On the construction of supergravity theories

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ON THE CONSTRUCTION OF SUPERGRAVITY THEORIES

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Aan allen die mijn studie mogelijk hebben gemaakt

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CHAPTER I

SUPERSYMMETRY AND SUPERGRAVITY I.

1. Introduction

Supergravity is a theory of fundamental interactions. Basically it describes gravitation and a new interaction mediated by a fermionic field, called the gravitino field. This results in a special symmetry between them, supersymmetry, whence supergravity derives its name.

Though all experimental data on gravitation are in good agreement with the general theory of relativity, we wish to point out, that there exist a number of good reasons for an attempt to extend this theory. We will discuss each of these in turn.

In the first place general relativity has not been shown to be a consistent quantum theory, as the other theories of fundamental interactions are. As a classical theory it describes very well the macroscopic phenomena of gravitation, but on the microscopic level its character and its relation to the other basic processes are unclear.

Three other fundamental forces are known in nature: the electromagnetic, the weak and the strong interactions. These interactions are thought to be transmitted by the quanta of certain fields. For the electromagnetic interactions these are the photons; for weak interactions they are the so called massive vector bosons, while the strong forces are mediated by massless gluons. The last ones are responsible for the binding of quarks inside the proton and other heavy particles.

In view of this it seems desirable that gravitation should also be described in terms of a quantized field. The quanta of this field will be called gravitons; they are massless and carry two units of spin. However, attempts to construct this theory directly from general relativity have not been very successful. In particular the technical complications of renormalization, which is a procedure necessary to avoid infinite results in quantum field theories, have not been overcome. In supergravity on the other hand these complications are often absent, at least in the lower order approximations.

A second reason to study supergravity lies in the prospect of constructing a theory which unifies gravitation with other interactions. The three types of fundamental forces discussed above are all described by so called gauge theories. These field theories exhibit very special symmetries, which may allow their unification into one theory. This has for example been accomplished successfully for the weak and electromagnetic interactions in the Weinberg-Salam model.

It is a very attractive idea to incorporate the strong interactions and gravitation in such a theory as well. This theory would then describe all known elementary processes in nature and would contain a minimum number of free parameters. Thus it would possess great predictive power. At present supergravity is the only theory which offers prospects in this direction.

Finally supergravity is a very interesting theory because it exhibits the property of supersymmetry. This is a symmetry between the two basic classes of particles in nature: the bosons, which carry an integral number of units of spin, and the fermions, with half integral spin. In all existing theories of fundamental processes there is a complete dichotomy between bosons and fermions. Only supersymmetry is able to overcome this and treat both on an equal basis.

In order to achieve supersymmetry one has to match the bosonic graviton of general relativity with a fermionic spin $\frac{3}{2}$ field. This field is called the gravitino field, and the combined theory of this field coupled to general relativity constitutes supergravity.

We would like to stress here, that supergravity encompasses general relativity, and therefore is not in contradiction with experimental evidence. In fact the new features of this theory lie not primarily in its description of gravitation, but in its possible incorporation of other interactions and the predictions on these interactions that may be derived from it. Thus it is essentially a theory of particle physics and its applications will presumably be found in the realm of microscopic phenomena.

2. Supersymmetry

In this section we will illustrate the concept of supersymmetry in a simple example. This will then be used to discuss some general features of supersymmetric field theories.

Supersymmetry is a symmetry between bosons and fermions. Therefore any supersymmetric theory must contain both kinds of fields. A simple model is provided by the scalar multiplet, which consists of a scalar field A, a pseudoscalar B and a real (Majorana) spinor field ψ *). Therefore this set of fields has two spin 0 and two spin $\frac{1}{2}$ degrees of freedom. The Lagrangian for these fields is simply

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}\bar{\psi}\not{\partial}\psi - \tfrac{1}{2}m^2A^2 - \tfrac{1}{2}m^2B^2 - \tfrac{1}{2}m\bar{\psi}\psi \ . \tag{2.1}$$

The action, the space-time integral of (2.1), is now invariant under the following set of infinitesimal transformations

$$\begin{aligned} \delta A &= \bar{\varepsilon}\psi \ , \\ \delta B &= i\bar{\varepsilon}\gamma_5\psi \ , \\ \delta\psi &= (\not{\partial}-m)(A + i\gamma_5 B)\varepsilon \ , \end{aligned} \tag{2.2}$$

where ε is a global, i.e. space-time independent, Majorana spinor, whose anticommuting components parametrize the supersymmetry transformations. Indeed, the Lagrangian (2.1) transforms into a total derivative:

$$\delta\mathcal{L} = \partial_\mu(-\tfrac{1}{2}\bar{\varepsilon}\gamma_\mu\not{\partial}(A + i\gamma_5 B)\psi). \tag{2.3}$$

Of course, the action is invariant under translations and Lorentz transformations as well.

This example demonstrates the main properties of supersymmetry. In the first place supersymmetry requires equal numbers of bosonic and fermionic degrees of freedom. For example, in the scalar multiplet we have two of each.

Secondly, in order to obtain the result (2.3), it is crucial that all fields have equal mass. This is a general property of supersymmetry, which has direct implications for phenomenology, as we will see.

The third feature that should be mentioned here is the commutator algebra of supersymmetry. When we take the commutator of two supersymmetry transformations (2.2), we obtain a translation with parameter $-2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2$, e.g.:

$$[\delta_2,\delta_1]A = 2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\partial_\mu A \ , \tag{2.4}$$

which is again an invariance of the action.

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*) For notations and conventions see appendix A.

However, an important remark has to be made. When we calculate the same commutator on ψ, we find besides this translation an additional term:

$$[\delta_2,\delta_1]\psi = 2\bar{\varepsilon}_1\gamma_\mu\varepsilon_2\partial_\mu\psi + (\bar{\varepsilon}_2\gamma_\mu\varepsilon_1)\gamma_\mu(\partial\!\!\!/+m)\psi. \tag{2.5}$$

This additional term vanishes upon use of the classical field equation for ψ, the Dirac equation:

$$(\partial\!\!\!/ + m)\psi = 0. \tag{2.6}$$

For this reason the commutator is said to close only on the classical level, or on shell, where this field equation may be inserted. With this proviso, however, the result (2.4) holds uniformly on all fields.

We will now discuss the consequences of these observations. We have seen that supersymmetric field theories must be based on multiplets containing equal numbers of boson and fermion states. However, in nature no such mass degenerate sets of bosons and fermions are known, except for the massless photon and neutrino. Therefore if supersymmetry is a property of the physical world, it must be realized in a broken manner, i.e. there must exist some mechanism by which bosons and fermions acquire different effective masses. This mechanism could be of the Higgs-Kibble type, where some field has a non-vanishing vacuum expectation value, which contributes to an apparent mass of another field. Or the mechanism might be of a dynamical character, the effective mass resulting from specific interactions.

Another reason why supersymmetry is not realized manifestly in the known physical world could be, that it only plays a role at the sub-quark level. In any case the energy at which supersymmetry becomes important must be much higher than the ones available at present.

Turning to the commutator algebra of supersymmetry, we first comment on its general structure. The commutation relations of two supersymmetry transformations have the same form on all fields which obey their corresponding equations of motion. Analogous results hold for commutators involving other infinitesimal symmetry transformations, such as translations and Lorentz transformations. Therefore there is a well defined algebraic structure underlying this whole set of infinitesimal transformations. In particular one can define generators Q_a of supersymmetry, together forming a Majorana spinor, which obey anticommutation relations:

$$\{Q_a,\bar{Q}_b\} = 2P\!\!\!\!/_{ab}\ , \tag{2.7}$$

Here P_μ is the generator of translations. Hence P_μ differs from the momentum k_μ by a factor $-i$:

$$P_\mu = -ik_\mu \ .$$

The abstract algebraic structure which involves the generators of supersymmetry, translations and Lorentz transformations is called the graded Poincaré algebra.

Secondly we comment on the fact that the graded Poincaré algebra is realized on fields satisfying the classical equations of motion only. This situation is characteristic of supersymmetry and will be encountered several more times in the following. One might ask, whether this restriction is essential. Indeed it is not and can be lifted here and in most cases by introducing so called auxiliary fields, i.e. fields which do not correspond to physical degrees of freedom. Whether this is always possible, is however not known.

Finally we want to point out what happens, if we consider local supersymmetry transformations, i.e. supersymmetry with space-time dependent parameters. There is a simple but powerful theorem which states, that the commutator of two infinitesimal symmetry transformations of an action is again a symmetry transformation of that action, modulo terms which vanish on shell. We have seen this in our example, where two supersymmetry transformations gave a translation.

Now the global transformations are a special case of the local ones. Hence the commutator of two local supersymmetry transformations, if these can be defined consistently, yields at least a local translation, or what amounts to the same, a general coordinate transformation. As a consequence local supersymmetry can only occur in theories which possess general coordinate invariance, and these theories necessarily contain the gravitational field. We conclude that a theory of local supersymmetry must incorporate a description of gravitation. Such a theory is therefore called a theory of supergravity.

We end this paragraph by mentioning that the Lagrangian density (2.1) can be extended in a supersymmetric fashion to include non-trivial interaction terms as well. For our basic observations they do not lead to anything new, however, and therefore we have not dealt with them here.

3. Extended supersymmetry

In the last paragraph we found, that supersymmetry is intimately connected with space-time symmetries. In this paragraph we will show that internal symmetries may be introduced in supersymmetric theories as well. In Poincaré supersymmetry, which has the graded Poincaré algebra as its basic commutator algebra, the maximal internal symmetry that can be accommodated consistently is SO(N). If such an internal symmetry is present, one speaks of extended or SO(N) supersymmetry *). We will demonstrate the principles in a simple SO(2) model.

In order to construct this, we first introduce another O(1) multiplet, the vector multiplet. It contains a massless vector field V_μ with spin 1 and a spin $\frac{1}{2}$ Majorana spinor field ϕ. Again we have two bosonic and two fermionic degrees of freedom. The Lagrangian density is

$$\mathcal{L} = -\tfrac{1}{4}F^2_{\mu\nu} - \tfrac{1}{2}\bar{\phi}\partial\!\!\!/\phi \ , \tag{3.1}$$

with

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \ .$$

It has the usual electromagnetic gauge invariance:

$$\delta V_\mu = \partial_\mu \Lambda \ , \tag{3.2}$$

while the supersymmetry transformations for the fields read:

$$\begin{aligned} \delta V_\mu &= \bar{\varepsilon}\gamma_\mu\phi \ , \\ \delta\phi &= -F_{\mu\nu}\sigma_{\mu\nu}\varepsilon \ . \end{aligned} \tag{3.3}$$

The commutator algebra is basically the same as that of the scalar multiplet, except for an extra term in the V_μ commutator:

$$[\delta_2,\delta_1]V_\mu = 2\bar{\varepsilon}_1\gamma_\nu\varepsilon_2\partial_\nu V_\mu - \partial_\mu(2\bar{\varepsilon}_1\gamma_\nu\varepsilon_2 V_\nu) \ . \tag{3.4}$$

This term represents a gauge transformation of V_μ and is perfectly allowable.

We can now fuse this multiplet with the massless scalar multiplet as follows. The total Lagrangian for the fields is:

$$\mathcal{L} = -\tfrac{1}{4}F^2_{\mu\nu} - \tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}\bar{\phi}\partial\!\!\!/\phi - \tfrac{1}{2}\bar{\psi}\partial\!\!\!/\psi \ . \tag{3.5}$$

*) The original supersymmetry without internal symmetry is often referred to as O(1), SO(1) or N=1 supersymmetry in analogy.

It is invariant under the combined supersymmetry transformations (2.2) and (3.3). However, it is obvious that the roles of ϕ and ψ can be interchanged. This implies that there exists a second set of supersymmetry transformations, where V_μ goes into ψ and A and B go into ϕ, etc. The full set of transformations thus becomes:

$$\begin{aligned}
\delta V_\mu &= \bar{\varepsilon}_1\gamma_\mu\phi + \bar{\varepsilon}_2\gamma_\mu\psi, \\
\delta A &= \bar{\varepsilon}_1\psi - \bar{\varepsilon}_2\phi, \\
\delta B &= i\bar{\varepsilon}_1\gamma_5\psi - i\bar{\varepsilon}_2\gamma_5\phi, \\
\delta\phi &= -F_{\mu\nu}\sigma_{\mu\nu}\varepsilon_1 - \not{\partial}(A+i\gamma_5B)\varepsilon_2, \\
\delta\psi &= \not{\partial}(A+i\gamma_5B)\varepsilon_1 - F_{\mu\nu}\sigma_{\mu\nu}\varepsilon_2 .
\end{aligned} \tag{3.6}$$

Notice that we have taken the opposite sign for A and B in the second set of transformations.

In fact, the Lagrangian (3.5) is invariant under two-dimensional rotations in the ψ-ϕ-plane and the spinors can be combined in a SO(2) doublet ψ^i, $i=1,2$. Under such rotations the transformation rules (3.6) are inert, if we rotate the parameters ε^i simultaneously by an equal amount. This means that we put the spinor parameters in a doublet of SO(2) also.

The Bose fields V_μ, A and B must be singlets under SO(2) and from (3.6) we see, that it is most conveniently done by putting V_μ in a SO(2) scalar representation, while assigning A and B to antisymmetric tensor representations A^{ij} and B^{ij}. In this notation eqs. (3.6) take the manifestly SO(2) invariant form:

$$\begin{aligned}
\delta V_\mu &= \bar{\varepsilon}^i\gamma_\mu\psi^i , \\
\delta A^{ij} &= \bar{\varepsilon}^{[i}\psi^{j]} , \\
\delta B^{ij} &= i\bar{\varepsilon}^{[i}\gamma_5\psi^{j]} , \\
\delta\psi^i &= -F_{\mu\nu}\sigma_{\mu\nu}\varepsilon^i - \not{\partial}(A+i\gamma_5B)^{ij}\varepsilon^j .
\end{aligned} \tag{3.7}$$

This multiplet is known as the SO(2) vector (gauge) multiplet. The Lagrangian (3.5) becomes:

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}^2 - \tfrac{1}{4}(\partial_\mu A^{ij})^2 - \tfrac{1}{4}(\partial_\mu B^{ij})^2 - \tfrac{1}{2}\bar{\psi}^i\not{\partial}\psi^i . \tag{3.8}$$

This example clearly demonstrates how SO(N) supersymmetry comes into existence. One starts with N independent sets of supersymmetry transformations on different multiplets of fields. One then tries to combine these in SO(N) representations, such that the N Majorana spinor parameters ϵ^i, $i=1,\ldots,N$, can be rotated into each other, while simultaneous transformations of the fields render the whole theory form invariant. If this procedure can be implemented, a SO(N) symmetry is obtained in the theory.

It is this possibility of fusing supersymmetry with internal symmetries that provides the framework for unification of supergravity with other interactions.

4. Supergravity

Any theory possessing local supersymmetry must include a description of the gravitational field and is therefore a theory of supergravity. We will now present the theory of O(1) Poincaré supergravity. It derives its name from the fact that the algebra of its symmetries forms a local version of the graded Poincaré algebra without any internal symmetries. The model consists of the usual Einstein-Cartan version of gravitation *) coupled to a real massless spin $\frac{3}{2}$ field, the gravitino field ψ_μ. The Lagrangian is given by

$$\mathcal{L}_{SG} = -\frac{e}{2\kappa^2} e^\mu_a e^\nu_b R^{ab}_{\mu\nu} - \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\rho\psi_\sigma. \qquad (4.1)$$

We will first explain the quantities appearing in this expression. The parameter κ is the gravitational coupling constant. It has the dimension $[m^{-1}]$ and is related to Newtons constant G by

$$\kappa^2 = 8\pi G. \qquad (4.2)$$

In our formulation gravitation is described by a single field e^a_μ $(\mu,a=1,\ldots,4)$, the vierbein field, with the property that the metric tensor is given by

$$g_{\mu\nu} = \delta_{ab}e^a_\mu e^b_\nu \quad . \qquad (4.3)$$

Contraction with a vierbein changes local Lorentz indices a into world indices μ and vice versa. These indices refer to the transformation character under local Lorentz and general coordinate transformations resp.; e^μ_a is the matrix inverse of e^a_μ and $e=\det e^a_\mu$ its determinant. $R^{ab}_{\mu\nu}$ is the curvature tensor, related to the usual Riemann tensor $R^\rho_{\sigma\mu\nu}$ by

*) In our conventions we follow ref. 3.

$$R^{ab}_{\mu\nu} = e^a_\rho e^{b\sigma} R^\rho_{\sigma\mu\nu} \quad . \tag{4.4}$$

It can be expressed in terms of another quantity, the spin connection ω^{ab}_μ. This is not an independent field, but is an expression in terms of the vierbeins and the gravitino field:

$$\begin{aligned} \omega_{\mu ab} = &-\tfrac{1}{2}\left\{e^\lambda_a(\partial_\mu e_{b\lambda} - \partial_\lambda e_{b\mu}) + e^\lambda_a e^\nu_b(\partial_\nu e_{\lambda c})e^c_\mu - (a\leftrightarrow b)\right\} \\ &- \frac{\kappa^2}{4}\left\{\bar\psi_\mu(\gamma_a\psi_b - \gamma_b\psi_a) + \bar\psi_a\gamma_\mu\psi_b\right\} . \end{aligned} \tag{4.5}$$

In terms of ω^{ab}_μ the curvature tensor is given by

$$R^{ab}_{\mu\nu} = \partial_\mu\omega^{ab}_\nu - \partial_\nu\omega^{ab}_\mu - (\omega^{ac}_\mu\omega_{\nu c}{}^b - \omega^{bc}_\mu\omega_{\nu c}{}^a). \tag{4.6}$$

Finally we have defined the covariant derivative

$$D_\rho\psi_\sigma = (\partial_\rho - \tfrac{1}{2}\omega^{ab}_\rho\sigma_{ab})\psi_\sigma. \tag{4.7}$$

This completes the definition of the Lagrangian (4.1). The corresponding action is invariant under the following local supersymmetry transformations:

$$\begin{aligned} \delta_S e^a_\mu &= \kappa\bar\varepsilon\gamma^a\psi_\mu, \\ \delta_S\psi_\mu &= \frac{2}{\kappa}D_\mu\varepsilon \ , \end{aligned} \tag{4.8}$$

where D_μ is defined as in (4.7) and ε is now space-time dependent. The action is also invariant under general coordinate and local Lorentz transformations with parameters ξ^λ and ε^{ab} respectively:

$$\begin{aligned} \delta e^a_\mu &= -e^a_\lambda\partial_\mu\xi^\lambda - \xi^\lambda\partial_\lambda e^a_\mu + \varepsilon^a{}_b e^b_\mu \ , \\ \delta\psi_\mu &= -\psi_\lambda\partial_\mu\xi^\lambda - \xi^\lambda\partial_\lambda\psi_\mu + \tfrac{1}{2}\varepsilon^{ab}\sigma_{ab}\psi_\mu. \end{aligned} \tag{4.9}$$

Having thus established the theory, we proceed by discussing its main properties. The field equations corresponding to $\hat{\mathcal{L}}_{SG}$ read:

$$\begin{aligned} R_\mu &= \frac{1}{e}\varepsilon_\mu{}^{\nu\rho\sigma}\gamma_5\gamma_\nu D_\rho\psi_\sigma = 0 \ . \\ R^\nu_\mu - \tfrac{1}{2}\delta^\nu_\mu R &= \frac{\kappa^2}{2e}\varepsilon^{\nu\lambda\rho\sigma}\bar\psi_\lambda\gamma_5\gamma_\mu D_\rho\psi_\sigma \ , \end{aligned} \tag{4.10}$$

with $R^\nu_\mu \equiv R^\nu{}_\mu{}^\lambda{}_\lambda$, $R \equiv R^\mu_\mu$.

The first equation is a covariantized version of the Rarita-Schwinger equation for free massles spin $\frac{3}{2}$ particles. The second equation is the usual Einstein equation with on the right-hand side a term representing the energy-momentum tensor of the gravitino field. Notice, however, that our curvature tensor is defined in terms of the ω_μ^{ab} given by (4.5), which differs from the ω_μ^{ab} encountered in the geometrical formulation of gravitation by the ψ_μ-dependent terms.

This ψ_μ-dependent part of ω_μ^{ab} is a manifestation of yet another property of supergravity, which is torsion. Physically torsion means that the curvature is not only determined by the mass density but also by the spin density. Obviously the spin density here is due to the gravitino field. Mathematically torsion manifests itself in non-symmetric affine connections $\Gamma^\rho_{\mu\nu}$. These quantities are defined in terms of the vierbein field by

$$\Gamma^\rho_{\mu\nu} = e^\rho_a D_\mu e^a_\nu = e^\rho_a(\partial_\mu e^a_\nu - \omega_\mu{}^a{}_b e_\nu{}^b) \ , \tag{4.11}$$

and in our case they satisfy

$$\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} = \frac{\kappa^2}{2}\bar{\psi}_\mu\gamma^\rho\psi_\nu \quad . \tag{4.12}$$

Obviously they are not symmetric.

In the action, torsion gives rise to ψ_μ^4-interactions. This one sees by substitution of the expression for ω_μ^{ab} into $R^{ab}_{\mu\nu}$. These interactions discriminate between supergravity and usual geometrical versions of the theory of gravitation. However, since spin $\frac{3}{2}$ fields have not yet been observed in any fundamental process, this does not lead to experimental consequences. Moreover these interactions are very weak, being short range and proportional to κ^2. Therefore there does not seem to be much hope in general for establishing torsion experimentally.

We finish this section on supergravity with an examination of the commutator algebra of the supersymmetry transformation (4.8). On the vierbein the commutator has the form

$$[\delta_2,\delta_1]e^a_\mu = [\delta_G(\xi^\lambda) + \delta_L(\xi^\lambda\omega_\lambda^{ab}) + \delta_S(\frac{\kappa}{2}\xi^\lambda\psi_\lambda)]\, e^a_\mu \quad , \tag{4.13}$$

where $\xi^\lambda = -2\bar{\epsilon}_1\gamma^\lambda\epsilon_2$, and δ_G, δ_L and δ_S represent general coordinate, local Lorentz and supersymmetry transformations respectively, as defined in (4.8) and (4.9), with the (field-dependent) parameters indicated.

We note that the commutator is different from that of global supersymmetry. This is to be expected, since also the algebra of space-time transformations changes. For example, two local translations do in general not commute, while global translations do.

For the gravitino field the commutator has the same structure as in (4.13), except for terms proportional to R_μ, which vanish on shell. This is familiar from previous discussions. However, its consequences reach much farther here, since it affects the definition of the quantum theory corresponding to (4.1). In the usual procedure for theories with a local invariance explicit use of the off shell closure of the algebra is made. Hence we either have to find a formulation of supergravity with closed algebra, or we have to invent a generalization of the quantization procedure where on shell closure of the algebra suffices. Both can be done and we will return at length to the problem later.

5. Synopsis

In this chapter we have introduced the concepts of supersymmetry and supergravity. We have seen, that supersymmetry is a powerful principle, which might solve some important problems in the physics of fundamental processes. In particular we have explained how local supersymmetry naturally leads to a theory of gravitation. It is this application with which we will be concerned mostly in later chapters.

The presentation of our material is organized as follows. In chapter II we will give a more precise and technical definition of supersymmetry. We will return to the examples presented in this chapter for a discussion of two important topics.

The first one concerns the off shell closure of the supersymmetry algebra using auxialiary fields. We have already briefly touched upon this problem in a discussion of the scalar multiplet.

The second one is the generalization of globally supersymmetric field multiplets to local ones, in order to couple them to supergravity.

In chapter III we present the theory of SO(2) supergravity. We construct the linearized version of the full multiplet, including auxiliary fields, of this theory, as well as of certain matter multiplets.

In chapter IV we give the extension of these results to all orders in the coupling constant κ. We also present the full Weyl multiplet for N=2 and discuss the coupling of matter multiplets.

In the last chapter, V, we will present the quantization procedure for theories with local gauge invariance and its generalization for theories with non-closing, or open, gauge algebra's.

We have included two appendices. In appendix A we give a summary of our notations and conventions. In appendix B we give an alternative to the graded Poincaré algebra: the graded conformal algebra. This supersymmetry algebra is of importance in connection with the Weyl multiplet.

References

We list here a number of general references on our subject. More specific references can be found there and at the end of each of the next chapters.

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CHAPTER II

SUPERSYMMETRY AND SUPERGRAVITY II.

1. Preliminary

Symmetries play an important role in field theories. For one thing, they can establish relations between different processes described by the theory. Also they may imply conservation laws for certain charges connected with the fields. And, very importantly, they may prescribe the form of interactions occurring in the theory.

A field theory possesses a symmetry, if its physical content is not affected by a certain set of field transformations. This means, that the action functional, from which one derives the dynamics of the fields, is invariant under these transformations. Usually, in order to obtain invariance, one must change the fields rigidly over all space-time. That is to say, the parameters of the transformations have a fixed value at all places and times, and the theory exhibits a global symmetry.

However, sometimes it is possible to realize the symmetry in such a way, that the strength of the field transformations may vary at different space-time points. We then have a local or gauge invariance. In general a gauge invariance is connected with the introduction of some kind of interaction into the theory.

Supersymmetry is a symmetry with the special feature, that its parameters are anticommuting Majorana spinors. As a result it generates conserved spinorial charges Q_a, which satisfy anticommutation relations. Furthermore it can be realized locally by introducing massless spin $\frac{3}{2}$ gauge fields in addition to gravitation. Thus one arrives at supergravity.

Some of these aspects were already discussed in chapter I. Others will be examined more closely in the present chapter. We start by formally defining the algebraic structure of supersymmetry, the graded Poincaré algebra. We analyse its representations in terms of particle states, paying special attention to their spin content. Then we give some realizations of the full closed algebra on multiplets of fields. This is an extension of results obtained in chapter I. In conclusion we discuss the coupling of supersymmetric matter multiplets to supergravity.

2. The graded Poincaré algebra

From the example in I.2 we have drawn the conclusion, that the infinitesimal supersymmetry transformations have a uniform commutator algebra on all fields. In particular we found, that the commutator of two such supersymmetry transformations yields an infinitesimal translation, which indicated that supersymmetry is intimately connected with space-time symmetries.

In this section we will formalize these results and discuss the abstract algebraic structure underlying supersymmetry and space-time transformations. These last ones obey a Lie algebra, known as the Poincaré algebra. Inclusion of supersymmetry promotes this Lie algebra to a graded Lie algebra. We will define these notions first [1] .

A Lie algebra L is a set of elements X_a, which span a linear vector space and which obey commutation relations

$$[X_a, X_b] = c_{ab}^{c} X_c. \tag{2.1}$$

Hence the commutator of two elements of L is again an element of L. The quantities c_{ab}^{c}, called structure constants, clearly obey

$$c_{ab}^{c} = -c_{ba}^{c}.$$

They are not arbitrary, but restricted by imposing on the X_a the Jacobi identity:

$$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0 . \tag{2.2}$$

It follows, that the structure constants have the property

$$c_{ak}^{\ell} c_{bc}^{k} + c_{bk}^{\ell} c_{ca}^{k} + c_{ck}^{\ell} c_{ab}^{k} = 0 , \tag{2.3}$$

which is equivalent to the statement, that the matrices $(c_a)_k^{\ell}$ must form a representation of the algebra (2.1).

A grading representation of a Lie algebra is defined as a set of anti-commuting elements Q_α , which transform under a representation of the Lie algebra and satisfy the following (anti-)commutation relations and Jacobi identities:

$$[X_a, Q_\alpha] = q^\beta_{a\alpha} Q_\beta,$$
$$\{Q_\alpha, Q_\beta\} = s^a_{\alpha\beta} X_a; \tag{2.4}$$

$$[X_a,[X_b,Q_\alpha]] + [X_b,[Q_\alpha,X_a]] + [Q_\alpha,[X_a,X_b]] = 0 ,$$
$$[X_a,\{Q_\alpha,Q_\beta\}] + \{Q_\alpha,[Q_\beta,X_a]\} + \{Q_\beta,[Q_\alpha,X_a]\} = 0 , \tag{2.5}$$
$$[Q_\alpha,\{Q_\beta,Q_\gamma\}] + [Q_\beta,\{Q_\gamma,Q_\alpha\}] + [Q_\gamma,\{Q_\alpha,Q_\beta\}] = 0 .$$

Substitution of the rules (2.1) and (2.4) into the identities (2.5) gives three constraints on the structure constants $q^\beta_{\cdot\alpha}$ and $s^a_{\alpha\beta}$, analogous to the relation (2.3).

The above definitions are quite general. We will now consider the special case of the Poincaré algebra [1,2]. The usual Poincaré algebra consists of the generators of infinitesimal translations P_μ, and Lorentz transformations $M_{\mu\nu}$. To obtain the graded Poincaré algebra, these are supplemented by a set of anticommuting elements Q_a, transforming as a Majorana spinor under Lorentz transformations. The full algebraic structure is:

$$[P_\mu, P_\nu] = 0,$$
$$[M_{\mu\nu}, P_\lambda] = \delta_{\nu\lambda} P_\mu - \delta_{\mu\lambda} P_\nu, \tag{2.6}$$
$$[M_{\mu\nu}, M_{\kappa\lambda}] = \delta_{\nu\kappa} M_{\mu\lambda} + \delta_{\mu\lambda} M_{\nu\kappa} - \delta_{\nu\lambda} M_{\mu\kappa} - \delta_{\mu\kappa} M_{\nu\lambda} .$$

$$[P_\mu, Q_a] = 0,$$
$$[M_{\mu\nu}, Q_a] = -(\sigma_{\mu\nu})_{ab} Q_b, \tag{2.7}$$
$$\{Q_a, \bar{Q}_b\} = 2\not{P}_{ab}.$$

It is straightforward to verify, that this algebra obeys the Jacobi identities (2.2), (2.5). Clearly the infinitesimal translations, Lorentz and supersymmetry transformations in our examples of I.2 satisfy the relations (2.6) and (2.7). Hence the grading elements of the Poincaré algebra can be identified as the generators of infinitesimal supersymmetry transformations.

One can extend the graded Poincaré algebra to include internal symmetries as well. This is done as follows. Consider N copies of the grading $\{Q_a\}$:

$Q_a^r, \qquad r=1,\dots,N.$

With these one can trivially generalize the graded Poincaré algebra to:

$$[P_\mu, Q_a^r] = 0,$$
$$[M_{\mu\nu}, Q_a^r] = -(\sigma_{\mu\nu})_{ab} Q_b^r, \tag{2.8}$$
$$\{Q_a^r, \bar{Q}_b^s\} = 2\not{P}_{ab}\delta^{rs}.$$

The key observation is now, that one may take $\{Q_a^r\}$ to be the grading representation of some internal symmetry algebra as well. When the generators of infinitesimal transformations of this symmetry are denoted by T_i, we define a graded Lie algebra:

$$[T_i, T_j] = f_{ij}^k T_k,$$
$$[T_i, Q_a^r] = t_i^{rs} Q_a^s. \tag{2.9}$$

At the same time the T_i transform as scalars under Poincaré transformations:

$$[P_\mu, T_j] = 0,$$
$$[M_{\mu\nu}, T_i] = 0. \tag{2.10}$$

The problem one has to solve is, whether one can find structure constants f_{ij}^k and t_i^{rs} in (2.9), such that the Jacobi identities (2.2), (2.5) hold for all combinations of elements T_i, P_μ, $M_{\mu\nu}$, Q_a^r. This is indeed possible and hence $\{Q_a^r\}$ can be a set of grading elements for both the Poincaré and an internal symmetry algebra simultaneously. However, the class of allowed internal symmetries is rather restricted. Substitution of eqs. (2.9) into (2.2) and (2.5) results in the following requirements on f_{ij}^k and t_i^{rs}:

a. the f_{ij}^k obey eq. (2.3):

$$f_{i\ell}^k f_{jm}^\ell - f_{j\ell}^k f_{im}^\ell = f_{ij}^\ell f_{\ell m}^k \;; \tag{2.11}$$

b. similarly the first of the relations (2.5) implies:

$$[t_i, t_j]^{rs} = f_{ij}^k t_k^{rs}; \tag{2.12}$$

c. the t_i^{rs} must be real and antisymmetric in r and s, if T_i is antihermitean.

Consequently the f_{ij}^k and t_i^{rs} both are matrix representations of the algebra (2.9). Furthermore, since there are only $\frac{1}{2}N(N-1)$ independent real antisymmetric $N \times N$ matrices, the t_i^{rs} can generate at most an SO(N) internal symmetry. Therefore also the T_i cannot belong to an algebra larger than

SO(N). In this way the Q_a^r become the generators of SO(N) extended supersymmetry, of which an example was given in I.3.

All possible modifications of the above scheme for a quantum field theory in a Hilbert space with positive definite metric were given by Haag, Lopuszanski and Sohnius [3]. They showed that the maximal algebra which can be realized by the charges of such a theory is the product of the graded Poincaré algebra and SO(N), the only possible modification being the occurrence of central charges. These are charges which commute with all other elements of the algebra. The modification therefore has the form:

$$\begin{aligned} &\{Q_a^r, \bar{Q}_b^s\} = 2\not{P}_{ab}\delta^{rs} + \delta_{ab}Z^{rs} + (\gamma_5)_{ab}Y^{rs}, \\ &[X, Z^{rs}] = 0, \quad [X, Y^{rs}] = 0, \quad \text{for all } X. \end{aligned} \tag{2.13}$$

This result holds for finite multiplets of massive fields. In the massless case a further generalization is possible. Here one may obtain a realization of the graded conformal algebra [4]. This algebra is discussed in appendix B. As shown there, also a chiral invariance exists. Hence the introduction of pseudoscalar charges is possible, which increases the number of internal symmetry generators allowed from $\frac{1}{2}N(N-1)$ to N^2. Thus the maximal internal symmetry becomes U(N), rather than SO(N). Central charges are allowed in massless representations as well.

3. Particle multiplets of supersymmetry

We will now determine the particle states in globally supersymmetric quantum field theories [1,2]. These are characterized by their quantum numbers, such as mass, spin, momentum etc. In contrast to usual relativistically invariant field theories, where the squared mass and spin are fixed for a given particle multiplet, we will find here, that particles of both integer and half integer spin are present within the same multiplet. In stead multiplets of supersymmetry are characterized by the squared mass and a new quantum number, called superspin.

Since we do not wish to distinguish particle states differing only in the value of the momentum $\vec{k}$, we will restrict ourselves to the manifold of states with fixed $\vec{k}$. In fact, because the squared mass, $m^2 = -k_\mu k_\mu$ *), is a Casimir invariant of the global graded Poincaré algebra, we only have to consider states with fixed four momentum k_μ. The condition for an

*) $k_\mu = iP_\mu$, see I.2.

operator X to leave the four momentum of a quantum state invariant is:

$$[P_\mu, X]|k_\mu\rangle = 0. \tag{3.1}$$

Obviously the operators satisfying (3.1) act as a subalgebra of the graded Poincaré algebra on the states $|k_\mu\rangle$. This is called the little algebra of the graded Poincaré algebra.

