12) and (9.13) respectively and it may be easily seen that, analogous to (I.5.9), one obtains

$$\Sigma^4_{\alpha,\beta=1}\, u_\alpha(\partial W^*_{\alpha\beta}/\partial x_\beta) = -\, c^{-1}\, \varrho'\mathrm{D}e' - c^{-1}\, \Sigma^4_{\beta=1}(\partial I^{(0)}_\beta/\partial x_\beta + c^{-1}\, I^{(0)}_\beta\, \mathrm{D}u_\beta) -$$
$$-\, c^{-1}\, \varrho' p'^* \mathrm{D}v' + \Sigma^4_{\alpha,\beta=1}\, P_{\alpha\beta}(\partial u_\beta/\partial x_\alpha). \qquad (9.17)$$

From (IV.7.2), (3.4), (9.5) and (9.6) we find

$$\Sigma_{\alpha=1}^{4} u_\alpha k_\alpha^* = -\Sigma_{k=1}^{n} \Sigma_{\alpha=1}^{4} I_\alpha^{(k)}(c^{-1} e^{(k)} \Sigma_{\beta=1}^{4} B_{\alpha\beta} u_\beta) + \Sigma_{\alpha=1}^{4} u_\alpha k_{(P)\alpha}^*. \quad (9.18)$$

Inserting the two preceding equations into (9.16) gives for the first law of thermodynamics

$$\varrho'(\mathrm{D}e' + p'^*\mathrm{D}v') = -\Sigma_{\beta=1}^{4}(\partial I_\beta^{(0)}/\partial x_\beta + c^{-1} I_\beta^{(0)} \mathrm{D}u_\beta) + c\,\Sigma_{\alpha,\beta=1}^{4} P_{\alpha\beta}(\partial u_\beta/\partial x_\alpha) +$$
$$+ \Sigma_{k=1}^{n} \Sigma_{\alpha=1}^{4} I_\alpha^{(k)}(e^{(k)} \Sigma_{\beta=1}^{4} B_{\alpha\beta} u_\beta) - c\,\Sigma_{\alpha=1}^{4} u_\alpha k_{(P)\alpha}^*. \quad (9.19)$$

We immediately see from (I.2.3), (8.1) and (9.11) that (8.3) and the above equation are identical.

Using (8.1) and (9.11) we can write for the second law of thermodynamics given by (8.4)

$$T'\mathrm{D}s' = \mathrm{D}e' + p'^*\mathrm{D}v' + cv'\,\Sigma_{\alpha=1}^{4} u_\alpha k_{(P)\alpha}^* - \Sigma_{j=1}^{n} \mu'^{(j)}\,\mathrm{D}c'^{(j)}. \quad (9.20)$$

Substituting (IV.6.4) and (9.19) into (9.20) leads to the entropy balance (IV.8.8). Thus, we obtain the same phenomenological equations among the same fluxes and "forces" (affinities). Hence, we see that with respect to thermodynamics it does not make any difference if we consider $W_{\alpha\beta}$ and $W_{(f)\alpha\beta}$ or $W_{\alpha\beta}^*$ and $W_{(f)\alpha\beta}^*$ as the energy-momentum tensors of the matter and the electromagnetic field respectively.

It should be remarked that the equivalence of the points of view of Kelvin and Helmholtz concerning the ponderomotive force in polarized media may be shown [2]) with the help of considerations which are analogous to those which we have given above.

We shall now give a second example. Analogous to (9.1) and (9.2) we now introduce

$$\widetilde{W}_{\alpha\beta} \equiv W_{\alpha\beta} - \widetilde{e}_{(v)}^*\, u_\alpha u_\beta, \qquad (\alpha, \beta = 1, \ldots, 4) \quad (9.21)$$

and

$$\widetilde{W}_{(f)\alpha\beta} \equiv W_{(f)\alpha\beta} + \widetilde{e}_{(v)}^*\, u_\alpha u_\beta, \qquad (\alpha, \beta = 1, \ldots, 4) \quad (9.22)$$

where $\widetilde{e}_{(v)}^*$ is a Lorentz invariant quantity having the dimension of an energy per unit volume. Further, we introduce

$$\widetilde{e}_{(v)}' \equiv e_{(v)}' - \widetilde{e}_{(v)}^* \quad (9.23)$$

as the new energy per unit volume of the matter measured by an observer in the barycentric Lorentz frame at position $\mathbf{r}'$ and at time t' and

$$\widetilde{e}\,' \equiv v'(e_{(v)}' - \widetilde{e}_{(v)}^*) - a \quad (9.24)$$

as the new specific energy of the matter measured by an observer in the barycentric Lorentz frame at position $\mathbf{r}'$ and at time t'. Analogous to

(9.5) and (9.6) we have

$$\tilde{k}_\alpha \equiv -\Sigma_{\beta=1}^4 \partial \tilde{W}_{(f)\alpha\beta}/\partial x_\beta = k_\alpha - \Sigma_{\beta=1}^4 \partial(\tilde{e}_{(v)}^* u_\alpha u_\beta)/\partial x_\beta, \qquad (\alpha = 1, .., 4) \quad (9.25)$$

and

$$\tilde{k}_{(P)\alpha} \equiv k_{(P)\alpha} - \Sigma_{\beta=1}^4 \partial(\tilde{e}_{(v)}^* u_\alpha u_\beta)/\partial x_\beta. \qquad (\alpha = 1, .., 4) \quad (9.26)$$

Using (IV.5.1) and (4.6) we get from (9.21) and (9.22)

$$\tilde{W}_{\alpha\beta} = \tilde{W}_{\beta\alpha}, \qquad (\alpha, \beta = 1, .., 4) \quad (9.27)$$

$$\tilde{W}_{(f)\alpha\beta} = \tilde{W}_{(f)\beta\alpha}. \qquad (\alpha, \beta = 1, .., 4) \quad (9.28)$$

With the help of (I.2.10), (I.2.12), (I.2.13), (I.2.15), (I.2.25), (I.3.13), (IV.6.3), (9.24) and (9.26) we derive

$$\mathrm{D}\tilde{e}' + v'c\,\Sigma_{\alpha=1}^4 u_\alpha \tilde{k}_{(P)\alpha} = \mathrm{D}e' + v'c\,\Sigma_{\alpha=1}^4 u_\alpha k_{(P)\alpha}. \quad (9.29)$$

From (I.3.12) and (9.21) we have with the help of (9.23)

$$\tilde{W}_{\alpha\beta} = \tilde{e}'_{(v)} u_\alpha u_\beta + c^{-1}(u_\beta I_\alpha^{(0)} + u_\alpha I_\beta^{(0)}) + w_{\alpha\beta}. \qquad (\alpha, \beta = 1, .., 4) \quad (9.30)$$

From (4.1), (9.21) and (9.22) we have

$$W_{(t)\alpha\beta} = \tilde{W}_{\alpha\beta} + \tilde{W}_{(f)\alpha\beta}. \qquad (\alpha, \beta = 1, .., 4) \quad (9.31)$$

It follows from the preceding equation that the laws of conservation of energy, momentum and angular momentum remain unchanged. On account of (9.27) and (9.28) we also have equations analogous to (4.11) and (4.12).

Using (9.21) and (9.25) we can also write for the balance equations for energy and momentum of the matter given by (IV.5.2)

$$\Sigma_{\beta=1}^4 \partial \tilde{W}_{\alpha\beta}/\partial x_\beta = \tilde{k}_\alpha. \qquad (\alpha = 1, .., 4) \quad (9.32)$$

Starting from the preceding equation it is easily seen that, analogous to (9.19), one obtains for the first law of thermodynamics

$$\varrho'(\mathrm{D}\tilde{e}' + p'\mathrm{D}v') = -\Sigma_{\beta=1}^4 (\partial I_\beta^{(0)}/\partial x_\beta + c^{-1} I_\beta^{(0)} \mathrm{D}u_\beta) + c\,\Sigma_{\alpha,\beta=1}^4 P_{\alpha\beta}(\partial u_\beta/\partial x_\alpha) +$$
$$+ \Sigma_{k=1}^n \Sigma_{\alpha=1}^4 I_\alpha^{(k)} (e^{(k)} \Sigma_{\beta=1}^4 B_{\alpha\beta} u_\beta) - c\,\Sigma_{\alpha=1}^4 u_\alpha \tilde{k}_{(P)\alpha}. \quad (9.33)$$

With the help of (I.2.3), (8.1) and (9.29) it is immediately seen that (8.3) and the preceding equation are identical.

