ANSWERS TO THE EXAM QUANTUM THEORY, 13 FEBRUARY 2017

each item gives 2 points for a fully correct answer, grade = total $\times\, 9/24 + 1$

1. *(a)*

$$\frac{d}{dt}\mathrm{Tr}\,\rho(t) = \frac{1}{i\hbar}\mathrm{Tr}\,[\hat{H},\hat{\rho}(t)] = 0.$$

(b) Define $\hat{F}(t) = \hat{\rho}^2(t) - \hat{\rho}(t)$, then calculate

$$i\hbar\frac{\partial\hat{F}}{\partial t} = \hat{\rho}[\hat{H},\hat{\rho}] + [\hat{H},\hat{\rho}]\hat{\rho} - [\hat{H},\hat{\rho}] = [\hat{H},\hat{F}],$$

so $\hat{F}(t) = e^{-i\hat{H}t/\hbar}\hat{F}(0)e^{i\hat{H}t/\hbar}$, and since $\hat{F}(0) = 0$ it follows that $\hat{F}(t) = 0$.
(c) $\hat{\rho} = |\psi\rangle\langle\psi|$, $\hat{\rho}\psi = \langle\psi|\psi\rangle\psi = \psi$.

2. *(a)* $\hat{P}$ is Hermitian if $\hat{P} = \hat{P}^\dagger$, with the Hermitian conjugate $\hat{P}^\dagger$ defined by $\langle\phi|\hat{P}\psi\rangle = \langle\hat{P}^\dagger\phi|\psi\rangle$; the parity operator is Hermitian because

$$\langle\phi|\hat{P}\psi\rangle = \int dx\,\phi^*(x)\hat{P}\psi(x) = \int dx\,\phi^*(x)\psi(-x) = \int dx\,\phi^*(-x)\psi(x) = \langle\hat{P}^\dagger\phi|\psi\rangle$$

(b) $\hat{P}^2 = I$ equals the identity operator I, so $\hat{P}^{-1} = \hat{P} = \hat{P}^\dagger$, which is the definition of a unitary operator; the eigenvalues satisfy $p^2 = 1 \rightarrow p = \pm 1$.
(c) If ψ is an eigenstate of $\hat{H}$ with a nondegenerate eigenvalue E, and $\hat{P}$ commutes with $\hat{H}$, then also $\hat{P}\psi$ is an eigenstate of $\hat{H}$ with the same eigenvalue E, since $\hat{H}\hat{P}\psi = \hat{P}\hat{H}\psi = E\hat{P}\psi$; if E is nondegenerate, the two eigenstates ψ and $\hat{P}\psi$ must be linearly related, so we must have $\hat{P}\psi = \lambda\psi$ for some number λ, hence ψ is an eigenfunction of the parity operator, hence either $\lambda = +1$ and $\psi(x) = \psi(-x)$ (even function) or $\lambda = -1$ and $\psi(x) = -\psi(-x)$ (odd function).

3. *(a)* $U = \exp(-(i/\hbar)ef(q))$, so $U^{-1}(p - eA)U = p - edf/dq - eA = p - e\tilde{A}$
(b) A unitary transformation leaves the eigenvalues unchanged, so the lowest energy of $\tilde{H}$ and H are the same; if we choose $f = -\frac{1}{2}A_0q^2$, the vector potential disappears from $\tilde{H}$, therefore $E_0 = 0$ for any A_0.
(c) The gauge transformation with $f = -qBR/2$ does remove the vector potential from the Hamiltonian, but the transformed wave function $\tilde{\psi}(q) = e^{(i/\hbar)eqBR/2}\psi(q)$ no longer satisfies the periodic boundary condition $\tilde{\psi}(q + 2\pi R) = \psi(q)$, unless $e\pi BR^2$ is a multiple of $2\pi\hbar$. The lowest energy is periodic in B with period $\Delta B = 2\pi\hbar/(e\pi R^2)$. Within one period the dependence on B is parabolic, $E_0 = (eBR/2)^2/2m$.

4. *(a)* $H^2 = v^2(p_x^2 + p_y^2)$ times the unit matrix, so $E^2 = v^2(p_x^2 + p_y^2)$; there are positive and negative energies, without a lowest energy.
(b) first choice:

$$H\psi_1 = i(eBx - \hbar k)f(x) - i\hbar f'(x) = 0 \Rightarrow f(x) = \exp(eBx^2/2\hbar - kx)f(0)$$

— fails because it is not normalizable; second choice

$$H\psi_2 = -i(eBx - \hbar k)f(x) - i\hbar f'(x) = 0 \Rightarrow f(x) = \exp(-eBx^2/2\hbar + kx)f(0)$$

is normalizable, so we have found our zero-energy eigenfunction.
(c) At zero energy the only phase shift accumulated along a periodic orbit is at the turning points, twice $-\pi/2$, plus the Berry phase of π from the circulating spin in graphene, so the net phase shift is zero, hence there is a bound state at zero energy.