We will now first analyse the case $m^2=0$. We will use the special representation of the Dirac matrices described in appendix A. In this representation the Majorana condition for the spinor Q_a^r, r=1,...,N, reads:

$$\begin{aligned} Q_1^r &= iQ_4^{r\dagger}, \\ Q_2^r &= -iQ_3^{r\dagger}. \end{aligned} \tag{3.2}$$

It is convenient to introduce the operators $Q_\pm^r$, defined by

$$Q_\pm^r = \tfrac{1}{2}(Q_1^r \pm iQ_2^{r\dagger}). \tag{3.3}$$

Taking for the four momentum of our states

$$k_\mu = (0,0,\omega,i\omega),$$

we find for the anticommutator of the Q's:

$$\{Q_a^r, \bar{Q}_b^s\} = -2i\omega(\gamma_3+i\gamma_4)_{ab}\delta^{rs}, \tag{3.4}$$

or equivalently:

$$\{Q_a^r, S_b^{s\dagger}\} = 2\omega\delta^{rs}(1-2i\sigma_{34})_{ab}. \tag{3.4a}$$

In terms of the operators $Q_\pm^r$ this becomes:

$$\{Q_-^r, Q_-^{s\dagger}\} = 2\omega\delta^{rs}, \tag{3.5}$$

while all other anticommutators, including $\{Q_+^r, Q_+^{s\dagger}\}$, vanish. From this last property it follows, that in the Hilbert spaces of physical states Q_+^r has the trivial representation

$$Q_+^r = 0,$$

and therefore we will leave it out of further consideration. Clearly, the Q_-^r satisfy (3.1), and hence they belong to the little algebra. The same holds for possible internal symmetry generators T_i. The full little algebra consists of the elements

$$\{Q_-^r, Q_-^{r\dagger}, J_3, N_1, N_2, T_i\}. \tag{3.6}$$

Here we have defined:

$$J_i = \frac{i}{2}\,\varepsilon_{ijk}M_{kj}, \qquad (i,j,k) = (1,2,3);$$
$$N_1 = -iJ_1 + M_{42}, \tag{3.7}$$
$$N_2 = -iJ_2 + M_{41}.$$

These statements may be verified by direct computation. From (3.7) we see, that the J_i are just the usual angular momentum operators. They satisfy:

$$[J_i, J_j] = i\varepsilon_{ijk}J_k \ ,$$
$$[P_\mu, J_i] = -i\varepsilon_{ijk}\delta_{k\mu}P_j \ . \tag{3.8}$$

Therefore J_3, the component of $\vec{J}$ in the direction of the three momentum $\vec{k}$, is the helicity operator. The complete algebra of the elements (3.6) reads:

$$\{Q_-^r, Q_-^{s\dagger}\} = 2\omega\delta^{rs} \ , \qquad [T_i, Q_-^r] = t_i^{rs}Q_-^s \ ,$$
$$[J_3, Q_-^r] = \tfrac{1}{2}Q_-^r, \qquad [J_3, N_1] = iN_2, \tag{3.9}$$
$$[J_3, Q_-^{r\dagger}] = -\tfrac{1}{2}Q_-^{r\dagger}, \qquad [J_3, N_2] = -iN_1.$$

All other (anti)commutators are zero on states $|k_\mu\rangle$. From this it follows that the eigenvalues of J_3 and those of the internal symmetry generators determine completely the particle content of the theory. Also from (3.9) one concludes that Q_-^r and $Q_-^{r\dagger}$ act as helicity raising and lowering operators respectively, changing the helicity of a state by $\pm\frac{1}{2}$ unit.

It is now easy to construct the finite multiplets of zero mass particle states. Suppose we have a singlet state $|k_\mu, \lambda\rangle$, λ representing the helicity, which acts as the vacuum state for the operators Q_-^r:

$$Q_-^r|k_\mu, \lambda\rangle = 0 \ .$$

Clearly λ is the maximal helicity for states constructed from $|k_\mu, \lambda\rangle$. By applying the operator $Q_-^{r\dagger}$ k times, we obtain a set of $\binom{N}{k}$ states with helicity $\lambda-\frac{1}{2}k$:

$$|k_\mu, \lambda-\tfrac{1}{2}k, [r_1 \ldots r_k]\rangle = (2\omega)^{-k/2}Q_-^{r_1\dagger}\ldots Q_-^{r_k\dagger}|k_\mu, \lambda\rangle \ . \tag{3.10}$$

The multiplicity $\binom{N}{k}$ is easily understood by realizing, that all Q_-^r's anticommute. Hence the states (3.10) are antisymmetric in $r_1, \ldots, r_k$, and form an irreducible representation of SO(N). The state with lowest helicity, $\lambda-\frac{1}{2}N$, is again a singlet. Furthermore the particle multiplets, which are finite by construction, consist of a total of

$$\sum_{k=0}^{N} \binom{N}{k} = 2^N$$

states. However, in Lagrangian field theories all states are accompanied by their CPT-conjugates. Hence in these theories their number usually doubles to 2^{N+1}. An exception occurs, when the particle multiplet is self-conjugate. This is the case if $\lambda=\frac{1}{4}N$, since for this λ all states occur in pairs of opposite helicity. Such a situation is found in SO(8) supergravity [5], which therefore is equivalent in particle content to the SO(7) theory.

We now turn to massive multiplets. We will describe these in the rest frame, where

$$k_\mu = (0,0,0,im) .$$

We define a new operator $\mathcal{J}_k$, called the superspin, by

$$\mathcal{J}_k = J_k - \frac{1}{8m} Q^{r\dagger}\sigma_k Q^r , \qquad (3.11)$$

where

$$\sigma_k = i\varepsilon_{ijk}\sigma_{ij} .$$

In an analogous way we construct a new operator from the internal symmetry generators:

$$\mathcal{T}_i = T_i + \frac{i}{4m} Q^{r\dagger} t_i^{rs} Q^s . \qquad (3.12)$$

These operators satisfy the usual angular momentum and internal symmetry commutation relations:

$$\begin{aligned} [\mathcal{J}_k, \mathcal{J}_\ell] &= i\varepsilon_{k\ell m}\mathcal{J}_m , \\ [\mathcal{T}_i, \mathcal{T}_j] &= f_{ij}^{k}\mathcal{T}_k , \\ [\mathcal{J}_k, \mathcal{T}_i] &= 0 . \end{aligned} \qquad (3.13)$$

However, in contrast to J_k and T_i they commute with the Q's:

$$[\mathcal{J}_k, Q_\pm^r] = 0, \qquad [\mathcal{T}_i, Q_\pm^r] = 0 . \qquad (3.14)$$

The little algebra now consists of the elements

$$\{Q_\pm^r, Q_\pm^{r\dagger}, \mathcal{J}_j, \mathcal{T}_i\} , \qquad (3.15)$$

while $\vec{\mathcal{J}}^2 = \sum_k \mathcal{J}_k^2$ and $\mathcal{T}^2 = \sum_i \mathcal{T}_i^2$ are Casimir invariants. In fact $\vec{\mathcal{J}}^2$ is the appropriate generalization of the Pauli-Lubanski operator W_μ^2, with

$$W_\mu = \varepsilon_{\mu\nu\rho\sigma} k_\nu M_{\rho\sigma} \ .$$

Therefore the massive representations of SO(N) supersymmetry are characterized by the squared mass m^2, by the quantum number j of the superspin $\mathcal{J}^2$, which has eigenvalues $j(j+1)$, and by the eigenvalues of $\mathcal{T}^2$. On the other hand the states in a multiplet are distinguished by the third component of the superspin $\mathcal{J}_3$ as well as of the ordinary spin J_3, and by the eigenvalues of the $\mathcal{T}_i$'s that can be diagonalized simultaneously.

In fact, the little algebra is given by the relations (3.13) and (3.14), complemented by the anticommutators

$$\{Q^r_+, Q^{s\dagger}_+\} = m\delta^{rs}, \qquad \{Q^r_-, Q^{s\dagger}_-\} = m\delta^{rs} \ . \tag{3.16}$$

All other anticommutators vanish as before. Also we have

$$[\mathcal{J}_3, J_3] = 0 \ , \qquad [J_3, Q^r_\pm] = \tfrac{1}{2} Q^r_\pm, \qquad [J_3, Q^{r\dagger}_\pm] = -\tfrac{1}{2} Q^{r\dagger}_\pm \ . \tag{3.17}$$

As a consequence of this (3.16) represents an algebra of spin-raising and -lowering operators, which is twice as large as in the massless case.

To construct the massive particle multiplets, we start with a (2j+1)-fold set of states $|k_\mu, \lambda\rangle$ $\lambda = -j, \ldots, +j$, on which $Q^{r\dagger}_+$ and Q^r_- give zero:

$$Q^{r\dagger}_+ |k_\mu, \lambda\rangle = 0 \ , \qquad Q^r_- |k_\mu, \lambda\rangle = 0 \ . \tag{3.18}$$

These states have the special property, that they are eigenstates of $\mathcal{J}_3$ and J_3 simultaneously, with the same eigenvalue λ. This is immediately seen by writing (3.11) in the form:

$$\mathcal{J}_3 = J_3 + \frac{1}{2m} (Q^{r\dagger}_- Q^r_- - Q^r_+ Q^{r\dagger}_+) \ . \tag{3.19}$$

Subsequent application of the spin-raising and -lowering operators Q^r_+, $Q^{r\dagger}_-$ on $|k_\mu, \lambda\rangle$ results in the construction of a complete multiplet. We find $(2j+1) \times \binom{N}{k} \times \binom{N}{\ell}$ states

$$|k_\mu, \lambda - \tfrac{1}{2}k + \tfrac{1}{2}\ell, [r_1 \ldots r_k, s_1 \ldots s_\ell]\rangle = m^{-(k+\ell)/2} Q^{s_1}_+ \ldots Q^{s_\ell}_+ Q^{r_1\dagger}_- \ldots Q^{r_k\dagger}_- |k \ \lambda\rangle \ . \tag{3.20}$$

They have a superspin component $\mathcal{J}_3 = \lambda$, while their ordinary spin in the z-direction is

$$J_3 = \lambda - \tfrac{1}{2}k + \tfrac{1}{2}\ell .$$

The whole multiplet consists of $(2j+1)2^{2N}$ states, with spin along the z-axis running from $\lambda - \frac{1}{2}N$ to $\lambda + \frac{1}{2}N$, and $\lambda = -j, \ldots, +j$. However, in a Lagrangian field theory only full spin multiplets with spin quantum numbers

$$\vec{J}^2 = s(s+1), \quad J_3 = -s,\ldots,+s,$$

occur. Hence we have to group the states into sets of $(2s+1)$ members, each set corresponding to a massive local field of spin s. As an example, consider the multiplet of states in $N=1$ supersymmetry which has superspin $j=\frac{3}{2}$. It has four subsets of states corresponding to the four values of $\mathcal{J}_3$:

$$\mathcal{J}_3 = \tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \ \tfrac{3}{2} .$$

Applying spin-raising and -lowering operators gives us a complete set of states with

$$J_3 = (2, \tfrac{3}{2}, \tfrac{3}{2}, 1, 1, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0, -\tfrac{1}{2}, -\tfrac{1}{2}, -1, -1, -\tfrac{3}{2}, -\tfrac{3}{2}, -2) ,$$

which corresponds to the components of one spin 2 field, two spin $\frac{3}{2}$ fields and one spin 1 field.

For convenience we have included two tables, in which are listed some massless and massive field multiplets of SO(1) and SO(2) supersymmetry. They are classified according to highest helicity λ for the massless fields (table 1), and superspin j in the massive case (table 2).

In conclusion we remark, that SO(N) Poincaré supergravity is always constructed from massless supermultiplets with helicities $2,\ldots,2-\frac{1}{2}N$, supplemented by CPT-conjugate states. As a result, fields with spin $s > 2$ become necessary for $N > 8$. For instance, by allowing spin $\frac{5}{2}$ fields one could realize SO(9) or SO(10) supersymmetry. Such an internal symmetry is large enough to account for the present phenomenology of elementary particles. Free field theories for spin $\frac{5}{2}$ have indeed been constructed [5]. Unfortunately, no complete interacting spin $\frac{5}{2}$ field theory is known to exist.

S	SO(1)				SO(2)			
2	1				1			
$\frac{3}{2}$	1	1			2	1		
1		1	1		1	2	1	
$\frac{1}{2}$			1	1		1	2	1
0				2			2	2
$\lambda\rightarrow$	2	$\frac{3}{2}$	1	$\frac{1}{2}$	2	$\frac{3}{2}$	1	$\frac{1}{2}$

table 1. Massless representations of SO(1) and SO(2) supersymmetry, classified according to spins and highest helicity λ.

S	SO(1)				SO(2)		
2	1				1		
$\frac{3}{2}$	2	1			4	1	
1	1	2	1		6	4	1
$\frac{1}{2}$		1	2	1	4	6	4
0			1	2	1	4	5
$j\rightarrow$	$\frac{3}{2}$	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0

table 2. Massive representations of SO(1) and SO(2) supersymmetry, classified according to spins and superspin j.

4. Auxiliary fields

In chapter I we have discussed several field theoretical models in which the graded Poincaré algebra was realized. In all these examples, however, one needed the classical field equations in order to obtain a closed commutator algebra. It is possible to remove this restriction on the field commutators by introducing auxiliary fields. Auxiliary fields do not correspond to dynamical degrees of freedom. They can be replaced both in the action and the transformation rules by their field equations to reproduce the original form of the theory, without any change in physical content. Their only function is to close the commutator algebra off shell. Moreover, since auxiliary fields are non-dynamical, they often have unphysical dimensions.

It has a number of advantages to have a fully closed commutator algebra.

In the first place, without auxiliary fields couplings between multiplets, and transformation rules, become explicitly model dependent. This is easily understood, when one realizes that the equations of motion by which the auxiliary fields are eliminated, will of necessity be model dependent. Because of this it is virtually impossible to construct general invariants without having the full multiplets at one's disposal. These invariants are important to provide the general form of Lagrangians, especially with interactions between coupled multiplets, and to find the quantum corrections to a certain classical action.

Secondly, for locally invariant field theories a closed gauge algebra facilitates considerably the construction of the corresponding quantum theory [7]. With closed algebra the usual Faddeev-Popov procedure applies. The generalization of this scheme to theories with open off shell algebra exists, but is rather complicated.

A further point of importance is raised by the existence of invariants of higher order in derivatives. In such invariants the dimensionalities of the fields are different from the original ones and former auxiliary fields may become propagating [8]. In that case they are no longer auxiliary and cannot be eliminated by their field equations.

Finally, the auxiliary fields also facilitate translations of supersymmetric field theories into superspace formalism. However, we will not be concerned with such formulations here.

We will illustrate the use of auxiliary fields in the scalar multiplet of N=1 supersymmetry. Here one needs one auxiliary scalar field F and one pseudoscalar G. The full transformation rules including these fields are:

$$\begin{aligned}
\delta A &= \bar{\varepsilon}\psi \ , \\
\delta B &= i\bar{\varepsilon}\gamma_5\psi \ , \\
\delta\psi &= \partial\!\!\!/(A+i\gamma_5 B)\varepsilon + (F+i\gamma_5 G)\varepsilon \ , \\
\delta F &= \bar{\varepsilon}\partial\!\!\!/\psi \ , \\
\delta G &= i\bar{\varepsilon}\gamma_5\partial\!\!\!/\psi \ .
\end{aligned} \tag{4.1}$$

From the rules one may verify that on all fields the commutator has the form (2.7):

$$[\,\delta_s(\varepsilon_2),\delta_s(\varepsilon_1)\,] = \delta_p(\xi_\lambda) \ , \tag{4.2}$$

where $\delta_p(\xi_\lambda)$ denotes the translation with parameter

$$\xi_\lambda = 2\bar{\varepsilon}_2\gamma_\lambda\varepsilon_1 \ .$$

as inv

This result holds without using any field equation. Therefore the algebra of the transformations closes independently of the Lagrangian for the fields. The specific Lagrangian (I.2.1) can be extended to become invariant under the transformations (4.1) as follows:

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}\bar{\psi}\not{\partial}\psi + \tfrac{1}{2}F^2 + \tfrac{1}{2}G^2 + m(AF+BG) - \frac{m}{2}\bar{\psi}\psi \quad . \qquad (4.3)$$

The field equations for the auxiliary fields are:

$$\begin{aligned} F &= -mA, \\ G &= -mB. \end{aligned} \qquad (4.4)$$

These equations are algebraic in character and hence F and G are not independent dynamical variables. Upon substitution of (4.4) into the transformation rules (4.1) and the Lagrangian (4.3), we reobtain eqs. (I.2.1) and (I.2.2). This shows the equivalence of the two formulations.

From the Lagrangian (4.3) we deduce, that the dimensionality of the auxiliary fields here is $[m^2]$. More generally we note that the relative dimensions of fields in a multiplet are fixed by the algebra and transformation rules. In respect to this we observe, that the dimension of ε is determined by (4.2) to be $[m^{-\frac{1}{2}}]$. The absolute dimensionality, however, depends on the action one takes for the fields. Also the division into physical and auxiliary fields for a multiplet can only be given after specifying the Lagrangian.

One can introduce auxiliary fields for the other multiplets we have mentioned in a similar way. Here we give the result for the vector multiplet. It has one pseudoscalar auxiliary field D, with the transformation rules:

$$\begin{aligned} \delta V_\mu &= \bar{\varepsilon}\gamma_\mu\phi \ , \\ \delta\phi_\mu &= -F_{\mu\nu}\sigma_{\mu\nu}\varepsilon + i\gamma_5 D\varepsilon \ , \\ \delta D &= i\bar{\varepsilon}\gamma_5\not{\partial}\phi \ . \end{aligned} \qquad (4.5)$$

The Lagrangian density becomes:

$$\mathcal{L} = -\tfrac{1}{4}F^2_{\mu\nu} - \tfrac{1}{2}\bar{\phi}\not{\partial}\phi + \tfrac{1}{2}D^2 \ . \qquad (4.6)$$

It can again be combined with the massless scalar multiplet in an SO(2) vector gauge multiplet, with auxiliary fields and closed algebra [9]. The result will be presented in chapter III.

We also give the full multiplet of N=1 Poincaré supergravity [10]. In this case we have a set of auxiliary fields consisting of one scalar S, one pseudoscalar P and one axial Lorentz vector A_a. The action is defined by

$$\mathcal{L}_{\text{Poincaré}} = -\frac{e}{2\kappa^2} R - \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\rho\psi_\sigma - \frac{e}{3}(S^2+P^2-A_a^2). \tag{4.7}$$

It is invariant, up to a total derivative, under the local supersymmetry transformations

$$\begin{aligned}
\delta e_\mu{}^a &= \kappa\bar{\varepsilon}\gamma^a\psi_\mu \,, \\
\delta\psi_\mu &= \frac{2}{\kappa} D_\mu\varepsilon + iA_\mu\gamma_5\varepsilon - \gamma_\mu\eta\varepsilon \,, \\
\delta S &= -\tfrac{1}{2}\bar{\varepsilon}\gamma\cdot R^P \,, \\
\delta P &= -\frac{i}{2}\bar{\varepsilon}\gamma_5\gamma\cdot R^P \,, \\
\delta A_a &= \frac{3i}{2}\bar{\varepsilon}\gamma_5(R_a^P - \frac{1}{3}\gamma_a\gamma\cdot R^P) \,.
\end{aligned} \tag{4.8}$$

In these equations we have used the notations

$$\begin{aligned}
\eta &= \frac{1}{3}(S + i\gamma_5 P - i\gamma_5 A\!\!\!/) \,, \\
R_\mu^P &= \frac{1}{3}\varepsilon_\mu{}^{\nu\rho\sigma}\gamma_5\gamma_\nu(D_\rho + \frac{i\kappa}{2}\gamma_5 A_\rho - \frac{\kappa}{2}\gamma_\rho\eta)\psi_\sigma \,.
\end{aligned} \tag{4.9}$$

R_μ^P is the supercovariantized version of the gravitino field equation. The commutator of two transformations (4.8) on any field has the form:

$$[\delta_2,\delta_1] = \delta_G(\xi_\lambda) + \delta_S(-\frac{\kappa}{2}\xi^\lambda\psi_\lambda) + \delta_L\big(\xi^\lambda\omega_{\lambda ab} - 2\kappa\bar{\varepsilon}_2(\sigma_{ab}\eta + \eta\sigma_{ab})\varepsilon_1\big) \,, \tag{4.10}$$

with ξ_λ as before.

This result is identical to the previous one (I.4.13), modulo the part of the Lorentz transformation depending on the auxiliary fields. However, this part vanishes on application of the field equations for S, P and A_a, as expected. Again we stress, that the transformation rules (4.8) are model independent and remain valid in particular, when one couples matter multiplets to the Poincaré Lagrangian. This example shows, that auxiliary fields can be found for locally supersymmetric field multiplets as well.

In the following we will often make use of the linearized version of supergravity. This is the free field part of (4.7), invariant under global transformations (4.8) in the limit κ=0. To take this limit correctly requires some care. It is done properly by first defining the tensor field $h_\mu{}^a$:

$$e_\mu{}^a = \delta_\mu^a + \frac{\kappa}{\sqrt{2}} h_\mu{}^a \,. \tag{4.11}$$

In terms of $h_\mu{}^a$ we have

$$\omega_{\mu ab} = \frac{\kappa}{\sqrt{2}}(\partial_a h_{b\mu} - \partial_b h_{a\mu}) + O(\kappa^2) ,$$
$$e = 1 + \frac{\kappa}{\sqrt{2}} h_{\mu\mu} + O(\kappa^2) . \tag{4.12}$$

Substituting this in the Lagrangian and transformation rules, and then taking the limit $\kappa \to 0$ gives:

$$\mathcal{L}_{\text{lin.}} = -\tfrac{1}{2}\partial_\lambda h_{\mu\lambda}\partial_\mu h_{\nu\nu} + \tfrac{1}{2}\partial_\lambda h_{\lambda\mu}\partial_\nu h_{\nu\mu} - \tfrac{1}{4}\partial_\lambda h_{\mu\nu}\partial_\lambda h_{\mu\nu}$$
$$+ \tfrac{1}{4}\partial_\lambda h_{\mu\mu}\partial_\lambda h_{\nu\nu} - \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu\partial_\rho\psi_\sigma - \tfrac{1}{3}(S^2 + P^2 - A_\mu^2) , \tag{4.13}$$

with the transformation rules:

$$\delta h_{\mu\nu} = \sqrt{2}\,\bar{\varepsilon}\gamma_\mu\psi_\nu ,$$
$$\delta\psi_\mu = -\tfrac{1}{2}\sqrt{2}\,(\partial_\lambda h_{\nu\mu})\sigma_{\lambda\nu}\varepsilon + i\gamma_5 A_\mu\varepsilon - \gamma_\mu\eta\varepsilon ,$$
$$\delta S = -\tfrac{1}{2}\bar{\varepsilon}\gamma_5\cdot R , \tag{4.14}$$
$$\delta P = -\frac{i}{2}\,\bar{\varepsilon}\gamma_5\gamma\cdot R ,$$
$$\delta A_\mu = \frac{3i}{2}\,\bar{\varepsilon}\gamma_5(R_\mu - \frac{1}{3}\,\gamma_\mu\gamma\cdot R) ,$$

where the linearized Rarita-Schwinger equation is:

$$R_\mu = \varepsilon_{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\partial_\rho\psi_\sigma .$$

However, in the following we will usually describe the linearized version in terms of the vierbein by resubstitution of eq. (4.11). The free field theory (4.13) and (4.14) is also called the flat space limit of (4.7) and (4.8), since the gravitational interaction has been switched off. In this case we need no longer distinguish world and Lorentz indices.

Two final remarks on auxiliary fields are in order here. In the first place, sometimes a commutator algebra of supersymmetry transformations closes without auxiliary fields. For later reference we give here the example of the so called tensor multiplet. It contains an antisymmetric tensor $T_{\mu\nu}$, a pseudoscalar B and a Majorana spinor ψ, with:

$$\delta T_{\mu\nu} = 2\bar{\varepsilon}\sigma_{\mu\nu}\psi ,$$
$$\delta B = i\bar{\varepsilon}\gamma_5\psi , \tag{4.15}$$
$$\delta\psi = i\partial\!\!\!/\gamma_5 B\varepsilon - \gamma_\mu\partial_\nu T_{\mu\nu}\varepsilon .$$

To these fields corresponds the Lagrangian:

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu T_{\mu\nu})^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}\bar{\psi}\partial\!\!\!/\psi . \tag{4.16}$$

It has the gauge invariance

$$\delta_T T_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma}\partial_\rho \Lambda_\sigma \ . \tag{4.17}$$

Because of this $T_{\mu\nu}$ represents a spin 0 field [11]. The gauge transformation (4.13) is also necessary to close the algebra. Notice, that in spin content this multiplet is equivalent to the massless scalar multiplet. However, in the massive case the tensor represents a spin 1 field, which is incompatible with the representations of Poincaré supersymmetry. Hence no massive version of this multiplet exists.

The second remark we want to make is, that auxiliary fields are in general not unique. We have given here only minimal sets of fields necessary to close the commutator algebra.

5. Matter coupling

In principle there are two ways to construct models in which supergravity interacts with matter. The first is based on combining the graviton and gravitino with the matter fields in one multiplet of extended supersymmetry. One takes a singlet spin 2 field, representing the graviton, and an N-tuplet of gravitino's. Then, for $N \geq 2$, one adds lower spin fields as necessary to complete the multiplet. If one succeeds in realizing local extended supersymmetry on these fields, one has found a theory of extended supergravity, which contains its own matter fields. This attractive scheme is unique to supersymmetric theories, since only supersymmetry combines fields of different spin in one multiplet.

The other possibility is to couple a multiplet of global supersymmetry to some supergravity model by generalizing it to local supersymmetry. This involves two things: on the one hand one has to find transformation rules for local supersymmetry, by introducing supergravity fields into them. On the other hand one has to extend the invariants of the global multiplet to the local case. This is the analogue of the minimal coupling procedure in electromagnetism and Yang-Mills theories.

However, in supergravity minimal coupling is by itself usually not sufficient. There are essentially two reasons for this. In the first place, as we have seen, the algebra of the supersymmetry transformations changes when going from global to local supersymmetry. In particular commuting translations have to be replaced by non-commuting general coordinate transformations. Secondly supersymmetry transformations involve derivatives of fields. In minimal coupling these are replaced by covariant derivatives. This introduces extra, non-linear terms in the transformation laws of the matter multiplet. Although these two mechanisms often conspire to make supercovariant derivatives a good first step towards extending a global multiplet to a local one, the resulting rules are generally not correct. Moreover, the non-linearity of the transformations implies that Lagrangians become complicated and certainly they are not obtained by minimal substitution.

In many cases, however, a multiplet of global supersymmetry can be coupled to supergravity using the so-called Noether procedure. In this procedure one starts from the global transformation rules and Lagrangian and adds terms to both, order by order in the coupling constant κ, so as to achieve invariance at every stage. The method can also be used to couple supergravity to itself, starting from the linearized theory. Both applications will be encountered in our later work. For this reason we illustrate the Noether procedure here for a very simple case [12]. From this example it will be clear how both the changes in the transformation rules and the Lagrangian, in going to local supersymmetry, are obtained.

Consider the massless scalar multiplet of global supersymmetry. The auxiliary field F transforms as a total derivative:

$$\delta F = \bar{\varepsilon}\not{\partial}\psi \ . \tag{5.1}$$

Therefore its space time integral is invariant and F can be taken as a Lagrangian density, invariant under the global supersymmetry rules (4.1). Of course, one may also start from the more complicated action (4.3), with m=0, but for our purpose the choice of F as a Lagrangian suffices. We will now extend F as well as the rules (4.1) to the case of local supersymmetry.

When ε becomes space-time dependent, F is no longer a good Lagrangian, since an extra term is needed:

$$\delta\mathcal{L} = (\partial_\mu\bar{\varepsilon})\gamma^\mu\psi \quad , \tag{5.2}$$

in $\delta\mathcal{L}$ to obtain a total derivative. From the transformation rules of the supergravity fields (4.8) one sees that the variation (5.2) is generated by a term

$$\frac{\kappa}{2}\,\bar{\psi}_\mu\gamma^\mu\psi \tag{5.3}$$

in the Lagrangian. This term of first order in κ is called a Noether term, since it has the form

$$\frac{\kappa}{2}\,\bar{\psi}_\mu J^\mu \ , \tag{5.4}$$

where J^μ is the Noether current of global supersymmetry:

$$J^\mu = \gamma^\mu\psi \ . \tag{5.5}$$

This is sufficient to achieve local invariance of the action up to order κ^0.

However, varying the Noether term (5.4) gives also rise to variations $\delta\mathcal{L}$ proportional to κ. We cancel these either by new terms in $\mathcal{L}$ or in the transformation rules of the matter fields. As a result we obtain a Lagrangian density:

$$\mathcal{L} = e\ F + \frac{\kappa}{2}\,\bar{\psi}\cdot\gamma\psi - \kappa(AS + BP) \ , \tag{5.6}$$

which is again invariant up to terms $\sim \partial_\mu\varepsilon$ under transformations:

$$\delta F = \bar{\varepsilon}\Big(\not\partial\psi - \frac{\kappa}{2}\,\gamma^\mu\big(\not\partial\,(A+i\gamma_5 B) + F + i\gamma_5 G\big)\psi_\mu + \big(\frac{i\kappa}{2}\,\gamma_5\not A + \kappa\eta\big)\psi\Big) \ . \tag{5.7}$$

The other fields still transform according to the global rules. We briefly indicate how (5.6) and (5.7) are determined. The factor e in front is necessary to obtain a proper coordinate invariant action. It is also needed to cancel a term $\kappa\bar{\varepsilon}\gamma\cdot\psi F$ from the variation of ψ in the Noether term. The variations $\delta\psi_\mu \sim S,P$ of this same term vanish with other ones coming from the new term $\kappa(AS+BP)$. All remaining variations of the Lagrangian (5.6) cancel either among themselves or against those from the new δF (5.7), except for one, which reads:

$$\delta\mathcal{L} = -2\kappa\bar{\psi}_\mu\,\sigma^{\mu\nu}(A+i\gamma_5 B)\partial_\nu\varepsilon. \tag{5.8}$$

To get rid of this one, we have to introduce a new Noether term in $\mathcal{L}$, of order κ^2:

$$\frac{\kappa^2}{2}\,\bar{\psi}_\nu\sigma^{\mu\nu}(A+i\gamma_5 B)\psi_\nu \ . \tag{5.9}$$

We now have succeeded in constructing a local invariant up to order κ. The whole procedure can be repeated in the next order, κ^2. No new terms in the Lagrangian are thereby found. It turns out to be sufficient to change the transformation rules of the scalar multiplet to:

$$\begin{aligned}
\delta A &= \bar{\varepsilon}\psi \ , \\
\delta B &= i\bar{\varepsilon}\gamma_5\psi \ , \\
\delta\psi &= \not\partial^p(A+i\gamma_5 B)\varepsilon + (F+i\gamma_5 G)\varepsilon \ , \\
\delta F &= \bar{\varepsilon}(\not\partial^p + \frac{i\kappa}{2}\,\gamma_5\not A + \kappa\eta)\psi \ , \\
\delta G &= i\bar{\varepsilon}\gamma_5(\not\partial^p + \frac{i\kappa}{2}\,\gamma_5\not A - \kappa\eta)\psi \ .
\end{aligned} \tag{5.10}$$

In (5.10) we have introduced the notation D^p_μ for supercovariant derivatives:

$$D^p_\mu = D_\mu - \frac{\kappa}{2}\,\delta_s(\psi_\mu) \;, \tag{5.11}$$

where $\delta_s(\psi_\mu)$ is a supersymmetry transformation with parameter ψ_μ. Explicitly:

$$\begin{aligned} D^p_\mu A &= \partial_\mu A - \frac{\kappa}{2}\,\bar\psi_\mu\psi \;, \\ D^p_\mu B &= \partial_\mu B - \frac{i\kappa}{2}\,\bar\psi_\mu\gamma_5\psi \;, \\ D^p_\mu\psi &= D_\mu\psi - \frac{\kappa}{2}\,\{\not{D}^p(A+i\gamma_5 B) + F + i\gamma_5 G\}\psi_\mu \;. \end{aligned} \tag{5.12}$$

One may verify also, that the transformations (5.10) satisfy the algebra (4.10). Hence we have succeeded not only in constructing a new, locally invariant Lagrangian:

$$\mathcal{L} = e\big[F + \frac{\kappa}{2}\,\bar\psi\cdot\gamma\psi - \kappa(AS+BP) + \frac{\kappa^2}{2}\,\bar\psi_\mu\sigma^{\mu\nu}(A+i\gamma_5 B)\psi_\nu\big] \;, \tag{5.13}$$

but also in obtaining a set of local supersymmetry transformations for the scalar multiplet with closed algebra. Actually, the transformation rule of G was found just by requiring the closure of the algebra, since it does not play a role in the variation of $\mathcal{L}$. Note, that minimal coupling is correct for A, B and ψ, but not for F and G. Evidently it is neither of any use in obtaining the Lagrangian (5.13). This Lagrangian may of course be completed still further by adding the Poincaré action (4.7) for the gauge-field multiplet.

We now comment on the uniqueness of the procedure. There are two sources of possible ambiguities. In the first place one may start from different Lagrangians in flat space. In general, this does not lead to different transformation rules for the fields, up to trivial field redefinitions, as follows from the closure of the algebra.

An exception to this occurs, when the Lagrangian has other local invariances beside space-time and supersymmetry. In that case one is liable to miss terms in the transformation laws which correspond to such a symmetry transformation. However, this immediately shows up in the commutators, where one finds extra terms of precisely this type. Therefore these ambiguities are not fundamental.

Of course this is no longer true when the auxiliary fields are not present. In that case the equations of motion, and consequently the Lagrangian, do play a role in the transformations and commutator algebra.

The second source of ambiguities are the Noether terms [13]. To see this, note that one may always add other conserved currents H^μ to it; if

$$\partial_\mu H^\mu = 0 \ , \tag{5.14}$$

then the "improved" Noether current

$$J'^\mu = J^\mu + H^\mu \tag{5.15}$$

can still be coupled to the gravitino field ψ_μ to cancel lower order variations with $\partial_\mu \varepsilon$:

$$\delta\left(\frac{\kappa}{2}\, \bar{\psi}_\mu J'^\mu \right) = (\partial_\mu \bar{\varepsilon}) J'^\mu = (\partial_\mu \bar{\varepsilon}) J^\mu \ . \tag{5.16}$$

The last step follows after partially integrating the term with H^μ and using (5.14). Such an improved Noether term may lead to completely different higher-order κ terms in the Lagrangian. We conclude therefore that a Lagrangian for a given multiplet follows uniquely from its flat space expression only in so far as the Noether terms are unambiguous.

We mention here, that an alternative procedure to find the transformation rules, e.g. (5.10), is to start with the global ones and try to add terms to obtain commutators of the required form (4.10). However, without any hint for the commutator algebra and at least some transformations this is almost impossible in practice, because of the large number of terms and coefficients one usually has to keep track off.

As a last remark we comment on the renormalizability of theories in which supergravity interacts with matter. Extended supergravity theories, just as N=1 supergravity itself, seem to have very good renormalizability properties. However, these results do not carry over to the case where external matter multiplets are coupled to them. For instance the coupling of an N=1 scalar or vector multiplet to Poincaré supergravity leads to irremediable divergencies [14].

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CHAPTER III

LINEARIZED N=2 SUPERGRAVITY

1. Summary

In this chapter we will describe the linearized version ($\kappa=0$) of SO(2) supergravity with auxiliary fields. We start by giving the full theory in terms of physical fields only. This is followed by a discussion of the two N=1 multiplets basic to the construction of the theory, the spin $(\frac{3}{2},1)$ multiplet and the linearized Poincaré supergravity multiplet, both with auxiliary fields. Next we present some global N=2 supermultiplets and their decomposition in terms of N=1 multiplets. The insights thus gained are then used to fuse the fields of linearized N=1 supergravity with those of the spin $(\frac{3}{2},1)$ multiplet into a multiplet of linearized SO(2) supergravity containing auxiliary fields. Its decomposition in terms of submultiplets is thereby found. We also discuss some properties of the linearized U(2) conformal supergravity theory and a non-minimal auxiliary field representation we find for N=1 Poincaré supergravity.

2. SO(2) supergravity

SO(2) supergravity [1] is a theory of Poincaré supergravity which displays a global SO(2) internal symmetry. It can be constructed by fusing the linearized N=1 supergravity multiplet with the spin $(\frac{3}{2},1)$ multiplet and generalizing the result to local supersymmetry.

The global spin $(\frac{3}{2},1)$ multiplet consists of a vector spinor ψ_μ and a vector B_μ, transforming as

$$\begin{aligned} \delta\psi_\mu &= -\tfrac{1}{2}\sqrt{2}\, F(B)_{\rho\sigma}\sigma_{\rho\sigma}\gamma_\mu\varepsilon \\ \delta B_\mu &= -\sqrt{2}\,\bar{\varepsilon}\psi_\mu , \end{aligned} \tag{2.1}$$

where $F(B)_{\rho\sigma} = \partial_\rho B_\sigma - \partial_\sigma B_\rho$.

The Lagrangian density for these fields is

$$\mathcal{L} = -\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu\partial_\rho\psi_\sigma - \tfrac{1}{4}F^2_{\mu\nu} . \tag{2.2}$$

Besides under the transformations (2.1), the action is also invariant under gauge transformations δ_R and δ_B of the fields:

$$\delta_R\psi_\mu = \partial_\mu\varepsilon,$$
$$\delta_B B_\mu = \partial_\mu\Lambda \ . \tag{2.3}$$

The N=1 supergravity multiplet and its linearized form were described in § II.4.