Using (8.1) and (9.29) we can write for the second law of thermodynamics given by (8.4)

$$T'\mathrm{D}s' = \mathrm{D}\tilde{e}' + p'\mathrm{D}v' + cv'\,\Sigma_{\alpha=1}^4 u_\alpha \tilde{k}_{(P)\alpha} - \Sigma_{j=1}^n \mu'^{(j)} \mathrm{D}c'^{(j)}. \quad (9.34)$$

Inserting (IV.6.4) and (9.33) into the preceding equation leads to the entropy balance (IV.8.8). Hence, we see that no changes enter into the thermodynamical results if we consider $\tilde{W}_{\alpha\beta}$ and $\tilde{W}_{(f)\alpha\beta}$ as the energy-momentum tensors of the matter and of the field respectively. It is easily seen that by taking $\tilde{e}^*_{(v)} = - \frac{1}{2} \Sigma^4_{\alpha,\beta=1} M_{\alpha\beta} B_{\alpha\beta}$ one obtains the expressions (8.13) and (8.14) for the first and second laws of thermodynamics respectively.

It should also be remarked that the form (3.14) for the energy-momentum tensor of the electromagnetic field does not follow uniquely from (3.1) and (3.13). For example, if the tensor $W^{**}_{(f)\alpha\beta}$ satisfies the relation

$$\Sigma^4_{\beta=1} \partial W^{**}_{(f)\alpha\beta}/\partial x_\beta = 0, \qquad (\alpha = 1, \ldots, 4) \tag{9.35}$$

we have from this equation and (3.1)

$$k_\alpha = - \Sigma^4_{\beta=1} \partial W^{***}_{(f)\alpha\beta}, \qquad (\alpha = 1, \ldots, 4) \tag{9.36}$$

where

$$W^{***}_{(f)\alpha\beta} \equiv W_{(f)\alpha\beta} + W^{**}_{(f)\alpha\beta}. \qquad (\alpha, \beta = 1, \ldots, 4) \tag{9.37}$$

If the tensor $W^{**}_{(f)\alpha\beta}$ vanishes if $\mathbf{E} = \mathbf{D} = 0$ and $\mathbf{B} = \mathbf{H} = 0$ we may consider $W^{***}_{(f)\alpha\beta}$ just as well as the energy-momentum tensor of the electromagnetic field.

§ 10. *Further discussion of the tensors of A b r a h a m and M i n k o w s k i.* We shall first discuss the tensor of A b r a h a m. Using (2.13) we have from (7.12) and (7.13)

$$W^A_{(f)\alpha\beta} = W_{(f)\alpha\beta} + \tfrac{1}{4} \delta_{\alpha\beta} \Sigma^4_{\gamma,\zeta=1} B_{\gamma\zeta} M_{\gamma\zeta}, \qquad (\alpha, \beta = 1, \ldots, 4) \tag{10.1}$$

or, with the help of (I.2.28),

$$W^A_{(f)\alpha\beta} = W_{(f)\alpha\beta} + \tfrac{1}{4} \Delta_{\alpha\beta} \Sigma^4_{\gamma,\zeta=1} B_{\gamma\zeta} M_{\gamma\zeta} - \tfrac{1}{4} u_\alpha u_\beta \Sigma^4_{\gamma,\zeta=1} B_{\gamma\zeta} M_{\gamma\zeta}. \tag{10.2}$$

$$(\alpha, \beta = 1, \ldots, 4)$$

Since $\Sigma^4_{\gamma,\zeta=1} B_{\gamma\zeta} M_{\gamma\zeta}$ is a Lorentz invariant quantity it is seen from the considerations in § 9 (Cf. (9.2), (9.22) and (10.2)) that we obtain an equivalent formalism by taking A b r a h a m's tensor as energy-momentum tensor of the electromagnetic field.

As may also be seen from the preceding section, the formalism with $W^A_{(f)\alpha\beta}$ involves new definitions for the hydrostatic pressure and for the internal energy of the matter measured by an observer moving with the barycentric velocity. Comparing (9.2), (9.22) and (10.2) we find from (9.3) and (9.23)

$$p'^A = p' - \tfrac{1}{4} \Sigma^4_{\gamma,\zeta=1} B_{\gamma\zeta} M_{\gamma\zeta} \tag{10.3}$$

for the new hydrostatic pressure and

$$e'^A_{(v)} = e'_{(v)} + \tfrac{1}{4} \Sigma^4_{\gamma,\zeta=1} B_{\gamma\zeta} M_{\gamma\zeta} \tag{10.4}$$

for the new energy of the matter per unit volume measured by an observer moving with the barycentric velocity. Analogous to (9.24) (Cf. also (9.23) and (10.4)) the new specific energy of the matter measured by an observer moving with the barycentric velocity is given by

$$e'^A = v'(e'_{(v)} + \tfrac{1}{4}\Sigma^4_{\gamma,\zeta=1} B_{\gamma\zeta} M_{\gamma\zeta}) - a, \tag{10.5}$$

or, using (I.3.13),

$$e'^A = e' + \tfrac{1}{4} v' \Sigma^4_{\gamma,\zeta=1} B_{\gamma\zeta} M_{\gamma\zeta}. \tag{10.6}$$

From the considerations in § 9 it can be seen that the new form for the second law of thermodynamics is obtained by inserting (10.3) and (10.6) into (8.8). This gives

$$\begin{aligned} T'\mathrm{D}s' = {} & \mathrm{D}e'^A + p'^A\mathrm{D}v' - \Sigma^n_{j=1}\mu'^{(j)}\mathrm{D}c'^{(j)} + \\ & + \tfrac{1}{4} v' \Sigma^4_{\alpha,\beta=1} M_{\alpha\beta}\mathrm{D}B_{\alpha\beta} + \tfrac{1}{2} v' \Sigma^4_{\alpha,\beta,\gamma=1} u_\alpha (M_{\alpha\gamma}B_{\gamma\beta} - B_{\alpha\gamma}M_{\gamma\beta})\mathrm{D}u_\beta - \\ & - \tfrac{1}{4} v' \Sigma^4_{\alpha,\beta=1} B_{\alpha\beta}\mathrm{D}M_{\alpha\beta} - \tfrac{1}{2} v' \Sigma^4_{\alpha,\beta,\gamma=1} u_\alpha (B_{\alpha\gamma}M_{\gamma\beta} - M_{\alpha\gamma}B_{\gamma\beta})\mathrm{D}u_\beta. \end{aligned} \tag{10.7}$$

Comparison of (IV.2.5) and (IV.4.19) shows that one can also write for (10.7)

$$\begin{aligned} T'\mathrm{D}s' = {} & \mathrm{D}e'^A + p'^A\mathrm{D}v' - \Sigma^n_{j=1}\mu'^{(j)}\mathrm{D}c'^{(j)} + \\ & + \tfrac{1}{4} v' \Sigma^4_{\alpha,\beta=1} (M'_{\alpha\beta}\mathrm{D}B'_{\alpha\beta} - B'_{\alpha\beta}\mathrm{D}M'_{\alpha\beta}). \end{aligned} \tag{10.8}$$

With the help of (2.7) and (2.11) we get for the preceding equation the form

$$\begin{aligned} T'\mathrm{D}s' = {} & \mathrm{D}e'^A + p'^A\mathrm{D}v' - \Sigma^n_{j=1}\mu'^{(j)}\mathrm{D}c'^{(j)} + \\ & + \tfrac{1}{2} v'(\mathbf{P}'\cdot\mathrm{D}\mathbf{E}' - \mathbf{E}'\cdot\mathrm{D}\mathbf{P}' + \mathbf{M}'\cdot\mathrm{D}\mathbf{B}' - \mathbf{B}'\cdot\mathrm{D}\mathbf{M}'). \end{aligned} \tag{10.9}$$

By dropping the primes and replacing D by $\mathrm{d}/\mathrm{d}t$ in (10.9) we get the corresponding non-relativistic second law of thermodynamics (Cf. § 2 of chapter IV). It is seen that this form for the non-relativistic second law is rather unusual.