The two spin $\frac{3}{2}$ fields ψ^i_μ, i=1,2, of these multiplets now combine in a doublet of SO(2), as do the spinor parameters ε^i. The graviton, represented by the vierbein e^a_μ, is an SO(2) scalar, while the vector field becomes an antisymmetric SO(2) tensor of rank 2, B^{ij}_μ. By the Noether procedure one can then generalize the result to local supersymmetry to obtain the transformation rules [2]:

$$\delta e^a_\mu = \kappa\bar{\varepsilon}^i\gamma^a\psi^i_\mu \ ,$$
$$\delta\psi^i_\mu = \frac{2}{\kappa}D_\mu\varepsilon^i + \tfrac{1}{2}\sqrt{2}\left(F(B)^{ij}_{\rho\sigma} + \frac{\kappa}{2}\sqrt{2}\ \bar{\psi}^{[i}_\rho\ \psi^{j]}_\sigma\right)\sigma^{\rho\sigma}\gamma_\mu\varepsilon^j \tag{2.4}$$
$$\delta B^{ij}_\mu = -\sqrt{2}\ \bar{\varepsilon}^{[i}\ \psi^{j]}_\mu \ .$$

The invariant action is constructed from the Lagrangian density

$$e^{-1}\mathcal{L} = -\frac{1}{2\kappa^2}R - \tfrac{1}{2}\bar{\psi}^i_\mu R^i_\mu - \frac{1}{8}\left(F(B)^{ij}_{\mu\nu}\right)^2 - \frac{\kappa}{4}\sqrt{2}\ \bar{\psi}^i_\mu(F^{\mu\nu} + \gamma_5\tilde{F}^{\mu\nu})^{ij}\psi^j_\nu$$
$$- \frac{\kappa^2}{8}\left(g^{\mu\rho}g^{\nu\sigma}\bar{\psi}^{[i}_\mu\ \psi^{j]}_\nu\bar{\psi}^i_\rho\psi^j_\sigma + \frac{1}{e}\ \varepsilon^{\mu\nu\rho\sigma}\bar{\psi}^i_\rho\gamma_5\psi^j_\sigma\bar{\psi}^i_\mu\psi^j_\nu\right). \tag{2.5}$$

In this expression R^i_μ denotes the covariantized Rarita-Schwinger equation:

$$R^{i\lambda} = \frac{1}{e}\ \varepsilon^{\lambda\nu\rho\sigma}\gamma_5\gamma_\nu D_\rho\psi^i_\sigma \ , \tag{2.6}$$

while $\tilde{F}^{ij}_{\mu\nu}$ is the dual of $F^{ij}_{\mu\nu}$:

$$\tilde{F}^{ij\mu\nu} = \frac{1}{2e}\varepsilon^{\mu\nu\rho\sigma}F^{ij}_{\rho\sigma} \ . \tag{2.7}$$

The commutators of the transformations (2.4) have the form

$$[\delta_S(\varepsilon_2),\delta_S(\varepsilon_1)] = \delta_G(\xi^\lambda) + \delta_S(\varepsilon^i_3) + \delta_L(\varepsilon_{ab}) + \delta_B(\Lambda^{ij}) \ , \tag{2.8}$$

where the parameters are defined as follows:
for the general coordinate transformation we have:

$$\xi^\lambda = -2\bar{\varepsilon}^i_1\gamma^\lambda\varepsilon^i_2 \ ; \tag{2.9}$$

for the supersymmetry transformation:

$$\varepsilon^i_3 = \frac{\kappa}{2}\ \xi^\lambda\psi_\lambda \ ; \tag{2.10}$$

for the local Lorentz transformation

$$e_{ab} = \xi^{\lambda}\omega_{\lambda ab} + \kappa\sqrt{2}\,\bar{\varepsilon}_1^{i}(F + \gamma_5\tilde{F})_{ab}^{ij}\varepsilon_2^{i} ; \qquad (2.11)$$

and for the gauge transformation on B_{μ}^{ij}:

$$\Lambda^{ij} = \xi^{\lambda}B_{\lambda}^{ij} - \frac{2}{\kappa}\sqrt{2}\,\bar{\varepsilon}_1^{[i}\,\varepsilon_2^{j]} . \qquad (2.12)$$

On the fields e_{μ}^{a} and B_{μ}^{ij} the algebra (2.8) closes off shell. For ψ_{μ}^{i} on the contrary, it holds upon use of the classical field equations only.

We wish to improve this situation by introducing auxiliary fields. However, to find a complete set of auxiliary fields is a considerable problem [2,3] . Its solution will constitute the main topic of this chapter.

We comment here on the physical interpretation of the theory. Besides the global SO(2) symmetry and the local supersymmetry the Lagrangian (2.5) possesses the local U(1) invariance (2.3), generated by the gauge transformations on the vector field B^{ij}. This gauge invariance is necessary to obtain the correct number of physical boson states. As a consequence the vector field can be interpreted as the electromagnetic field and SO(2) supergravity as the unification of supergravity and electromagnetism. Also the spin $\frac{3}{2}$ fields can be combined in a complex Dirac spinor, by which procedure the global SO(2) becomes an equivalent global U(1) symmetry. However, under the local U(1) this complex spinor has zero charge.

The advantage of this theory above the usual Einstein-Maxwell form of gravitation and electromagnetism is found in its quantum properties. For example, the photon-photon scattering amplitude, which diverges in the Einstein-Maxwell theory, is finite in SO(2) supergravity on the one loop level [4].

It is also possible to gauge the SO(2) internal symmetry of the theory, i.e. to promote it to a local symmetry [5]. In this case the vector field becomes the gauge field of SO(2), making use of the isomorphy of the SO(2) and U(1) groups. In fact one makes the global U(1) symmetry of the complex spin $\frac{3}{2}$ field local and identifies it with the local U(1) of the vector field. Hence the spinor can now couple to the vector field with non-zero charge.

This theory can again be interpreted as a unification of gravitation and electromagnetism. However, it contains a masslike term for the spin $\frac{3}{2}$ fields, with mass proportional to the charge q:

$$m = \frac{q}{\kappa} ,$$

while there is a cosmological constant also. This cosmological constant gives rise to difficulties in the interpretation and quantization of the theory [5,6] . In particular the concept of mass is problematic. Furthermore, if one takes the charge to be that of the electron,

$$\frac{q^2}{4\pi} \approx \frac{1}{137} ,$$

the value of the cosmological constant exceeds all observational upper limits by orders of magnitude.

A possible solution to this problem is to break local supersymmetry by taking the cosmological constant to be zero [6]. This procedure then leads to a consistent theory of massive charged spin $\frac{3}{2}$ fields, in the sense that no anomalous propagation occurs, and coupling to electromagnetism and gravitation takes place in a flat background space. However, since this goes at the expense of giving up supersymmetry, this is no longer a theory of supergravity.

3. Basic N=1 multiplets

The procedure we will follow in constructing the auxiliary field formulation of SO(2) supergravity is in principle analogous to the one we described for the theory without auxiliary fields in the preceding section. We start with the linearized versions of the N=1 supergravity and spin $(\frac{3}{2},1)$ multiplets with auxiliary fields. We fuse these into a multiplet of linearized SO(2) Poincaré supergravity and then complete the construction by extending the results to all orders in the coupling constant κ.

In this chapter we implement the first part of the program, to obtain a linearized theory. As a preparation we discuss here the full N=1 multiplets with auxiliary fields.

Of the spin $(\frac{3}{2},1)$ multiplet there exist two versions. The original one was found by Ogievetski and Sokatchev [7]. However, it cannot be combined with the N=1 supergravity multiplet [8]. In order to accomplish this, a change in the field content of the multiplet is necessary. In this way the second multiplet was found [2,3]. These are the only versions of the multiplet [9] .

The second multiplet, the only one relevant to us, contains two Majorana spinors χ and λ, of dimension $\frac{5}{2}$ and $\frac{3}{2}$ respectively, one Majorana vector spinor ψ_μ, a vector gauge field B_μ, an axial vector field A_μ, a vector field V_μ, an antisymmetric tensor $t_{\mu\nu}$, a scalar M and two pseudo-slavars N and P. The invariant action is given by the Lagrangian

$$\mathcal{L}_{(3/2,1)} = -\tfrac{1}{2}\bar{\psi}_\mu R_\mu - \tfrac{1}{4}\big(F(B)_{\mu\nu}\big)^2 + 2\bar{\chi}\lambda + 2\bar{\lambda}\partial\!\!\!/\lambda + \tfrac{1}{4}t^2_{\mu\nu} + + \tfrac{1}{2}A^2_\mu + \tfrac{1}{2}V^2_\mu - \tfrac{1}{2}M^2 - \tfrac{1}{2}N^2 - P^2. \tag{3.1}$$

Here ψ_μ and B_μ represent the physical spin $\frac{3}{2}$ and 1 fields, while all other ones are auxiliary. R_μ is the linearized version of (2.6). The Lagrangian (3.1) is invariant up to a total derivative under the local gauge transformations (2.3):

$$\begin{aligned} \delta_R\psi_\mu &= \partial_\mu\varepsilon \;, \\ \delta_B B_\mu &= \partial_\mu\Lambda \;, \end{aligned} \tag{3.2}$$

as well as under rigid supersymmetry transformations:

$$\begin{aligned}
\delta\psi_\mu &= -\tfrac{1}{2}\sqrt{2}\,F(B)_{\rho\sigma}\sigma_{\rho\sigma}\gamma_\mu\varepsilon - (V_\mu + i\gamma_5 A_\mu)\varepsilon - \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\gamma_5\gamma_\nu t_{\rho\sigma}\varepsilon + i\gamma_5\gamma_\mu P\varepsilon, \\
\delta B_\mu &= -\sqrt{2}\,\bar{\varepsilon}\psi_\mu \;, \\
\delta\lambda &= \tfrac{1}{2}\sigma_{\rho\sigma}t_{\rho\sigma}\varepsilon + i\gamma_5 P\varepsilon + \tfrac{1}{2}V\!\!\!/\varepsilon - \frac{i}{2}\gamma_5 A\!\!\!/\varepsilon + \tfrac{1}{2}(M + i\gamma_5 N)\varepsilon, \\
\delta\chi &= \partial\!\!\!/\sigma_{\rho\sigma}t_{\rho\sigma}\varepsilon - \sigma_{\rho\sigma}\big(F(V)_{\rho\sigma} + i\gamma_5 F(A)_{\rho\sigma}\big)\varepsilon + 2i\gamma_5\partial\!\!\!/ P\varepsilon, \\
\delta V_\mu &= -\bar{\varepsilon}R_\mu + \bar{\varepsilon}\gamma_\mu\chi + 2\bar{\varepsilon}\partial_\mu\lambda, \\
\delta A_\mu &= -i\bar{\varepsilon}\gamma_5 R_\mu + i\bar{\varepsilon}\gamma_5\gamma_\mu\chi + 2i\bar{\varepsilon}\gamma_5\partial_\mu\lambda \;, \\
\delta t_{\mu\nu} &= -2\bar{\varepsilon}\sigma_{\mu\nu}\chi - \varepsilon_{\mu\nu\rho\sigma}\bar{\varepsilon}\gamma_5\gamma_\rho R_\sigma, \\
\delta M &= \bar{\varepsilon}(\chi + 2\partial\!\!\!/\lambda), \\
\delta N &= i\bar{\varepsilon}\gamma_5(\chi + 2\partial\!\!\!/\lambda), \\
\delta P &= i\bar{\varepsilon}\gamma_5(\chi - \tfrac{1}{2}\gamma\cdot R).
\end{aligned} \tag{3.3}$$

If we count the number of field components and subtract the gauge degrees of freedom, we find that there are 20 bosonic and 20 fermionic components, denoted by 20 + 20 for short. That the numbers are equal follows from the non-singularity of the supersymmetry transformations.

The multiplet described in (3.1) - (3.3) corresponds to an irreducible representation of Poincaré supersymmetry. However, from the field transformations (3.3) one sees, that it is possible to extract three different submultiplets. Here we use the term submultiplet to denote any set of field components transforming only among themselves. These submultiplets correspond to other irreducible representations of the graded Poincaré algebra and are described by different Lagrangians. As a result some of the auxiliary fields of the $(\frac{3}{2},1)$ multiplet become propagating as members of such a submultiplet. We will now describe these submultiplets of (3.3).

The ones that are most crucial to our construction of SO(2) supergravity are an N=1 tensor multiplet and a scalar multiplet. Both were discussed in II.4, where also their quadratic Lagrangians were given. The tensor multiplet is generated by the following field components:

$$\begin{aligned} T_{\mu\nu} &= -t_{\mu\nu} + \tfrac{1}{2}\sqrt{2}\, F(B)_{\mu\nu} , \\ B &= P, \\ \psi &= \chi - \tfrac{1}{2}\gamma\cdot R . \end{aligned} \tag{3.4}$$

One may verify from (3.3), that they transform exactly according to (II.4.15), modulo a gauge transformation of the type (II.4.17).

The scalar multiplet is defined by the components:

$$\begin{aligned} A &= M, \qquad & F &= \partial\cdot V, \\ B &= N, \qquad & G &= \partial\cdot A, \\ \psi &= \chi + 2\not{\partial}\lambda, \end{aligned} \tag{3.5}$$

transforming as in (II.4.1).

Both these multiplets contain 4 + 4 bosonic and fermionic components. This leaves 12 + 12 components of the original $(\frac{3}{2},1)$ multiplet. They do not correspond to a standard multiplet, but to a multiplet of U(1) conformal supersymmetry.*) Such a superconformal multiplet does of course contain Poincaré submultiplets. However, these cannot be described in terms of the fields themselves, but only in terms of their higher spin components, which are obtained by applying non-local projection operators on the the fields. Hence there do not exist local Lagrangians for these Poincaré submultiplets. For this same reason the superconformal multiplet must be described by a Lagrangian which is of higher order in derivatives:

$$\begin{aligned} \mathcal{L} = &-\epsilon_{\mu\nu\rho\sigma}\overline{(R_\mu - \tfrac{1}{3}\gamma_\mu\gamma\cdot R)}\gamma_5\gamma_\nu\partial_\rho(R_\sigma - \tfrac{1}{3}\gamma_\sigma\gamma\cdot R) - 3\overline{(\chi - \tfrac{1}{3}\gamma\cdot R)}\not{\partial}(\chi - \tfrac{1}{3}\gamma\cdot R) \\ &- \tfrac{1}{2}F(A)^2_{\mu\nu} - \tfrac{1}{2}F(V)^2_{\mu\nu} + \tfrac{1}{2}\big(\partial_\mu(t_{\mu\nu} - \sqrt{2}F(B)_{\mu\nu})\big)^2 - \tfrac{1}{2}(\partial_\mu\tilde{t}_{\mu\nu})^2 , \end{aligned} \tag{3.6}$$

where $\tilde{t}_{\mu\nu}$ is the dual of $t_{\mu\nu}$ (cf. 2.6).

---

*) Conformal supersymmetry is described in appendix B.

We now turn to the SO(1) supergravity multiplet. It has 12 + 12 components. Its action and transformation laws were given in II.4. We will restrict ourselves here to the linearized version only. We see that it has two submultiplets [10,11] . The first one is an N=1 scalar multiplet consisting of the auxiliary fields S and P, the scalar Riemann curvature R, the divergence of the axial vector field $\partial\cdot A$, and the contracted Rarita-Schwinger field equation $\gamma\cdot R$. The assignment of the 4 + 4 components corresponding to (II.4.1) is:

$$
\begin{aligned}
&A = S, \qquad F = -\frac{R}{2\kappa}, \\
&B = P, \qquad G = \partial\cdot A . \\
&\psi = -\tfrac{1}{2}\gamma\cdot R.
\end{aligned}
\tag{3.7}
$$

The remaining 8 + 8 components together form the multiplet of U(1) conformal supergravity [12]. This theory is the supersymmetric extension of the Weyl theory of gravitation. It consists of the highest spin components of the fields e_μ^a, ψ_μ and A_μ, whose transformation rules were given in (II.4.8). The linearized Lagrangian for this theory is:

$$
\begin{aligned}
\mathcal{L}_W^{(1)} = {} & \frac{1}{\kappa^2}\left(R_{\mu\nu}^2 - \tfrac{1}{3}R^2\right) - \varepsilon_{\mu\nu\rho\sigma}\overline{\left(R_\mu - \tfrac{1}{3}\gamma_\mu\gamma\cdot R\right)}\gamma_5\gamma_\nu\partial_\rho\left(R_\sigma - \tfrac{1}{3}\gamma_\sigma\gamma\cdot R\right) - \\
& - \tfrac{1}{3}\left(F(A)_{\mu\nu}\right)^2 .
\end{aligned}
\tag{3.8}
$$

As explained in more detail in § 6, this Lagrangian describes in fact the transversal and traceless parts of $R_{\mu\nu}$ and R_μ, denoted by $R^T_{\mu\nu}$ and R^T_μ :

$$
\begin{aligned}
&\partial_\mu R^T_{\mu\nu} = 0, \qquad R^T_{\mu\mu} = 0 , \\
&\partial_\mu R^T_\mu = 0, \qquad \gamma_\mu R^T_\mu = 0 .
\end{aligned}
\tag{3.9}
$$

One sees, that both (3.6) and (3.8) are quadratic in the corresponding SO(1) field equations.

4. Multiplets of SO(2) supersymmetry

Before we explain the construction of the complete SO(2) supergravity multiplet, it is convenient to have some results concerning other SO(2) supermultiplets at our disposal. Two such multiplets and their quadratic Lagrangians will be presented in this section.

The first multiplet is the SO(2) vector-gauge multiplet [13]. Its transformation rules follow from combining the scalar and vector multiplets given in II.4 by the procedure described in I.3. One finds:

$$
\begin{aligned}
\delta V_\mu &= \bar{\varepsilon}^i \gamma_\mu \psi^i \ , \\
\delta A^{ij} &= \bar{\varepsilon}^{[i} \psi^{j]} \ , \\
\delta B^{ij} &= i\bar{\varepsilon}^{[i} \gamma_5 \psi^{j]} \ , \\
\delta \psi^i &= -\sigma_{\rho\sigma} F(V)_{\rho\sigma} \varepsilon^i - \not\partial (A^{ij} + i\gamma_5 B^{ij})\varepsilon^j - (F^{ij} - i\gamma_5 G^{ij})\varepsilon^j \ , \\
\delta F^{ij} &= \bar{\varepsilon}^{[i} \not\partial \psi^{j]} \ , \\
\delta G^{ij} &= i\bar{\varepsilon}^i \gamma_5 \not\partial \psi^j + (i \leftrightarrow j \ ; \text{traceless}),
\end{aligned}
\tag{4.1}
$$

where $F(V)_{\mu\nu}$ is again the field strength of V_μ. The Lagrangian for this multiplet reads:

$$
\mathcal{L}_V = -\tfrac{1}{4} F(V)^2_{\mu\nu} - \tfrac{1}{4}(\partial_\mu A^{ij})^2 - \tfrac{1}{4}(\partial_\mu B^{ij})^2 - \tfrac{1}{2}\bar{\psi}^i \not\partial \psi^i + \tfrac{1}{4} F^{ij2} + \tfrac{1}{4} G^{ij2} \ . \tag{4.2}
$$

The commutator algebra closes off shell, yielding the usual translation δ_p and a gauge transformation δ_V on the vector field V_μ:

$$
[\delta_2, \delta_1] = \delta_p(\xi_\lambda) + \delta_V\left(-2\bar{\varepsilon}^i_1 \not V \varepsilon^i_2 - 2\bar{\varepsilon}^i_1 (A^{ij} + i\gamma_5 B^{ij})\varepsilon^j_2\right) \ , \tag{4.3}
$$

with

$$
\xi_\lambda = 2\bar{\varepsilon}^i_2 \gamma_\lambda \varepsilon^i_1 \ .
$$

Because V_μ is a gauge field, the multiplet contains $8+8$ components. However, the equations of motion show, that there are $4+4$ physical degrees of freedom, corresponding to massless particles of spin $(1, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0)$.

A related multiplet is the SO(2) tensor-gauge multiplet. It consists of a tensor field $T^{ij}_{\mu\nu}$, antisymmetric in both pairs of indices, a singlet Lorentz scalar A, a symmetric traceless pseudo-scalar B^{ij}, and a spinor doublet χ^i, with two auxiliary fields: a scalar F and a pseudo-scalar G. These fields transform as follows:

$$
\begin{aligned}
\delta T_{\mu\nu}^{\ ij} &= 2\bar{\varepsilon}^{[i} \sigma_{\mu\nu} \chi^{j]}, \\
\delta A &= \bar{\varepsilon}^i \chi^i \ , \\
\delta B^{ij} &= i\bar{\varepsilon}^i \gamma_5 \chi^j + (i \leftrightarrow j ; \text{traceless}) \ , \\
\delta \chi^i &= \not\partial (A\delta^{ij} + i\gamma_5 B^{ij})\varepsilon^j + \gamma_\mu \partial_\nu T^{ij}_{\mu\nu} \varepsilon^j + (F + i\gamma_5 G)\varepsilon^i \ , \\
\delta F &= \bar{\varepsilon}^i \not\partial \chi^i \ , \\
\delta G &= i\bar{\varepsilon}^i \gamma_5 \not\partial \chi^i \ .
\end{aligned}
\tag{4.4}
$$

We can reduce this multiplet to its constituent N=1 multiplets by consistently setting $\varepsilon^2=0$ everywhere. In this way one may verify, that the tensor-gauge multiplet (4.4) is a fusion of the N=1 tensor multiplet (II.4.15) with a massless scalar multiplet (II.4.1). It has the same gauge invariance

$$\delta_T T^{ij}_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma}\partial_\rho \Lambda^{ij}_\sigma \ , \tag{4.5}$$

as we found in the N=1 case. With this gauge invariance the commutators of the transformations (4.4) close on all fields, including $T^{ij}_{\mu\nu}$:

$$[\delta_2,\delta_1] = \delta_\rho(\xi_\lambda) + \delta_T\big(-\tfrac{1}{2}\varepsilon_{\sigma\mu\nu\lambda}\xi_\lambda T^{ij}_{\mu\nu} + 2\bar{\varepsilon}_1^{[i}\gamma_5\gamma_\sigma(A\delta^{j]k} + i\gamma_5 B^{j]k})\varepsilon_2^k\big). \tag{4.6}$$

Again (4.5) implies that there are 8 + 8 components in the multiplet, while the spin content is $(\frac{1}{2},\frac{1}{2},0,0,0,0)$. This corresponds precisely to the massless particle multiplet with highest helicity $\lambda=\frac{1}{2}$ of table II.3.1, provided we take a doublet highest helicity state in stead of a singlet. The spin content also follows from the Lagrangian:

$$\mathcal{L}_T = \tfrac{1}{4}(\partial_\mu T^{ij}_{\mu\nu}) - \tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{4}(\partial_\mu B^{ij})^2 - \tfrac{1}{2}\bar{\chi}^i \partial\!\!\!/ \chi^i + \tfrac{1}{2}F^2 + \tfrac{1}{2}G^2 \ . \tag{4.7}$$

It is equivalent to the massless scalar-spinor multiplet described in refs. [13,14] . However, contrary to the scalar-spinor multiplet, it cannot be used to describe the massive case. This is analogous to the N=1 case.

5. The construction of the SO(2) supergravity multiplet

We are now prepared to come to the main problem of this chapter, the construction of the full linearized SO(2) supergravity multiplet. In order to accomplish this we take the N=1 linearized supergravity and spin $(\frac{3}{2},1)$ multiplets and try to combine these in an SO(2) covariant fashion. We may expect some problems, since these multiplets have different numbers of components. However, a good strategy will be to start with the fusion of some of their submultiplets, which contain equal numbers of components. Later one may then try to find any missing components by extending the remaining N=1 submultiplets to the N=2 case in a suitable manner. We will now describe this procedure in detail.

In first instance the only submultiplet of O(1) Poincaré supergravity which comes into consideration is the scalar multiplet

$$(S,P,\ -\tfrac{1}{2}\gamma\cdot R,\ -\frac{R}{2\kappa},\ \partial\cdot A) \ , \tag{5.1}$$

presented in (3.7). On the other hand, the spin $(\frac{3}{2},1)$ multiplet has the tensor submultiplet (3.4). As we have shown, it is possible to combine these two N=1 submultiplets in an SO(2) tensor-gauge multiplet (4.4). This immediately suggests the SO(2) assignments of the fields: the axial vector field A_μ and the scalar S from O(1) supergravity become SO(2) singlets, the pseudo-scalars of (5.1) and (3.4) fuse to form a symmetric, traceless tensor, and both $t_{\mu\nu}$ and B_μ are antisymmetric SO(2) singlets. To combine the two spinors into a doublet seems troublesome at first sight, but it should be realized, that (3.4) and (5.1) may still originate from a common field representation, since they are presumably to be obtained from that by different reduction procedures. Therefore we simply take the multiplet (3.3) and double the spinors ψ_μ and χ into doublets, choosing $\chi^i - \frac{1}{2}\gamma\cdot R^i$ as the spinor components of the SO(2) tensor multiplet.

One then finds, that the scalar curvature cannot fully represent the F-component of this multiplet, since the supersymmetry variation of R will never yield terms proportional to $\bar{\epsilon}\not\partial\chi^i$. As a result the F-component must be a linear combination of R and some other field, which is as yet unknown.

We can summarize the above by stating, that one of the submultiplets of SO(2) supergravity will be a tensor multiplet with components:

$$
\begin{aligned}
T^{ij}_{\mu\nu} &= -t^{ij}_{\mu\nu} + \sqrt{2}F(B)^{ij}_{\mu\nu} \,, \\
A &= S \,, \\
B^{ij} &= P^{ij} \,, \\
\psi^i &= \chi^i - \tfrac{1}{2}\gamma\cdot R^i \,, \\
F &= -\frac{R}{2\kappa} + D \,, \\
G &= \partial\cdot A \,,
\end{aligned}
\tag{5.2}
$$

where D is still to be determined. At this stage we may note, that indeed the $(2,\frac{3}{2})$ and $(\frac{3}{2},1)$ multiplets of N=1 do not contain a sufficient set of auxiliary fields for the SO(2) supergravity multiplet. We have introduced an extra spinor χ and an unknown scalar component D, which neither originate in the $(\frac{3}{2},1)$ nor in the $(2,\frac{3}{2})$ multiplet. The reason for this, as noted, is that the minimal representation of SO(1) supergravity, with 12 + 12 components, cannot be fused with the 20 + 20 components of the spin $(\frac{3}{2},1)$ multiplet. Instead we have to use a non-minimal representation, which also contains 20 + 20 components. The missing 8 + 8 components have to be found by direct construction.

Therefore our next step is to further generalize the multiplet (3.3) to SO(2). To obtain closure on the vector and axial vector fields it turns out to be necessary that V_μ in (3.3) is assigned to an antisymmetric SO(2) singlet representation, whereas A_μ must be extended to a symmetric traceless tensor. At this point one then has found the transformation rules for

$$\psi^i_\mu,\ \chi^i,\ t^{ij}_{\mu\nu},\ B^{ij}_\mu,\ V^{ij}_\mu,\ A^{ij}_\mu,\ A_\mu,\ S \text{ and } P,$$

with a few arbitrary coefficients. For certain values of these coefficients one can show by explicit calculation, that the algebra closes on these fields modulo gauge transformations on ψ^i_μ, B^{ij}_μ, V^{ij}_μ and A^{ij}_μ. This gives the following partial results for the SO(2) supergravity multiplet:

$$\begin{aligned}
\delta e^a_\mu &= \kappa\bar\varepsilon^i\gamma^a\psi^i_\mu \ ,\\
\delta\psi^i_\mu &= -\tfrac{1}{2}\big(t^{ij}_{\rho\sigma} - \sqrt{2}F(B)^{ij}_{\rho\sigma}\big)\sigma_{\rho\sigma}\gamma_\mu\varepsilon^j + (V^{ij}_\mu - i\gamma_5A^{ij}_\mu)\varepsilon^j + i\gamma_5A_\mu\varepsilon^i - \gamma_\mu\eta^{ij}\varepsilon^j \ ,\\
\delta B^{ij}_\mu &= -\sqrt{2}\ \bar\varepsilon^{[i}\psi^{j]}_\mu \ ,\\
\delta\chi^i &= -2\not\partial\eta^{ij}\varepsilon^j + D\varepsilon^i + \big(F(V)^{ij}_{\mu\nu} - i\gamma_5F(A)^{ij}_{\mu\nu}\sigma_{\mu\nu}\varepsilon^j \ ,\\
\delta t^{ij}_{\mu\nu} &= -2\bar\varepsilon^{[i}\sigma_{\mu\nu}\chi^{j]} - \varepsilon_{\mu\nu\rho\sigma}\bar\varepsilon^{[i}\gamma_5\gamma_\rho R^{j]}_\sigma \ ,\\
\delta A_\mu &= \frac{i}{2}\ \bar\varepsilon^i\gamma_5R^i_\mu + i\bar\varepsilon^i\gamma_5\gamma_\mu(\chi^i - \tfrac{1}{2}\gamma\cdot R^i) \ ,\\
\delta A^{ij}_\mu &= -i\bar\varepsilon^i\gamma_5R^j_\mu + i\bar\varepsilon^i\gamma_5\gamma_\mu\chi^j + (i\leftrightarrow j;\ \text{traceless}) \ ,\\
\delta V^{ij}_\mu &= -\bar\varepsilon^{[i}R^{j]}_\mu + \bar\varepsilon^{[i}\gamma_\mu\chi^{j]} \ ,\\
\delta S &= \bar\varepsilon^i(\chi^i - \tfrac{1}{2}\gamma\cdot R^i) \ ,\\
\delta P^{ij} &= i\bar\varepsilon^i\gamma_5(\chi^j - \tfrac{1}{2}\gamma\cdot R^j) + (i\leftrightarrow j;\ \text{traceless}) \ ,\\
\delta D &= \bar\varepsilon^i\not\partial\chi^i \ ,
\end{aligned} \qquad (5.3)$$

where

$$\eta^{ij} = S\delta^{ij} + i\gamma_5P^{ij} - i\gamma_5\not{A}\delta^{ij} + \tfrac{1}{2}\sigma_{\rho\sigma}t^{ij}_{\rho\sigma} \ . \qquad (5.4)$$

It will turn out that the commutator algebra of (5.3) closes, and the field components form two irreducible multiplets, one of which is the tensor multiplet (5.2), the other the multiplet of U(2) conformal supergravity. Again this superconformal multiplet only contains the highest-spin field components, as is evidenced by the gauge invariances introduced for the spin $\frac{3}{2}$ and vector fields.

The field components of the $(\frac{3}{2},1)$ multiplet that have not yet been incorporated in an SO(2) multiplet are exactly those of the scalar multiplet (3.5), which corresponds to the lower spins contained in these fields. Since χ^i is an SO(2) doublet, λ will have to be extended to a doublet also in order to obtain the spinor components $\chi^i + 2\not\partial\lambda^i$. The auxiliary field components $\partial\cdot V^{ij}$ and $\partial\cdot A^{ij}$ form an antisymmetric singlet and traceless symmetric SO(2) tensor respectively. Furthermore one demands, that the SO(2) fields into which λ^i is to transform do not affect the closure on V_μ^{ij} and A_μ^{ij} already obtained. In this way one obtains uniquely that the full SO(2) completion of (3.5) is a vector-gauge multiplet (4.1).

As a consequence M and N become antisymmetric singlets of SO(2), while a new vector field V_μ has to be introduced. The SO(2) vector-gauge multiplet thus consists of the components

$$(V_\mu,\ M^{ij},\ N^{ij},\ \chi^i + 2\not\partial\lambda^i,\ \partial\cdot V^{ij},\ \partial\cdot A^{ij}) \ . \tag{5.5}$$

The only field component not yet assigned to a SO(2) submultiplet is now $\partial\cdot V$, but one sees immediately that it fits exactly the role of D in (5.3).

This completes our construction. To summarize the results we give here the full set of fields and transformation laws of linearized SO(2) supergravity. It contains the vierbein e_μ^a, a doublet of Majorana vectorial spinors ψ_μ^i and the vector gauge field B_μ^{ij} as physical components, and furthermore the following auxiliary fields: two doublets of spinors χ^i and λ^i, an antisymmetric tensor $t_{\mu\nu}^{ij}$, three axial vector fields A_μ and A_μ^{ij}, two vector fields V_μ and V_μ^{ij}, two scalars S and M^{ij}, and three pseudo-sclars P^{ij} and N^{ij}. In all there are 40 + 40 components. The fields B_μ^{ij}, $t_{\mu\nu}^{ij}$, V_μ^{ij}, M^{ij} and N^{ij} are antisymmetric SO(2) singlets, whereas A_μ^{ij} and P^{ij} are assigned to the symmetric traceless representation of SO(2). Under rigid SO(2) supersymmetry transformations these fields transform as follows:

$$\begin{aligned}
\delta e_\mu^a &= \kappa\bar\varepsilon^i\gamma^a\psi_\mu^i \ , \\
\delta\psi_\mu^i &= \frac{2}{\kappa} D_\mu\varepsilon^i - \tfrac{1}{2}\left(t_{\rho\sigma}^{ij} - \sqrt{2}F(B)_{\rho\sigma}^{ij}\right)\sigma_{\rho\sigma}\gamma_\mu\varepsilon^j + (V_\mu^{ij} - i\gamma_5 A_\mu^{ij})\varepsilon^j + i\gamma_5 A_\mu\varepsilon^i - \gamma_\mu\eta^{ij}\varepsilon^j , \\
\delta B_\mu^{ij} &= -\sqrt{2}\ \bar\varepsilon^{[i}\psi_\mu^{\ j]} \ , \\
\delta\lambda^i &= -\tfrac{1}{2}(\not V^{ij} + i\gamma_5\not A^{ij})\varepsilon^j - \tfrac{1}{2}\not V\varepsilon^i - \tfrac{1}{2}(M^{ij} + i\gamma_5 N^{ij})\varepsilon^j + \eta^{ij}\varepsilon^j \ , \\
\delta\chi^i &= -2\not\partial\eta^{ij}\varepsilon^j + \partial\cdot V\varepsilon^i + \left(F(V)_{\mu\nu}^{ij} - i\gamma_5 F(A)_{\mu\nu}^{ij}\right)\sigma_{\mu\nu}\varepsilon^j \ , \\
\delta t_{\mu\nu}^{ij} &= -2\bar\varepsilon^{[i}\sigma_{\mu\nu}\chi^{j]} - \varepsilon_{\mu\nu\rho\sigma}\bar\varepsilon^{[i}\gamma_5\gamma_\rho R_\sigma^{j]} \ , \\
\delta A_\mu &= \frac{i}{2}\ \bar\varepsilon^i\gamma_5 R_\mu^i + i\bar\varepsilon^i\gamma_5\gamma_\mu(\chi^i - \tfrac{1}{2}\gamma\cdot R^i) \ ,
\end{aligned} \tag{5.6}$$

((5.6) continued)

$$\begin{aligned}
\delta A^{ij}_\mu &= -i\bar\varepsilon^i\gamma_5 R^j_\mu + i\bar\varepsilon^i\gamma_5\gamma_\mu\chi^j + 2i\bar\varepsilon^i\gamma_5\partial_\mu\lambda^j + (i\leftrightarrow j;\ \text{traceless}), \qquad (5.6)\\
\delta V^{ij}_\mu &= -\bar\varepsilon^{[i} R^{j]}_\mu + \bar\varepsilon^{[i}\gamma_\mu\chi^{j]} + 2\bar\varepsilon^{[i}\partial_\mu\lambda^{j]} ,\\
\delta V_\mu &= \bar\varepsilon^i\gamma_\mu(\chi^i + 2\not\partial\lambda^i) - 2\bar\varepsilon^i\partial_\mu\lambda^i ,\\
\delta S &= \bar\varepsilon^i(\chi^i - \tfrac12\gamma\cdot R^i) ,\\
\delta P^{ij} &= i\bar\varepsilon^i\gamma_5(\chi^j - \tfrac12\gamma\cdot R^j) + (i\leftrightarrow j;\ \text{traceless}) ,\\
\delta M^{ij} &= \bar\varepsilon^{[i}(\chi^{j]} + 2\not\partial\lambda^{j]}) ,\\
\delta N^{ij} &= i\bar\varepsilon^{[i}\gamma_5(\chi^{j]} + 2\not\partial\lambda^{j]}) .
\end{aligned}$$

Although (5.6) refers to global supersymmetry, we have included a term $\partial_\mu\varepsilon^i$ in $\delta\psi^i_\mu$ to indicate the gauge invariance of ψ^i_μ under local Rarita-Schwinger transformations. The Lagrangian density for the multiplet reads:

$$\mathcal{L}_p = -\frac{e}{2\kappa^2}R - \tfrac12\bar\psi^i_\mu R^i_\mu - \tfrac18\big(F(B)^{ij}_{\mu\nu}\big)^2 + 2\bar\chi^i\lambda^i + 2\bar\lambda^i\not\partial\lambda^i + \tfrac18(t^{ij}_{\mu\nu})^2$$
$$+ A^2_\mu + \tfrac14(A^{ij}_\mu)^2 + \tfrac14(V^{ij}_\mu)^2 - \tfrac12 V^2_\mu - \tfrac14(M^{ij})^2 - \tfrac14(N^{ij})^2 - S^2 - \tfrac12(P^{ij})^2 . \qquad (5.7)$$

The first term has to be linearized as in (II.4.13).

The full SO(2) theory with auxiliary fields will be invariant under local supersymmetry, Maxwell and Lorentz transformations, as well as general coordinate transformations. At the linearized level the algebra closes with the infinitesimal supersymmetry transformations (5.6). In particular the commutator of two supersymmetry transformations acting on the fields is given by:

$$[\delta_2,\delta_1] = \delta_G(\xi_\lambda) + \delta_S(\varepsilon^i_3) + \delta_L(\varepsilon_{ab}) + \delta_B(\Lambda^{ij}) , \qquad (5.8)$$

as in (2.8), but now with parameters:

$$\begin{aligned}
\xi_\lambda &= 2\bar\varepsilon^i_2\gamma_\lambda\varepsilon^i_1 ,\\
\varepsilon^i_3 &= -\kappa(\bar\varepsilon^j_1\gamma^\lambda\varepsilon^j_2)(\psi^i_\lambda + \gamma_\lambda\lambda^i) + \tfrac34\kappa(\bar\varepsilon^{[i}_1\varepsilon^{j]}_2 + \bar\varepsilon^{[i}_1\gamma_5\varepsilon^{j]}_2\gamma_5)\lambda^j\\
&\quad + \frac{\kappa}{4}\,\big((\bar\varepsilon^i_1\gamma^\lambda\varepsilon^j_2 + \bar\varepsilon^j_1\gamma^\lambda\varepsilon^i_2)\gamma_\lambda + \bar\varepsilon^{[i}_1\gamma^\lambda\gamma_5\varepsilon^{j]}_2\gamma_\lambda\gamma_5\big)\lambda^j\\
&\quad - \frac{\kappa}{2}(\bar\varepsilon^i_1\sigma^{\rho\sigma}\varepsilon^j_2 + \bar\varepsilon^j_1\sigma^{\rho\sigma}\varepsilon^i_2)\sigma_{\rho\sigma}\lambda^j , \qquad (5.9)\\
\varepsilon_{ab} &= -2\bar\varepsilon^i_1\gamma^\lambda\varepsilon^i_2\omega_{\lambda ab} - \kappa\bar\varepsilon^i_1\Big(t^{ij}_{ab} - \sqrt2 F(B)^{ij}_{ab} + \gamma_5\big(\tilde t^{ij}_{ab} - \sqrt2 F(B)^{ij}_{ab}\big)\Big)\varepsilon^j_2\\
&\quad - 2\kappa\bar\varepsilon^i_1(\sigma_{ab}\eta^{ij} + \eta^{ij}\sigma_{ab})\varepsilon^j_2,\\
\Lambda^{ij} &= -2\bar\varepsilon^k_1\gamma^\lambda\varepsilon^k_2 B^{ij}_\lambda + \frac{2}{\kappa}\sqrt2\,\bar\varepsilon^{[i}_1\varepsilon^{j]}_2 .
\end{aligned}$$

The Lagrangian (5.7), the transformation rules (5.6) and the algebra all reduce to the ones given in § 2, when the field equations of the auxiliary fields are imposed.

6. Discussion

We have presented in (5.6) the full spin $(2, \frac{3}{2}, \frac{3}{2}, 1)$ multiplet with auxiliary fields of linearized SO(2) supergravity. At the level $\kappa=0$ the commutator algebra of these transformations closes. The construction was done by fusing the N=1 spin $(\frac{3}{2},1)$ multiplet with a multiplet of N=1 supergravity. This was not the usual minimal multiplet of the theory, since more components were needed to match those of the spin $(\frac{3}{2},1)$ multiplet. However, we did not know the required *non-minimal* multiplet in advance. Hence we had to construct it in the course of our work. For this it was very important, that we could go step by step, first generalizing N=1 submultiplets independently to the N=2 case and putting all information together afterwards to obtain the complete SO(2) supergravity set of fields.

As a byproduct of this procedure we have immediately found a decomposition of the SO(2) multiplet into submultiplets. These submultiplets are the following. There is the N=2 tensor-gauge multiplet (5.2) with components:

$$\left(-t^{ij}_{\mu\nu} + \tfrac{1}{2}\sqrt{2}F(B)^{ij}_{\mu\nu},\ S,\ P^{ij},\ \chi^i - \tfrac{1}{2}\gamma\cdot R^i,\ -\frac{R}{2\kappa} + \partial\cdot V,\ \partial\cdot A\right)\ , \tag{6.1}$$

transforming as in (4.4). The general Lagrangian was given in (4.7) and becomes in terms of the fields (6.1):

$$\mathcal{L}_T = \tfrac{1}{4}\Big(\partial_\mu\big(t^{ij}_{\mu\nu} - \tfrac{1}{2}\sqrt{2}F(B)^{ij}_{\mu\nu}\big)\Big)^2 - \tfrac{1}{2}(\partial_\mu S)^2 - \tfrac{1}{4}(\partial_\mu P^{ij})^2 - \tfrac{1}{2}\overline{(\chi^i - \tfrac{1}{2}\gamma\cdot R^i)}\,\partial\!\!\!/\,(\chi^i - \tfrac{1}{2}\gamma\cdot R^i)$$
$$+ \tfrac{1}{2}(\partial\cdot V - \frac{R}{2\kappa})^2 + \tfrac{1}{2}(\partial\cdot A)^2\ . \tag{6.2}$$

It contains 8 + 8 components. Then there also is the vector-gauge multiplet (5.5). Its transformation rules were given in (4.1) and its Lagrangian becomes:

$$\mathcal{L}_V = -\tfrac{1}{4}\big(F(V)_{\mu\nu}\big)^2 - \tfrac{1}{4}(\partial_\mu M^{ij})^2 - \tfrac{1}{4}(\partial_\mu N^{ij})^2 - \tfrac{1}{2}\overline{(\chi^i + 2\partial\!\!\!/\lambda^i)}\,\partial\!\!\!/\,(\chi^i + 2\partial\!\!\!/\lambda^i)$$
$$+ \tfrac{1}{4}(\partial\cdot V^{ij})^2 + \tfrac{1}{4}(\partial\cdot A^{ij})^2\ . \tag{6.3}$$

It contains 8 + 8 components as well.

Finally, the remaining 12 + 12 components form the N=2 version of conformal supergravity. It describes the highest spin components of the graviton e^a_μ, the gravitino ψ^i_μ, the axial vectors A_μ and A^{ij}_μ and the vector field V^{ij}_μ, as well as the tensor field $t^{ij}_{\mu\nu} - \sqrt{2}F(B)^{ij}_{\mu\nu}$ and its dual, and finally the scalar component $\partial\cdot V - \frac{R}{3\kappa}$. Their transformation rules are given in (5.6), while the quadratic Lagrangian becomes:

$$\begin{aligned}\mathcal{L}^{(2)}_W = {} & \frac{1}{\kappa^2}\,(R^2_{\mu\nu} - \tfrac{1}{3}R^2) - \varepsilon_{\mu\nu\rho\sigma}\overline{(R^i_\mu - \tfrac{1}{3}\gamma_\mu\gamma\cdot R^i)}\gamma_5\gamma_\nu\;\partial_\rho(R^i_\sigma - \tfrac{1}{3}\gamma_\sigma\gamma\cdot R^i) \\ & - 3\overline{(\chi^i - \tfrac{1}{3}\gamma\cdot R^i)}\,\partial\!\!\!/\,(\chi^i - \tfrac{1}{3}\gamma\cdot R^i) - \left(F(A)_{\mu\nu}\right)^2 - \tfrac{1}{4}\left(F(A)^{ij}_{\mu\nu}\right)^2 - \tfrac{1}{4}\left(F(V)^{ij}_{\mu\nu}\right)^2 \\ & + \tfrac{1}{2}\left(\partial_\mu\left(t^{ij}_{\mu\nu} - \sqrt{2}F(B)^{ij}_{\mu\nu}\right)\right)^2 - \tfrac{1}{2}(\partial_\mu\tilde{t}^{ij}_{\mu\nu})^2 + 3(\partial\cdot V - \frac{R}{3\kappa})^2 .\end{aligned} \tag{6.4}$$

Thus we have found four invariants for the fields of the linearized SO(2) Poincaré multiplet. The Lagrangians $\mathcal{L}_T$, $\mathcal{L}_V$ and $\mathcal{L}^{(2)}_W$ are all of higher order in derivatives compared to Poincaré supergravity and in fact quadratic in the field equations of this theory. Still they are of importance for Poincaré supergravity also, since they might occur as one-loop counter terms to the Poincaré Lagrangian $\mathcal{L}_P$ in a quantum theory with interactions. In that case the particle content of the theory changes drastically. This phenomenon was discussed for N=1 supergravity in ref. (11). In our case the most general linearized one-loop Lagrangian has the form:

$$\mathcal{L} = \alpha\mathcal{L}^{(2)}_W + \beta\mathcal{L}_T + \gamma\mathcal{L}_V + m^2\mathcal{L}_P , \tag{6.5}$$

with arbitrary parameters α, β, γ and m^2. In general the states arising in $\mathcal{L}$ are <u>massive</u> particle states. Therefore the physical states described by (6.5) must cover twice as large a range of spins as their massless counterparts. An analysis for general N, based on the assumed existence of massive multiplets as contained in $\mathcal{L}$, was carried out in ref. [15]. The ideas presented there can be verified by our results.

The particle content of $\mathcal{L}$ is found by studying wave equations and propagators derived from it [11,16]. One finds a massless spin $(2, \frac{3}{2}, \frac{3}{2}, 1)$ multiplet as expected. However, one also finds a massive multiplet with superspin j=1, as described in table II.3.2. This multiplet is realized by the following components:

$$\left(\frac{1}{\kappa}R^T_{\mu\nu},\ R^{iT}_\mu,\ F(V)^{ij}_{\mu\nu},\ F(A)^{ij}_{\mu\nu},\ F(A)_{\mu\nu}, \partial_\mu(t^{ij}_{\mu\nu} - \sqrt{2}F^{ij}_{\mu\nu}), \partial_\mu\tilde{t}^{ij}_{\mu\nu}, \chi^i - \tfrac{1}{3}\gamma\cdot R^i, \partial\cdot V - \frac{R}{3\kappa}\right). \tag{6.6}$$

Here the superscript T denotes the highest spin components:

$$R^T_{\mu\nu} = [\tfrac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \tfrac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma}]F_{\rho\sigma} \tag{6.7}$$

with

$$\theta_{\mu\nu} \equiv \delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box} .$$

Hence

$$\partial_\mu R^T_{\mu\nu} = 0, \quad R^T_{\mu\mu} = 0. \tag{6.8}$$

Also one has:

$$\begin{aligned} R^{iT}_\mu &= R^i_\mu - \tfrac{1}{3}\theta_{\mu\nu}\gamma_\nu \gamma\cdot R^i , \\ \partial\cdot R^{iT} &= 0, \quad \gamma\cdot R^{iT} = 0. \end{aligned} \tag{6.9}$$

The fields (6.6) all satisfy the wave equation:

$$(\Box + \frac{m^2}{2\alpha})\phi = 0 . \tag{6.10}$$

Hence they only have a real mass if $\alpha<0$. However, one may show that with this restriction the residues of the propagators for these fields become negative [11,16] . Therefore the states in this multiplet represent unphysical ghosts.

There are two other multiplets of massive particle states contained in $\mathcal{L}$. They correspond to the massive versions of the N=2 vector and tensor multiplets respectively. Both have positive norm states and represent physical degrees of freedom. The first one, generated by the field components:

$$\left(F(V)_{\mu\nu}, \chi^i + 2\partial\!\!\!/\lambda^i, \partial\cdot V^{ij}, \partial\cdot A^{ij}, M^{ij}, N^{ij}\right) , \tag{6.11}$$

satisfies the wave equation:

$$(\Box - \frac{m^2}{\gamma})\phi = 0, \quad \gamma>0 . \tag{6.12}$$

The other set consists of the components:

$$\left(\partial_\mu(t^{ij}_{\mu\nu} - \frac{1}{\sqrt{2}} F^{ij}_{\mu\nu}), \chi^i - \tfrac{1}{2}\gamma\cdot R^i, \partial\cdot V - \frac{R}{2\kappa}, \partial\cdot A, S, P^{ij}\right) . \tag{6.13}$$

They obey the Klein-Gordon equation:

$$(\Box - \frac{2m^2}{\beta})\phi = 0 , \quad \beta>0 . \tag{6.14}$$

Contrary to the massless case, both multiplets (6.11) and (6.13) have the same spin content, corresponding to j=0 (cf. table II.3.2):

$(1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0^+, 0^+, 0^-, 0^-, 0^-)$.

The basic observation in this respect is, that the massive tensor field represents a spin 1 state, while in the massless case it has spin 0. This discontinuity is crucial in obtaining the correct number of spin 1 states in $\mathcal{L}$, as predicted in [15].

Furthermore we have twice as many spin $\frac{1}{2}$ states as in the massless multiplets, because the spinors in (6.11) and (6.13) satisfy Klein-Gordon equations, rather than Dirac equations. Their propagator reads:

$$\frac{\not{\partial}}{\Box - \lambda^2} = \tfrac{1}{2}\left(\frac{1}{\not{\partial} + \lambda} + \frac{1}{\not{\partial} - \lambda}\right) , \tag{6.15}$$

which represent two positive norm states with mass $|\lambda| > 0$. This concludes our analysis of the linearized SO(2) Poincaré supergravity multiplet.

We now turn to consider in a little more detail the U(2) Weyl multiplet, consisting of the fields:

$$\left(e_\mu^a,\ \psi_\mu^i,\ V_\mu^{ij},\ A_\mu^{ij},\ A_\mu,\ t_{\mu\nu}^{ij} - \sqrt{2}F_{\mu\nu}^{ij},\ \tilde{t}_{\mu\nu}^{ij} - \sqrt{2}\tilde{F}_{\mu\nu}^{ij},\ \chi^i - \tfrac{1}{3}\gamma\cdot R^i,\ \partial\cdot V - \frac{R}{3\kappa}\right) . \tag{6.16}$$

Under supersymmetry they transform as in (5.6), provided we interpret the vector fields as gauge fields. The U(2) internal symmetry of this multiplet is realized on the chiral components of the fields (6.16). To make it manifest, we introduce chiral spinors:

$$\begin{aligned} \epsilon^i &\equiv \tfrac{1}{2}(1+\gamma_5)\epsilon^i , & \epsilon_i &\equiv \tfrac{1}{2}(1-\gamma_5)\epsilon^i , \\ \bar{\epsilon}^i &\equiv \tfrac{1}{2}\bar{\epsilon}^i(1+\gamma_5) , & \bar{\epsilon}_i &\equiv \tfrac{1}{2}\bar{\epsilon}^i(1-\gamma_5) , \end{aligned} \tag{6.17}$$

where upper and lower indices denote a transformation character according to the 2 and $\bar{2}$ representations respectively:

$$\begin{aligned} A^i &\to U^i_j A^j , \\ A_i &\to A_j U^{\dagger j}_{\ \ i} , \end{aligned} \tag{6.18}$$

with U^i_j a unitary 2×2 matrix. For the chiral components of the spinors ψ_μ^i and χ_c^i defined by:

$$\chi_c^i \equiv \chi^i - \tfrac{1}{3}\gamma\cdot R^i , \tag{6.19}$$

we can use the same notations (6.17), since they transform as in (6.18) also. The tensor fields are in antisymmetric singlet representations of SU(2), while under U(1) the tensor $t_{\mu\nu}^{ij} - \sqrt{2}F_{\mu\nu}^{ij}$ transforms into its dual and vice versa. Hence we introduce the (anti) self dual combinations:

$$T^{ij}_{\mu\nu} = (t^{ij}_{\mu\nu} - \sqrt{2}F^{ij}_{\mu\nu}) - (\tilde{t}^{ij}_{\mu\nu} - \sqrt{2}\tilde{F}^{ij}_{\mu\nu}),\ T_{\mu\nu ij} = (t^{ij}_{\mu\nu} - \sqrt{2}F^{ij}_{\mu\nu}) - (\tilde{t}^{ij}_{\mu\nu} - \sqrt{2}\tilde{F}^{ij}_{\mu\nu}) , \tag{6.20}$$

whose U(2) character is in accordance with (6.18). The auxiliary field of the multiplet is:

$$D_c \equiv \partial\cdot V - \frac{R}{3\kappa} . \tag{6.20}$$

Like the vierbein e^a_μ, it does not transform under U(2). Finally we turn to the vector fields A_μ, A^{ij}_μ, V^{ij}_μ. A_μ does not transform under the global U(2) symmetry. However, it does possess the gauge invariance:

$$\delta A_\mu = \partial_\mu \Lambda . \tag{6.21}$$

Hence it can, and will, become the gauge field of U(1) in the local theory. Similarly, the gauge fields A^{ij}_μ and V^{ij}_μ combine in an SU(2) vector representation:

$$A^i_{\mu j} = A^{ij}_\mu + iV^{ij}_\mu , \tag{6.22}$$

under global transformations, but become SU(2) gauge fields in the local version of the Weyl multiplet. Thus, under local SU(2) transformations:

$$\delta A^i_{\mu j} = \partial_\mu \Lambda^i_j + \Lambda^i_k A^k_{\mu\ell} + A^i_{\mu k}\Lambda^{\dagger k}_{\ j} , \tag{6.23}$$

where Λ^i_j is a traceless, hermitean 2×2 matrix. The justification for these U(2) assignments can be found in the invariance of the Lagrangian. In our new notations it becomes:

$$\begin{aligned}\mathcal{L}^{(2)}_W = {} & \frac{1}{\kappa^2}\,(R^2_{\mu\nu} - \tfrac{1}{3}R^2) + \varepsilon_{\mu\nu\rho\sigma}\overline{(R^i_\mu - \tfrac{1}{3}\gamma_\mu\gamma\cdot R^i)}\gamma_\nu\partial_\rho(R_{i\sigma} - \tfrac{1}{3}\gamma_\sigma\gamma\cdot R_i) \\ & - \varepsilon_{\mu\nu\rho\sigma}\overline{(R_{i\mu} - \tfrac{1}{3}\gamma_\mu\gamma\cdot R_i)}\gamma_\nu\partial_\rho(R^i_\sigma - \tfrac{1}{3}\gamma_\sigma\gamma\cdot R^i) - \big(F(A)_{\mu\nu}\big)^2 - \tfrac{1}{4}\left|F(A)^i_{\mu\nu j}\right|^2 \\ & + \tfrac{1}{2}(\partial_\mu T_{\mu\nu ij})(\partial_\lambda T^{ij}_{\lambda\nu}) + 3D_c^2 - 3\bar{\chi}^i_c\not{\partial}\chi_{ci} - 3\bar{\chi}_{ci}\not{\partial}\chi^i_c .\end{aligned} \tag{6.24}$$

Moreover, the supersymmetry transformations turn out to be in accordance with this U(2) transformation character of the fields also, e.g.:

$$\delta\chi^i_c = -\tfrac{1}{3}\gamma_\mu\partial_\nu T^{ij}_{\mu\nu}\varepsilon_j + D_c\varepsilon^i - \tfrac{i}{3}\,\sigma\cdot F(A)^i{}_j\varepsilon^j - \tfrac{2}{3}i\sigma\cdot F(A)\varepsilon^i , \tag{6.25}$$

$$\delta T^{ij}_{\mu\nu} = -4\bar{\varepsilon}^{[i}\sigma_{\mu\nu}\chi^{j]}_c + \tfrac{4}{3}\bar{\varepsilon}^{[i}\big(\partial_{[\mu}\psi^{j]}_{\nu]} - \varepsilon_{\mu\nu\rho\sigma}\partial_\rho\psi^{j]}_\sigma - \sigma_{\mu\rho}\partial_{[\rho}\psi^{j]}_{\nu]} + \sigma_{\nu\rho}\partial_{[\rho}\psi^{j]}_{\mu]}\big).$$

This completes our discussion of linearized N=2 supergravity.

7. The N=1 reduction

We have stressed already several times, that the N=1 reduction of the SO(2) Poincaré multiplet does not give back the minimal N=1 multiplet of Poincaré supergravity (II.4.8). Instead we find a larger multiplet containing 20 + 20 components, which we will present now.

In addition to the vierbein e^a_μ, the ψ_μ and the auxiliary fields A_μ, S and P, there are two Majorana spinors χ and λ, a vector V_μ and an axial vector A'_μ, all of which are auxiliary. They are contained in the following linearized Lagrangian:

$$\mathcal{L} = -\frac{e}{2\kappa^2} R - \tfrac{1}{2}\bar{\psi}_\mu R_\mu + A_\mu^2 - S^2 - P^2 + 2\bar{\chi}\lambda + 2\bar{\lambda}\not{\partial}\lambda + \tfrac{1}{2}A'^2_\mu - \tfrac{1}{2}V_\mu^2 \,. \tag{7.1}$$

This Lagrangian is invariant up to a total derivative under rigid supersymmetry transformations:

$$\begin{aligned}
\delta e^a_\mu &= \kappa\bar{\varepsilon}\gamma^a\psi_\mu \,, \\
\delta\psi_\mu &= \frac{2}{\kappa} D_\mu\varepsilon + i\gamma_5 A_\mu\varepsilon - i\gamma_5 A'_\mu\varepsilon - \gamma_\mu(S + i\gamma_5 P - i\gamma_5\not{A})\varepsilon \,, \\
\delta\lambda &= -\tfrac{1}{2}(\not{V} + i\gamma_5\not{A}')\varepsilon + (S + i\gamma_5 P - i\gamma_5\not{A})\varepsilon \,, \\
\delta\chi &= \partial\cdot V\varepsilon - i\gamma_5\sigma\cdot F(A')\varepsilon - 2\not{\partial}(S + i\gamma_5 P - i\gamma_5\not{A})\varepsilon \,, \\
\delta V_\mu &= \bar{\varepsilon}\gamma_\mu(\chi + 2\not{\partial}\lambda) - 2\bar{\varepsilon}\partial_\mu\lambda \,, \\
\delta A'_\mu &= -i\bar{\varepsilon}\gamma_5 R_\mu + i\bar{\varepsilon}\gamma_5\gamma_\mu\chi + 2i\bar{\varepsilon}\gamma_5\partial_\mu\lambda \,, \\
\delta A_\mu &= \frac{i}{2}\bar{\varepsilon}\gamma_5 R_\mu + i\bar{\varepsilon}\gamma_5\gamma_\mu(\chi - \tfrac{1}{2}\gamma\cdot R) \,, \\
\delta S &= \bar{\varepsilon}(\chi - \tfrac{1}{2}\gamma\cdot R) \,, \\
\delta P &= i\bar{\varepsilon}\gamma_5(\chi - \tfrac{1}{2}\gamma\cdot R) \,.
\end{aligned} \tag{7.2}$$

We remark, that the quantity

$$2\bar{\lambda}(\chi + \not{\partial}\lambda) + \tfrac{1}{2}A'^2_\mu - \tfrac{1}{2}V^2_\mu \,, \tag{7.3}$$

appearing in (7.1), is a mass Lagrangian for the N=1 spinor multiplet:

$$(V_\mu,\ A'_\mu,\ \lambda,\ \chi+\not{\partial}\lambda) \,.$$

Such a multiplet is characterized by the transformation rules:

$$\begin{aligned}
\delta V_\mu &= \bar{\varepsilon}\gamma_\mu(\chi + \not{\partial}\lambda) - \bar{\varepsilon}\not{\partial}\gamma_\mu\lambda \,, \\
\delta A'_\mu &= i\bar{\varepsilon}\gamma_5\gamma_\mu(\chi + \not{\partial}\lambda) + i\bar{\varepsilon}\gamma_5\not{\partial}\gamma_\mu\lambda \,, \\
\delta\lambda &= -\tfrac{1}{2}(\not{V} + i\gamma_5\not{A}')\varepsilon \,, \\
\delta(\chi + \not{\partial}\lambda) &= \tfrac{1}{2}\partial_\mu(\not{V} + i\gamma_5\not{A}')\gamma_\mu\varepsilon \,.
\end{aligned} \tag{7.4}$$

This multiplet is not a proper submultiplet, since the transformation rules (7.2) for V_μ, A'_μ, λ and $\chi + \not\partial\lambda$ contain terms depending on the other fields in addition to (7.4). Furthermore the minimal auxiliary field representation of SO(1) supergravity is not a submultiplet of (7.2) either. Hence this set of auxiliary fields for N=1 is not reducible to the minimal one. However, a connection between them was found by Siegel [17], who showed, that there is a one-parameter family of auxiliary fields for SO(1) supergravity, which contains both the minimal set and the set (7.2) as special cases.

Apart from this, the multiplet (7.2) can be decomposed into submultiplets. In fact there are four of them, which we will list here. The first is the scalar multiplet:

$$(S,\ P,\ \chi - \tfrac{1}{2}\gamma\cdot R,\ \partial\cdot V - \frac{R}{2\kappa},\ \partial\cdot A)\ . \tag{7.5}$$

It was fused with the tensor submultiplet (3.4) of the spin $(\frac{3}{2},1)$ multiplet into the SO(2) tensor-gauge multiplet (5.2).

Then there is a vector submultiplet, generated by:

$$(V_\mu,\ \chi + 2\not\partial\lambda,\ \partial\cdot A')\ , \tag{7.6}$$

which was combined with the scalar submultiplet (3.5) to form the SO(2) vector gauge multiplet (5.5). Both (7.5) and (7.6) contain 4 + 4 components.

Another 4 + 4 of these can be fitted in an axial vector multiplet. This is just a vector multiplet of reversed parity:

$$\begin{aligned}
&\delta\,\tfrac{1}{3}(A'_\mu - 2A_\mu) = i\bar{\varepsilon}\gamma_5\gamma_\mu(\chi - \tfrac{1}{3}\gamma\cdot R)\ ,\\
&\delta(\chi - \tfrac{1}{3}\gamma\cdot R) = -\tfrac{i}{3}\,\gamma_5\sigma\cdot F(A'-2A)\varepsilon + (\partial\cdot V - \tfrac{R}{3\kappa})\varepsilon\ ,\\
&\delta(\partial\cdot V - \tfrac{R}{3\kappa}) = \bar{\varepsilon}\not\partial(\chi - \tfrac{1}{3}\gamma\cdot R)\ .
\end{aligned} \tag{7.7}$$

The last 8 + 8 components form the multiplet of U(1) conformal supergravity (3.8), with components $(e_\mu^{\ a},\ \psi_\mu,\ A_\mu + A'_\mu)$. These, with the fields (7.7), can be extended to the N=2 Weyl multiplet by combining them with the spin $(\frac{3}{2},1)$ multiplet described in (3.6).

8. Conclusion

We have obtained the linearized multiplets and Lagrangians for N=2 supergravity, both the Poincaré and Weyl theories, with auxiliary fields. These multiplets, as well as their N=1 reductions, contain many submultiplets. The submultiplets were discussed in the first place in order to show, how the N=2 multiplet could be understood, and constructed, in terms of N=1 multiplets. Another application of the submultiplets is, that they may be considered as abstract matter multiplets, which may be coupled to the supergravity fields, if their extension to the local theory exists. This subject will be discussed in the next chapter. For convenience of the reader we collect here the N=2 multiplets that we have discussed. In the last two columns we list the places, where the transformation rules and Lagrangians can be found.

Table 3: multiplets of N=2 supergravity.

Multiplet	fields	transf. rules	Lagrangian
Poincaré	$e_\mu^a, \psi_\mu^i, B_\mu^{ij}, \lambda^i, \chi^i, A_\mu, A_\mu^{ij}, V_\mu^{ij}, V_\mu, t_{\mu\nu}^{ij}$ S, P, M^{ij}, N^{ij}	(5.6)	(5.7)
Weyl	$e_\mu^a, \psi_\mu^i, V_\mu^{ij}, A_\mu^{ij}, A_\mu, t_{\mu\nu}^{ij}-\sqrt{2}F_{\mu\nu}^{ij}, \tilde{t}_{\mu\nu}^{ij}-\sqrt{2}\tilde{F}_{\mu\nu}^{ij}$ $\chi^i-\frac{1}{3}\gamma\cdot R^i,\ \partial\cdot V-\frac{R}{3\kappa}$	(5.6)	(6.4)
Vector	$V_\mu, M^{ij}, N^{ij}, \chi^i+2\not\partial\lambda^i, \partial\cdot V^{ij}, \partial\cdot A^{ij}$	(4.1)	(6.3)
Tensor	$t_{\mu\nu}^{ij}-\frac{1}{\sqrt{2}}F_{\mu\nu}^{ij}, S, P^{ij}, \chi^i-\frac{1}{2}\gamma\cdot R^i, \partial\cdot V-\frac{R}{2\kappa}, \partial\cdot A$	(4.4)	(6.2)

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CHAPTER IV

THE FULL N=2 SUPERGRAVITY THEORY

1. Introduction

In the foregoing chapter we have constructed the linearized SO(2) supergravity multiplet with a minimal set of auxiliary fields and its quadratic Lagrangian. We have discussed its transformation rules and submultiplets under global supersymmetry. Using this as a starting point we will now derive the fully coupled SO(2) supergravity theory, with this auxiliary field configuration [1-4] . Our derivation will be order by order in the gravitational coupling constant κ, following procedures outlined in chapter II. The results are fairly complicated, but can be understood more easily by the introduction of superconformal notions [4,5] . The insight thus gained is also useful in obtaining the generalization of the submultiplets to local supersymmetry, as will be illustrated for the vector multiplet [4].

2. The self-coupling of SO(2) supergravity to order κ

In this section we will present the extension of the global SO(2) supergravity multiplet to a local one, to first order in the gravitational coupling constant κ. Our derivation goes in three steps. First we discuss the commutator algebra, which can be found to order κ from the global theory as in (III.5.8) and (III.5.9). By imposing this algebra we can find the transformation rules of the fields. Finally we extend the Poincaré Lagrangian (III.5.7) by the Noether procedure to an invariant action for the locally supersymmetric theory. We stress, that the results obtained here generally refer to order κ, even when we do not state this explicitly all the time.

We start with the commutator algebra (III.5.8) and (III.5.9). The parameters of the general coordinate, Lorentz en supersymmetry transformations $\delta_G(\xi_\lambda), \delta_L(\epsilon_{ab})$ and $\delta_s(\epsilon_3^i)$ are obtained to order κ from the transformation rules of the global multiplet. This is possible, because the linearized supersymmetry transformation of e_μ^a contains an explicit κ and that of ψ^i_μ a term $\frac{1}{\kappa}\,\partial_\mu\varepsilon^i$. In the global theory this last term is to be interpreted as an independent gauge transformation of ψ^i_μ. However, it is an essential part of the supersymmetry transformation law, if one considers the linearized theory as the zeroth-order version of the locally supersymmetric one. Using this, one can calculate $\delta_G(\xi_\lambda)$ and $\delta_L(\varepsilon_{ab})$ to order κ from the commutator on the vierbein; to find ε^i_3 to this order, a calculation of $\delta_S(\varepsilon^i_3)$ in zeroth order from the commutator on ψ^i_μ suffices. When we assume the commutator algebra to hold uniformly on all the fields, we have thus obtained it to order κ for the whole multiplet.

It now becomes possible to find the transformation rules of the fields by requiring them to satisfy these commutation relations.

We begin by making explicit a few assumptions which we will hold understood throughout our derivation. We also present arguments for them, but their final justification is of course, that they lead to consistent results. In the first place we interpret the space-time indices of the gauge fields of SO(2) supergravity as world indices, while those of the other fields are taken to be local-Lorentz indices. The reason for this is, that we want only the gauge fields to transform with derivatives on the parameters. Hence we have the following set of fields:

$$(e^a_\mu,\psi^i_\mu,B^{ij}_\mu,t^{ij}_{ab},A_a,A^{ij}_a,V^{ij}_a,V_a,S,P^{ij},M^{ij},N^{ij},\lambda^i,\chi^i)\ . \tag{2.1}$$

Related to this is the assumption, that the gauge fields appear in the transformation rules of the auxiliary fields only in covariant combinations with respect to their various local symmetries. This is to avoid terms with derivatives on the parameters in the gauge algebra. For this same reason we will supercovariantize all derivatives and curvatures in the transformation laws of the auxiliary fields.

Finally, we assume that terms of a certain power in the fields occur in the field variations only in a fixed order of κ. This we argue as follows. The linearized supersymmetry transformations are of order $\frac{1}{\kappa}$, when not field dependent, and of order κ^0 when linear in the fields. In the commutator they give rise to transformations which contain one more power, both of κ and of the fields. To close such a commutator on all fields, one needs to introduce order κ variations which are quadratic in the fields. These can again be used, as we will do, to calculate the commutator to next higher order in κ, which is then automatically also next higher order in the fields. Continuing this argument one finds, that field transformations of order κ^n are always of the $(n+1)^{st}$ power in the fields. This observation is especially important, when we have to write down all allowed variations of a field in a certain order of κ, for the dimensionality and the prescribed power of the fields put severe restrictions on the possible terms.