The four-vector representing the total force per unit volume exerted by the electromagnetic field on the matter if we use the formalism with A b r a h a m's tensor will be denoted by k^A_α. We have

$$k^A_\alpha = -\Sigma^4_{\beta=1}\partial W^A_{(f)\alpha\beta}/\partial x_\beta. \qquad (\alpha = 1, \ldots, 4) \tag{10.10}$$

Using (IV.5.3), (3.1) and (10.1) we find from this equation

$$k^A_\alpha = \Sigma^n_{j=1}\varrho^{(j)}_{(0)}K^{(j)}_\alpha + k_{(P)\alpha} - \tfrac{1}{4}(\partial/\partial x_\alpha)(\Sigma^4_{\beta,\gamma=1}B_{\beta\gamma}M_{\beta\gamma}). \tag{10.11}$$

$$(\alpha = 1, \ldots, 4)$$

We now define the new ponderomotive force per unit volume by

$$k^A_{(P)\alpha} \equiv k_{(P)\alpha} - \tfrac{1}{4}\,(\partial/\partial x_\alpha)\,(\Sigma^4_{\beta,\gamma=1}\,B_{\beta\gamma}\,M_{\beta\gamma}). \qquad (\alpha = 1, \ldots, 4) \tag{10.12}$$

It is easily seen that this definition is in agreement with (9.6) and (9.26). From (8.5) and (10.12) we have

$$\begin{aligned} k^A_{(P)\alpha} = {} & \tfrac{1}{4}\,\Sigma^4_{\beta,\gamma=1}\,M_{\beta\gamma}(\partial B_{\beta\gamma}/\partial x_\alpha) - \tfrac{1}{4}\,\Sigma^4_{\beta,\gamma=1}\,B_{\beta\gamma}\,(\partial M_{\beta\gamma}/\partial x_\alpha) + \\ & + \Sigma^4_{\beta,\gamma,\zeta=1}\,u_\beta\,(M_{\beta\gamma}\,B_{\gamma\zeta} - B_{\beta\gamma}\,M_{\gamma\zeta})\,(\partial u_\zeta/\partial x_\alpha) + \\ & + \varrho'\Sigma^4_{\beta=1}\,\Delta_{\alpha\beta}\,\Phi_\beta. \qquad (\alpha = 1, \ldots, 4) \end{aligned} \tag{10.13}$$

We define the four-vectors $k^{*A}_{(P)\alpha}$ and $k^{**A}_{(P)\alpha}$ $(\alpha = 1, \ldots, 4)$ by

$$\begin{aligned} k^{*A}_{(P)\alpha} \equiv {} & \varrho'\{\tfrac{1}{4}\,v'\,\Sigma^4_{\beta,\gamma=1}\,M_{\beta\gamma}\,(\partial B_{\beta\gamma}/\partial x_\alpha) + \Sigma^4_{\beta=1}\,\Delta_{\alpha\beta}\,\Phi_\beta + \\ & + \tfrac{1}{2}v'\,\Sigma^4_{\beta,\gamma,\zeta=1}\,u_\beta\,(M_{\beta\gamma}\,B_{\gamma\zeta} - B_{\beta\gamma}\,M_{\gamma\zeta})\,(\partial u_\zeta/\partial x_\alpha)\}, \qquad (\alpha = 1, \ldots, 4) \end{aligned} \tag{10.14}$$

$$\begin{aligned} k^{**A}_{(P)\alpha} \equiv {} & -\,\varrho'\{\tfrac{1}{4}\,v'\,\Sigma^4_{\beta,\gamma=1}\,B_{\beta\gamma}\,(\partial M_{\beta\gamma}/\partial x_\alpha) + \\ & + \tfrac{1}{2}v'\,\Sigma^4_{\beta,\gamma,\zeta=1}\,u_\beta\,(B_{\beta\gamma}\,M_{\gamma\zeta} - M_{\beta\gamma}\,B_{\beta\gamma})\,(\partial u_\zeta/\partial x_\alpha)\}. \qquad (\alpha = 1, \ldots, 4) \end{aligned} \tag{10.15}$$

Using (I.2.3) we have from the three preceding equations

$$k^A_{(P)\alpha} = k^{*A}_{(P)\alpha} + k^{**A}_{(P)\alpha}. \qquad (\alpha = 1, \ldots, 4) \tag{10.16}$$

Comparison of (IV.8.2) with (10.14) and (10.15) shows that each of the two four-vectors $k^{*A}_{(P)\alpha}$ and $k^{**A}_{(P)\alpha}$ has the required form. This corresponds to the fact that on the right hand side of (10.8) two terms occur having the form $\Sigma^4_{\alpha,\beta=1}\,G'_{\alpha\beta}\,\mathrm{D}Z'_{\alpha\beta}$ whereas on the right hand side of (IV.2.5) only one term occurs of this form.

We now consider the tensor of Minkowski. We first remark that we cannot obtain a formalism with this tensor by means of a procedure analogous to those given in the preceding section. In passing to other formalisms by means of the procedures given in § 9 the energy-momentum tensor of the system, $W_{(t)\alpha\beta}$, remains unchanged. Hence, if we should try to pass to a formalism with Minkowski's tensor by means of such a procedure the energy-momentum tensor of the matter would become asymmetric since $W_{(t)\alpha\beta}$ is symmetric and Minkowski's tensor is asymmetric. This, however, would give rise to difficulties because the symmetry of the energy-momentum tensor of the matter has been a basic assumption in our thermodynamical considerations.

Also if we should assume that the energy-momentum tensor of the matter is symmetric we get into difficulties, since $W_{(t)\alpha\beta}$ would become asymmetric and hence, (Cf. § 4), the macroscopic angular momentum of the system would not be conserved.

Moreover, we get a different thermodynamical formalism with M i n k o w s k i's tensor. To show this, we first remark that we have from M i n k o w s k i's tensor for the four-vector, k_α^M, representing the total force per unit volume exerted by the electromagnetic field on the matter

$$k_\alpha^M = - \Sigma_{\beta=1}^4 \, \partial W_{(f)\alpha\beta}^M / \partial x_\beta. \qquad (\alpha = 1, \ldots, 4) \qquad (10.17)$$

With the help of (2.8), (2.10) and (2.15) we find the relation

$$\Sigma_{\beta,\gamma=1}^4 \, (\partial B_{\alpha\gamma} / \partial x_\beta) \, H_{\gamma\beta} = - \tfrac{1}{2} \Sigma_{\beta,\gamma=1}^4 \, H_{\beta\gamma} (\partial B_{\beta\gamma} / \partial x_\alpha). \qquad (10.18)$$

$$(\alpha = 1, \ldots, 4)$$

Using (2.14), (3.8), (7.14) and (10.18) we derive from (10.17)

$$k_\alpha^M = \Sigma_{j=1}^n \, \varrho_{(0)}^{(j)} \, K_\alpha^{(j)} + \tfrac{1}{4} \Sigma_{\beta,\gamma=1}^4 \, B_{\beta\gamma} \, (\partial H_{\beta\gamma} / \partial x_\alpha) -$$

$$- \tfrac{1}{4} \Sigma_{\beta,\gamma=1}^4 \, H_{\beta\gamma} \, (\partial B_{\beta\gamma} / \partial x_\alpha). \qquad (\alpha = 1, \ldots, 4) \qquad (10.19)$$

Hence, the four-vector, $k_{(P)\alpha}^M$, representing the ponderomotive force per unit volume, is given by (Cf. also (IV.5.3))

$$k_{(P)\alpha}^M = \tfrac{1}{4} \Sigma_{\beta,\gamma=1}^4 \, B_{\beta\gamma} (\partial H_{\beta\gamma} / \partial x_\alpha) - \tfrac{1}{4} \Sigma_{\beta,\gamma=1}^4 \, H_{\beta\gamma} \, (\partial B_{\beta\gamma} / \partial x_\alpha), \qquad (10.20)$$

$$(\alpha = 1, \ldots, 4)$$

or, with the help of (2.13),

$$k_{(P)\alpha}^M = \tfrac{1}{4} \Sigma_{\beta,\gamma=1}^4 \, M_{\beta\gamma} \, (\partial B_{\beta\gamma} / \partial x_\alpha) - \tfrac{1}{4} B_{\beta\gamma} \, (\partial M_{\beta\gamma} / \partial x_\alpha). \qquad (10.21)$$

$$(\alpha = 1, \ldots, 4)$$

From the considerations in chapter IV it is seen that one obtains from (10.21) for the form of the special terms occurring in the relativistic second law and connected with polarization and magnetization

$$\tfrac{1}{4} v' \, \Sigma_{\alpha,\beta=1}^4 \, M_{\alpha\beta} \, \mathrm{D} B_{\alpha\beta} - \tfrac{1}{4} v' \, \Sigma_{\alpha,\beta=1}^4 \, B_{\alpha\beta} \, \mathrm{D} M_{\alpha\beta}$$

(Cf. (I.2.25), (8.1) and (8.4)). This expression for the special terms has not the required form $\Sigma_{\alpha,\beta=1}^4 G'_{\alpha\beta} \, \mathrm{D} Z'_{\alpha\beta}$, neither is it a sum of such forms (Cf. § 2 of chapter IV). From a pure mathematical point of view it would be possible to determine a set of tensors $G_{\alpha\beta}^i$ and $Z_{\alpha\beta}^i$, given by expressions analogous to (IV.3.23) and (IV.3.24), such that

$$\Sigma_i \, \Sigma_{\alpha,\beta=1}^4 \, G_{\alpha\beta}'^i \, \mathrm{D} Z_{\alpha\beta}'^i = \tfrac{1}{4} v' \, \Sigma_{\alpha,\beta=1}^4 \, M_{\alpha\beta} \, \mathrm{D} B_{\alpha\beta} - \tfrac{1}{4} v' \, \Sigma_{\alpha,\beta=1}^4 \, B_{\alpha\beta} \, \mathrm{D} M_{\alpha\beta}.$$

It appears that this equation can only be satisfied if the quantities $\lambda'_{(k)}$ occurring in (IV.3.23) and (IV.3.24) depend on the acceleration of the matter. This, however, is rather unsatisfactory since these quantities occur in the relativistic second law of thermodynamics.