With these assumptions in mind we begin the calculation of commutators in order to find the field transformation laws. The first one we examine is that of ψ_μ^i. We notice, that the result

$$[\delta_2,\delta_1]\psi_\mu^i = \frac{1}{\kappa}\,\partial_\mu \varepsilon_3^i + \dots , \tag{2.2}$$

with ε_3^i as in (III.5.9), is obtained only, if the parameters $\varepsilon_{1,2}^i$ are space-time independent. To find the same result, in lowest order, for local parameters requires variations of $\delta\psi_\mu^i$ which have the schematic structure:

$$\delta\psi_\mu^i \to \lambda\partial_\mu\varepsilon,\ \psi_\nu\partial_\mu\varepsilon, \tag{2.3}$$

(cf. III.5.9). This is realized by the introduction of the following new terms in $\delta\psi_\mu^i$:

$$\begin{aligned}\delta\psi_\mu^i = \delta\psi_\mu^i\Big|_{\text{lin.}} &+ \frac{\kappa}{2}\,\sigma^{\rho\sigma}\gamma_\mu\varepsilon^j(\bar\psi_\rho^{[i}\psi_\sigma^{\ j]}) - \kappa(\bar\varepsilon^j\gamma_\nu\psi_\mu^j)\gamma^\nu\lambda^i \\ &+ \tfrac{3}{4}\kappa(\bar\varepsilon^{[i}\psi_\mu^{j]})\lambda^j + \tfrac{3}{4}\kappa(\bar\varepsilon^{[i}\gamma_5\psi_\mu^{j]})\gamma_5\lambda^j + \frac{\kappa}{4}(\bar\varepsilon^{[i}\gamma_5\gamma_\nu\psi_\mu^{j]})\gamma_5\gamma^\nu\lambda^j \\ &+ \tfrac{1}{4}\kappa(\bar\varepsilon^i\gamma_\nu\psi_\mu^j + \bar\varepsilon^j\gamma_\nu\psi_\mu^{\ i})\gamma^\nu\lambda^j - \frac{\kappa}{2}(\bar\varepsilon^i\sigma_{\rho\sigma}\psi_\mu^j + \bar\varepsilon^j\sigma_{\rho\sigma}\psi_\mu^i)\sigma^{\rho\sigma}\lambda^j ,\end{aligned} \tag{2.4}$$

where $\delta\psi_\mu^i\big|_{\text{lin.}}$ is the linearized transformation (III.5.6). The term with $\bar\psi_\rho^i\psi_\sigma^j$ can be absorbed in a covariantization of the electromagnetic field strength with respect to supersymmetry:

$$\hat F(B)_{\mu\nu}^{ij} = F(B)_{\mu\nu}^{ij} + \frac{\kappa}{2}\sqrt{2}\,\bar\psi_\mu^{[i}\psi_\nu^{j]} . \tag{2.5}$$

The result (2.4) can be further simplified by performing some Fierz rearrangements. This leads to:

$$\begin{aligned}\delta\psi_\mu^i = \frac{2}{\kappa}\,D_\mu\varepsilon^i &+ (V_\mu^{ij} + \kappa\bar\psi_\mu^{[i}\lambda^{j]})\varepsilon^j - i\gamma_5\big(A_\mu^{ij} + i\kappa(\bar\psi_\mu^i\gamma_5\lambda^j)_s\big)\varepsilon^j + i\gamma_5 A_\mu\varepsilon^i \\ &- \tfrac{1}{2}(t_{\rho\sigma}^{ij} - \sqrt{2}\,\hat F_{\rho\sigma}^{ij})\sigma^{\rho\sigma}\gamma_\mu\varepsilon^j - \gamma_\mu\eta^{ij}\varepsilon^j \\ &- \kappa(\bar\varepsilon^{[i}\lambda^{j]})\psi_\mu^j - \kappa(\bar\varepsilon^i\gamma_5\lambda^j)_s\gamma_5\psi_\mu^j .\end{aligned} \tag{2.6}$$

The quantity η^{ij} was defined in (III.5.4). The surprising result of these modifications is, that the commutators of e_μ^a and B_μ^{ij} close completely in order κ with their original transformation rules:

$$\delta e_\mu^a = \kappa \bar{\varepsilon}^i \gamma^a \psi_\mu^i ,$$
$$\delta B_\mu^{ij} = -\sqrt{2}\, \bar{\varepsilon}^{[i} \psi_\mu^{j]} . \tag{2.7}$$

Therefore we do not have to change these rules, while that of ψ_μ^i can be modified further only to the extent, that this closure is not affected. The only possible terms in $\delta\psi_\mu^i$ which still need examination are of the type:

$$\psi \to \kappa\lambda\lambda\varepsilon. \tag{2.8}$$

However, in order to maintain the closure on e_μ^a and B_μ^{ij}, such terms are necessarily of a form, that they can always be absorbed in a redefinition of the auxiliary Bose fields: V_a^{ij}, A_a^{ij}, A_a^i, etc. Hence we may choose (2.6) to represent the complete variation of ψ_μ^i.

Our next step is then to try to close the commutator of ψ_μ^i itself. By concentrating on specific terms in it we may obtain information on the transformations of λ^i and the auxiliary Bose fields, which we will henceforth denote collectively by B. In particular we can find how λ^i transforms into a bilinear combination of itself. Schematically:

$$\lambda \to \kappa\lambda\lambda\varepsilon. \tag{2.9}$$

This can be seen as follows. Under supersymmetry ψ_μ^i transforms into:

$$\psi \to \kappa\psi\lambda\varepsilon,$$

as in (2.4). Therefore we will get terms in the commutator, which are quadratic in λ^i and linear in ψ_μ^i, from the sequence of variations:

$$\psi \to \kappa\psi\eta\varepsilon_1 \to \kappa^2(\psi\lambda\varepsilon_2)\lambda\varepsilon_1. \tag{2.10}$$

Because ε_3^i depends on λ^i, we do indeed need such terms to give:

$$[\delta_2,\delta_1]\psi_\mu^i = \ldots + \kappa(\bar{\psi}_\mu^{[i}\lambda^{j]})\varepsilon_3^j + \kappa(\bar{\psi}_\mu^i\gamma_5\lambda^j)_s\gamma_5\varepsilon_3^j - \kappa(\bar{\varepsilon}_3^{[i}\lambda^{j]})\psi_\mu^j - \kappa(\bar{\varepsilon}_3^i\gamma_5\lambda^j)\gamma_5\psi_\mu^j . \tag{2.11}$$

However, the structure of the terms from (2.10) does not coincide with (2.11). Hence we need variations of other fields to contribute in (2.11) as well. These can come either from the λ^i variation of the type (2.9) by:

$$\psi \to \kappa\psi\lambda\varepsilon_1 \to \kappa^2\psi(\lambda\lambda\varepsilon_2)\varepsilon_1 , \tag{2.12}$$

or from transformations of the Bose fields, of the form:

$$B \to \kappa^2\psi\lambda\lambda\varepsilon. \tag{2.13}$$

As argued before, we expect terms with ψ_μ^i, like (2.13), to come only from supercovariantizations; in this case from covariant derivatives of λ^i in the Bose-field variations, where we replace:

$$\partial_\mu \lambda^i \to D_\mu \lambda^i - \frac{\kappa}{2}\,\delta_s(\psi_\mu)\lambda^i \equiv D^P_\mu \lambda^i \ . \tag{2.14}$$

Our strategy is therefore, to parametrize all possible transformations (2.9), simultaneously using these in the supercovariant derivatives (2.14), to find the variations (2.13). We then calculate the commutator of ψ^i_μ and choose the parameters in $\delta\lambda^i$ in such a way, that the result (2.11) holds. This establishes the quadratic terms (2.9) in $\delta\lambda^i$ uniquely:

$$\delta\lambda^i = \delta\lambda^i\Big|_{\mathrm{lin.}} - \kappa\lambda^i(\bar\lambda^j\varepsilon^j) \ . \tag{2.15}$$

Adhering to our principle, that no ψ_μ's are allowed in the transformation rules of the auxiliary fields, except in supercovariant quantities, we see that no other variations of λ^i are possible. Hence (2.15) represents the complete transformation of λ^i.

We now continue with the derivation of the transformation rules for the auxiliary Bose fields. Two types of variations are possible in order κ:

$$B \to \kappa B\psi\varepsilon, \quad \kappa B\lambda\varepsilon \ . \tag{2.16}$$

The first one must come from supercovariantization of the Rarita-Schwinger equations:

$$\begin{aligned}
R^{i\mu} \to R^{i\mu P} = \frac{1}{e}\,\varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\Big\{ & D_\rho\psi^i_\sigma + \frac{\kappa}{2}\big(V^{ij}_\rho + \kappa\bar\psi^{[i}_\rho\lambda^{j]}\big)\psi^j_\sigma \\
& -\frac{i}{2}\kappa\gamma_5\big(A^{ij}_\rho + i\kappa(\psi^i_\rho\gamma_5\lambda^j)_s\big)\psi^j_\sigma \\
& +\frac{i}{2}\kappa\gamma_5 A_\rho\psi^j_\sigma - \frac{\kappa}{4}(t^{ij}_{\alpha\beta} - \sqrt{2}\,\hat F^{ij}_{\alpha\beta})\sigma^{\alpha\beta}\gamma_\rho\psi^j_\sigma - \frac{\kappa}{2}\gamma_\rho\eta^{ij}\psi^j_\sigma \\
& +\frac{\kappa^2}{4}(\bar\psi^{[i}_\rho\lambda^{j]})\psi^j_\sigma + \frac{\kappa^2}{4}(\bar\psi^i_\rho\gamma_5\lambda^j)_s\gamma_5\psi^j_\sigma\Big\} \ .
\end{aligned} \tag{2.17}$$

The second one must be determined from the commutators on ψ^i_μ and λ^i. In particular the commutator of ψ^i_μ is not allowed to produce terms of the form $\kappa B\lambda\varepsilon_1\varepsilon_2$, while those in the commutator of λ^i have to add up to the Lorentz transformation $\delta_L(\varepsilon_{ab})$, with field-dependent parameter ε_{ab} as described in (III.5.9). As a result one uniquely finds the rules:

$$
\begin{aligned}
\delta t^{ij}_{ab} &= -2\bar{\varepsilon}^{[i}\sigma_{ab}\chi^{j]} - \varepsilon_{ab}{}^{cd}\bar{\varepsilon}^{[i}\gamma_5\gamma_c R^{j]P}_d \ , \\
\delta A_a &= i\bar{\varepsilon}^i\gamma_5\gamma_a\chi^i - i\bar{\varepsilon}^i\gamma_5\sigma_{ab}R_b{}^{iP} \ , \\
\delta V^{ij}_a &= \bar{\varepsilon}^{[i}\gamma_a\chi^{j]} + 2\bar{\varepsilon}^{[i}D^P_a\lambda^{j]} - \bar{\varepsilon}^{[i}R_a{}^{j]P} + \kappa\bar{\varepsilon}^{[i}(V+i\gamma_5A)^{j]k}_a\lambda^k \\
&\quad - i\kappa\bar{\varepsilon}^{[i}\gamma_5\lambda^{j]}A_a - \tfrac{1}{2}\kappa\bar{\varepsilon}^{[i}(T+\gamma_5\tilde{T})^{j]k}_{ab}\gamma_b\lambda^k - \kappa\bar{\varepsilon}^k\eta^{k[i}\gamma_a\lambda^{j]} \ , \\
\delta A^{ij}_a &= \big(i\bar{\varepsilon}^i\gamma_5\gamma_a\chi^j + 2i\bar{\varepsilon}^i\gamma_5D^P_a\lambda^j - i\bar{\varepsilon}^i\gamma_5R^{jP}_a + i\kappa\bar{\varepsilon}^i\gamma_5(V+i\gamma_5A)^{jk}_a\lambda \\
&\quad + \kappa\bar{\varepsilon}^i\lambda^jA_a - \tfrac{1}{2}i\kappa\bar{\varepsilon}^i(\gamma_5T+\tilde{T})^{jk}_{ab}\gamma_b\lambda^k + i\kappa\bar{\varepsilon}^k\eta^{ki}\gamma_5\gamma_a\lambda^j\big)_S \ , \\
\delta S &= \bar{\varepsilon}^i(\chi - \tfrac{1}{2}\gamma\cdot R^P)^i + 2i\kappa\bar{\varepsilon}^k\gamma_5\lambda^jP^{kj} \ , \qquad\qquad (2.18) \\
\delta P^{ij} &= \big(i\bar{\varepsilon}^i\gamma_5(\chi - \tfrac{1}{2}\gamma\cdot R^P)^j - 2i\kappa\bar{\varepsilon}^i\gamma_5\lambda^jS - \kappa\bar{\varepsilon}^{[i}\lambda^{k]}P^{kj}\big)_S \ , \\
\delta V_a &= \bar{\varepsilon}^i\gamma_a\big(\chi^i + 2\not{D}^P\lambda^i - \tfrac{1}{2}\kappa\sigma\cdot T^{ij}\lambda^j + \kappa(M-i\gamma_5N)^{ij}\lambda^j - \kappa\not{V}\lambda^i\big) - 2\bar{\varepsilon}^iD^P_a\lambda^i \\
&\quad + \kappa\bar{\varepsilon}^i\eta^{ij}\gamma_a\lambda^j - \kappa\bar{\varepsilon}^i(V+i\gamma_5A)^{ij}_a\lambda^j - 2i\kappa\bar{\varepsilon}^i\gamma_5\sigma_{ab}\lambda^iA_b \ , \\
\delta M^{ij} &= \bar{\varepsilon}^{[i}\big(\chi^{j]} + 2\not{D}^P\lambda^{j]} - \kappa\sigma\cdot T^{j]k}\lambda^k + \kappa(M-i\gamma_5N)^{j]k}\lambda^k - \kappa(\not{V}-i\gamma_5\not{A})\lambda^{j]}\big) \ , \\
N^{ij} &= i\bar{\varepsilon}^{[i}\gamma_5\big(\chi^{j]} + 2\not{D}^P\lambda^{j]} - \kappa\sigma\cdot T^{j]k}\lambda^k + \kappa(M-i\gamma_5N)^{j]k}\lambda^k - \kappa(\not{V}-i\gamma_5\not{A}(\lambda^{j]}\big) \ ,
\end{aligned}
$$

with

$$T^{ij}_{\mu\nu} = t^{ij}_{\mu\nu} - \sqrt{2}\,\hat{F}^{ij}_{\mu\nu} \ . \qquad (2.19)$$

According to our earlier discussion, all space-time indices in (2.18) are Lorentz indices. The consistency of this assignment may be verified from the commutators on these Bose fields themselves, which yield precisely the general coordinate and local Lorentz transformations corresponding to such a transformation character.

The only field variation we have not yet determined is that of χ^i. This we will obtain now by requiring closure of the commutator on the auxiliary Bose fields. As a first step we supercovariantize again all derivatives and curvatures in the linearized result. After that still many other types of variations are possible in order κ, and unfortunately they all appear:

$$\chi^i \to \kappa\lambda(\partial\psi)\varepsilon,\ \kappa\lambda\chi\varepsilon,\ \kappa\lambda(\partial\lambda)\varepsilon,\ \kappa BB\varepsilon. \qquad (2.20)$$

To parametrize these and calculate the coefficients from the Bose field commutators, is not an easy matter. We have however gone through the whole procedure to find the result:

$$
\begin{aligned}
\delta\chi^{i} = & -2\not{D}^{P}\eta^{ij}\epsilon^{j} + D^{P}\cdot V\epsilon^{i} + 2(D^{P}_{a}V^{ij}_{b} - i\gamma_5 D^{P}_{a}A^{ij}_{b} - \frac{\kappa}{2}A^{ik}_{a}A^{kj}_{b} - i\kappa\gamma_5\bar{V}^{ik}_{a}A^{kj}_{b})\cdot\sigma^{ab}\epsilon^{j} \\
& - \kappa\Big(\bar{\lambda}^{[i}(D_{[\mu}\psi^{j]}_{\nu]})^{P} - \big(\bar{\lambda}^{i}\gamma_5(D_{[\mu}\psi_{\nu]})^{P}\big)_{S}\gamma_5\Big)\sigma^{\mu\nu}\epsilon^{j} \\
& + \kappa\epsilon^{i}\big(\bar{\lambda}^{j}(2\chi^{j} + 2\not{D}^{P}\lambda^{j} - \tfrac{1}{2}\gamma\cdot R^{jP}) - 2A^{2}_{a} + \tfrac{1}{4}(V^{jk}_{a})^{2} + \tfrac{1}{4}(A^{jk}_{a})^{2} \\
& - \tfrac{1}{2}V^{2}_{a} - \tfrac{1}{4}(M^{jk})^{2} - \tfrac{1}{4}(N^{jk})^{2}\big) + 2i\kappa\gamma_5(S\not{A}^{ij} - P^{ik}\not{V}^{kj})\epsilon^{j} \\
& + \kappa(P^{ik}\not{A}^{kj} + \not{A}^{ik}P^{kj})\epsilon^{j} - i\kappa\gamma_5\sigma\cdot T^{ij}\not{A}\epsilon^{j} - 4\kappa(S\delta^{ij} - i\gamma_5 P^{ij} - \tfrac{1}{4}\sigma\cdot T^{ij})\eta^{jk}\epsilon^{k} \\
& + \frac{\kappa}{2}t^{ij}_{ab}(T + \gamma_5\tilde{T})^{jk}_{ab}\epsilon^{k} - \kappa(\bar{\epsilon}^{[i}\lambda^{j]})\chi^{j} - \kappa(\bar{\epsilon}^{i}\gamma_5\lambda^{j})_{S}\gamma_5\chi^{j} \;. \qquad (2.21)
\end{aligned}
$$

Esq. (2.6), (2.7), (2.15), (2.18) and (2.21) define the local SO(2) supergravity multiplet to first order in κ. One may verify, that the commutator algebra indeed has the form (III.5.8), (III.5.9) on all the fields.

We now have to extend the Lagrangian (III.5.7) to first order in κ. This is done by the Noether procedure. First one adds to the global Poincaré Lagrangian a Noether term:

$$
\mathcal{L}_{\text{Noether}} = -\frac{\kappa}{2}\,\bar{J}^{i\mu}\psi^{i}_{\mu} \;, \qquad (2.22)
$$

to absorb variations proportional to $\partial_\mu\epsilon^{i}$ in lowest order. Here $\bar{J}^{i\mu}$ is given by:

$$
\begin{aligned}
\bar{J}^{i\mu} = & \tfrac{1}{2}\sqrt{2}\,\bar{\psi}^{j}_{\nu}\big(F(B) + \gamma_5\tilde{F}(B)\big)^{ji\nu\mu} + \bar{\lambda}^{j}\big(2\gamma^{\mu}\eta^{ji} + (\not{V} - i\gamma_5\not{A})^{ji}\gamma^{\mu} - \not{V}\gamma^{\mu}\delta^{ji} \\
& + (M^{ji} - i\gamma_5 N^{ji})\gamma^{\mu}\big) \;. \qquad (2.23)
\end{aligned}
$$

It is not the exact Noether current $\bar{J}^{i\mu}_{\text{Noether}}$, defined as the coefficient of $\partial_\mu\epsilon^{i}$ in the variation of the global Lagrangian:

$$
\delta\mathcal{L}_{\text{Poincaré}} = \bar{J}^{i\mu}_{\text{Noether}}\;\partial_\mu\epsilon^{i} + \text{total derivative} \;. \qquad (2.24)
$$

The reason for this is, that the Noether current itself contains ψ^{i}_{μ}, and hence would generate extra variations with $\partial_\mu\epsilon^{i}$ in (2.22). To compensate for this, the corresponding part of $\bar{J}^{i\mu}$ has a factor $\frac{1}{2}$ in front.

We now continue with the Lagrangian:

$$
\mathcal{L}_{\text{Poincaré}}\;(\text{global}) + \mathcal{L}_{\text{Noether}} \;,
$$

and add terms to it such that it transforms as a total derivative under the new transformation rules in order κ. Again this is only possible up to terms proportional to $\partial_\mu\epsilon^{i}$. To cancel these we have to introduce a Noether term in order κ^2. The complete result of this procedure reads:

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\text{Poincaré}} = & -\tfrac{1}{2}\kappa^{-2}R - \tfrac{1}{2}\bar{\psi}_\mu^i R^{\mu i} - \tfrac{1}{8}\big(F_{\mu\nu}^{ij}(B)\big)^2 - \tfrac{1}{4}\sqrt{2}\kappa\bar{\psi}_\mu^i\big(F(B) + \gamma_5\tilde{F}(B)\big)^{\mu\nu ij}\psi_\nu^j \\
& - \tfrac{1}{8}\kappa^2\bar{\psi}_\mu{}^{[i}\psi_\nu^{j]}(\bar{\psi}^{\mu i}\psi^{\nu j} + \tfrac{1}{2}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\rho^i\gamma_5\psi_\sigma^j) \\
& - S^2 - \tfrac{1}{2}(P^{ij})^2 + \tfrac{1}{8}(t_{ab}^{ij})^2 + A_a^2 + \tfrac{1}{4}(A_a^{ij})^2 \\
& + \tfrac{1}{4}(V_a^{ij})^2 - \tfrac{1}{2}V_a^2 - \tfrac{1}{4}(M^{ij})^2 - \tfrac{1}{4}(N^{ij})^2 \\
& + \bar{\lambda}^i[2\chi^i + 2\not{D}^P\lambda^i - \kappa\gamma^\mu\eta^{ij}\psi_\mu^j - \tfrac{1}{2}\kappa(\not{V} - i\gamma_5\not{A})^{ij}\gamma\cdot\psi^j \\
& + \tfrac{1}{2}\kappa\not{V}\gamma\cdot\psi^i - \tfrac{1}{2}\kappa(M - i\gamma_5 N)^{ij}\gamma\cdot\psi^j + i\kappa\gamma_5\not{A}\lambda^i - \kappa\sigma\cdot T^{ij}\lambda^j] \\
& - 2\kappa^2\bar{\psi}_\mu^i\lambda^i\bar{\lambda}^j\sigma^{\mu\nu}\psi_\nu^j - \tfrac{1}{4}\kappa^2 e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu^i\gamma_\rho\psi_\nu^i\bar{\lambda}^j\gamma_5\gamma_\sigma\lambda^j\,. \qquad (2.25)
\end{aligned}
$$

3. The self-coupling of the SO(2) supergravity multiplet to all orders

We now extend our results to all orders in κ. As before we use the field variations to order κ to calculate $\delta_G(\xi_\lambda)$, $\delta_L(\varepsilon_{ab})$ and $\delta_S(\varepsilon_3^i)$ in the commutator to order κ^2. it turns out, that only ε_{ab} gets an order κ^2 contribution, which comes from the covariantization of $F(B)_{\mu\nu}^{ij}$ and can be absorbed in the combination $T_{\mu\nu}^{ij}$, defined in (2.19):

$$
\varepsilon_{ab} = -2\bar{\varepsilon}_1^i\gamma^\lambda\varepsilon_2^i\omega_{\lambda ab} - \kappa\bar{\varepsilon}_1^i(T_{ab}^{ij} + \gamma_5\tilde{T}_{ab}^{ij})\varepsilon_2^j - 2\kappa\bar{\varepsilon}_1^i(\sigma_{ab}\eta^{ij} + \eta^{ij}\sigma_{ab})\varepsilon_2^j. \quad (3.1)
$$

The other parameters remain unchanged. To close the commutators on e_μ^a and B_μ^{ij} the same rule $\delta\psi_\mu^i$, eq. (2.6), suffices. In fact, it turns out that with this algebra and this variation of ψ_μ^i they close to all orders in κ. Using eq. (2.15) for $\delta\lambda^i$ and full supercovariantizations to all orders in the Bose-field transformation rules, the commutators on ψ_μ^i and λ^i themselves close as well. This suggests, that we do not have to introduce new variations for the Bose fields either, beside this full covariantization. Therefore we try to close the algebra on all fields by only adjusting $\delta\chi^i$. This is indeed possible and one finds:

$$
\delta\chi^i = \delta\chi^i\big|_\kappa + \kappa^2\bar{\varepsilon}^i(i\lambda^j\gamma_5\not{A}\lambda^j - \bar{\lambda}^j\sigma\cdot T^{jk}\lambda^k)\,, \qquad (3.2)
$$

where $\delta\chi^i|_\kappa$ refers to the result (2.21), but now with full supercovariantizations and the complete T_{ab}^{ij}.

One may check, that with these changes in the transformation rules the Lagrangian (2.25) still transforms into a total derivative. Hence it is already complete. Collecting our results, we are now in a position to present the full set of transformations:

$$
\begin{aligned}
\delta e_\mu^a &= \kappa\bar{\epsilon}^i\gamma^a\psi_\mu^i\,, \qquad\qquad \delta B_\mu^{ij} = -\sqrt{2}\,\bar{\epsilon}^{[i}\psi_\mu^{j]}\,,\\
\delta\psi_\mu^i &= 2\kappa^{-1}D_\mu\epsilon^i + iA_\mu\gamma_5\epsilon^i + \left(V_\mu^{ij} + \kappa\bar{\psi}_\mu^{[i}\lambda^{j]}\right)\epsilon^j\\
&\quad - i\left(A_\mu^{ij} + i\kappa(\bar{\psi}_\mu^i\gamma_5\lambda^j)_S\right)\gamma_5\epsilon^j - \tfrac{1}{2}\sigma\cdot T^{ij}\gamma_\mu\epsilon^j\\
&\quad - \gamma_\mu\eta^{ij}\epsilon^j - \kappa\bar{\epsilon}^{[i}\lambda^{j]}\psi_\mu^j - \kappa(\bar{\epsilon}^i\gamma_5\lambda^j)_S\gamma_5\psi_\mu^j\,,\\
\delta\lambda^i &= \eta^{ij}\epsilon^j - \tfrac{1}{2}(\not{V} + i\gamma_5\not{A})^{ij}\epsilon^j - \tfrac{1}{2}\not{V}\epsilon^i - \tfrac{1}{2}(M + i\gamma_5 N)^{ij}\epsilon^j - \kappa\lambda^i(\bar{\lambda}^j\epsilon^j)\,,\\
\delta\chi^i &= -2\not{D}^P\eta^{ij}\epsilon^j + D^P\cdot V\epsilon^i + 2\left(D_a^P(V - i\gamma_5 A)_b^{ij} - \tfrac{1}{2}\kappa A_a^{ik}A_b^{kj} - \kappa i V_a^{ik}A_b^{kj}\gamma_5\right)\sigma^{ab}\epsilon^j\\
&\quad - \kappa\left(\bar{\lambda}^{[i}(D_{[\mu}\psi_{\nu]}^{j]})^P - \left(\lambda^i\gamma_5(D_{[\mu}\psi_{\nu]}^j)^P\right)_S\gamma_5\right)\sigma^{\mu\nu}\epsilon^j\\
&\quad + \kappa\epsilon^i\big[\bar{\lambda}^j(2\chi^j + 2\not{D}^P\lambda^j - \tfrac{1}{2}\gamma\cdot R^{jP} + i\kappa\gamma_5\not{A}\lambda^j - \kappa T^{jk}\cdot\sigma\lambda^k)\\
&\quad - 2A_a^2 + \tfrac{1}{4}(V_a^{jk})^2 + \tfrac{1}{4}(A_a^{jk})^2 - \tfrac{1}{2}V_a^2 - \tfrac{1}{4}(M^{jk})^2 - \tfrac{1}{4}(N^{jk})^2\big]\\
&\quad + 2i\kappa\gamma_5(S\not{A}^{ij} - P^{ik}\not{V}^{kj})\epsilon^j + \kappa(P^{ik}\not{A}^{kj} + \not{A}^{ik}P^{kj})\epsilon^j\\
&\quad - i\kappa\gamma_5\sigma\cdot T^{ij}\not{A}\epsilon^j - 4\kappa(S\delta^{ij} - i\gamma_5 P^{ij} - \tfrac{1}{4}\sigma\cdot T^{ij})\eta^{jk}\epsilon^k\\
&\quad + \tfrac{1}{2}\kappa t_{ab}^{ij}(T + \gamma_5\tilde{T})_{ab}^{jk}\epsilon^k - \kappa\bar{\epsilon}^{[i}\lambda^{j]}\chi^j - \kappa(\bar{\epsilon}^i\gamma_5\lambda^j)_S\gamma_5\chi^j\,,\\
\delta t_{ab}^{ij} &= -2\bar{\epsilon}^{[i}\sigma_{ab}\chi^{j]} - \epsilon_{ab}{}^{cd}\bar{\epsilon}^{[i}\gamma_5\gamma_c R_d^{j]P}\,,\\
\delta A_a &= i\bar{\epsilon}^i\gamma_5\gamma_a\chi^i - i\bar{\epsilon}^i\gamma_5\sigma_{ab}R_b^{iP}\,, && (3.3)\\
\delta V_a^{ij} &= \bar{\epsilon}^{[i}\gamma_a\chi^{j]} + 2\bar{\epsilon}^{[i}D_a^P\lambda^{j]} - \bar{\epsilon}^{[i}R_a^{j]P} + \kappa\bar{\epsilon}^{[i}(V + i\gamma_5 A)_a^{j]k}\lambda^k\\
&\quad - i\kappa\bar{\epsilon}^{[i}\gamma_5\lambda^{j]}A_a - \tfrac{1}{2}\kappa\bar{\epsilon}^{[i}(T + \gamma_5\tilde{T})_{ab}^{j]k}\gamma_b\lambda^k - \kappa\bar{\epsilon}^k\eta^{k[i}\gamma_a\lambda^{j]}\,,\\
\delta A_a^{ij} &= \big(i\bar{\epsilon}^i\gamma_5\gamma_a\chi^j + 2i\bar{\epsilon}^i\gamma_5 D_a^P\lambda^j - i\bar{\epsilon}^i\gamma_5 R_a^{jP} + i\kappa\bar{\epsilon}^i\gamma_5(V + i\gamma_5 A)_a^{jk}\lambda^k\\
&\quad + \kappa\bar{\epsilon}^i\lambda^j A_a - \tfrac{1}{2}i\kappa\bar{\epsilon}^i(\gamma_5 T + \tilde{T})_{ab}^{jk}\gamma_b\lambda^k + i\kappa\bar{\epsilon}^k\eta^{ki}\gamma_5\gamma_a\lambda^j\big)_S\,,\\
\delta S &= \bar{\epsilon}^i(\chi - \tfrac{1}{2}\gamma\cdot R^P)^i + 2i\kappa\bar{\epsilon}^k\gamma_5\lambda^j P^{kj}\,,\\
\delta P^{ij} &= \left(i\bar{\epsilon}^i\gamma_5(\chi - \tfrac{1}{2}\gamma\cdot R^P)^j - 2i\kappa\bar{\epsilon}^i\gamma_5\lambda^j S - \kappa\bar{\epsilon}^{[i}\lambda^{k]}P^{kj}\right)_S\,,\\
\delta V_a &= \bar{\epsilon}^i\gamma_a\left(\chi^i + 2\not{D}^P\lambda^i - \tfrac{1}{2}\kappa\sigma\cdot T^{ij}\lambda^j + \kappa(M - i\gamma_5 N)^{ij}\lambda^j - \kappa\not{V}\lambda^i\right) - 2\bar{\epsilon}^i D_a^P\lambda^i\\
&\quad + \kappa\bar{\epsilon}^i\eta^{ij}\gamma_a\lambda^j - \kappa\bar{\epsilon}^i(V + i\gamma_5 A)_a^{ij}\lambda^j - 2i\kappa\bar{\epsilon}^i\gamma_5\sigma_{ab}\lambda^i A_b\,,\\
\delta M^{ij} &= \bar{\epsilon}^{[i}\left(\chi^{j]} + 2\not{D}^P\lambda^{j]} - \kappa\sigma\cdot T^{j]k}\lambda^k + \kappa(M - i\gamma_5 N)^{j]k}\lambda^k - \kappa(\not{V} - i\gamma_5\not{A})\lambda^{j]}\right)\,,\\
\delta N^{ij} &= i\bar{\epsilon}^{[i}\gamma_5\left(\chi^{j]} + 2\not{D}^P\lambda^{j]} - \kappa\sigma\cdot T^{j]k}\lambda^k + \kappa(M - i\gamma_5 N)^{j]k}\lambda^k - \kappa(\not{V} - i\gamma_5\not{A})\lambda^{j]}\right)\,.
\end{aligned}
$$

The commutator of two such transformations is given by (III.5.8) with parameters ξ_λ and e_3^i as defined in (III.5.9) and ε_{ab} as in (3.1). The Lagrangian density is given by (2.25). These results constitute the complete theory of SO(2) Poincaré supergravity.

4. Concerning the structure of the Poincaré multiplet

Equations (3.3) and (2.25) form the central result of this chapter: the transformation rules and classical action of SO(2) Poincaré supergravity. In principle one could now go on to construct the corresponding quantum theory, and study aspects like renormalizability, BRS-transformations, symmetry breaking, etc. However, our theory is rather complicated and it would be very useful, if we had some way of gaining more insight into its structure on the classical level.

In fact we have some cues to a better understanding of the Poincaré theory. To begin with, we know that the linearized multiplet contains a number of submultiplets, each with their own Lagrangian. One way to obtain more information on the coupled SO(2) supergravity multiplet is to analyse what happens to these submultiplets in the local theory.

Moreover, we know from N=1 supergravity, that the results, in particular for the gauge fields, become much more transparent when considered in the light of the superconformal theory [5]. This suggests, that also in N=2 supergravity we focus our attention initially on the components of the Weyl multiplet, and especially on the gauge fields it contains. On the linearized level these were:

$$e_\mu^a,\ \psi_\mu^i,\ A_\mu,\ A_\mu^{ij},\ V_\mu^{ij}\ . \tag{4.1}$$

They gauge general coordinate invariance, Q-supersymmetry and chiral U(2) transformations respectively. The superconformal theory contains more local invariances (see app. B): local Lorentz invariance, S-supersymmetry, dilatations and special conformal transformations. However, their gauge fields either drop out of the theory completely (as for dilatations), or they are not independent fields, but can be expressed in terms of the other fields, as ω_μ^{ab} is determined in terms of e_μ^a and ψ_μ^i [6].

One now has to find, which components of the local Poincaré multiplet correspond to the gauge fields of the local Weyl multiplet, as represented on the linearized level by (4.1). If one has identified these, it becomes possible to construct the full Weyl multiplet by just rewriting the Poincaré transformation rules. This will be done below. Although it is not necessary to be familiar with the superconformal algebra, it may be useful to know that in the Weyl multiplet one may expect the appearance of two independent local supersymmetries, called Q- and S-supersymmetry, chiral-U(2) transformations and scale transformations, or dilatations. These will be identified in the derivation of the Weyl multiplet. In fact we will find, that it is possible to decompose the Poincaré transformations of the field components belonging to the Weyl submultiplet into field dependent superconformal transformations of the kinds mentioned above. This statement can also be reversed to say, that the superconformal transformations of the Weyl multiplet are reflected in the variations of the corresponding Poincaré components. Once this is recognized, the Poincaré multiplet becomes much more manageable.

Consider first the fields V_μ^{ij} and A_μ^{ij}, which are the gauge fields of SU(2) on the linearized level. They have the following Poincaré supersymmetry transformation:

$$\delta(A_\mu^{ij} + iV_\mu^{ij}) = 2i\left((\bar{\varepsilon}^i\gamma_5\partial_\mu\lambda^j)_S + \bar{\varepsilon}^{[i}\partial_\mu\lambda^{j]}\right) + \ldots \tag{4.2}$$

Eq. (4.2) can be interpreted as an SU(2) gauge transformation of these fields with hermitean parameter:

$$\Lambda^{ij} = i\kappa\left((\bar{\varepsilon}^i\gamma_5\lambda^j)_S + \bar{\varepsilon}^{[i}\lambda^{j]}\right) , \tag{4.3}$$

when ε^i is a global spinor parameter. In the local case this is impossible, since one needs variations with $\lambda\partial_\mu\varepsilon$ also. To provide these, we redefine the vector fields as follows:

$$\begin{aligned} \mathcal{V}_\mu^{ij} &= V_\mu^{ij} + \kappa\bar{\psi}_\mu^{[i}\lambda^{j]} , \\ \mathcal{A}_\mu^{ij} &= A_\mu^{ij} + i\kappa(\bar{\psi}_\mu^i\gamma_5\lambda^j)_S , \end{aligned} \tag{4.4}$$

which leads to the desired result:

$$\delta(\mathcal{A}_\mu^{ij} + i\,\mathcal{V}_\mu^{ij}) = \frac{2}{\kappa}\,\partial_\mu\Lambda^{ij} + \ldots \tag{4.5}$$

Hence we interpret $\mathcal{V}_\mu^{ij}$ and $\mathcal{A}_\mu^{ij}$ as the gauge fields of chiral SU(2) in the local Weyl multiplet. For A_μ, which does not transform under the SU(2) transformation (4.3), a redefinition of this type is not necessary, and we take

$$\mathcal{A}_\mu = A_\mu \tag{4.6}$$

as the U(1) gauge field. The consistency of the assignments (4.4), (4.6) is shown by rewriting the transformation rule of ψ^i_μ:

$$\delta\psi^i_\mu = \frac{2}{\kappa} D^{CH}_\mu\varepsilon^i - \tfrac{1}{2}\sigma\cdot T^{ij}\gamma_\mu\varepsilon^j - \frac{1}{\kappa}\gamma_\mu(\kappa\eta^{ij}\varepsilon^j) - \kappa\left(\bar\varepsilon^{[i}\lambda^{j]} + (\bar\varepsilon^i\gamma_5\lambda^j)_S\gamma_5\right)\psi^j_\mu \tag{4.7}$$

where D^{CH}_μ is the U(2) covariant derivative:

$$D^{CH}_\mu\varepsilon^i = D_\mu\varepsilon^i + \frac{\kappa}{2}(\mathcal{V}^{ij}_\mu - i\gamma_5\mathcal{A}^{ij}_\mu + i\gamma_5\mathcal{A}_\mu\delta^{ij})\varepsilon^j . \tag{4.8}$$

Note, that (4.7) contains a chiral SU(2) transformation on ψ^i_μ with the same field-dependent parameter (4.3) as before. Explicitly we can write it as:

$$\delta_{SU(2)}\Psi^i_\mu = i\Lambda^i_j\Psi^j_\mu , \quad \delta_{SU(2)}\Psi_{\mu i} = -i\Psi_{\mu j}\Lambda^{\dagger j}_{\ i} , \tag{4.9}$$

with

$$\Psi^i_\mu \equiv \tfrac{1}{2}(1+\gamma_5)\psi^i_\mu , \quad \Psi_{\mu i} \equiv \tfrac{1}{2}(1-\gamma_5)\psi^i_\mu , \tag{4.10}$$

(see III.6). As is clear from this, the chiral U(2) transformations of ψ^i_μ in the Weyl theory can indeed be recognized in its Poincaré transformation rule. The other terms in $\delta\psi^i_\mu$ will be identified later as a Q-supersymmetry transformation:

$$\delta_Q\psi^i_\mu = \frac{2}{\kappa} D^{CH}_\mu\varepsilon^i_Q - \tfrac{1}{2}\sigma\cdot T^{ij}\gamma_\mu\varepsilon^j_Q \tag{4.11}$$

with parameter

$$\varepsilon^i_Q = \varepsilon^i , \tag{4.12}$$

and an S-supersymmetry transformation:

$$\delta_S\psi^i_\mu = -\frac{1}{\kappa}\gamma_\mu\varepsilon^i_S , \tag{4.13}$$

with parameter

$$\varepsilon^i_S = \kappa\eta^{ij}\varepsilon^j . \tag{4.14}$$

In this approach ψ^i_μ clearly is the gauge field of Q-supersymmetry. Next we turn to the variation of the U(2) gauge fields (4.4) and (4.6). They read:

$$\begin{aligned}
\delta\mathcal{V}^{ij}_\mu &= \bar\varepsilon^{[i}\gamma_\mu(\chi - \tfrac{1}{3}\gamma\cdot R^P)^{j]} - \bar\varepsilon^{[i}(R^{CH}_\mu - \tfrac{1}{3}\gamma_\mu\gamma\cdot R^{CH})^{j]} + \bar\psi^{[i}_\mu(\kappa\eta^{j]k}\varepsilon^k) + \frac{2}{\kappa}D^{CH}(\kappa\bar\varepsilon^{[i}\lambda^{j]}),\\
\delta\mathcal{A}^{ij}_\mu &= i\bar\varepsilon^i\gamma_5\gamma_\mu(\chi - \tfrac{1}{3}\gamma\cdot R^P)^j - i\bar\varepsilon^i\gamma_5(R^{CH}_\mu - \tfrac{1}{3}\gamma_\mu\gamma\cdot R^{CH})^j + i\bar\psi^i_\mu\gamma_5(\kappa\eta^{jk}\varepsilon^k)\\
&\quad + \frac{2i}{\kappa}D^{CH}_\mu(\kappa\bar\varepsilon^i\gamma_5\lambda^j) + (i\leftrightarrow j;\ \text{traceless}),\\
\delta\mathcal{A}_\mu &= i\bar\varepsilon^i\gamma_5\gamma_\mu(\chi - \tfrac{1}{3}\gamma\cdot R^P)^i - i\bar\varepsilon^i\gamma_5(R^{CH}_\mu - \tfrac{1}{3}\gamma_\mu\gamma\cdot R^{CH})^i - \frac{i}{2}\bar\psi^i_\mu\gamma_5(\kappa\eta^{ik}\varepsilon^k) .
\end{aligned} \tag{4.15}$$

Here we use the notation:

$$R^{i\mu CH} = \frac{1}{e}\,\varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu(D^{CH}_\rho\psi^i_\sigma + \frac{\kappa}{4}\,\sigma\cdot T^{ij}\gamma_\sigma\psi^j_\rho)\ , \tag{4.16}$$

with the same definition of the chiral-U(2) covariant derivative as in (4.8). This shows, that ψ^i_μ and ε^i are in the same chiral representation, as is χ^i. The chiral derivative on λ^i is defined by:

$$D^{CH}_\mu\lambda^i = D_\mu\lambda^i + \frac{\kappa}{2}\,(\mathcal{V}^{ij}_\mu + i\gamma_5\mathcal{A}^{ij}_\mu - i\gamma_5\,\mathcal{A}_\mu\delta^{ij})\lambda^j\ . \tag{4.17}$$

Using this we can verify, that $\mathcal{A}^{ij}_\mu + i\,\mathcal{V}^{ij}_\mu$ transforms indeed precisely as a Yang-Mills gauge field of SU(2):

$$\delta(\mathcal{A}^{ij}_\mu + i\,\mathcal{V}^{ij}_\mu) = \frac{2}{\kappa}\,D^{CH}_\mu\Lambda^{ij} + (\text{Q-, S-supersymm. transf.}). \tag{4.18}$$

The results simplify further if we use the notations (4.12) and (4.14), and define:

$$\begin{aligned} \chi^i_c &= \chi^i - \tfrac{1}{3}\gamma\cdot R^{iP}\ , \\ \phi^i_\mu &= R^{iCH}_\mu - \tfrac{1}{3}\gamma_\mu\gamma\cdot R^{iCH} + \tfrac{1}{2}\gamma_\mu\chi^i_c\ , \end{aligned} \tag{4.19}$$

which leads for example to:

$$\delta\,\mathcal{V}^{ij}_\mu = \frac{3}{2}\,\bar{\varepsilon}^{[i}_Q\gamma_\mu\chi^{j]}_c - \bar{\varepsilon}^{[i}_Q\phi^{j]}_\mu + \bar{\psi}^{[i}_\mu\varepsilon^{j]}_S - \frac{i}{\kappa}\,D^{CH}_\mu\Lambda^{[ij]}. \tag{4.20}$$

The fields χ^i_c and T^{ij}_{ab} are the "matter" fields of the Weyl multiplet, as opposed to the gauge fields. The field ϕ^i_μ is not an independent variable, since it is expressed in terms of the other fields of the Weyl multiplet. To give it a meaning, we note that the principal term in its transformation rule is:

$$\delta\phi^i_\mu = \frac{2}{\kappa}\,D^{CH}_\mu\varepsilon^i_S + \ldots \tag{4.21}$$

Hence it acts as the gauge field of the S-supersymmetry transformations we have encountered above. This interpretation is supported by our further results. In particular we can write the transformation rules of the fields χ^i_c and T^{ij}_{ab} as:

$$\begin{aligned} \delta T^{ij}_{ab} &= 2\varepsilon^\mu_a e^\nu_b\,\bar{\varepsilon}^{[i}_Q(\mathcal{D}_{[\mu}\psi^{j]}_{\nu]})^c\ , \\ \delta\chi^i_c &= -\tfrac{1}{3}\sigma^{ab}\not{\mathcal{D}}^c T^{ij}_{ab}\varepsilon^j_Q + \tfrac{1}{3}\,(\mathcal{R}(\mathcal{V})^{ij}_{\mu\nu} - i\gamma_5\mathcal{R}(\mathcal{A})^{ij}_{\mu\nu} - 2i\gamma_5\mathcal{R}(\mathcal{A})_{\mu\nu}\delta^{ij})\cdot\sigma^{\mu\nu}\varepsilon^j_Q \\ &\quad + D_c\varepsilon^i_Q + \tfrac{1}{3}\sigma\cdot T^{ij}\varepsilon^j_S + \tfrac{i}{2}\,(\Lambda^{[ij]} + (\Lambda^{ij})_S\gamma_5)\chi^j\ . \end{aligned} \tag{4.22}$$

Here $\mathcal{D}^c_\mu$ stands for the covariant derivative with respect to Lorentz and chiral-U(2) symmetry, as well as Q- and S-supersymmetry:

$$\mathcal{D}^c_\mu = D^{CH}_\mu - \frac{\kappa}{2}\,\delta_Q(\psi_\mu) - \frac{\kappa}{2}\,\delta_S(\phi_\mu)\ . \tag{4.23}$$

Likewise $(\mathcal{D}_{[\mu}\psi^i_{\nu]})^c$ means the completely covariantized curvature of ψ^i_μ, and the quantities $\mathcal{R}_{\mu\nu}$ are those of the U(2)-gauge fields:

$$\begin{aligned}
\mathcal{R}(\mathcal{V})^{ij}_{\mu\nu} &= F(\mathcal{V})^{ij}_{\mu\nu} + \frac{\kappa}{2}\,\bar\psi^{[i}_{[\mu}\phi^{j]}_{\nu]} - \frac{3}{4}\,\bar\psi^{[i}_{[\mu}\gamma_{\nu]}\chi^{j]}_c\ ,\\
\mathcal{R}(\mathcal{A})^{ij}_{\mu\nu} &= F(\mathcal{A})^{ij}_{\mu\nu} + \frac{i}{2}\kappa\,(\bar\psi^i_{[\mu}\gamma_5\phi^j_{\nu]})_S + \frac{3i}{4}\kappa\,(\bar\psi^i_{[\mu}\gamma_{\nu]}\gamma_5\chi^j_c)_S\ ,\\
\mathcal{R}(\mathcal{A})_{\mu\nu} &= F(\mathcal{A})_{\mu\nu} - \frac{i}{4}\kappa\,\bar\psi^i_{[\mu}\gamma_5\phi^i_{\nu]} + \frac{3i}{8}\kappa\,\bar\psi^i_{[\mu}\gamma_{\nu]}\gamma_5\chi^i_c\ .
\end{aligned} \tag{4.24}$$

The $F_{\mu\nu}$ are the ordinary Yang-Mills field strenghts.

Finally D_c is the auxiliary field of the Weyl multiplet. In terms of Poincaré fields it reads:

$$\begin{aligned}
D_c = D^P\cdot V - \frac{\kappa}{3}R^P + \frac{\kappa}{4}\Big\{&(V^{ij}_a)^2 + (A^{ij}_a)^2 - 2V^2_a - (M^{ij})^2 - (N^{ij})^2\Big\}\\
+ \kappa\bar\lambda^i\Big\{&2\chi^i + 2\not{D}^P\lambda^i - \tfrac{1}{2}\gamma\cdot R^{iP} + i\kappa\gamma_5\not{A}\lambda^i - \kappa\sigma\cdot T^{ij}\lambda^j\Big\}\ .
\end{aligned} \tag{4.25}$$

Its transformation is given by:

$$\delta D_c = \bar\varepsilon^i_Q\,\not{\mathcal{D}}^c\chi^i_c\ . \tag{4.26}$$

$\mathcal{D}^c_\mu$ is the complete supercovariant derivative of χ^i, the covariantization with respect to U(2) being identical with that of ε^i, eq. (4.8). With this result we have obtained a complete multiplet, with the field content:

$$(e^a_\mu,\ \psi^i_\mu,\ \mathcal{V}^{ij}_\mu,\ \mathcal{A}^{ij}_\mu,\ \mathcal{A}_\mu,\ T^{ij}_{ab},\ \chi^i_c,\ D_c)\ , \tag{4.27}$$

transforming according to:

$$\begin{aligned}
\delta e^a_\mu &= \kappa\bar\varepsilon^i_Q\gamma^a\psi_\mu\ ,\\
\delta\psi^i_\mu &= \frac{2}{\kappa}D^{CH}_\mu\varepsilon^i_Q - \tfrac{1}{2}\sigma\cdot T^{ij}\gamma_\mu\varepsilon^j_Q - \frac{1}{\kappa}\,\gamma_\mu\varepsilon^i_S + \frac{i}{2}\kappa\big(\Lambda^{[ij]} + (\Lambda^{ij})_S\gamma_5\big)\psi^j_\mu\ ,\\
\delta\mathcal{V}^{ij}_\mu &= \frac{3}{2}\bar\varepsilon^{[i}_Q\gamma_\mu\chi^{j]}_c - \bar\varepsilon^{[i}_Q\phi^{j]}_\mu + \bar\psi_\mu{}^i\varepsilon^j_S - \frac{i}{\kappa}\,D^{CH}_\mu\Lambda^{[ij]}\ ,\\
\delta\mathcal{A}^{ij}_\mu &= \frac{3}{2}\,i\bar\varepsilon^i_Q\gamma_5\gamma_\mu\chi^j_c - i\bar\varepsilon^i_Q\gamma_5\phi^j_\mu + i\bar\psi^i_\mu\gamma_5\varepsilon^j_S + \frac{1}{\kappa}\,D^{CH}_\mu\Lambda^{ij} + (i\leftrightarrow j;\ \text{traceless}),\\
\delta\mathcal{A}_\mu &= \frac{3i}{4}\,\bar\varepsilon^i_Q\gamma_5\gamma_\mu\chi^i_c + \frac{i}{2}\,\bar\varepsilon^i_Q\gamma_5\phi^i_\mu - \frac{i}{2}\,\bar\psi^i_\mu\gamma_5\varepsilon^i_S\ ,\\
\delta T^{ij}_{ab} &= 2\bar\varepsilon^{[i}_Q(\mathcal{D}_{[\mu}\psi^{j]}_{\nu]})^c e^\mu_a e^\nu_b\ ,
\end{aligned} \tag{4.28}$$

((4.28) continued)

$$\delta\chi_c^i = -\tfrac{1}{3}\sigma^{ab}(\not{\mathcal{D}}^c T_{ab}^{ij})\epsilon_Q^j + \tfrac{1}{3}\left(\mathcal{R}(\mathcal{V})_{\mu\nu}^{ij} - i\gamma_5\mathcal{R}(\mathcal{A})_{\mu\nu}^{ij} - 2i\gamma_5\mathcal{R}(\mathscr{A})_{\mu\nu}\delta^{ij}\right)\sigma^{\mu\nu}\epsilon_Q^j$$
$$+ D_c\epsilon_Q^i + \tfrac{1}{3}\sigma\cdot T^{ij}\epsilon_S^j + \tfrac{i}{2}\left(\Lambda^{[ij]} + (\Lambda^{ij})_S\gamma_5\right)\chi^j ,$$

$$\delta D_c = \bar{\epsilon}_Q^i\not{\mathcal{D}}^c\chi_c^i .$$

If one substitutes for ϵ_S^i and Λ^{ij} their definitions, (4.14) and (4.3), one reobtains the Poincaré transformation rules. However, the fields in (4.28) form a real multiplet by themselves if their transformation rules hold for arbitrary, field-independent parameters ϵ_Q^i, ϵ_S^i and Λ^{ij}, and have a closed commutator algebra. This turns out to be the case. From (4.28) and the definition of ϕ_μ^i, eq. (4.19), one may verify, that the commutators yield the same result on all fields:

$$[\delta_Q(2), \delta_Q(1)] = \delta_G(\xi^\lambda) + \delta_L(\epsilon_{ab}) + \delta_Q(\epsilon_3^i) + \delta_S(\epsilon_S^i) + \delta_{CH}(\Lambda^{ij}) ,$$
$$[\delta_S, \delta_Q] = \delta_L(\epsilon'_{ab}) + \delta_S(\epsilon'^i_S) + \delta_D(\Lambda_D) + \delta_{CH}(\Lambda'^{ij}) , \qquad (4.29)$$
$$[\delta_S, \delta_S] = 0 ,$$

with

$$\xi^\lambda = -2\bar{\epsilon}_1^i\gamma\lambda\epsilon_2^i ,$$
$$\epsilon_{ab} = \xi^\lambda\omega_{\lambda ab} - \kappa\bar{\epsilon}_1^i(T_{ab} + \gamma_5\tilde{T}_{ab})^{ij}\epsilon_2^j ,$$
$$\epsilon'_{ab} = 2\bar{\epsilon}_S^i\sigma_{ab}\epsilon_Q^j ,$$
$$\epsilon_3^i = \frac{\kappa}{2}\,\xi^\lambda\psi_\lambda^i , \qquad (4.29a)$$
$$\epsilon_3^i = \frac{\kappa}{2}\,\xi^\lambda\phi_\lambda^i + \kappa\chi^j(\bar{\epsilon}_2^{[i}\epsilon_1^{j]}) - \kappa\gamma_5\chi^j(\bar{\epsilon}_2^{[i}\gamma_5\epsilon_1^{j]}) - \frac{\kappa}{4}\,\gamma_\lambda\chi^j(\bar{\epsilon}_2^{(i}\gamma^\lambda\epsilon_1^{j)})$$
$$+ \frac{\kappa}{4}\,\gamma_5\gamma_\lambda\chi^j(\bar{\epsilon}_2^{[i}\gamma_5\gamma_\lambda\epsilon_1^{j]}) ,$$
$$\epsilon'^i_S = -\frac{\kappa}{2}\,\gamma^\lambda\epsilon_Q^i\bar{\epsilon}_S^j\psi_\lambda^j ,$$
$$\Lambda_D = \bar{\epsilon}_S^j\epsilon_Q^j ,$$

while $\delta_{CH}(\Lambda^{ij})$ and $\delta_{CH}(\Lambda'^{ij})$ stand for the following chiral U(2) transformations:

$$
\begin{aligned}
\delta_{CH}(\Lambda^{ij}) &= \delta_{SU(2)}\big(\xi^{\lambda}(\mathcal{V}_{\lambda}^{ij} - i\mathcal{A}_{\lambda}^{ij})\big) + \delta_{U(1)}(\xi^{\lambda}\mathcal{A}_{\lambda}) \, , \\
\delta_{CH}(\Lambda'^{ij}) &= \delta_{SU(2)}\big(-i\bar{\varepsilon}_S^{[i}\varepsilon_Q^{j]} - i(\bar{\varepsilon}_S^{i}\gamma_5\varepsilon_Q^{j})_S\big) + \delta_{U(1)}(-\tfrac{i}{2}\,\bar{\varepsilon}_S^{j}\gamma_5\varepsilon_Q^{j}) \, .
\end{aligned}
\tag{4.29b}
$$

On the right-hand side of the commutator $[\delta_S, \delta_Q]$, one finds a new type of transformation, the dilatation δ_D. This transformation multiplies the field by a real number:

$$
\delta_D(\Lambda)\phi = \alpha\Lambda\phi \, . \tag{4.30}
$$

The strength of the transformation is determined by the number α, which is called the Weyl weight of the field. This Weyl weight may actually be found for the fields by calculating the commutator of S- and Q-supersymmetry. For example, one finds:

$$
\begin{aligned}
\delta_D(\Lambda)e_\mu^a &= -\Lambda e_\mu^a \, , \\
\delta_D(\Lambda)\psi_\mu^i &= -\tfrac{1}{2}\Lambda\psi_\mu^i \, .
\end{aligned}
\tag{4.31}
$$

The eqs. (4.28) imply an important result: for all the components of the Poincaré multiplet which enter into the Weyl submultiplet, the Poincaré supersymmetry transformations can be decomposed into transformations of the superconformal group, with field dependent parameters, as follows:

$$
\delta_{\text{Poincaré}}(\varepsilon^i) = \delta_Q(\varepsilon^i) + \delta_S(\kappa\eta^{ij}\varepsilon^j) + \delta_{SU(2)}\Big(i\kappa\big(\bar{\varepsilon}^{[i}\lambda^{j]} + (\bar{\varepsilon}^i\gamma_5\lambda^j)_S\big)\Big) \, . \tag{4.32}
$$

Moreover, the Weyl multiplet itself turns out to transform <u>linearly</u> under the full, local transformation group, except for pure covariantizations. How much of this can be carried over to the other components of the Poincaré multiplet, will be the subject of the next section.

5. Some results for the submultiplets

In our analysis of the structure of the N=2 Poincaré multiplet of supergravity, we have discovered a local version of the Weyl submultiplet, which has much simpler transformation rules than the original Poincaré fields. Moreover, using notions borrowed from the superconformal theory, we found, that the Poincaré transformation rules could be decomposed into a number of simpler variations, at least for the fields occurring in the Weyl submultiplet. However, we are also interested in what happens to the other components, belonging in the linearized theory to either the vector or the tensor submultiplets. It is the purpose of this section to show, that one may profit from the results, obtained for the Weyl multiplet, here also. We first must point out, that the notion of submultiplet acquires a different meaning in the local theory from that in the global one.

The Weyl multiplet, which is a multiplet of supergravity gauge fields, is a real submultiplet in the sense, that its fields transform only among themselves. This is the case at least, when one takes into account all the local symmetries of this submultiplet. For the vector and tensor multiplets, which can also be viewed as abstract "matter" multiplets, this is no longer true. Their transformation rules become non-linear in the coupled theory, and these non-linear terms contain fields of the original gauge multiplet, which do not belong to the submultiplet proper. We interpret this result as an indication that the extension of each submultiplet to local supersymmetry contains the Poincaré fields of N=2 supergravity manifestly as background fields. Such an extension of a global-supersymmetry multiplet to a local one with coupling to the supergravity fields, will still be called a matter multiplet. If only the field components of the Weyl multiplet enter as background fields, we will use the term "conformal matter multiplet". Because they contain more symmetries, conformal multiplets will in general be simpler as far as their supersymmetry structure is concerned. Both the vector and tensor multiplets turn out to be conformal matter multiplets.

To illustrate the above ideas, we present here the vector multiplet of the locally supersymmetric theory. As a submultiplet of Poincaré supergravity it is realized by the components:

$$
\begin{aligned}
W_\mu &= V_\mu - \kappa\bar{\psi}^i_\mu\lambda^i , \\
A^{ij} &= M^{ij} , \\
B^{ij} &= N^{ij} , \\
\psi^i &= \chi^i + 2\not{D}^P\lambda^i + \kappa(M^{ij} - i\gamma_5 N^{ij} - \not{V}\delta^{ij} + i\gamma_5\not{A}\delta^{ij} - \sigma\cdot T^{ij})\lambda^j , \\
F^{ij} &= D^P\cdot V^{ij} + 2\kappa\bar{\lambda}^{[i}(\chi + \not{D}^P\lambda - \tfrac{1}{4}\gamma\cdot R^P)^{j]} - \kappa V\cdot V^{ij} \\
&\quad + \kappa^2(-M^{ij}\bar{\lambda}^k\lambda^k + iN^{ij}\bar{\lambda}^k\gamma_5\lambda^k + 2\bar{\lambda}^i\not{V}\lambda^j) , \\
G^{ij} &= D^P\cdot A^{ij} - 2i\kappa\left(\bar{\lambda}^i\gamma_5(\chi + \not{D}^P\lambda - \tfrac{1}{4}\gamma\cdot R^P)^j\right)_S - \kappa V\cdot A^{ij} \\
&\quad + i\kappa^2(-M^{ik}\bar{\lambda}^k\gamma_5\lambda^j + iN^{ik}\bar{\lambda}^k\lambda^j + \bar{\lambda}^i\gamma_5\not{V}\lambda^j)_S .
\end{aligned}
\tag{5.1}
$$

One can see, that this is a conformal matter multiplet from its Poincaré transformations, which can be decomposed into Q- and S-supersymmetry and a chiral-SU(2) transformation by the same rule (4.32) as for the Weyl multiplet. We will first give the transformation laws for the multiplet (5.1), and then comment on its derivation. In terms of the fields defined in (5.1) and § 4 we find:

$$
\begin{aligned}
\delta W_\mu &= \bar{\varepsilon}^i_Q \gamma_\mu \psi^i - \kappa \bar{\varepsilon}^i_Q (A + i\gamma_5 B)^{ij} \psi^j_\mu \, , \\
\delta A^{ij} &= \bar{\varepsilon}^{[i}_Q \psi^{j]} \, , \\
\delta B^{ij} &= i\bar{\varepsilon}^{[i}_Q \gamma_5 \psi^{j]} \, , \\
\delta \psi^i &= -\sigma^{ab} \mathcal{F}_{ab} \varepsilon^i_Q - \not{\mathcal{D}}^c (A + i\gamma_5 B)^{ij} \varepsilon^j_Q - (F - i\gamma_5 G)^{ij} \varepsilon^j_Q \\
&\quad - (A - i\gamma_5 B)^{ij} \varepsilon^j_S + \frac{i}{2} \left(\Lambda^{[ij]} + (\Lambda^{ij})_S \gamma_5\right) \psi^j \, , \\
\delta F^{ij} &= \bar{\varepsilon}^{[i}_Q \not{\mathcal{D}}^c \psi^{j]} + \tfrac{1}{2} (\Lambda^{[ik})_S G^{kj]} \, , \\
\delta G^{ij} &= i\bar{\varepsilon}^i_Q \gamma_5 \not{\mathcal{D}}^c \psi^j + \frac{i}{2} \Lambda^{[ik]} G^{kj} - \tfrac{1}{2} (\Lambda^{ik})_S F^{kj} + (i \leftrightarrow j;\ \text{traceless}).
\end{aligned}
\tag{5.2}
$$

We have used $\mathcal{D}^c_\mu$ as before for the covariant derivative with respect to all the symmetries of the Weyl multiplet. Furthermore we introduced a generalized field strength of W_μ:

$$
\begin{aligned}
\mathcal{F}_{ab} &= F_{ab}(W) - \frac{\kappa}{2} \bar{\psi}^i_{[a} \gamma_{b]} \psi^i + \frac{\kappa^2}{2} \bar{\psi}^i_a (A + i\gamma_5 B)^{ij} \psi^j_b \\
&\quad + \frac{\kappa}{4} \left(A^{jk} T^{jk}_{ab} - iB^{ik} \tilde{T}^{jk}_{ab}\right) .
\end{aligned}
\tag{5.3}
$$

Again, if one substitutes expression (4.3) and (4.14) for Λ^{ij} and ε^i_S, one obtains the Poincaré transformations of the fields. We see, that the rules (5.2) are still linear in the fields of the vector multiplet itself, the non-linearities being almost entirely the result of covariantizations. The single exception to this is the transformation rule of W_μ. However, if we express it through its field strength (5.3), its transformation rule becomes a covariant expression also:

$$
\delta \mathcal{F}_{ab} = -\bar{\varepsilon}^i_Q \gamma_{[a} \mathcal{D}^c_{b]} \psi^i - 2\bar{\varepsilon}^i_S \sigma_{ab} \psi^i \, . \tag{5.4}
$$

The derivation of the rules (5.2) involved using the Poincaré rules to find the expression for δW_μ, δA^{ij} and δB^{ij} from (5.1). At that point it was convenient to abstract from the specific representation of the vector multiplet. We found $\delta\Psi^i$, δF^{ij} and δG^{ij} from the linearized results and the requirement of closure of the commutator of two Poincaré transformations. Subsequently we could decompose the Poincaré transformations by the same rule as for the Weyl multiplet. This established the result (5.2). The identifications of Ψ^i, F^{ij} and G^{ij} to all orders, as in (5.1), were only made afterwards.

6. Conclusions and outlook

We have constructed the full interacting theory of SO(2) Poincaré supergravity with a minimal set of auxiliary fields. This supergravity multiplet has complicated transformation rules, but can be understood much better, when we recognize its decomposition into submultiplets. This is analogous to the results obtained for the global theory. These submultiplets themselves appear very simple, when formulated in terms of superconformal symmetries. In particular their transformation rules turn out to be proper covariantizations of the linearized ones with respect to all local symmetries of the Weyl multiplet.

One can also view the vector and tensor multiplets as independent matter multiplets, which can be coupled to supergravity, either as conformal or as Poincaré multiplets. In that case one needs to construct invariant actions for them, as we have done in the linearized theory. However, the tensor multiplet turns out to have the wrong Weyl weight and the Lagrangian (III.6.2) cannot be generalized to the interacting case [4]. On the other hand, for the vector multiplet it is possible to extend the global action (III.6.3) to a local invariant by the Noether procedure.

To find the most general invariants of N=2 multiplets requires however a set of multiplication rules for these multiplets and a corresponding action formula, as exist in N=1 supergravity [7]. Such a multiplet calculus then allows the study of renormalizability and the Higgs mechanism for local N=2 supersymmetric field theories. Hence the theory becomes more interesting phenomenologically, especially with a suitable set of vector multiplets coupled to that of supergravity. A drawback of this scheme is, that coupling of such matter fields spoils the renormalizability believed to exist for the pure supergravity theory [8].

Another reason to study these aspects of the theory lies in the hope, that some features of higher N extended supergravity theories, which have more realistic phenomenological properties, can be studied in this mathematically simpler model. For all these applications, however, it is crucial to have a formulation of the theory with closed gauge algebra, as is provided by our set of auxiliary fields.

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CHAPTER V

COVARIANT QUANTIZATION OF SUPERGRAVITY

1. Introduction

In this chapter we will discuss the quantization of supergravity. We will do this in the framework of the covariant quantization procedure [1]. This procedure is centered around the construction of a generating functional for Greens functions, called the path integral. Since physical quantities calculated from this path integral are to be gauge independent, the path integral has to be a gauge invariant object itself. However, in the usual prescription for the quantization of gauge theories the gauge invariance is not manifest, but is implied by the existence of a special global invariance of the generating functional, called B.R.S. invariance [2]. For the Greens functions this B.R.S. invariance results in a set of diagrammatic identities, known as generalized Ward-Takahashi identities. These identities are crucial in the proofs of unitarity and renormalizability of gauge theories in the context of perturbation theory.

Because B.R.S. invariance plays such an important role, it can be used as a guiding principle in the construction of path integrals for gauge theories. This will be exploited in the following. We will show, that the usual covariant quantization procedure, as established for gauge theories of the Yang-Mills type, is not correct for theories with an open gauge algebra, such as supergravity without auxiliary fields [3,4] . But we can modify this procedure by imposing a generalized form of B.R.S. invariance on the theory. This generalized B.R.S. invariance on the one hand determines the path integral, up to the usual freedom in choice of gauge fixing condition; on the other hand it leads to the correct Ward-Takahashi identities, ensuring gauge invariance of the S-matrix elements.

This chapter is organized as follows. In section 2 we outline the standard procedure for quantization of gauge theories with closed commutator algebra. In section 3 we apply it to the theory of SO(2) supergravity in the formulation with auxiliary fields. We then discuss, what happens when the auxiliary fields are eliminated. This leads to the conclusion, that in a formulation without auxiliary fields the standard quantization procedure cannot be applicable. The reason can be traced back to the non-closure of the commutators off-shell in this formulation, which is related to the existence of so-called equation-of-motion symmetries. A general discussion of equation-of-motion symmetries and their role in field theory is the subject of section 4.

Sections 5 and 6 are devoted to the establishment of the correct quantization procedure for gauge theories with open gauge algebra. The proof of generalized B.R.S. invariance of the path integral goes by induction with respect to the number of ghost fields. The lowest order results, which suffice for the case of supergravity, are presented in section 5. They can be verified explicitly for SO(2) supergravity by comparison with section 3. The induction step, necessary to prove B.R.S. invariance to all orders, is given in section 6. In V.7 the gauge invariance of the theory is proven and some properties of the B.R.S. transformations are discussed. Finally conclusions are drawn, and some general remarks made, in section 8.

2. The covariant quantization procedure

We will first review the standard covariant quantization procedure for gauge theories. For details we may refer to many available expositions, such as refs. [1]. Suppose we have a theory with fields ϕ^i, where the index i denotes all parameters on which the fields depend, e.g. space-time parameters, Lorentz indices, internal symmetry indices, etc. Classically the dynamics of the fields is derived from an action $S[\phi]$, which is minimized to give the field equations:

$$\frac{\delta}{\delta\phi^i} S \equiv S_{,i} = 0 \,. \tag{2.1}$$

The corresponding quantum theory is defined by the path integral:

$$Z[J_i] = N \int D\phi \, \exp i(S[\phi] + J_i\phi^i) \,. \tag{2.2}$$

Here the functions J_i are external sources for the fields ϕ^i; $D\phi$ denotes the functional integration measure and N is a normalization factor. The expectation value of an arbitrary function of the fields, $O[\phi]$, in the presence of the sources J_i is calculated from $Z[J_i]$ by:

$$\langle O[\phi] \rangle_J = N \int D\phi \, O[\phi] \, \exp i(S[\phi] + J_i\phi^i) \,. \tag{2.3}$$

Suppose now, that the classical action $S[\phi]$ possesses an invariance under local transformations, parametrized by n independent parameters ξ^{α}:

$$\begin{aligned} \delta\phi^{i} &= R^{i}_{\alpha}\xi^{\alpha} , \\ S_{,i}R^{i}_{\alpha} &= 0 , \qquad \alpha=1,\dots,n. \end{aligned} \tag{2.4}$$

We assume, that the gauge algebra closes, i.e. that the commutator of two transformations (2.4) is again such a transformation, possibly with field dependent transformation parameter:

$$[R^{i}_{\alpha,j}, R^{j}_{\beta}\} \equiv R^{i}_{\alpha,j}R^{j}_{\beta} - (-1)^{a_{\alpha}a_{\beta}}R^{i}_{\beta,j}R^{j}_{\alpha} = R^{i}_{\gamma}f^{\gamma}_{\beta\alpha} , \tag{2.5}$$

with

$$a_{\alpha} = \begin{cases} 0 , & \text{if } \xi^{\alpha} \text{ is a commuting parameter,} \\ 1 , & \text{if } \xi^{\alpha} \text{ is an anticommuting parameter.} \end{cases} \tag{2.6}$$

The function $f^{\gamma}_{\beta\alpha}$, which may depend on the fields, is known as the structure function of the transformations (2.4).

In constructing the path integral one must now deal with the problem of how to treat the superfluous gauge components of the fields in the functional integral. The solution of this problem is laid down in the following prescription. One replaces the action in (2.2) by an effective action, in which all field components, including the unphysical gauge components, appear. However, this effective action is constructed in such a way, that all unphysical components either decouple from the theory or are cancelled by a set of so-called ghost fields. These ghost fields have unphysical statistics: they are anticommuting, when they have integer spin, and commuting in case they have half integer spin.

Explicitly this prescription takes the following form. To the classical action one adds a gauge fixing term of the form:

$$S_{fix} = -\tfrac{1}{2}F^{2}_{\alpha} , \tag{2.7}$$

where F_{α} is a set of n independent functions of the gauge fields. This term breaks the gauge invariance of the action, and introduces the unphysical field components into the theory. Hence the theory now violates unitarity. This is restored by adding a term containing ghost fields c^{α}, $c^{*\alpha}$, which has the form of a gauge transformation on F_{α} with parameter c^{β}, multipled by an antighost $\bar{c}^{*\alpha}$, which acts as a Lagrange multiplier:

$$S_{\text{ghost}} = \bar{c}^{*\alpha} F_{\alpha,i} R^i_\beta c^\beta \ . \tag{2.8}$$

That the effective action only describes the physical components of the theory is formally a result of the ghost field equations:

$$F_{\alpha,i} R^i_\beta c^\beta = 0 \ , \tag{2.9}$$

which assures the gauge invariance of F_α and consequently of the effective action as well. The complete effective action now reads:

$$S_{\text{eff}} = S[\phi] - \tfrac{1}{2} F^2_\alpha + \bar{c}^{*\alpha} F_{\alpha,i} R^i_\beta c^\beta \ . \tag{2.10}$$

Furthermore the functional integration measure has to be extended to include the ghost fields as well:

$$D\phi \rightarrow D\phi Dc Dc^* \ . \tag{2.11}$$

Eqs. (2.10) and (2.11) define the path integral for the gauge theory under consideration.

Although the effective action (2.10) has lost its manifest gauge invariance, it does possess an invariance under a set of transformations with a global anticommuting parameter Λ, defined by:

$$\begin{aligned} \delta\phi^i &= R^i_\alpha c^\alpha \Lambda \ , \\ \delta c^\alpha &= -\tfrac{1}{2} f^\alpha_{\beta\gamma} c^\beta \Lambda c^\gamma \ , \\ \delta c^{*\alpha} &= -\Lambda F^\alpha \ . \end{aligned} \tag{2.12}$$

These are the B.R.S. transformations [2]. On ϕ^i they have precisely the form of a gauge transformation with parameter $c^\alpha \Lambda$. Hence they reflect the original gauge invariance in the full effective action, S_{eff}. The important step in the proof of B.R.S. invariance of (2.10) is the cancellation of the terms:

$$-\tfrac{1}{2} \bar{c}^{*\alpha} F_{\alpha,i} \left(R^i_\beta f^\beta_{\mu\lambda} - R^i_{\lambda,j} R^j_\mu + (-1)^{a_\lambda a_\mu} R^i_{\mu,j} R^j_\lambda \right) c^\mu \Lambda c^\lambda \ , \tag{2.13}$$

in the variation of the ghost action. This is guaranteed by the closure of the gauge algebra (2.5). Hence this closure is crucial for the correctness of the above prescription. Actually one may prove that the B.R.S. invariance (2.12) is a necessary condition for the gauge invariance of the S-matrix elements, calculated with the path integral defined by (2.10), (2.11).