Hence, we conclude that from the point of view of the developed theory Abraham's tensor is preferable to Minkowski's tensor.

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LIST OF SYMBOLS

Roman superscripts between parentheses indicate chemical components. Greek subscripts without parentheses indicate tensor components. The meaning of a primed quantity is explained in § 2 of chapter I. The meaning of the symbols // and $\perp$ used as subscripts is explained in § 2 of chapter II for unprimed quantities and in § 4 of chapter II for primed quantities. Three-dimensional vectors are denoted in bold face type. In the following list one finds the symbols used in the text, their meaning and the sections where they have first been introduced.

ROMAN SYMBOLS

A	arbitrary Lorentz frame (IV.1)
A	affinity of De Donder (I.6)
$A^{(j)(k)}$	transformation matrix (II.9)
$A^{-1(j)(k)}$	transformation matrix (II.9)
a	constant fixing the zero point of the specific energy of the matter (I.3)
$a_{\alpha\beta}$	coefficients of a pure Lorentz transformation (II.4)
$\mathrm{B}_{\mathbf{r},t}$	barycentric Lorentz frame assigned to the position $\mathbf{r}$ at the time t (IV.1)
$B_{\alpha\beta}$	four-dimensional tensor describing the electromagnetic field (III.2)
$B^*_{\alpha\beta}$	auxiliary four-dimensional tensor (IV.3)
B^*_{α}	auxiliary four-vector (IV.3)
$\mathbf{B}$	magnetic field vector (III.2)
B_{α}	components of $\mathbf{B}$ (III.2)
c	velocity of light (I.2)
$c^{(j)}$	concentration of the chemical component j (I.2)
$\mathbf{D}$	electric field vector (III.2)
D_{α}	components of $\mathbf{D}$ (V.2)
$\mathbf{E}$	electric field vector (III.2)
E_{α}	components of $\mathbf{E}$ (III.2)
e	specific energy of the matter derived from $W_{\alpha\beta}$ (I.3)

$e_{(v)}$	density of the energy of the matter derived from $W_{\alpha\beta}$ (I.3)
$\bar{e}$	specific energy of the matter (related to e by (V.8.11)) (V.8)
$\tilde{e}$	specific energy of the matter derived from $\widetilde{W}_{\alpha\beta}$ (V.9)
$\tilde{e}_{(v)}$	density of the energy of the matter derived from $\widetilde{W}_{\alpha\beta}$ (V.9)
e^A	specific energy of the matter using Abraham's tensor (V.10)
$e^A_{(v)}$	density of the energy of the matter using Abraham's tensor (V.10)
$e_{(f)(v)}$	density of the energy of the electromagnetic field (V.5)
$\tilde{e}^*_{(v)}$	auxiliary Lorentz invariant quantity having the dimension of an energy per unit volume (V.9)
$e^{(j)}$	charge per unit of rest mass of the chemical component j (III.2)
$F^{(j)}_{\alpha\beta}$	four-dimensional tensor representing the "local" electric and magnetic fields to which the ions of the chemical component j are subjected (IV.5)
$\mathbf{F}^{(j)}$	force per unit of rest mass acting on the chemical component j (I.4)
$F^{(j)}_{\alpha}$	components of $\mathbf{F}^{(j)}$ (I.4)
$G_{\alpha\beta}$	four-dimensional tensor occurring in the second law as intensive variable (IV.2)
$G^*_{\alpha\beta}$	auxiliary four-dimensional tensor (IV.3)
G^*_{α}	auxiliary four-vector (IV.3)
$G^i_{\alpha\beta}$	auxiliary four-dimensional tensors (V.10)
$\mathfrak{g}$	density of momentum of the matter (I.3)
g_α	components of $\mathfrak{g}$ (I.3)
$\mathfrak{g}_{(f)}$	density of momentum of the electromagnetic field (V.4)
$g_{(f)\alpha}$	components of $\mathfrak{g}_{(f)}$ (V.4)
$\mathfrak{g}_{(t)}$	density of momentum of the system (III.3)
$H_{\alpha\beta}$	four-dimensional tensor describing the electromagnetic field (V.2)
$H^*_{\alpha\beta}$	auxiliary four-dimensional tensor (V.2)
H^*_{α}	auxiliary four-vector (V.2)
$\mathbf{H}$	magnetic field vector (III.2)
H_α	components of $\mathbf{H}$ (V.2)
$h^{(j)}$	partial specific enthalpy of the chemical component j (II.7)
$I^{(0)}_\alpha$	four-vector representing the density of heat flow (I.3)
$I^{(j)}_\alpha$	four-vector representing the density of the relative flow of matter of the chemical component j (I.2)
$I_{(s)\alpha}$	four-vector representing the density of the conductive flow of entropy (II.5)
$\mathbf{I}^{(0)}$	vector with components $I^{(0)}_1$, $I^{(0)}_2$ and $I^{(0)}_3$ (II.2)
$\mathbf{I}^{(j)}$	vector with components $I^{(j)}_1$, $I^{(j)}_2$ and $I^{(j)}_3$ (II.2)
$\mathbf{i}_\alpha$	unit vector in the direction of the positive α-axis in ordinary space (II.2)
i	imaginary unit (I.2)
$\mathbf{J}_{(e)}$	density of the energy flow of the matter (I.3)

$J_{(e)a}$ components of $\mathbf{J}_{(e)}$ (I.3)

$\mathbf{J}_{(s)}$ density of the conductive flow of entropy (II.5)

$J_{(s)a}$ components of $\mathbf{J}_{(s)}$ (II.5)

$\mathbf{J}_{(P)}$ Poynting vector (V.5)

$J_{(P)a}$ components of $\mathbf{J}_{(P)}$ (V.5)

$\mathbf{J}^{(0)}$ density of the heat flow (related to $\mathbf{I}^{(0)}$ by (II.2.14)) (II.2)

$J_a^{(0)}$ components of $\mathbf{J}^{(0)}$ (II.2)

$\mathbf{J}^{(j)}$ relative flow of matter of the chemical component j (related to $\mathbf{I}^{(j)}$ by (II.2.6)) (II.2)

$J_a^{(j)}$ components of $\mathbf{J}^{(j)}$ (II.2)

$\mathbf{J}^{*(0)}$ density of the heat flow (identical with $\mathbf{J}^{(0)}$) (II.9)

$J_a^{*(0)}$ components of $\mathbf{J}^{*(0)}$ (II.9)

$\mathbf{J}^{*(j)}$ density of the relative flow of matter of the chemical component j using $\mathbf{v}^*$ as reference velocity (II.9)

$J_a^{*(j)}$ components of $\mathbf{J}^{*(j)}$ (II.9)

$\mathbf{J}_{(q)}$ density of the heat flow (related to $\mathbf{J}^{(0)}$ by (II.2.16)) (II.2)

$J_{(c)}$ chemical reaction rate in mass per unit volume and per unit time (I.4)

$\mathbf{j}$ density of the electric current (III.2)

j_a components of $\mathbf{j}$ (III.2)

$K_a^{(j)}$ four-vector representing the force per unit mass on component j and the work done by this force per unit time (I.4)

k_a four-vector representing the force per unit volume exerted by the electromagnetic field on the matter and the work done by the electromagnetic field on the matter per unit volume and per unit time (IV.5)

k_a^* four-vector representing the force per unit volume exerted by the electromagnetic field on the matter and the work done by the electromagnetic field on the matter per unit volume and per unit time (related to k_a by (V.9.5)) (V.9)

$\tilde{k}_a$ four-vector representing the force per unit volume exerted by the electromagnetic field on the matter and the work done by the electromagnetic field on the matter per unit volume and per unit time (related to k_a by (V.9.25)) (V.9)

k_a^A four-vector representing the force per unit volume exerted by the electromagnetic field on the matter and the work done by the electromagnetic field on the matter per unit volume and per unit time (related to k_a by (IV.5.3) and (V.10.11)) (V.10)

k_a^M four-vector representing the force per unit volume exerted by the electromagnetic field on the matter and the work done by the electromagnetic field on the matter per unit volume and per unit time (related to k_a by (IV.5.3), (V.3.9) and (V.10.19)) (V.10)

$k_{(P)a}$ four-vector representing the ponderomotive force per unit volume

	and the work done on the matter by the ponderomotive forces per unit volume and per unit time (IV.5)
$k^*_{(P)a}$	four-vector representing the ponderomotive force per unit volume and the work done on the matter by the ponderomotive forces per unit volume and per unit time (related to $k_{(P)a}$ by (V.9.6)) (V.9)
$\tilde{k}_{(P)a}$	four-vector representing the ponderomotive force per unit volume and the work done on the matter by the ponderomotive forces per unit volume and per unit time (related to $k_{(P)a}$ by (V.9.26)) (V.9)
$k^A_{(P)a}$	four-vector representing the ponderomotive force per unit volume and the work done on the matter by the ponderomotive forces per unit volume and per unit time (related to $k_{(P)a}$ by (V.10.12)) (V.10)
$k^M_{(P)a}$	four-vector representing the ponderomotive force per unit volume and the work done on the matter by the ponderomotive forces per unit volume and per unit time (related to $k_{(P)a}$ by (V.3.9) and (V.10.21)) (V.10)
$k^{*A}_{(P)a}$	auxiliary four-vector (V.10)
$k^{**A}_{(P)a}$	auxiliary four-vector (V.10)
$\mathbf{k}$	vector with components k_1, k_2 and k_3 (V.6)
$\mathbf{k}_{(P)}$	vector with components $k_{(P)1}$, $k_{(P)2}$ and $k_{(P)3}$ (V.6)
$\mathbf{k}_{(\mathbf{EP})}$	force per unit volume exerted by the electric field on the matter in consequence of the polarization of the medium (V.6)
$\mathbf{k}_{(\mathbf{BM})}$	force per unit volume exerted by the magnetic field on the matter in consequence of the magnetization of the medium (V.6)
$\mathbf{k}_{(\mathbf{EM})}$	force per unit volume exerted by the electric field on the matter in consequence of the magnetization of the medium (V.6)
$\mathbf{k}_{(\mathbf{BP})}$	force per unit volume exerted by the magnetic field on the matter in consequence of the polarization of the medium (V.6)
L	Lorentz invariant phenomenological coefficient for the chemical reaction (I.7)
$L_{(c)(p)}$	Lorentz invariant phenomenological coefficient for visco-chemical effects (I.7)
$L_{(p)(c)}$	Lorentz invariant phenomenological coefficient for visco-chemical effects (I.7)
$L^{(j)(k)}$	Lorentz invariant phenomenological coefficients for vectorial fluxes (I.7)
$L^{(j)(k)}_{\alpha\beta}$	four-dimensional phenomenological tensors for the vectorial fluxes $I^{(j)}_\alpha$, using $Y^{(j)}_\alpha$ as affinity (I.7)
$\tilde{L}^{*(j)(k)}_{\alpha\beta}$	four-dimensional phenomenological tensors for the vectorial fluxes $I^{(j)}_\alpha$, using $\tilde{Y}^{(j)}_\alpha$ as affinity (IV.9)
$\tilde{L}^{(j)(k)}_{\alpha\beta}$	four-dimensional phenomenological tensors for the vectorial fluxes $I^{(j)}_\alpha$, using $\tilde{Y}^{(j)}_\alpha$ as affinity (related to $\tilde{L}^{*(j)(k)}_{\alpha\beta}$ by (IV.9.8)) (IV.9)

$\bar{L}^{(j)(k)}_{\alpha\beta}$ three-dimensional phenomenological tensors for the vectorial fluxes $\mathbf{J}^{(j)}$ $(j = 0, 1, \ldots, n)$ (II.7)

$L^{*(j)(k)}_{\alpha\beta}$ three-dimensional phenomenological tensors for the vectorial fluxes $\mathbf{J}^{*(j)}$ $(j = 0, 1, \ldots, n)$ (II.9)

$L_{\alpha\beta\gamma\zeta}$ four-dimensional phenomenological tensor for viscous flow using $(\partial u_\gamma/\partial x_\zeta)$ as affinity (I.7)

$L^{*}_{\alpha\beta\gamma\zeta}$ four-dimensional phenomenological tensor for viscous flow using $Y_{\gamma\zeta}$ as affinity (IV.10)

$L^{**}_{\alpha\beta\gamma\zeta}$ four-dimensional phenomenological tensor for viscous flow using $Y_{\gamma\zeta}$ as affinity (related to $L^{*}_{\alpha\beta\gamma\zeta}$ by (IV.10.11)) (IV. 10)

$L^{***}_{\alpha\beta\gamma\zeta}$ four-dimensional phenomenological tensor for viscous flow using $Y_{\gamma\zeta}$ as affinity (related to $L^{**}_{\alpha\beta\gamma\zeta}$ by (IV.10.9)) (IV.10)

$M_{\alpha\beta}$ four-dimensional tensor representing the polarization and magnetization of the matter (IV.2)

$M^{*}_{\alpha\beta}$ auxiliary four-dimensional tensor (IV.3)

M^{*}_{α} auxiliary four-vector (IV.3)

$M^{(j)}_{(0)(k)}$ rest mass of particle k of the chemical component j (I.2)

$\mathbf{M}$ polarization vector (IV.2)

M_α components of $\mathbf{M}$ (IV.2)

m auxiliary Lorentz invariant quantity (I.2)

m_α four-vector representing the density of the total flow of rest mass and the total density of rest mass (I.2)

$m^{(j)}_{\alpha}$ four-vector representing the density of the flow of rest mass of the chemical component j and the density of rest mass of the chemical component j (I.2)

$N^{(j)}$ number of atomic particles of the chemical component j per unit volume (I.2)

n number of the chemical components (I.2)

P auxiliary quantity having the dimension of a pressure (II.2)

$P_{\alpha\beta}$ four-dimensional tensor representing the viscous stresses (I.3)

$\bar{P}_{\alpha\beta}$ four-dimensional tensor representing the ordinary viscous stresses (I.6)

$\mathbf{P}$ polarization vector (IV.2)

P_α components of $\mathbf{P}$ (IV.2)

p hydrostatic pressure following from $W_{\alpha\beta}$ (I.3)

p^* hydrostatic pressure following from $W^{*}_{\alpha\beta}$ (V.9)

p^A hydrostatic pressure following from $W^{A}_{(f)\alpha\beta}$ (V.10)

$\bar{p}$ auxiliary quantity having the dimension of a pressure (V.9)

$Q^{(j)}$ heat of transfer of the chemical component j using the flows $m^{(j)}_{\alpha}$ $(j = 1, \ldots, n)$ (II.6)

$Q^{*(j)}$ heat of transfer of the chemical component j using the fluxes $\mathbf{J}^{(j)}$ $(j = 1, \ldots, n-1)$ (II.6)

$\bar{Q}^{(j)}$ heat of transfer of the chemical component j using the flows $\varrho^{(j)}\,\mathbf{v}^{(j)}$ $(j = 1, \ldots, n)$ (II.6)