3. Quantization of SO(2) supergravity

The classical field equations for the auxiliary fields in supergravity are algebraic in character. In fact all auxiliary fields are zero classically. Hence they can be eliminated from the action, without changing the physical content of the theory. However, this is no longer true in the quantum theory. In the effective action the ghost terms appear and because of this the auxiliary field equations are modified. We will treat here the example of SO(2) supergravity. In the formulation with auxiliary fields, the gauge algebra closes and we may use the quantization procedure described in V.2. We will restrict ourselves to supersymmetry and take the gauge fixing term:

$$F^{i} = \gamma\cdot\psi^{i} \ . \tag{3.1}$$

The corresponding ghost action becomes *):

$$\begin{aligned} S_{\text{ghost}} = {} & 2\bar{c}^{*i}\not{D}c^{i} - \kappa^{2}\bar{c}^{*i}\gamma^{\lambda}\psi^{i}_{\mu}\bar{c}^{j}\gamma^{\mu}\psi^{j}_{\lambda} + \kappa\bar{c}^{*i}(\not{V}^{ij} - i\gamma_{5}\not{A}^{ij} + i\gamma_{5}\not{A}\delta^{ij} - 4\eta^{ij})c^{j} \\ & + \kappa^{2}\bar{\psi}^{[i}_{\mu}\lambda^{j]}\bar{c}^{*i}\gamma^{\mu}c^{j} + \kappa^{2}(\bar{\psi}^{i}_{\mu}\gamma_{5}\lambda^{j})_{S}\bar{c}^{*i}\gamma^{\mu}\gamma_{5}c^{j} \\ & - \kappa^{2}\bar{c}^{[i}\lambda^{j]}\bar{c}^{*i}\gamma\cdot\psi^{j} - \kappa^{2}(\bar{c}^{i}\gamma_{5}\lambda^{j})_{S}\bar{c}^{*i}\gamma^{\mu}\gamma_{5}\psi^{j}_{\mu} \ . \end{aligned} \tag{3.2}$$

Here c^{i} and c^{*i} are the Majorana spinor ghost and antighost respectively, while the bar denotes conjugation as usual. The field equations for the auxiliary fields now read:

$$\begin{aligned} S &= -2\kappa e^{-1}\bar{c}^{*i}c^{i} \ , \\ P^{ij} &= -2i\kappa e^{-1}(\bar{c}^{*i}\gamma_{5}c^{j})_{S} \ , \\ A_{a} &= \frac{3i}{2}\kappa e^{-1}\bar{c}^{*i}\gamma_{a}\gamma_{5}c^{i} \ , \\ A^{ij}_{a} &= i\kappa e^{-1}(\bar{c}^{*i}\gamma_{a}\gamma_{5}c^{j})_{S} \ , \\ V^{ij}_{a} &= -\kappa e^{-1}\bar{c}^{*[i}\gamma_{a}c^{j]} \ , \\ t^{ij}_{ab} &= 4\kappa e^{-1}\bar{c}^{*[i}\sigma_{ab}c^{j]} \ , \\ \lambda^{i} &= V_{a} = M^{ij} = N^{ij} = 0 \ . \end{aligned} \tag{3.3}$$

---

*) We ignore the problem of Nielsen ghosts [6], since they are not relevant to our discussion

The equation for χ^i is complicated, but of no importance to us here. Inserting (3.3) back into the action one obtains the following modified ghost terms [5] :

$$\begin{aligned} S_{ghost} &= 2\bar{c}^{*i}\not{\partial} c^i - \kappa^2\bar{c}^{*i}\gamma^\lambda\psi^i_\mu\bar{c}^j\gamma^\mu\psi^j_\lambda \\ &- \tfrac{3}{4}\kappa^2 e^{-1}\left[(\bar{c}^{*i}c^{*j})(\bar{c}^i c^j) - (\bar{c}^{*i}\gamma_5 c^{*j})(\bar{c}^{\,i}\gamma_5 c^j)\right] \\ &+ \tfrac{1}{8}\kappa^2 e^{-1}\left[-(\bar{c}^{*i}\gamma^a c^{*j})(\bar{c}^i\gamma_a c^j) + (\bar{c}^{*i}\gamma^a\gamma_5 c^{*j})(\bar{c}^i\gamma_a\gamma_5 c^j)\right] \\ &- \tfrac{1}{2}\kappa^2 e^{-1}(\bar{c}^{*i}\gamma^a c^{*i})(\bar{c}^j\gamma_a c^j) \ . \end{aligned} \tag{3.4}$$

Thus elimination of the auxiliary fields introduces <u>quartic</u> ghost terms in the effective action. Such terms can never be obtained by applying our previous covariant quantization procedure to the theory formulated in terms of physical fields only. Failure of this procedure was to be expected, since the gauge algebra does not close in this formulation of the theory. Hence it is clear, that the quantization procedure has to be modified in this case. Such a procedure for quantization of a gauge theory with open gauge algebra is in principle even more general, since it could also be used, when no closed version of the algebra exists. To develop this procedure will be the subject of the rest of this chapter.

4. <u>Equation of motion symmetries</u>

It has become clear from the foregoing discussion, that non-closure of the gauge algebra is an important feature in the quantization of field theories. Therefore we will devote this section to an explanation of the role of equation-of-motion terms in the gauge algebra.

Suppose again, that we have a field theory of fields ϕ^i described by a classical action $S[\phi]$, which is invariant under gauge transformations $R^i_\alpha\xi^\alpha$, as in (2.4). The classical field equations are

$$S_{,i} = 0 \ . \tag{4.1}$$

Because of this we will refer to $S_{,i}$ as an "equation of motion", even when the fields are evaluated off-shell, i.e. when (4.1) does not hold. We now come to an important point: besides the gauge transformations R^i_α, there exist infinitely many other invariances of the action, which are of the type:

$$\delta\phi^i = S_{,j}\eta^{ji} \ , \tag{4.2}$$

where η^{ji} is an arbitrary function of the fields, except for the requirement:

$$\eta^{ji} = (-1)^{1+a_i a_j} \eta^{ij} \ . \tag{4.3}$$

Here the numbers a_i are defined analogously to (2.6):

$$a_i = \begin{cases} 0 \ , & \text{if } \phi^i \text{ is a commuting field,} \\ 1 \ , & \text{if } \phi^i \text{ is an anticommuting field.} \end{cases} \tag{4.4}$$

As a consequence of (4.2), the equation:

$$S_{,i} X^i = 0 \ , \tag{4.5}$$

has the general solution:

$$X^i = R^i_\alpha Y^\alpha + S_{,j} \eta^{ji} \ , \tag{4.6}$$

with arbitrary Y^α and η^{ij} .

The transformations (4.2) do not correspond to superfluous degrees of freedom, since they vanish on shell. In this respect they do not present a problem in the quantization of the theory. However, we note that one can always add a transformation of the equation-of-motion type to the gauge invariances R^i_α :

$$R'^i_\alpha = R^i_\alpha + S_{,j} \eta^{ji}_\alpha \ . \tag{4.7}$$

With η^{ji}_α satisfying (4.3) this is still an invariance of the classical action. Clearly, different choices of R^i_α lead to different ghost Lagrangians and therefore to different effective actions.

Another problem is posed by the commutator algebra of the transformations. Suppose we perform two successive gauge transformations on the classical action. The invariance, expressed by (2.4), implies:

$$S_{,i} R^i_{\alpha,j} R^j_\beta + (-1)^{a_\beta(a_\alpha + a_i)} S_{,ij} R^j_\beta R^i_\alpha = 0 \ . \tag{4.8}$$

Next we interchange the two transformations and subtract the result from (4.8). Thus we find the generalized commutator of two gauge transformations on S (cf. (2.5)):

$$S_{,i} [R^i_{\alpha,j} R^j_\beta\} = 0 \ . \tag{4.8}$$

This means that $[R^i_{\alpha,j} R^j_\beta\}$ is itself an invariance of the action, and by (4.6) it can be written:

$$[R^i_{\alpha,j} R^j_\beta\} = R^i_\gamma f^\gamma_{\beta\alpha} + 2S_{,j}\eta^{ji}_{\beta\alpha} \quad . \tag{4.10}$$

This is the generalization of (2.5) and states, that the commutators of <u>any</u> two gauge transformations of a theory always close modulo equation-of-motion terms. The quantities $f^\gamma_{\beta\alpha}$ and $\eta^{ji}_{\beta\alpha}$ in (4.10) are called generalized structure functions.

Sometimes it is possible to remove the second term on the right-hand side of (4.10) by a redefinition of the gauge transformations, as in (4.7). Then this is clearly a reasonable thing to do, because quantization becomes straightforward. However, it is not always possible to do this, as is shown by the supergravity theories formulated in terms of physical fields only. We have seen, that our previous quantization procedure then fails. In general it can also be seen from eq. (2.13), which leads to the result:

$$\delta_{BRS} S_{eff} = \bar{c}^{*\alpha} F_{\alpha,i} S_{,j} \eta^{ji}_{\beta\gamma} c^\beta \Lambda c^\gamma \ , \tag{4.11}$$

where we have used (4.10). Eq. (4.11) does not vanish off-shell and B.R.S. invariance is violated.

Summarizing we conclude, that equation-of-motion symmetries pose two kinds of problems in the quantization of gauges theories. In the first place, even for a fixed gauge condition F_α, the ghost action is not uniquely defined. Secondly, with the quantization prescription given in V.2 we do not obtain B.R.S. invariant effective actions. We will now solve this last problem. We will derive a new prescription for the construction of S_{eff}, which is invariant under generalized B.R.S. transformations. We will also show, that this theory leads to a gauge independent S-matrix. Hence the first problem is solved implicitly.

5. <u>Quantization of gauge theories with open gauge algebra</u>

As one may expect from the example of SO(2) supergravity, the solution to the problems described above is to be found in the introduction of higher order ghost interactions in the effective Lagrangian, accompanied by a suitable extension of the B.R.S. transformations. Indeed we will prove the following theorem:

There exist quantities $M^{i_n \ldots i_1}_{\alpha_n \ldots \alpha_1}(n)$, $X^{i_n \ldots i_1}_{\alpha_n \ldots \alpha_1}{}^{\lambda}_{\mu\nu}(n)$, obtainable solely in terms of the gauge transformations and their commutators, such that the effective action for any gauge theory is:

$$S_{eff} = S - \tfrac{1}{2}F_\alpha^2 + \bar{F}_{i_1}\Big(R^{i_1}_{\alpha_1} + \sum_{n\geq 2}\frac{1}{n}\bar{F}_{i_2}\cdots\bar{F}_{i_n}M^{i_n\cdots i_1}_{\alpha_n\cdots\alpha_1}(n)c^{\alpha_n}\cdots c^{\alpha_2}\Big)c^{\alpha_1}, \quad (5.1)$$

and that this effective action is invariant under a generalization of the B.R.S. transformations:

$\alpha_1\alpha^1$ a_1

$$\begin{aligned}
\delta\phi^{i_1} &= \Big(R^{i}_{\alpha_1} + \sum_{n\geq 2}\bar{F}_{i_2}\cdots\bar{F}_{i_n}M^{i_n\cdots i_1}_{\alpha_n\cdots\alpha_1}(n)c^{\alpha_n}\cdots c^{\alpha_2}\Big)c^{\alpha_1}\Lambda\,, \\
\delta c^{\alpha} &= \Big(-\tfrac{1}{2}f^{\alpha}_{\beta\gamma} + \sum_{n\geq 1}\bar{F}_{i_1}\cdots\bar{F}_{i_n}X^{i_n\cdots i_1\,\alpha}_{\alpha_n\cdots\alpha_1\,\beta\gamma}(n)c^{\alpha_n}\cdots c^{\alpha_1}\Big)c^{\beta}\Lambda c^{\gamma}\,, \qquad (5.2) \\
\delta c^{*\alpha} &= -\Lambda F^{\alpha}\,.
\end{aligned}$$

Here and in the following we use the notation:

$$\begin{aligned}
\bar{F}_i &= \bar{c}^{*\alpha}F_{\alpha,i}\,, \\
\bar{F}_{ij} &= \bar{c}^{*\alpha}F_{\alpha,ij} = (-1)^{\alpha_i\alpha_j}\bar{F}_{ji}\,.
\end{aligned} \qquad (5.3)$$

The proof of this theorem goes by induction with respect to the number of ghost fields *). In this section we will present it up to terms quadratic in $\bar{F}_i$. This shows the general procedure and is moreover completely sufficient for the case of SO(2) supergravity. We suspect it to be sufficient for higher N supergravity theories as well. The generalization of our proof to all orders will be discussed in the next section.

We begin by noting, that S_{eff} and the B.R.S. transformations reduce to the usual ones, (2.10), (2.12), in lowest order in $\bar{F}_i$. Hence in this order we find again (4.11), which can be written:

$$\delta^{(1)}_{BRS}S_{eff} = S_{,j}\bar{F}_i\eta^{ji}_{\beta\gamma}c^{\beta}\Lambda c^{\gamma}(-1)^{a_j(1+a_i)}\,. \qquad (5.4)$$

Clearly this variation can be cancelled by a new term in $\delta\phi^i$, as in (5.2); we only have to define

$$M^{ij}_{\beta\gamma}(2) = (-1)^{a_i+a_\gamma+1}\eta^{ij}_{\beta\gamma}\,. \qquad (5.5)$$

However, such an extension of $\delta\phi^i$ introduces other new variations in S_{eff}, besides one that cancels (5.4). Using (5.1) and (5.2) the complete variation of S_{eff} to second order in the antighosts becomes:

*) For S-matrix elements this is equivalent to induction with respect to the number of loops in the diagrams.

$$\delta^{(2)}_{BRS}S_{eff} = \bar{F}_{,ij}\bar{F}_{,k}M^{kj}_{\beta\gamma}(2)c^{\beta}c^{\gamma}\Lambda R^{i}_{\alpha}c^{\alpha} + \bar{F}_{,i}R^{i}_{\alpha,j}\bar{F}_{,k}M^{kj}_{\beta\gamma}(2)c^{\beta}c^{\gamma}\Lambda c^{\alpha}$$
$$+ \bar{F}_{,ji}R^{i}_{\alpha}c^{\alpha}\,\bar{F}_{,k}M^{kj}_{\beta\gamma}(2)c^{\beta}c^{\gamma} + \tfrac{1}{2}\bar{F}_{,i}\bar{F}_{,j}M^{ji}_{\beta\gamma}(2)_{,k}R^{k}_{\alpha}c^{\alpha}\Lambda c^{\beta}c^{\gamma}$$
$$- \tfrac{1}{2}\bar{F}_{,i}\bar{F}_{,j}M^{ji}_{\beta\lambda}(2)c^{\beta}f^{\lambda}_{\gamma\alpha}c^{\gamma}\Lambda c^{\alpha} + \bar{F}_{,i}R^{i}_{\lambda}\bar{F}_{,j}X^{j\lambda}_{\beta\gamma\alpha}(1)c^{\beta}c^{\gamma}\Lambda c^{\alpha}$$
$$+ S_{,k}\bar{F}_{,i}\bar{F}_{,j}M^{jik}_{\gamma\beta\alpha}(3)c^{\gamma}c^{\beta}c^{\alpha}\Lambda\ . \tag{5.6}$$

In fact, the last term is a new variation of ϕ^i, which has to cancel the net effect of the other variations with $S_{,k}$ in (5.6) in the same way the variation with M(2) had to cancel (5.4). Similarly, the term with X(1) comes from a new variation of the ghost, designed to cancel the other terms in (5.6), which are not explicitly proportional to an equation of motion. We will now show this cancellation mechanism in some more detail.

In the first place the variation of $-\tfrac{1}{2}F^2_{\alpha}$ vanishes with those of the antighosts $c^{*\alpha}$, as may easily be verified to all orders in $\bar{F}_{,i}$. Hence we do not consider these. Furthermore, the terms with $\bar{F}_{,ij}$ in (5.6) cancel after some rewriting to cast them into the same form, using (5.3). Taking M(2) as in (5.5), we can now write $\delta^{(2)}_{BRS}S_{eff}$ as:

$$\delta^{(2)}_{BRS}S_{eff} = (-1)^{a_j+a_\beta}\bar{F}_{,i}\bar{F}_{,j}\Big\{(-1)^{a_j+a_\beta+a_i(a_j+a_k)}S_{,i}M^{kji}_{\gamma\beta\alpha}(3) +$$
$$-\tfrac{1}{2}(-1)^{a_\alpha+a_\beta+a_\lambda+a_k(a_j+a_\lambda)}R^{j}_{\lambda}X^{k}_{\gamma\beta\alpha}(1) + \tfrac{1}{2}(-1)^{a_\alpha+a_\beta+a_\lambda(1+a_j)}R^{k}_{\lambda}X^{j}_{\gamma\beta\alpha}(1)$$
$$+\tfrac{1}{2}\Big(\eta^{kj}_{\lambda\alpha}f^{\lambda}_{\gamma\beta} - [\eta^{kj}_{\beta\alpha,i}R^{i}_{\gamma}\} - (-1)^{a_ja_k+a_\gamma(a_k+a_\alpha+a_\beta)}R^{j}_{\gamma,i}\eta^{ik}_{\beta\alpha}\Big)\Big\}\cdot c^{\gamma}c^{\beta}c^{\alpha}\Lambda\ , \tag{5.7}$$

with

$$[\eta^{kj}_{\beta\alpha,i}R^{i}_{\gamma}\} \equiv \eta^{kj}_{\beta\alpha,i}R^{i}_{\gamma} - (-1)^{a_\gamma(a_\alpha+a_\beta+a_j)}R^{k}_{\gamma,i}\eta^{ij}_{\beta\alpha}\ . \tag{5.8}$$

This is the commutator of a gauge transformation and a transformation of the type:

$$\delta_\eta\phi^i = \eta^{ij}_{\beta\alpha}\xi^{\beta\alpha}_{j}, \tag{5.9}$$

where $\xi^{\beta\alpha}_{j}$ is an arbitrary parameter. Note however, that (5.9) is not an invariance of the action in general. We will now prove, that one can define M(3) and X(1) such that (5.7) vanishes. This is done by evaluating the commutator $[\eta,R\}$, (5.8), and substituting the result into (5.7). The evaluation is carried out by calculating the Jacobi identity for three gauge transformations. Using its cyclic nature and contracting it with ghosts, it reads:

$$(-1)^{a_\beta}\left(\{R^i_{\alpha,j}R^j_\beta\}_{,k}R^k_\gamma - R^i_{\alpha,j}\{R^j_{\beta,k}R^k_\gamma\}\right)c^\gamma c^\beta c^\alpha = 0 \ . \qquad (5.9)$$

Inserting (4.10) this leads to:

$$(-1)^{a_\beta}\Big(R^i_\lambda(f^\lambda_{\beta\alpha,k}R^k_\gamma - f^\lambda_{\mu\alpha}f^\mu_{\beta\gamma}) + 2S_{,j}\big(-\eta^{ji}_{\lambda\alpha}f^\lambda_{\gamma\beta} + \{\eta^{ji}_{\beta\alpha,k}R^k_\gamma\} +$$
$$+ (-1)^{a_i a_j + a_\gamma(a_j+a_\alpha+a_\beta)}R^i_{\gamma,k}\eta^{kj}_{\beta\alpha}\big)\Big)\,c^\gamma c^\beta c^\alpha = 0 \ . \qquad (5.10)$$

Eq. (5.10) has the generic form:

$$R^i_\lambda\, A^\lambda + S_{,j}B^{ji} = 0 \ . \qquad (5.11)$$

In order to solve it, it is necessary to make a clear cut separation between gauge invariances and equation-of-motion symmetries. We do this by defining gauge transformations to be invariances of the action which do not vanish on shell. Therefore the R^i_λ can not be proportional to a field equation. With this convention, we proceed to solve eq. (5.11). On shell this reduces to:

$$R^i_\lambda A^\lambda = 0 \ , \qquad (5.12)$$

which, in view of the above, implies that A^λ has the structure:

$$A^\lambda = S_{,j}A^{j\lambda} \ . \qquad (5.13)$$

Then (5.11) becomes:

$$S_{,j}\left((-1)^{a_j(a_i+a_\lambda)}R^i_\lambda A^{j\lambda} + B^{ji}\right) = 0 \ . \qquad (5.14)$$

By eq. (4.6) this has the solution:

$$\begin{aligned}&(-1)^{a_j(a_i+a_\lambda)}R^i_\lambda A^{j\lambda} + B^{ji} = R^j_\lambda C^{i\lambda} + S_{,k}D^{kji} \ ,\\ &D^{kji} = (-1)^{1+a_k a_j}D^{jki} \ .\end{aligned} \qquad (5.15)$$

Applying this result to (5.10) leads to the equations:

$$(-1)^{a_\beta}\left(f^\lambda_{\beta\alpha,k}R^k_\gamma - f^\lambda_{\mu\alpha}f^\mu_{\beta\gamma} + 2S_{,j}U^{j\lambda}_{\gamma\beta\alpha}(-1)^{a_j(a_i+a_\lambda)}\right)c^\gamma c^\beta c^\alpha = 0 \ , \qquad (5.16)$$

$$-\eta^{ji}_{\lambda\alpha} f^{\lambda}_{\gamma\beta} + \{\eta^{ji}_{\beta\alpha,k} R^{k}_{\gamma}\} + (-1)^{a_i a_j + a_\gamma (a_j + a_\alpha + a_\beta)} R^{i}_{\gamma,k} \eta^{kj}_{\beta\alpha} =$$

$$= R^{i}_{\lambda} U^{j\lambda}_{\gamma\beta\alpha} + (-1)^{1+a_i a_j} R^{j}_{\lambda} U^{i\lambda}_{\gamma\beta\alpha} + S_{,k} \eta^{kji}_{\gamma\beta\alpha} \quad , \tag{5.17}$$

which are the analogues of (5.13) and (5.15). However, we have used an extra piece of information, which is that the left-hand side of (5.17) is multiplied by a factor $(-1)^{1+a_i a_j}$ upon interchange of i and j; hence on the right-hand side of (5.17) this symmetry has been imposed as well. This results in a complete cyclic symmetry of $\eta^{kji}_{\gamma\beta\alpha}$:

$$\eta^{kji}_{\gamma\beta\alpha} = (-1)^{a_k a_j + 1} \eta^{jki}_{\gamma\beta\alpha} = (-1)^{a_i a_j + 1} \eta^{kij}_{\gamma\beta\alpha} \; . \tag{5.18}$$

Finally we can insert (5.17) into (5.7) and find, that $\delta^{(2)}_{BRS} S_{eff}$ vanishes if we identify:

$$(-1)^{a_\beta + a_j (a_k + 1)} M^{ijk}_{\gamma\beta\alpha}(3) = -\tfrac{1}{2} \eta^{ijk}_{\gamma\beta\alpha} \; , \tag{5.19}$$

$$(-1)^{a_\alpha + a_\beta + a_\lambda (1 + a_j)} X^{j\lambda}_{\gamma\beta\alpha}(1) = U^{j\lambda}_{\gamma\beta\alpha} \; , \tag{5.20}$$

which was what we set out to prove. As a result of our analysis we have also found the relations (5.16) and (5.17). Eq. (5.16) expresses the fact, that the structure functions form a representation of the algebra (4.10), where $U^{j\lambda}_{\gamma\beta\alpha}$ plays the role of $\eta^{ji}_{\gamma\beta}$. This provides an interpretation of X(1). Eq. (5.17) gives an expression for the commutator $\{\eta, R\}$. Such a commutator, and its generalization $\{M(n), R\}$ for $n>2$, will play an important role in the proof of invariance to all orders.

It has been shown by Kallosh [4], that the results derived above suffice for N=1 supergravity. This follows from the vanishing of M(n), X(n-1) for $n \geq 3$. Townsend [5], has calculated the quartic ghost terms for SO(2) supergravity by this procedure and arrived at our result (3.4). He has also calculated the quartic ghost terms for all higher N theories ($N \leq 8$), and they turn out to have the same form (3.4). However, he has not shown the vanishing of the higher order quantities M(n), X(n). Hence we do not know whether the result is complete for $N \geq 3$.

6. Results to all orders

We will now continue the proof of B.R.S. invariance to arbitrary order. In order to keep the discussion clear of intractable minus signs, we will for definiteness assume that we have only Bose fields and gauge parameters. Hence we have anticommuting ghosts and $\bar{F}_i$. The general case is then obtained by a consistent introduction of minus signs for Fermi fields and parameters everywhere, but this does not alter the various steps of the proof as outlined below.

We will assume the n^{th} order variation of S_{eff}, (5.1), to be zero for all $n \leq N-1$, where N is a given integer. This variation reads:

$$\begin{aligned}\delta^{(n)}_{BRS} S_{eff} = \bar{F}_{i_1} \cdots \bar{F}_{i_n} \Big\{ & -R_\mu^{[i_1} X^{i_n \cdots i_2]}_{\alpha_n \cdots \alpha_2}{}^{\mu}_{\alpha_1 \beta}(n-1) + n M^{j i_n \cdots i_1}_{\beta \alpha_n \cdots \alpha_1}(n+1) S_{,j} + \\ & + M^{i_1 \cdots i_n}_{\alpha_1 \cdots \alpha_n}(n)_{,q} R^q_\beta - R^{[i_1}_{\beta,q} M^{i_n \cdots i_2]q}_{\alpha_n \cdots \alpha_1}(n) - \frac{n}{2} f^\mu_{\alpha_1\alpha_2} M^{i_n \cdots i_1}_{\alpha_n \cdots \alpha_3 \beta\mu}(n) \\ & + \sum_{k=2}^{n-1} \frac{(k-1)!(n-k)!}{(n-1)!} \Big(M^{[i_k \cdots i_1}_{\alpha_k \cdots \alpha_1}(k)_{,q} M^{i_n \cdots i_{k+1}]q}_{\alpha_n \cdots \alpha_{k+1}\beta}(n) + \\ & + k M^{[i_k \cdots i_1}_{\alpha_k \cdots \alpha_3 \beta\mu} X^{i_n \cdots i_{k+1}]}_{\alpha_n \cdots \alpha_{k+1}}{}^{\mu}_{\alpha_1\alpha_2}(n-k) \Big) \Big\} c^\beta c^{\alpha_n} \cdots c^{\alpha_1} \Lambda . \qquad (6.1)\end{aligned}$$

Square brackets denote complete antisymmetrization of all independent combinations of indices, except for indices that carry a hat "^". To show, that one can find quantities X(N-1), M(N+1), such that (6.1) also vanishes for n=N, we proceed as follows. We assume, that for all $n \leq N-1$, the quantities X(n-1) and M(n+1) satisfy:

$$\begin{aligned}\Big\{ & M^{i_n \cdots i_1}_{\alpha_n \cdots \alpha_1}(n)_{,q} R^q_\beta - R^{[i_1}_{\beta,q} M^{i_n \cdots i_2]q}_{\alpha_n \cdots \alpha_1}(n) - \frac{n}{2} f^\mu_{\alpha_1\alpha_2} M^{i_n \cdots i_1}_{\alpha_n \cdots \alpha_3\beta\mu}(n) + \\ & \sum_{k=2}^{n-1} \frac{(k-1)!(n-k)!}{(n-1)!} \Big(M^{[i_k \cdots i_1}_{\alpha_k \cdots \alpha_1}(k)_{,q} M^{i_n \cdots i_{k+1}]q}_{\alpha_n \cdots \alpha_{k+1}\beta}(n-k+1) \\ & + k M^{[i_k \cdots i_1}_{\alpha_k \cdots \alpha_3\beta\mu}(k) X^{i_n \cdots i_{k+1}]}_{\alpha_n \cdots \alpha_{k+1}}{}^{\mu}_{\alpha_1\alpha_2}(n-k) \Big) \Big\} c^\beta c^{\alpha_n} \cdots c^{\alpha_1} = \\ & = \Big\{ R_\mu^{[i_1} X^{i_n \cdots i_2]}_{\alpha_n \cdots \alpha_2}{}^{\mu}_{\alpha_1\beta}(n-1) - n M^{j i_n \cdots i_1}_{\beta\alpha_n \cdots \alpha_1}(n+1) S_{,j} \Big\} c^\beta c^{\beta_n} \cdots c^{\alpha_1}, \qquad (6.2)\end{aligned}$$

and

$$\Big\{ R_{\mu}^{[i_1}\Big(A^{\mu}_{\gamma\beta\alpha_1}{}^{i_n\ldots i_2]}_{\alpha_n\ldots\alpha_2}(n-1) + nX^{i_n\ldots i_2]j}_{\alpha_n\ldots\alpha_2\gamma}\,{}^{\mu}_{\alpha_1\beta}(n)S_{,j}\Big) +$$

$$+ \sum_{k=1}^{n-2} \frac{k!(n-k)!}{(n-1)!} M^{[i_{n-k}\ldots i_1}_{\alpha_{n-k}\ldots\alpha_2\mu}(n-k)\Big(A^{\mu}_{\gamma\beta\alpha_1}{}^{i_n\ldots i_{n-k+1}]}_{\alpha_n\ldots\alpha_{n-k+1}}(k) +$$

$$+ (k+1)X^{ji_n\ldots i_{n-k+1}]}_{\gamma\alpha_n\ldots\alpha_{n-k+1}}\,{}^{\mu}_{\alpha_1\beta}\, S_{,j}\Big)\Big\} c^{\gamma}c^{\beta}c^{\alpha_n}\ldots c^{\alpha_1} = 0\ . \tag{6.3}$$

The quantity $A(n)$ above is defined by:

$$A^{\mu}_{\alpha\beta\gamma}{}^{i_n\ldots i_1}_{\alpha_n\ldots\alpha_1}(n) = R^{q}_{\beta}X^{i_n\ldots i_1}_{\alpha_n\ldots\alpha_1}\,{}^{\mu}_{\alpha\gamma}(n)_{,q} - R^{[i_n}_{\beta,q}\, X^{\hat{q}i_{n-1}\ldots i_1]}_{\alpha_n\ldots\alpha_1}\,{}^{\mu}_{\alpha\gamma}(n)$$

$$- \frac{n+2}{2} f^{\lambda}_{\alpha\gamma}X^{i_n\ldots i_1}_{\alpha_n\ldots\alpha_1}\,{}^{\mu}_{\lambda\beta}(n) - f^{\mu}_{\lambda\beta}X^{i_n\ldots i_1}_{\alpha_n\ldots\alpha_1}\,{}^{\lambda}_{\alpha\gamma}(n) - \tfrac{1}{2}f^{\mu}_{\alpha\gamma,q}M^{i_n\ldots i_1 q}_{\alpha_n\ldots\alpha_1\beta}(n+1) +$$

$$+ \sum_{k=2}^{n} \frac{(k-1)!(n-k+1)!}{n!}\Big\{ M^{[i_{k-1}\ldots i_1\hat{q}}_{\alpha_{k-1}\ldots\alpha_1\gamma}(k)X^{i_k\ldots i_n]}_{\alpha_k\ldots\alpha_n}\,{}^{\mu}_{\alpha\beta}(n+1-k)_{,q}$$

$$- M^{[i_{k-1}\ldots i_1 i_n}_{\alpha_{k-1}\ldots\alpha_1\gamma}(k)_{,q}\, X^{\hat{q}i_{n-1}\ldots i_k]}_{\alpha_n\ldots\alpha_k}\,{}^{\mu}_{\alpha\beta}(n+1-k)$$

$$+ (k+1)X^{[i_{k-1}\ldots i_1}_{\alpha_{k-1}\ldots\alpha_1}\,{}^{\mu}_{\lambda\gamma}(k-1)X^{i_n\ldots i_k]}_{\alpha_n\ldots\alpha_k}\,{}^{\lambda}_{\alpha\beta}(n+1-k)\Big\} \tag{6.4}$$

with

$$A(0) = \tfrac{1}{2}(f^{\lambda}_{\alpha\gamma}f^{\mu}_{\lambda\beta} - R^{q}_{\beta}f^{\mu}_{\alpha\gamma,q})\ . \tag{6.5}$$

We must now prove, that (6.2), and (6.3) are also satisfied for $n=N$. Since we know them to hold for $n=1$, we have then completed their proof to all orders by induction. The essential element of the proof is the calculation of the Jacobi identity for two gauge transformations and a transformation of the type:

$$\delta\phi^{i} = M^{k\ldots ji}_{\beta\ldots\alpha}(n)\xi^{\alpha\ldots\beta}_{j\ldots k}(n)\ , \tag{6.6}$$

with arbitrary $\xi(n)$. This Jacobi identity reads:

$$\Big\{\big[[R^{i}_{\beta,j}R^{j}_{\gamma}]_{,k}M^{i_n\ldots i_1 k}_{\alpha_n\ldots\alpha_1}(n)\big] + 2\big[[R^{i_1}_{\gamma,k}M^{i_n\ldots i_2 k}_{\alpha_n\ldots\alpha_1}(n)]_{,j}R^{j}_{\beta}\big]\Big\}c^{\gamma}c^{\beta}c^{\alpha_n}\ldots c^{\alpha_1} = 0. \tag{6.7}$$

When we evaluate it for $n=N-1$, and use expression (4.10) for $[R,R]$ and (6.2) for the commutator:

$$[R^{i}_{\gamma,k} M^{i_n \cdots i_2 k}_{\alpha_n \cdots \alpha_1}(n)] \equiv R^{i}_{\gamma,k} M^{i_n \cdots i_2 k}_{\alpha_n \cdots \alpha_1}(n) - M^{i_n \cdots i_1}_{\alpha_n \cdots \alpha_1}(n)_{,k} R^{k}_{\gamma} \, , \tag{6.8}$$

we find:

$$\begin{aligned}
S_{,i_N} D^{i_N \cdots i_1}(N) \equiv\ & S_{,i_N} \Big\{ M^{i_N \cdots i_1}_{\alpha_N \cdots \alpha_1}(N)_{,q} R^{q}_{\beta} - R^{[i_1}_{\beta,q} M^{i_N \cdots i_2]q}_{\alpha_N \cdots \alpha_1}(N) \\
& - \tfrac{1}{2} N f^{\mu}_{\alpha_1 \alpha_2} M^{i_N \cdots i_1}_{\alpha_N \cdots \alpha_3 \beta \mu}(N) \\
& + \sum_{k=2}^{N-1} \frac{(k-1)!(N-k)!}{(N-1)!} \Big(M^{[i_k \cdots i_1}_{\alpha_k \cdots \alpha_1}(k)_{,q} M^{i_N \cdots i_{k+1}]q}_{\alpha_N \cdots \alpha_{k+1} \beta}(N+1-k) \\
& - k M^{[i_k \cdots i_1}_{\alpha_k \cdots \alpha_3 \beta \mu}(k) X^{i_N \cdots i_{k+1}]}_{\alpha_N \cdots \alpha_{k+1}}{}^{\mu}_{\alpha_1 \alpha_2}(N-k) \Big) \Big\} \cdot c^{\beta} c^{\alpha_N} \cdots c^{\alpha_1} = \\
= & \Big\{ \frac{1}{N-1} R^{[i_1}_{\mu} A^{\mu}_{\alpha_N \alpha_1 \beta}{}^{i_{N-1} \cdots i_2]}_{\alpha_{N-1} \cdots \alpha_2}(N-2) + \sum_{k=2}^{N-2} \frac{k!(N-k-1)!}{(N-1)!} M^{[i_k \cdots i_1}_{\alpha_k \cdots \alpha_2 \mu}(k) \cdot \\
& \cdot \Big(A^{\mu}_{\alpha_N \alpha_1 \beta}{}^{i_{N-1} \cdots i_{k+1}]}_{\alpha_{N-1} \cdots \alpha_{k+1}}(N-k-1) \\
& - (N-k) X^{\hat{j} i_{N-1} \cdots i_{k+1}]}_{\beta \alpha_{N-1} \cdots \alpha_{k+1}}{}^{\mu}_{\alpha_1 \alpha_N}(N-k) S_{,j} \Big) \Big\} c^{\beta} c^{\alpha_N} \cdots c^{\alpha_1} \, .
\end{aligned} \tag{6.9}$$

Since the summation in the right-hand side of (6.9) involves only $k \geq 2$, we may use eq. (6.3) to evaluate the quantities:

$$\begin{aligned}
B^{\mu}_{\alpha_N \alpha_1 \beta}{}^{i_{N-1} \cdots i_{k+1}}_{\alpha_{N-1} \cdots \alpha_{k+1}}(N-k-1) \equiv\ & A^{\mu}_{\alpha_N \alpha_1 \beta}{}^{i_{N-1} \cdots i_{k+1}}_{\alpha_{N-1} \cdots \alpha_{k+1}}(N-k-1) \\
& - (N-k) X^{\hat{j} i_{N-1} \cdots i_{k+1}}_{\beta \alpha_{N-1} \cdots \alpha_{k+1}}{}^{\mu}_{\alpha_1 \alpha_N}(N-k) S_{,j} \, .
\end{aligned} \tag{6.10}$$