$\mathbf{r}$	position vector (II.5)
S_a	four-vector representing the density of the flow of entropy and the density of entropy (II.5)
s	specific entropy (I.4)
$s_{(v)}$	density of entropy (II.5)
$s^{(j)}$	partial specific entropy of the chemical component j (II.7)
T	temperature (I.4)
t	time (I.2)
$t_{\alpha\beta}$	three-dimensional stress tensor (I.3)
u_a	four-vector representing the barycentric velocity (I.2)
V	volume (II.5)
$\mathbf{v}$	barycentric velocity (I.2)
v_a	components of $\mathbf{v}$ (I.2)
$\mathbf{v}^{(j)}$	velocity of the chemical component j (I.2)
$v_a^{(j)}$	components of $\mathbf{v}^{(j)}$ (I.2)
$\mathbf{v}_{(r)}$	relative velocity of two Lorentz frames (II.4)
$v_{(r)a}$	components of $\mathbf{v}_{(r)}$ (II.4)
$\mathbf{v}^*$	linear combination of the velocities of the chemical components (II.9)
$\mathbf{v}^{**}$	linear combination of the velocities of the chemical components (II.9)
v	specific volume (I.2)
$v^{(j)}$	partial specific volume of the chemical component j (II.7)
$W_{\alpha\beta}$	energy-momentum tensor of the matter (I.3)
$W^*_{\alpha\beta}$	energy-momentum tensor of the matter (related to $W_{\alpha\beta}$ by (V.9.1)) (V.9)
$\widetilde{W}_{\alpha\beta}$	energy-momentum tensor of the matter (related to $W_{\alpha\beta}$ by (V.9.21)) (V.9)
$W_{(f)\alpha\beta}$	energy-momentum tensor of the electromagnetic field (III.3)
$W^*_{(f)\alpha\beta}$	energy-momentum tensor of the electromagnetic field (related to $W_{(f)\alpha\beta}$ by (V.9.2)) (V.9)
$\widetilde{W}_{(f)\alpha\beta}$	energy-momentum tensor of the electromagnetic field (related to $W_{(f)\alpha\beta}$ by (V.9.22)) (V.9)
$W^A_{(f)\alpha\beta}$	energy-momentum tensor of Abraham (V.7)
$W^M_{(f)\alpha\beta}$	energy-momentum tensor of Minkowski (V.7)
$W^{**}_{(f)\alpha\beta}$	auxiliary four-dimensional tensor (V.9)
$W^{***}_{(f)\alpha\beta}$	energy-momentum tensor of the electromagnetic field (related to $W_{(f)\alpha\beta}$ by (V.9.37)) (V.9))
$W_{(t)\alpha\beta}$	energy-momentum tensor of the system (III.3)
$w_{\alpha\beta}$	four-dimensional tensor representing the stresses, derived from $W_{\alpha\beta}$ (I.3)
$w^*_{\alpha\beta}$	four-dimensional tensor representing the stresses, derived from $W^*_{\alpha\beta}$ (V.9)

$\mathbf{X}^{(0)}$	affinity conjugate to $\mathbf{J}^{(0)}$ (II.3)
$X_\alpha^{(0)}$	components of $\mathbf{X}^{(0)}$ (II.3)
$\mathbf{X}^{(j)}$	affinity conjugate to $\mathbf{J}^{(j)}$ (II.3)
$X_\alpha^{(j)}$	components of $\mathbf{X}^{(j)}$ (II.3)
$\overline{\mathbf{X}}^{(0)}$	affinity conjugate to $\mathbf{J}^{(0)}$ (related to $\mathbf{X}^{(0)}$ by (II.5.20)) (II.5)
$\overline{X}_\alpha^{(0)}$	components of $\overline{\mathbf{X}}^{(0)}$ (II.7)
$\overline{\mathbf{X}}^{(j)}$	affinity conjugate to the relative flow of matter $\mathbf{J}^{(j)}$ (related to $\mathbf{X}^{(j)}$ by (II.5.20)) (II.5)
$\overline{X}_\alpha^{(j)}$	components of $\overline{\mathbf{X}}^{(j)}$ (II.7)
$\mathbf{X}^{*(0)}$	identical with $\overline{\mathbf{X}}^{(0)}$ (II.9)
$X_\alpha^{*(0)}$	components of $\mathbf{X}^{*(0)}$ (II.9)
$\mathbf{X}^{*(j)}$	affinity conjugate to $\mathbf{J}^{*(j)}$ (II.9)
$X_\alpha^{*(j)}$	components of $\mathbf{X}^{*(j)}$ (II.9)
x_α	four-vector representing position and time in the four-dimensional space-time continuum (I.2)
$Y_\alpha^{(0)}$	four-vector conjugate as affinity to $I_\alpha^{(0)}$ (I.6)
$Y_\alpha^{(j)}$	four-vector conjugate as affinity to $I_\alpha^{(j)}$ (I.6)
$\tilde{Y}_\alpha^{(0)}$	four-vector conjugate as affinity to $I_\alpha^{(0)}$ (related to $Y_\alpha^{(0)}$ by (IV.9.2)) (IV.9)
$\tilde{Y}_\alpha^{(j)}$	four-vector conjugate as affinity to $I_\alpha^{(j)}$ (related to $Y_\alpha^{(j)}$ by (IV.9.2)) (IV.9)
$Y_{\alpha\beta}$	four-dimensional tensor conjugate as affinity to the ordinary viscous pressure tensor (IV.10)
$Z_{\alpha\beta}$	four-dimensional tensor occurring in the second law of thermodynamics as extensive variable (IV.2)
$Z_{\alpha\beta}^*$	auxiliary four-dimensional tensor (IV.3)
Z_α^*	auxiliary four-vector (IV.3)
$Z_{\alpha\beta}^i$	auxiliary four-dimensional tensors (V.10)

GREEK SYMBOLS

Γ	auxiliary Lorentz invariant quantity (IV.3)
$\Delta_{\alpha\beta}$	auxiliary four-dimensional tensor (I.2)
$\delta_{\alpha\beta}$	Kronecker tensor (three-dimensional and four-dimensional) (I.2)
$\delta(\alpha;\beta)$	Kronecker symbol (I.2)
ε	dielectric constant (IV.3)
$\zeta^{(j)}$	auxiliary quantity belonging to the chemical component j (II.9)
η	ordinary viscosity (I.7)
$\eta_{(v)}$	volume viscosity (I.7)
$\vartheta'^{(j)}$	angle between $\mathbf{v}'^{(j)}$ and $\mathbf{v}$ measured by an observer in the barycentric Lorentz frame (II.8)
$\varkappa$	auxiliary quantity (II.4)
λ	coefficient of heat conduction in the stationary state (II.6)

$\lambda_{(j)}$ auxiliary quantities (IV.3)

$\lambda^{(j)}_{(k)}$ auxiliary quantities (IV.5)

μ magnetic permeability (IV.3)

$\mu^{(j)}$ partial specific Gibbs function of the chemical component j (I.4)

$\nu^{(j)}$ Lorentz invariant quantity proportional to the stoechiometric number of component j in the chemical reaction (I.4)

Ξ arbitrary quantity (II.5)

$\xi^{(j)}$ auxiliary quantity occurring in the expression for $\mathbf{v}^*$ (II.9)

Π Lorentz invariant viscous pressure (I.6)

ϱ total density of rest mass (I.2)

$\varrho^{(j)}$ density of rest mass of the chemical component j (I.2)

$\varrho^{(j)}_{(0)}$ density of rest mass of the chemical component j measured by an observer moving with this component (I.2)

$\varrho_{(el)}$ density of electric charge (III.2)

σ entropy production per unit volume and per unit time (II.5)

$\sigma_{(v)}$ contribution of the viscous flow to σ (IV.10)

$\sigma_{(h)(d)}$ contribution of heat conduction and diffusion to σ (II.5)

$\sigma_{(c)(v)}$ contribution of the chemical reaction and the volume viscosity to σ (II.5)

Φ_a four-vector occurring in the expression for $k_{(P)a}$ (IV.8)

φ that part of the work done by the electromagnetic field on the medium per unit volume and per unit time which is used to change the internal energy e' of the matter (V.8)

χ electric conductivity (III.8)

Ψ_a four-vector of Minkowski ("Ruhstrahlvector") (V.7)

Ψ^*_a auxiliary four-vector (V.7)

Ω auxiliary Lorentz invariant quantity (IV.3)

$\omega^{(j)}$ auxiliary Lorentz invariant quantity belonging to the chemical component j (I.5)

VECTOR NOTATION AND OPERATORS

$\mathbf{a}\,.\,\mathbf{b} = \Sigma_{a=1}^{3}\ a_a b_a$

$\mathbf{a} \wedge \mathbf{b} = (a_2 b_3 - a_3 b_2)\mathbf{i}_1 + (a_3 b_1 - a_1 b_3)\mathbf{i}_2 + (a_1 b_2 - a_2 b_1)\mathbf{i}_3$

$|\mathbf{a}| = (\Sigma_{a=1}^{3}\ a_a^2)^{\frac{1}{2}}$

$\text{grad}\ \Xi = \Sigma_{a=1}^{3}\ (\partial \Xi/\partial x_a)\mathbf{i}_a$

$\text{div}\ \mathbf{a} = \Sigma_{a=1}^{3}\ (\partial a_a/\partial x_a)$

$\text{rot}\ \mathbf{a} = \{(\partial a_3/\partial x_2) - (\partial a_2/\partial x_3)\}\,\mathbf{i}_1 + \{(\partial a_1/\partial x_3) - (\partial a_3/\partial x_1)\}\,\mathbf{i}_2$
$\quad + \{(\partial a_2/\partial x_1) - (\partial a_1/\partial x_2)\}\,\mathbf{i}_3$

(Grad $\mathbf{a}$) . $\mathbf{b}$ is defined by (V.6.10)

D substantial derivative with respect to time defined by (I.2.25)

d/dt substantial derivative with respect to time defined by (II.3.10) (except in the formulae (III.3.10), (V.4.11), (V.4.12) and (V.4.13) where (d/dt) is the ordinary derivative with respect to time)

SOMMAIRE

Nous nous proposons dans cette thèse de développer en premier lieu la thermodynamique relativiste des phénomènes irréversibles dans un mélange continu d'un nombre arbitraire de constituants chimiques. De plus nous étudions le tenseur d'impulsion et d'énergie du champ électromagnétique macroscopique.

Nous nous limitons à la théorie de la relativité restreinte et nous supposons qu'il y a ni création ni annihilation de particules atomaires. La validité de la théorie est limitée par la condition que, pour un observateur se déplaçant avec la vitesse barycentrique, les variations en température, pression etc. doivent être petites sur des distances comparables au libre parcours moyen des molécules.

Dans le premier chapitre nous donnons la théorie de systèmes qui sont soumis à l'action de forces ne dépendant pas des vitesses des constituants chimiques. La théorie est présentée en forme tensorielle quadridimensionnelle. En premier lieu nous introduisons quelques notions servant de point de départ pour le développement de la théorie. Les quadrivecteurs qui représentent les flux relatifs de la matière et le flux calorifique sont définis de telle façon qu'ils sont perpendiculaires au quadrivecteur représentant la vitesse barycentrique. Les tenseurs quadridimensionnels qui représentent les tensions mécaniques possèdent des propriétés d'orthogonalité semblables. Le bilan d'entropie est déduit des lois relativistes fondamentales de la physique macroscopique. Nous montrons que dans des systèmes isotropes les relations d'Onsager sont invariantes pour des transformations de Lorentz. De plus nous trouvons un effet croisé nouveau entre la diffusion et la conduction thermique. Il apparaît que, par suite de cet effet, le phénomène de diffusion est influencé par le mouvement barycentrique.

Les résultats obtenus dans le chapitre premier sont discutés en plus grand détail dans le second chapitre. En outre ce chapitre contient des considérations sur les chaleurs de transport et sur quelques quantités qui sont presque invariantes pour des transformations de Lorentz. Les résultats de la théorie concernant la conduction thermique et la diffusion sont reformulés en forme de tenseurs tridimensionnels à l'aide de quantités qui sont utilisées aussi dans la théorie non-relativiste. Dans la théorie présentée ici la densité d'entropie

est la quatrième composante d'un quadrivecteur et il apparaît qu'en général l'entropie dans un élément de volume n'est pas invariante pour des transformations de Lorentz. Nous discutons aussi la relation entre les différentes chaleurs de transport qui figurent dans la littérature. Une formulation de la théorie à l'aide de flux relatifs de la matière, qui sont définis par rapport à une vitesse différente de la vitesse barycentrique, est déduite du formalisme développé.

Dans le troisième chapitre nous considérons des systèmes sans polarisation ou magnétisation dans un champ électromagnétique. Nous arrivons au bilan d'entropie par une méthode qui diffère un peu de celle utilisée dans le premier chapitre. Les relations phénoménologiques pour des milieux isotropes sont données en forme des tenseurs tridimensionnels et quadridimensionnels. La loi relativiste d'Ohm est un cas spécial des équations générales obtenues pour les phénomènes de diffusion. Il apparaît que le courant électrique ne dépend pas seulement des vecteurs des champs électriques et magnétiques ainsi que des gradients de la température et des potentiels chimiques des constituants, mais aussi des dérivées partielles par rapport au temps des deux dernières quantités et de l'accélération barycentrique.

La théorie thermodynamique des systèmes polarisés et magnétisés est donnée dans le quatrième chapitre. Nous ne discutons que les systèmes qui sont isotropes quant à leur polarisation et leur magnétisation. Au cas où le milieu est polarisé et magnétisé, des termes supplémentaires s'ajoutent à la formule non-relativiste de Gibbs. Nous donnons en premier lieu la formule relativiste de Gibbs pour le cas considéré. Si l'on veut déduire une forme satisfaisante pour le bilan d'entropie il apparaît que l'expression explicite des forces pondéromotrices doit avoir un rapport étroit avec l'expression de la formule de Gibbs. Les relations phénoménologiques et les relations d'Onsager sont données pour des systèmes anisotropes quant aux phénomènes irréversibles.

Dans le cinquième chapitre nous considérons le tenseur d'impulsion et d'énergie du champ électromagnétique macroscopique. Nous continuons aussi la discussion des deux principes de la thermodynamique et des forces macroscopiques que le champ électromagnétique exerce sur la matière. Comme dans le quatrième chapitre nous nous bornons aux systèmes qui sont isotropes quant à la polarisation et la magnétisation. Nous montrons qu'il est possible de trouver un tenseur symétrique d'impulsion et d'énergie pour le champ électromagnétique macroscopique. Les éléments non-diagonaux de ce tenseur sont égaux aux éléments correspondants du tenseur d'A b r a h a m. Nous montrons ensuite que le tenseur d'A b r a h a m correspond à un formalisme en tous points équivalent. Enfin nous montrons que le tenseur d'A b r a h a m est préférable à celui-ci de M i n k o w s k i du point de vue de notre théorie.

SAMENVATTING

Het hoofddoel van dit proefschrift is de relativistische thermodynamica te ontwikkelen van de irreversibele processen in een continu mengsel, dat bestaat uit een willekeurig aantal chemische componenten. Het nevendoel is de energie-impulstensor te onderzoeken van het macroscopische electromagnetische veld.

We hebben ons beperkt tot de speciale relativiteitstheorie. Verder hebben we aangenomen, dat er geen atomistische deeltjes gecreëerd worden of verdwijnen. De geldigheid van de thermodynamische theorie wordt begrensd door de voorwaarde, dat, voor een waarnemer, die met de barycentrische snelheid meebeweegt, de verschillen in druk, temperatuur enz. klein moeten zijn over een afstand, die vergelijkbaar is met de gemiddelde vrije weglengte van de moleculen.

In hoofdstuk I ontwikkelen we de theorie voor systemen, die beïnvloed worden door krachten, die niet afhangen van de snelheden van de chemische componenten. De theorie wordt geformuleerd met behulp van vierdimensionale tensorrekening. De vierdimensionale vectoren, die de relatieve materiestromen en de warmtestroom representeren, worden zo gedefinieerd, dat ze loodrecht staan op de vierdimensionale vector, die de barycentrische snelheid voorstelt. De tensoren, die de druk representeren, bezitten eveneens dergelijke orthogonaliteitseigenschappen. Uit de relativistische macroscopische fundamentele wetten (te weten: de tweede hoofdwet van de thermodynamica en de balansvergelijkingen voor rustmassa, impuls en energie) wordt de entropiebalans afgeleid. De fenomenologische vergelijkingen worden gegeven voor isotrope media en er wordt aangetoond, dat de Onsager-relaties Lorentz-invariant zijn. Een nieuw kruiseffect wordt gevonden, dat voortkomt uit een relativistische term in de affiniteit geconjugeerd aan de warmtestroom. Het blijkt, dat door dit kruiseffect de diffusieverschijnselen beïnvloed worden door de barycentrische beweging.

Voor zover het de warmtegeleiding, de diffusie en de entropie betreft, worden de resultaten van de in hoofdstuk I gegeven theorie verder uitgewerkt in hoofdstuk II. Bovendien bevat dit hoofdstuk beschouwingen over de transportwarmten en enkele grootheden, die bijna Lorentz-invariant zijn. De resultaten van de in hoofdstuk I gegeven theorie betreffende warmtegeleiding en

diffusie worden geformuleerd in driedimensionale tensorvorm met behulp van grootheden, die in de niet-relativistische theorie gebruikt worden. Er worden formules afgeleid waaruit het verschil tussen de resultaten van de relativistische en de niet-relativistische theorie gemakkelijk kan worden overzien. Uit de ontwikkelde theorie volgt, dat de entropie in een klein volume-element in het algemeen niet een Lorentz-invariante grootheid is. Dit resultaat verschilt van dat van Planck en Einstein volgens hetwelk de entropie in een klein volume-element wel Lorentz-invariant is. Het verband tussen de verschillende in de literatuur voorkomende definities van de transportwarmten wordt afgeleid. Enkele grootheden, die in het formalisme voorkomen, blijken bijna Lorentz-invariant te zijn. Een formulering van de theorie, die gebruik maakt van relatieve materiestromen, die gedefinieerd zijn ten opzichte van een andere referentiesnelheid dan de barycentrische snelheid, wordt afgeleid uit het ontwikkelde formalisme.