Eq. (6.3) is actually a recurrence relation for the quantities $B(n)$. Since $B(0) = 0$, they would imply all $B(n)$ to vanish, if no antisymmetrizations were involved. Due to these antisymmetrizations, however, non-trivial solutions exist of the form:

$$B^{\mu}_{\gamma\beta\alpha}{}^{i_1}_{\alpha_1}(1) = R^{i_1}_{\lambda} Y^{\mu\lambda}_{\gamma\beta\alpha\alpha_1}(0) \, , \qquad \text{for } n=1 \, ,$$

$$\begin{aligned}
B^{\mu}_{\gamma\beta\alpha}{}^{i_n \cdots i_1}_{\alpha_n \cdots \alpha_1}(n)\, c^{\gamma} c^{\beta} c^{\alpha} c^{\alpha_n} \cdots c^{\alpha_1} = & \Big\{ R^{[i_1}_{\lambda} Y^{\mu\lambda}_{\gamma\beta\alpha}{}^{i_n \cdots i_2]}_{\alpha_n \cdots \alpha_1}(n-1) \\
& + \sum_{k=2}^{n} \frac{k!(n-k+1)!}{n!} M^{[i_k \cdots i_1}_{\lambda \alpha_{k-1} \cdots \alpha_1}(k) Y^{\mu\lambda}_{\gamma\beta\alpha}{}^{i_n \cdots i_{k+1}]}_{\alpha_n \cdots \alpha_k}(n-k) \Big\} \cdot c^{\gamma} c^{\beta} c^{\alpha} c^{\alpha_n} \cdots c^{\alpha_1} \, , \text{ for } n \geq 2.
\end{aligned} \tag{6.11}$$

Here the quantities $Y^{\mu\lambda}(n)$ are <u>symmetric</u> in μ and λ. When one substitutes this result into (6.9), one now makes the important observation, that owing to the symmetries of the factors $Y^{\mu\lambda}(n)$, all terms of the form:

$$M_{\mu\ldots}(k)\, M_{\lambda\ldots}(\ell) Y^{\mu\lambda}(N-k-\ell-1) \quad , \quad k,\ell \geq 2,$$

cancel. This leaves us with the result:

$$S_{,i_N} D^{i_N\ldots i_1}(N) = \frac{1}{N-1} R_\mu^{[i_1}\Big(A^{\mu}_{\alpha_N\alpha_1\beta}{}^{i_{N-1}\ldots i_2]}_{\alpha_{N-1}\ldots\alpha_2}(N-2) + \sum_{k=2}^{N-2}\frac{k!(N-k-1)!}{(N-2)!} M^{i_{k+1}\ldots i_2}_{\alpha_k\ldots\alpha_2\lambda}(k)\cdot$$
$$\cdot\, Y^{\mu\lambda}_{\alpha_N\alpha_1\beta}{}^{i_{N-1}\ldots i_{k+2}]}_{\alpha_{N-1}\ldots\alpha_{k+1}}(N-k-2)\Big)c^\beta c^{\alpha_N}\ldots c^{\alpha_1} \equiv \frac{1}{N-1} R_\mu^{[i_1} C^{\mu i_{N-1}\ldots i_2]}(N-2)\ . \tag{6.12}$$

This has a form, which is a direct generalization of eq. (5.11) and can be solved analogously. It leads to the desired result, that one can write:

$$\Big(M^{i_N\ldots i_1}_{\alpha_N\ldots\alpha_1}(N)_{,q}\, R^q_\beta - R^{[i_1}_{\beta,q}\, M^{i_N\ldots i_2]q}_{\alpha_N\ldots\alpha_1}(N) - \tfrac{1}{2}N f^{\mu}_{\alpha_1\alpha_2}\, M^{i_N\ldots i_1}_{\alpha_N\ldots\alpha_3\beta\mu}(N) +$$
$$+ \sum_{k=2}^{N-1}\frac{(k-1)!(N-k)!}{(N-1)!}\Big(M^{[i_k\ldots i_1}_{\alpha_k\ldots\alpha_1,q} M^{i_N\ldots i_{k+1}]q}_{\alpha_N\ldots\alpha_{k+1}\beta}(N+1-k)$$
$$- kM^{[i_k\ldots i_1}_{\alpha_k\ldots\alpha_3\beta\mu}(k) X^{i_N\ldots i_{k+1}]}_{\alpha_N\ldots\alpha_{k+1}}{}^{\mu}_{\alpha_1\alpha_2}(N-k)\Big)\Big)\cdot c^\beta c^{\alpha_N}\ldots c^{\alpha_1} =$$
$$= \Big(R_\mu^{[i_1} X^{i_N\ldots i_2]}_{\alpha_N\ldots\alpha_2}{}^{\mu}_{\alpha_1\beta}(N-1) - NM^{\hat{j} i_N\ldots i_1}_{\beta\alpha_N\ldots\alpha_1}(N+1) S_{,j}\Big)c^\beta c^{\alpha_N}\ldots c^{\alpha_1}\ . \tag{6.13}$$

Choosing this for X(N-1) and M(N+1) in (5.1) and (5.2), we have indeed:

$$\delta^{(N)}_{BRS} S_{eff} = 0\ . \tag{6.14}$$

Substitution of (6.13) into the Jacobi identity (6.7) with n=N, finally proves eq. (6.3) in next order. Hence the proof of (6.2) and (6.3) by induction is complete.

7. <u>Discussion</u>

We have proven, that the effective action (5.1) is invariant under the generalized B.R.S transformations (5.2). Of course this is not sufficient to ensure, that S_{eff} defines the correct path integral. We must still show, that B.R.S. invariance guarantees the gauge independence of the quantum theory. The proof is simple and completely analogous to the one for theories with a closed gauge algebra [1].

We will first derive the following general theorem:

> a B.R.S. transformation on the expectation value of any function 0 of the fields can always be absorbed in a redefinition of the source terms in the path integral.

To prove this statement, we construct a generalized path integral:

$$Z[J,H] = \int D\phi \exp i(S_{eff} + J_i\phi^i + HO) . \tag{7.1}$$

By performing a B.R.S. transformation on it and assuming the integration measure to be gauge invariant, we find:

$$-i\Lambda \langle J_i \delta_{BRS}\phi^i + H\delta_{BRS}O \rangle_{J,H} = 0 . \tag{7.2}$$

Taking the derivative with respect to H and evaluating the result at H=0 then gives:

$$\langle \delta_{BRS}O \rangle_J = - \langle J_i(\delta_{BRS}\phi^i)O \rangle_J \quad , \tag{7.3}$$

which is the desired result.

The following step is now to show, that an infinitesimal change of gauge in the path integral can be written as a B.R.S. transformation on a certain function. Explicitly, if we change the gauge:

$$F_\alpha \to F_\alpha + \lambda G_\alpha , \tag{7.4}$$

then the change in the path integral is, to first order in λ:

$$Z[J_i] \to Z[J_i] +$$

$$+ i\lambda \langle -F^\alpha G_\alpha + \bar{c}^{*\alpha} G_{\alpha,i_1} \Big(R^{i_1}_{\alpha_1} + \sum_{n\geq 2} \bar{F}_{i_2}\ldots\bar{F}_{i_n} M^{i_n\ldots i_1}_{\alpha_n\ldots\alpha_1}(n) c^{\alpha_n}\ldots c^{\alpha_2}\Big) c^{\alpha_1} \rangle_J . \tag{7.5}$$

One sees immediately, that (7.5) can be written:

$$\delta Z[J_i] = i\lambda \frac{\partial}{\partial\Lambda} \langle \delta_{BRS}(\bar{c}^{*\alpha}G_\alpha) \rangle_J . \tag{7.6}$$

By our theorem this can be compensated by a simultaneous redefinition of the source terms:

$$\delta Z[J_i] = i\lambda \langle J_i \frac{\partial}{\partial\Lambda} (\delta_{BRS}\, \phi^i)\bar{c}^{*\alpha}G_\alpha \rangle_J . \tag{7.7}$$

Clearly this is zero, when the sources are B.R.S. invariant:

$$J_i(\delta_{BRS}\phi^i) = 0 \ . \tag{7.8}$$

However, we may also invoke a general theorem in field theory, that path integrals, which differ only in the source terms, give rise to the same S-matrix elements [1]. This establishes the gauge invariance of the quantum theory, provided we make some restrictions. The above argument presupposes that B.R.S. invariance is not spontaneously broken. Furthermore the sources need to be physical, i.e. they must satisfy:

$$\langle F^\alpha J_i \rangle = 0 \ , \tag{7.9}$$

for arbitrary gauge fixing functions F^α (see ref. [1.b]) .

We now want to discuss a property of the B.R.S. transformations themselves. If we take the commutator of two B.R.S. transformations, the argument which also led to (4.10) shows, that it closes upon use of the effective field equations. However, it turns out that a stronger condition holds, which is, that the B.R.S. transformations are nilpotent upon use of the effective equations of motion. Indeed one may verify, that:

$$\begin{aligned}
\delta^2_{BRS}\phi^i &= \Lambda_1\Lambda_2 \left\{ \frac{\delta S_{eff}}{\delta\phi^k}\left(M^{ik}_{\alpha_2\alpha_1}(2) + \sum_{n\geq 2} n\bar{F}_{i_2}\cdots\bar{F}_{i_n} M^{i_n\cdots i_2 ik}_{\alpha_{n+1}\cdots\alpha_1}(n+1)c^{\alpha_{n+1}}\cdots c^{\alpha_3}\right)c^{\alpha_2\alpha_1} \right. \\
&\quad \left. + \frac{\delta S_{eff}}{\delta c^\alpha}\left(-X^{i\alpha}_{\nu_1\mu\lambda}(1) + \sum_{n\geq 2}(-1)^n n\bar{F}_{i_2}\cdots\bar{F}_{i_n} X^{i_n\cdots i_2 i}_{\nu_n\cdots\nu_1}{}^{\alpha}_{\mu\lambda}(n)c^{\nu_n}\cdots c^{\nu_1}\right)c^\mu c^\lambda \right\}, \\
\delta^2_{BRS}c^\alpha &= \Lambda_1\Lambda_2 \left\{ -\frac{\delta S_{eff}}{\delta\phi^k}\left(X^k_{\nu_1}{}^{\alpha}_{\mu\lambda}(1) + \sum_{n\geq 2} n\bar{F}_{i_2}\cdots\bar{F}_{i_n} X^{i_n\cdots i_2 k}_{\nu_n\cdots\nu_1}{}^{\alpha}_{\mu\lambda}(n)c^{\nu_n}\cdots c^{\nu_2}\right)c^{\nu_1}c^\mu c^\lambda \right. \\
&\quad \left. + \frac{\delta S_{eff}}{\delta\phi^k}\ (\text{terms containing } Y^{\mu\lambda}(n)\ ) \right\}, \qquad (7.10) \\
\delta^2_{BRS}c^{*\alpha} &= \Lambda_1\Lambda_2 \frac{\delta S_{eff}}{\delta c^*_\alpha} \ .
\end{aligned}$$

This is to be compared with the strict nilpotency of $\delta_{BRS}\phi^i$ and $\delta_{BRS}c^\alpha$ in Yang-Mills theory. However, the nilpotency condition (7.10) for $c^{*\alpha}$ holds in both cases. These observations may be important in restricting possible higher order counter terms to S_{eff} [1.a].

8. Conclusion

We have shown how to quantize supergravity, both in a formulation with and without auxiliary fields. Moreover we have generalized the covariant quantization procedure to arbitrary gauge theories with open gauge algebra. This shows, that a closed commutator algebra of the gauge transformations is no prerequisite for obtaining a consistent quantum theory. The most conspicuous feature of this procedure is the introduction of higher order ghost terms, both in the effective action and the B.R.S. transformations. Incidentally this, and the gauge algebra, show that theories with open gauge algebra are most likely to be found among theories with dimensionful coupling constants, such as gravity.

It would be interesting if we were able to understand these higher order corrections from a more general point of view. For example one might ask whether there is a relation between open gauge algebra's and auxiliary fields. At present we know only, that elimination of auxiliary fields leads to open gauge algebra's. Whether there always corresponds a closed algebra with auxiliary fields to any open gauge algebra is an unanswered question.

Finally there is an interesting observation by Otten [7], who showed that the generating functional for proper vertices has a set of local invariances with open commutator algebra. This generating functional is the Legendre transform with respect to the sources of the logarithm of the path integral:

$$\Gamma_{eff}[\phi] = \ln Z[J_i] - J_i\phi^i \, , \tag{8.1}$$

with

$$\phi^i \equiv \frac{\delta}{\delta J_i}(\ln Z[J_i]) \, . \tag{8.2}$$

It turns out that Γ_{eff} has precisely the form (5.1), with the quantities $M(n)$ generated by the algebra of local invariances, as discussed.

With this remark we conclude our discussion of the quantization procedure for theories with an open commutator algebra of local gauge transformations.

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APPENDIX A. Notations and conventions

In this place we collect the notations and conventions used throughout the text concerning representations of the Lorentz group and Dirac algebra. When we consider global Lorentz invariance, we denote vectors by Greek indices μ, ν, ..., and spinors by Latin indices a, b,... . Both run from 1 to 4. We use the Pauli metric

$$\delta_{\mu\nu} = \text{diag.}\ (+,+,+,+), \tag{A.1}$$

with imaginary time components of four-vectors:

$$p_\mu = (\vec{p},p_4) = (\vec{p},ip^0)\ . \tag{A.2}$$

Hence there is no need for distinguishing upper and lower indices. The four-dimensional Levi-Civita tensor is defined in terms of the permutation symbol $\delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}$:

$$\begin{aligned} \varepsilon_{\mu\nu\rho\sigma} = \delta^{1234}_{\mu\nu\rho\sigma} = &+1, \quad (\mu\nu\rho\sigma) = \text{even permutation of } (1234), \\ &-1, \quad (\mu\nu\rho\sigma) = \text{odd permutation of } (1234), \\ &\ \ 0, \quad \text{otherwise}\ . \end{aligned} \tag{A.3}$$

Three-dimensional ordinary space vectors carry Latin indices i, j,..., running from 1 to 3. The three-dimensional Levi-Civita tensor is

$$\varepsilon_{ijk} = \delta^{123}_{ijk} = \varepsilon_{ijk4}.$$

In all cases repeated indices will imply a summation, unless explicitly stated otherwise.

When we discuss local Lorentz invariance, we need to distinguish between world indices $\mu,\nu,...$ and local Lorentz indices a,b,... . In this case we will suppress spinor indices, hence no confusion with local Lorentz indices can arise. Vierbeins e^a_μ and their matrix inverse e^μ_a convert local Lorentz tensors into world tensors and vice versa, as discussed in I.4. Consequently we have world tensors with upper and lower indices, related by a contraction with the metric tensor

$$g_{\mu\nu} = \delta_{ab} e^a_\mu e^b_\nu, \tag{A.4}$$

or its inverse $g^{\mu\nu}$. The summation convention for world indices implies a contraction over $g_{\mu\nu}$.

The Levi-Civita symbol with world indices is defined as in (A.3):

$$\varepsilon_{\mu\nu\rho\sigma} = \delta^{1234}_{\mu\nu\rho\sigma} = \frac{1}{e}\ \varepsilon_{abcd}\ e^a_\mu e^b_\nu e^c_\rho e^d_\sigma\ , \tag{A.5}$$

where

$$e = \det e^a_\mu\ .$$

Hence $\varepsilon_{\mu\nu\rho\sigma}$ is no longer a tensor, but a tensor density.

Next we discuss the Dirac algebra. We will do this in the context of global Lorentz invariance. When considering the local case the Dirac algebra remains unchanged when one defines all elements with local Lorentz indices.

With our metric convention (A.1) the Dirac algebra is defined by

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}\ . \tag{A.6}$$

Hence

$$\gamma^2_\mu = 1\ , \quad \text{(no summation over } \mu)\ . \tag{A.7}$$

The standard irreducible representation of this algebra is four-dimensional and has hermitean γ-matrices:

$$\gamma^\dagger_\mu = \gamma_\mu . \tag{A.8}$$

From these elements we define the following quantities:

$$\begin{aligned} &\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4\ , \qquad \gamma_5^\dagger = \gamma_5, \\ &\sigma_{\mu\nu} = \tfrac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\ ,\ \sigma^\dagger_{\mu\nu} = -\sigma_{\mu\nu}\ . \end{aligned} \tag{A.9}$$

The $\sigma_{\mu\nu}$ form a representation of the Lorentz algebra:

$$[\sigma_{\mu\nu},\sigma_{\alpha\beta}] = \delta_{\mu\beta}\sigma_{\nu\alpha} + \delta_{\nu\alpha}\sigma_{\mu\beta} - \delta_{\mu\alpha}\sigma_{\nu\beta} - \delta_{\nu\beta}\sigma_{\mu\alpha}\ . \tag{A.10}$$

The set of sixteen 4×4-matrices

$$\{O^J\} = \{1,\gamma_5,\gamma_\mu,i\gamma_\mu\gamma_5,i\sqrt{2}\ \sigma_{\mu\nu}\}$$

is complete. Hence any 4×4-matrix X can be expanded in terms of the O^J:

$$X = \sum_J \tfrac{1}{4}\ \mathrm{Tr}(XO^J)O^J\ . \tag{A.11}$$

In particular all O^J except 1 are traceless:

$$\mathrm{Tr}\ O^J = (O^J)_{aa} = 0\ .$$

We define a charge conjugation matrix C by

$$\begin{aligned} &C\gamma_\mu{}^T = -\gamma_\mu C\ , \\ &C^2 = 1,\quad C = -C^T. \end{aligned} \tag{A.12}$$

Here the superscript T denotes transposition. From (A.12) we derive:

$$\begin{aligned} C\gamma_5{}^T C^{-1} &= \gamma_5, \\ C\gamma_\mu{}^T C^{-1} &= -\gamma_\mu, \\ C(\gamma_5\gamma_\mu)^T C^{-1} &= \gamma_5\gamma_\mu, \\ C\sigma_{\mu\nu}{}^T C^{-1} &= -\sigma_{\mu\nu}. \end{aligned} \tag{A.13}$$

The four-dimensional representation space of the Dirac algebra is called spinor space. The elements of this space, the spinors, transform covariantly under the representation of the Lorentz group generated by $\sigma_{\mu\nu}$:

$$\delta\psi_a = \tfrac{1}{2}\epsilon^{\mu\nu}(\sigma_{\mu\nu})_{ab}\psi_b \ . \tag{A.14}$$

As a consequence they can be used to construct relativistically invariant field theories, in which they represent fields with spin $\frac{1}{2}$. For consistency they have to be anticommuting, i.e. they are elements of a Grassmann algebra:

$$\psi_a\psi_b + \psi_b\psi_a = 0 \ . \tag{A.15}$$

Free spinor fields satisfy the Dirac equation:

$$(\not\partial+m)\psi = 0 \ , \tag{A.16}$$

where we use the notation

$$\not a = a_\mu\gamma_\mu \ . \tag{A.17}$$

The Pauli conjugate $\bar\psi$ of a spinor is defined by

$$\bar\psi = \psi^\dagger\gamma_4 \ . \tag{A.18}$$

It satisfies the equation

$$\bar\psi(\overleftarrow{\not\partial}-m) = 0 \ . \tag{A.19}$$

A Majorana spinor is defined as a self conjugate spinor:

$$\psi_a = C_{ab}\bar\psi_b \ . \tag{A.20}$$

Using (A.13) and (A.15) we find for Majorana spinors:

$$\bar\psi\gamma_\mu\psi = 0 = \bar\psi\sigma_{\mu\nu}\psi \ . \tag{A.21}$$

The completeness relation (A.11) may be used to expand the direct product of two spinors:

$$\psi_a\bar{\phi}_b = -\tfrac{1}{4}(\bar{\phi}\psi)1_{ab} - \tfrac{1}{4}(\bar{\phi}\gamma_5\psi)(\gamma_5)_{ab} - \tfrac{1}{4}(\bar{\phi}\gamma_\mu\psi)(\gamma_\mu)_{ab}$$
$$+ \tfrac{1}{4}(\bar{\phi}\gamma_5\gamma_\mu\psi)(\gamma_5\gamma_\mu)_{ab} + \tfrac{1}{2}(\bar{\phi}\sigma_{\mu\nu}\psi)(\sigma_{\mu\nu})_{ab} . \tag{A.22}$$

This is sometimes called the Fierz rearrangement formula. A number of useful (anti-)commutation relations is given below. Writing

$$\begin{aligned} [A,B] &= AB-BA, \\ \{A,B\} &= AB+BA, \end{aligned} \tag{A.23}$$

one may derive:

$$\begin{aligned}
&\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu} \quad , \quad [\gamma_\mu,\gamma_\nu] = 4\sigma_{\mu\nu}, \\
&\{\gamma_\mu,\gamma_5\} = 0 \quad , \quad [\sigma_{\mu\nu},\gamma_5] = 0, \\
&\{\gamma_\mu,\sigma_{\nu\lambda}\} = \varepsilon_{\mu\nu\lambda\rho}\gamma_5\gamma_\rho \quad , \quad [\gamma_\mu,\sigma_{\nu\lambda}] = \gamma_\lambda\delta_{\mu\nu} - \gamma_\nu\delta_{\mu\lambda}, \\
&\{\sigma_{\mu\nu},\sigma_{\rho\sigma}\} = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\gamma_5 - \tfrac{1}{2}(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}), \\
&[\sigma_{\mu\nu},\sigma_{\rho\sigma}] = \delta_{\nu\rho}\sigma_{\mu\sigma} + \delta_{\mu\sigma}\sigma_{\nu\rho} - \delta_{\mu\rho}\sigma_{\nu\sigma} - \delta_{\nu\sigma}\sigma_{\mu\rho}.
\end{aligned} \tag{A.24}$$

Sometimes it is convenient to have an explicit representation of the Dirac algebra. In terms of the Pauli matrices σ_i :

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{A.25}$$

there is a representation:

$$\gamma_i = \begin{pmatrix} \theta & -i\sigma_i \\ i\sigma_i & \theta \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 1 & \theta \\ \theta & -1 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} \theta & -1 \\ -1 & \theta \end{pmatrix} , \tag{A.26}$$

with

$$\sigma_{\mu\nu} = \begin{cases} \frac{i}{2}\,\varepsilon_{ijk}\begin{pmatrix} \sigma_k & \theta \\ \theta & \sigma_k \end{pmatrix} , & \text{when } (\mu\nu) = (ij) , \\[2ex] \frac{i}{2}\begin{pmatrix} \theta & \sigma_i \\ \sigma_i & \theta \end{pmatrix} , & \text{when } (\mu\nu) = (i4) , \end{cases} \tag{A.27}$$

$$\text{and } C = i\gamma_4\gamma_2 = \begin{pmatrix} \theta & \sigma_2 \\ \sigma_2 & \theta \end{pmatrix} . \tag{A.28}$$

We end this appendix by explaining some notations employed in manipulating internal symmetry indices. As with the space-time indices, repeated internal symmetry indices will imply a summation. We have the following conventions on symmetrization:

antisymmetrization is denoted by square brackets:

$$X^{[i}\,Y^{j]} = X^iY^j - X^jY^i \;;$$

symmetrization is denoted by round brackets:

$$X^{(i}\,Y^{j)} = X^iY^j + X^jY^i \;;$$

finally the symmetric traceless part is denoted by applying an s to the symmetrization bracket:

$$X^{(i}\,Y^{j)s} = X^iY^j + X^jY^i - \frac{2}{N}\,\delta^{ij}\,X^kY^k \,,$$

when $i,j,\ldots = 1,\ldots,N$. This is also written explicitly:

$$X^{(i}\,Y^{j)s} = X^iY^j + (i\leftrightarrow j;\ \text{traceless}).$$

APPENDIX B. The superconformal algebra

In this appendix we will describe a supersymmetry algebra more general than the graded Poincaré algebra. This is the graded conformal, or superconformal, algebra which includes the Poincaré algebra as a subalgebra. However, it can only be realized in field theories which do not contain an intrinsic mass. This is due to the scale invariance present in superconformal theories. The nice feature of this algebra is, that is allows incorporation of chiral U(N) internal symmetries, which seem preferable to the SO(N) symmetries of the Poincaré theories.

The graded conformal algebra is based on the ordinary Lie algebra of conformal space-time transformations. The elements of this Lie algebra are the space-time translations P , the Lorentz transformations M , conformal boosts K and dilatations D. Their commutation relations read:

$$\begin{aligned} &[P_\mu, M_{\nu\lambda}] = \delta_{\mu\nu}P_\lambda - \delta_{\mu\lambda}P_\nu, && [K_\mu, M_{\nu\lambda}] = \delta_{\mu\nu}K_\lambda - \delta_{\mu\lambda}K_\nu, \\ &[P_\mu, D] = P_\mu, && [K_\mu, D] = -K_\mu, \\ &[P_\mu, K_\nu] = 2(\delta_{\mu\nu}D - M_{\mu\nu}) , && \\ &[M_{\mu\nu}, M_{\kappa\lambda}] = \delta_{\mu\lambda}M_{\nu\kappa} + \delta_{\nu\kappa}M_{\mu\lambda} - \delta_{\mu\kappa}M_{\nu\lambda} - \delta_{\nu\lambda}M_{\mu\kappa} . && \end{aligned} \tag{B.1}$$

All other commutators vanish.

A grading of this algebra can be obtained by adding two Majorana spinor elements, Q and S, and a U(1) chiral charge A. The Q's form the grading representation of the ordinary Poincaré subalgebra contained in (B.1), while the S play a similar role with respect to the subalgebra of conformal boosts K and Lorentz transformations M. If these subalgebra's are combined in a non-trivial way as in (B.1), the dilatation D is necessary to close the algebra. To obtain the full graded algebra, we have to include the chiral U(1) transformations as well. This leads us to the following set of (anti-) commutators, in addition to (B.1):

$$\begin{aligned} &[P_\mu, S_a] = -(\gamma_\mu Q)_a , && [K_\mu, Q_a] = (\gamma_\mu S)_a, \\ &[M_{\mu\nu}, S_a] = -(\sigma_{\mu\nu} S)_a, && [M_{\mu\nu}, Q_a] = -(\sigma_{\mu\nu} Q)_a, \\ &[D, S_a] = \tfrac{1}{2}S_a, && [D, Q_a] = -\tfrac{1}{2}Q_a, \\ &[A, S_a] = -3(\gamma_5 S)_a, && [A, Q_a] = 3(\gamma_5 Q)_a, \\ &\{S_a, S_b\} = 2(\not{K}C)_{ab}, && \{Q_a, Q_b\} = -2(\not{P}C)_{ab}, \\ &\{S_a, Q_b\} = 2DC_{ab} + 2(\sigma_{\mu\nu}C)_{ab}M_{\mu\nu} + (\gamma_5 C)_{ab}A . && \end{aligned} \tag{B.2}$$

Again the other commutators are tacitly understood to be zero.

Exactly as in the case of Poincaré supersymmetry one can include other symmetries, besides the U(1) transformations, by extending Q and S to grading representations of some internal symmetry algebra as well. Making use of chiral invariance a reasoning similar to that in II.2 shows, that this extra internal symmetry is SU(N), at least when no central charges are present. Hence the total internal symmetry is $U(1) \times SU(N) = U(N)$. However, in contrast to the Poincaré case, the U(N) charges A, B^r_s appear in the graded conformal algebra itself, since they are necessary to close the anticommutator of Q and S, as in (B.2). Writing out the commutation relations on a chiral basis,

$$Q^r = \tfrac{1}{2}(1-\gamma_5)Q^r, \qquad S^r = \tfrac{1}{2}(1+\gamma_5)S^r,$$
$$Q_r = \tfrac{1}{2}(1+\gamma_5)Q^r, \qquad S_r = \tfrac{1}{2}(1-\gamma_5)S^r, \tag{B.3}$$

we obtain:

$$[A,S_{r,a}] = \left(\frac{4}{N}-1\right)S_{r,a}, \qquad [A,Q_{r,a}] = \left(\frac{4}{N}-1\right)Q_{r,a},$$
$$[A,S^r_a] = -\left(\frac{4}{N}-1\right)S^r_a, \qquad [A,Q^r_a] = -\left(\frac{4}{N}-1\right)Q^r_a,$$
$$[B^r_s,S_{t,a}] = +\delta^r_t S_{s,a} - \frac{1}{N}\delta^r_s S_{t,a}, \qquad [B^r_s,Q_{t,a}] = +\delta^r_t Q_{s,a} - \frac{1}{N}\delta^r_s Q_{t,a},$$
$$[B^r_s,S^t_a] = -\delta^t_s S^r_a + \frac{1}{N}\delta^r_s S^t_a, \qquad [B^r_s,Q^t_a] = -\delta^t_s Q^r_a + \frac{1}{N}\delta^r_s Q^t_a. \tag{B.4}$$

$$\{S^t_a,Q_{r,b}\} = \left(\tfrac{1}{2}(1+\gamma_5)\left(\delta^t_r(2DC + 2(\sigma_{\mu\nu}C)M_{\mu\nu} + CA) + 4CB^t_r\right)\right)_{ab},$$
$$\{S_{t,a},Q^r_b\} = \left(\tfrac{1}{2}(1-\gamma_5)\left(\delta^r_t(2DC - 2(\sigma_{\mu\nu}C)M_{\mu\nu} - CA) - 4CB^r_t\right)\right)_{ab}.$$

We have written the SU(N) generators in terms of the fundamental representation:

$$[B^r_s,B^m_n] = -\delta^m_s B^r_n + \delta^r_n B^m_s; \quad (r,s,m,n) = 1,\ldots,N; \; B^r_r = 0. \tag{B.5}$$

Notice, that for N=4 the chiral charge A becomes a central charge, appearing only in the anticommutator of Q and S. Hence in this case the internal symmetry is reduced to SU(4).

SAMENVATTING

Supergravitatie is de naam van een aantal theorieën die de fundamentele wisselwerking tussen elementaire deeltjes pogen te beschrijven. De eenvoudigste hiervan beschrijft alleen de zwaartekracht, of gravitatie, en één nieuwe hypothetische wisselwerking, die overgebracht wordt door een massaloos zgn. ijkdeeltje, het gravitino. De bijzonderheid van het gravitino is, dat het een halftallig aantal eenheden spin bezit, nl. $\frac{3}{2}$. Zulke deeltjes noemt men fermionen. Alle bekende wisselwerkingen in de natuur worden overgebracht door ijkdeeltjes met een heeltallige spin, bosonen geheten. De beschrijving van de gravitatie in deze theorie, als klassieke veldentheorie, is dezelfde als in de algemene relativiteitstheorie van Einstein (1916). Door het uitbreiden van deze theorie met het gravitino krijgt hij echter een aantal bijzondere eigenschappen.

De eerste hiervan is, dat supergravitatie tot een consistente quantumtheorie van de gravitatie kan leiden. Dit houdt in, dat de zwaartekracht kan worden beschouwd als een fundamentele wisselwerking op microscopisch niveau, die het gevolg is van de uitwisseling van zgn. quanta tussen elementaire deeltjes. Deze quanta zijn energiepakketjes, behorend bij een bepaald type veld. De quanta van het gravitatieveld worden gravitonen genoemd. Dat een dergelijke beschrijving van de zwaartekracht mogelijk is, is geenszins triviaal. In het bijzonder is het in een quantumtheorie van de gravitatie nooit eerder mogelijk gebleken een consistente wiskundige procedure te definiëren, die tot eindige resultaten leidt.

Een tweede eigenschap die supergravitatie onderscheidt van de algemene relativiteitstheorie is, dat de eerstgenoemde een bijzondere symmetrie bezit, supersymmetrie. Hieraan ontleent supergravitatie zijn naam. Deze symmetrie houdt in, dat het graviton en gravitino kunnen worden opgevat als twee aspecten van eenzelfde wisselwerking. In het bijzonder zijn de natuurkundige wetten, die een waarnemer vindt in een wereld die door supergravitatie wordt beschreven, onafhankelijk van hoe hij de wisselwerking opsplitst in een graviton- en een gravitinocomponent. Dit wordt uitgedrukt door te zeggen, dat supergravitatie de ijktheorie van supersymmetrie is.

De derde bijzondere eigenschap van supergravitatie is gelegen in de beschrijving van wisselwerkingen met de materie. Zulk een beschrijving is

essentieel in iedere gravitatietheorie. In supergravitatie kan men de ijkdeeltjes, die verantwoordelijk zijn voor de andere fundamentele interacties, en de deeltjes waaruit de materie is opgebouwd opnemen als verdere componenten van hetzelfde stel velden, dat ook het graviton en gravitino beschrijft. Supergravitatie is dus mogelijk een geünificeerde theorie van alle deeltjes en wisselwerkingen, die er in de natuur zijn. Deze bijzondere en tot op heden unieke eigenschappen maken het zeer de moeite waard supergravitatie te bestuderen.

In dit proefschrift wordt uiteengezet, hoe men theorieën van supergravitatie construeert. Na twee algemene inleidende hoofdstukken wordt in het bijzonder de theorie uitgewerkt die supergravitatie unificeert met een vorm van elektromagnetisme, de zgn. SO(2) supergravitatie. De volledige Lagrangiaan, waaruit de veldvergelijkingen volgen, wordt gegeven, alsmede de transformatieregels voor de velden onder supersymmetrie. Deze laten de actie, de integraal van de Lagrangiaan over ruimte en tijd, invariant. De regels zijn gecompliceerd, maar kunnen beter begrepen worden in termen van een grotere symmetriegroep, nl. conforme supersymmetrie, waaronder deelverzamelingen van de velden transformeren. Tenslotte wordt besproken, hoe men uitgaande van de Lagrangiaan een consistente quantumtheorie kan definiëren.

Een deel van het in dit proefschrift beschreven onderzoek is gepubliceerd in Nucl.Phys.B. De in hoofdstuk IV beschreven resultaten werden verkregen mede in samenwerking met dr.A.Van Proeyen van de Kath. Universiteit te Leuven, België. Mevr.S.Hélant Muller-Soegies verzorgde het typewerk. De omslag werd ontworpen door dhr.W.Verzantvoort.

CURRICULUM VITAE

Jan Willem van Holten, geboren 12 mei 1952 te 's-Gravenhage, behaalde in 1970 het einddiploma gymnasium B aan het Huygens Lyceum te Voorburg. In het academisch jaar 1970-1971 studeerde hij natuurkunde en wiskunde aan Hamline University te St.Paul, U.S.A. Daartoe werd hij in staat gesteld door een beurs, verkregen met medewerking van het Nederland-Amerika Instituut te Amsterdam. Vanaf 1971 zette hij deze studie voort aan de Rijksuniversiteit te Leiden, waar hij in 1974 het kandidaatsexamen natuurkunde en wiskunde met bijvak scheikunde aflegde. Gedurende twee jaren verrichtte hij vervolgens experimenteel onderzoek naar de magneto-calorische eigenschappen van type-II supergeleiders, onder leiding van Dr.P.H.Kes en wijlen Dr.D.de Klerk op het Kamerlingh Onnes Laboratorium. In juni 1976 behaalde hij het doctoraalexamen natuurkunde gemengde richting, met bijvak wiskunde. Vanaf september 1976 werkt hij als medewerker aan het Instituut-Lorentz voor theoretische natuurkunde in Leiden onder leiding van Dr.F.A.Berends en Dr.B.de Wit op het gebied van de veldentheorie en hoge-energiefysica. Sinds 1978 doet hij dit binnen de werkgroep H-th-L van de Stichting voor Fundamenteel Onderzoek der Materie. Naast het onderzoek verricht hij ook een aantal onderwijstaken, zoals het geven van werkcolleges en een studentenseminarium. Ter ondersteuning van het onderzoek bezocht hij een aantal zomerscholen en conferenties, deels met financiële steun van de Stichting F.O.M. en de Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek (Z.W.O.). Vanaf 1 oktober 1980 zal hij als Fellow zijn verbonden aan het Europese centrum voor hoge-energiefysica, CERN, te Genève.

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