Hoofdstuk III handelt over systemen, zonder polarisatie en magnetisatie, die beïnvloed worden door een electromagnetisch veld. De entropiebalans moet voor dit geval uit de fundamentele relativistische vergelijkingen worden afgeleid door middel van een methode, die in sommige opzichten verschilt van die welke in hoofdstuk I gebruikt werd. De fenomenologische vergelijkingen worden gegeven in vierdimensionale en driedimensionale tensorvorm. Ook de Onsager-relaties worden weer besproken. De relativistische wet van Ohm blijkt een speciaal geval te zijn van de algemene vergelijkingen, die verkregen zijn voor de diffusieverschijnselen. Het blijkt, dat de electrische stroom niet alleen afhangt van de electrische en magnetische veldvectoren en van de gradienten van de temperatuur en van de partiële specifieke Gibbs-potentialen van de chemische componenten, maar, dat hij ook afhangt van de locale afgeleiden naar de tijd van de beide laatstgenoemde grootheden en van de barycentrische versnelling.

De thermodynamische theorie voor systemen met polarisatie en magnetisatie wordt ontwikkeld in hoofdstuk IV. We hebben ons beperkt tot systemen, die isotroop zijn voor zover het polarisatie en magnetisatie betreft. In het geval, dat het systeem gepolariseerd en gemagnetiseerd is, treden er in de niet-relativistische tweede hoofdwet van de thermodynamica termen op, die een gevolg zijn van de polarisatie en de magnetisatie van de materie. In dit hoofdstuk leiden we nu eerst de relativistische tweede hoofdwet af voor het beschouwde geval. Het blijkt, dat er een nauw verband moet bestaan tussen de expliciete uitdrukking voor de ponderomotorische kracht en de gedaante van de relativistische tweede hoofdwet van de thermodynamica indien men een bevredigende vorm wil verkrijgen voor de entropiebalans. De fenomenologische vergelijkingen en de Onsager-relaties worden gegeven voor media, die anisotroop zijn voor zo ver het de irreversibele processen betreft.

In hoofdstuk V zijn de resultaten gegeven van het onderzoek, dat betrekking heeft op de energie-impulstensor van het macroscopische electromagnetische veld. Bovendien bevat dit hoofdstuk verdere discussies over de eerste en tweede hoofdwet van de thermodynamica. Als in hoofdstuk IV hebben we ons beperkt tot systemen, die isotroop zijn voor zover het polarisatie en magnetisatie betreft. Om onze beschouwingen zo algemeen mogelijk te houden voerden we in hoofdstuk IV verschillende grootheden in, die we niet nader specificeerden. Het blijkt nu, dat het mogelijk is om zodanige keuzen te maken voor de bovenbedoelde grootheden, dat een explicite uitdrukking kan worden afgeleid voor een symmetrische energie-impulstensor van het macroscopische electromagnetische veld. De niet-diagonaalelementen van de op deze wijze gevonden tensor zijn gelijk aan de corresponderende elementen van de energie-impulstensor, die Abraham aan het electromagnetische veld toekent. Er wordt verder aangetoond, dat Abraham's tensor tot een gelijkwaardig formalisme leidt. Het blijkt echter, dat de vorm voor de relativistische tweede hoofdwet, die volgt uit het formalisme met Abraham's tensor, correspondeert met een tamelijk ongebruikelijke gedaante voor de niet-relativistische tweede hoofdwet van de thermodynamica. Tenslotte wordt er aangetoond, dat, vanuit het gezichtspunt van de ontwikkelde theorie, Abraham's tensor te prefereren is boven de energie-impulstensor, die Minkowski aan het macroscopische electromagnetische veld toekent.

STELLINGEN

I

De opmerkelijk hoge waarden voor log *ft*, die men vindt bij de β-desintegratie van ^{56}Co en ^{60}Co, voor overgangen met $\Delta I = 0$ of ± 1 (zonder pariteitsverandering), wijzen er op, dat er, behalve de gewone selectieregels, nog bijzondere selectieregels moeten zijn. Het is niet uitgesloten, dat hier nog een andere selectieregel een rol speelt dan die voor het ,,benaderde'' quantumgetal *l*.

II

Bij de gebruikelijke wijze waarop men de door een gloeiend gas geëmitteerde spectraallijnen berekent, veronderstelt men, dat een atoom in het gas aanvankelijk in een stationaire toestand is. Het behoeft echter nadere verklaring waarom deze veronderstelling tot het juiste resultaat leidt.

III

Het is zowel uit didactisch als uit methodisch oogpunt ongewenst om in een leerboek over quantummechanica aanvankelijk als ladingwolk te interpreteren hetgeen als waarschijnlijkheidsdichtheid behoort te worden opgevat. De winst aan ,,aanschouwelijkheid'' weegt niet op tegen het offer van de juistheid der interpretatie.

F. Hund, Materie als Feld, Springer, Berlin (1954).

IV

Het zou voor het inzicht in het verband tussen de moleculaire polarisatie en de diëlectrische constanten van stoffen met zeer grote dichtheden van belang zijn het model van het molecuul in een doosje aan te vullen met een statistische celtheorie.

A. Michels, J. de Boer en A. Bijl, Physica **4** (1937) 981.
S. R. de Groot en C. A. ten Seldam, Physica **12** (1946) 669.

V

In tegenstelling met hetgeen Tamm beweert, kan de dichtheid van de macroscopische kracht, die het electromagnetische veld op de materie uitoefent, niet opgevat worden als een gemiddelde van de dichtheid van de microscopische kracht, die door het electromagnetische veld op de materie wordt uitgeoefend

C. Møller, The theory of relativity, Oxford University Press, London (1952).

VI

Om de energie-impulstensor van het macroscopische electromagnetische veld in anisotrope media te kunnen bestuderen is het wenselijk eerst de in hoofdstuk IV van dit proefschrift gegeven thermodynamische theorie uit te breiden tot systemen, die, ook wat betreft polarisatie en magnetisatie, anisotroop zijn.

VII

De energie-impulstensor welke Abraham aan het macroscopische electromagnetische veld toekent verdient de voorkeur boven die, welke Minkowski er aan toekent.

Hoofdstuk V van dit proefschrift.

VIII

Pauli merkt op, dat de door hem gegeven vorm voor de energie-impulstensor van Abraham geldt voor homogene isotrope media. Deze uitdrukking is echter ook geldig voor inhomogene isotrope media.

W. Pauli, Relativitätstheorie (Encyclopädie der Mathematischen Wissenschaften, Band V, Teil 2), B. G. Teubner, Leipzig (1904–1922).
Hoofdstuk V van dit proefschrift.

IX

In de relativistische thermostatica van Planck en Einstein is de entropie binnen een volume-element een Lorentz-invariante grootheid. Dit is echter niet meer het geval indien er warmtegeleiding of diffusie optreedt.

Hoofdstuk II van dit proefschrift.

X

Voor de in de tweede hoofdwet van de thermodynamica (relatie van Gibbs) optredende extensieve variabelen, die aan balansvergelijkingen voldoen, is er voor de wijze van transformeren bij Lorentz-transformaties een geringere mate van vrijheid dan voor de intensieve variabelen, die voorkomen in de genoemde wet.

XI

De afleiding van Callen, Barasch en Jackson voor de Onsager-relaties leidt niet alleen tot de verlangde betrekking $L_{jk} = L_{kj}$, doch tevens tot $L_{jk} = 0$. De redenering, die ze geven om aan dit ongewenste resultaat te ontkomen, bereikt dat doel niet.

H. B. Callen, M. L. Barasch en J. L. Jackson, Phys. Rev. **88** (1952) 1382.

XII

De thermodynamica van de irreversibele processen in enkelvoudige stoffen met inwendig impulsmoment kan worden uitgebreid tot stoffen, die uit een willekeurig aantal chemische componenten bestaan.

XIII

Het verdient aanbeveling om na te gaan of het mogelijk is in ons land supra-universitaire cursussen te organiseren om het steeds groter wordende verschil te kunnen overbruggen tussen de stand van de natuurkundige wetenschap en het wetenschappelijk niveau, dat vereist is voor het afleggen van een doctoraal examen met natuurkunde als hoofdvak.

XIV

Het is onjuist om in de Europese Gemeenschap voor Kolen en Staal (Plan Schuman) een voorbeeld te zien van een ontwikkeling naar internationale beheers- respectievelijk bezitsorganen.

Irène Scizier, De Europese Kolen- en Staalgemeenschap (Cahier uit de serie Documentatie over Europa).

De weg naar vrijheid, N.V. De Arbeiderspers, Amsterdam (1951).

XV

De verdere ontwikkeling van de muziek zal waarschijnlijk eerder gaan in de richting van nieuwe modulaties binnen het kader van de oude tonaliteit en in de richting van nieuwe toonladders dan in die van bi- of atonaliteit. (Vergelijk respectievelijk de latere werken van Fauré en Badings.)