II. Entanglement

1 Entanglement measures

The state $\ket{\Psi}$ of two qubits $A$ and $B$ can be constructed out of the building blocks $\ket{0}_A, \ket{1}_A, \ket{0}_B, \ket{1}_B$ of the individual qubits,

$$\ket{\Psi} = c_{00}\ket{0}_A\ket{0}_B + c_{01}\ket{0}_A\ket{1}_B + c_{10}\ket{1}_A\ket{0}_B + c_{11}\ket{1}_A\ket{1}_B.$$  

a) Show that the condition of normalisation can be written as $\text{tr} c c^\dagger = 1$, with $c$ the $2 \times 2$ matrix with coefficients $c_{ij}$.

b) Why is it always possible to choose a basis such that

$$\ket{\Psi} = c'_{00}\ket{0}_A\ket{0}_B + c'_{11}\ket{1}_A\ket{1}_B,$$

with real positive $c'_{00}, c'_{11}$. (This is called the Schmidt decomposition.)

If I only do measurements on qubit $A$, then it suffices to know the reduced density matrix

$$\rho_A = \text{tr}_B \ket{\Psi}\bra{\Psi}.$$  

c) Derive that

$$(\rho_A)_{ij} = \sum_k c_{ik}c_{jk}^*, \text{ hence } \rho_A = cc^\dagger.$$  

The qubits $A$ and $B$ are called “entangled” if $\rho_A$ describes a mixed (not pure) state.

d) Show that $\det \rho_A \neq 0$ (det = determinant) is a necessary and sufficient condition for entanglement.

e) Why is it equivalent to determine the entanglement from $\rho_A$ or from $\rho_B = \text{tr}_A \ket{\Psi}\bra{\Psi}$?

A widely used measure of entanglement is the concurrence

$$\mathcal{C} = 2\sqrt{\det \rho_A}.$$  

f) Derive that $0 \leq \mathcal{C} \leq 1$.

2 Teleportation

The basis $\ket{00}, \ket{01}, \ket{10}, \ket{11}$ of two qubits is not entangled. Sometimes it is more convenient to use another basis,

$$\ket{\beta_{00}} = \frac{\ket{00} + \ket{11}}{\sqrt{2}}, \quad \ket{\beta_{01}} = \frac{\ket{01} + \ket{10}}{\sqrt{2}}, \quad \ket{\beta_{10}} = \frac{\ket{00} - \ket{11}}{\sqrt{2}}, \quad \ket{\beta_{11}} = \frac{\ket{01} - \ket{10}}{\sqrt{2}}.$$
which is entangled. This basis is called the Bell basis.

a) Show that each of the four Bell states is maximally entangled (has concurrence = 1).

You can produce these states starting from a non-entangled basis, by using two gates: The Hadamard gate and the \texttt{cnot} gate. The Hadamard gate rotates the first qubit, \( |x\rangle \rightarrow 2^{-1/2}(|0\rangle + (-1)^x|1\rangle) \). The \texttt{cnot} gate (controlled \texttt{not}) exchanges \( 0 \rightarrow 1 \) at the second qubit, but only if the first qubit (the control qubit) is 1.

b) Show that \( |i j\rangle \rightarrow |\beta_{i j}\rangle \).

Suppose that Alice and Bob have met to produce the state \( |\beta_{00}\rangle \) and then have separated, each taking 1 qubit. Alice wants to use this entangled pair to transmit to Bob the unknown state \( \alpha|0\rangle + \beta|1\rangle \) of a second qubit in her possession.

c) Why can't Alice just measure \( \alpha \) and \( \beta \) and send the result to Bob?

The state of the 3 qubits (2 with Alice, 1 with Bob) reads

\[ |\Psi\rangle = 2^{-1/2} \left[ \alpha(|00\rangle + |11\rangle)|0\rangle + \beta(|00\rangle + |11\rangle)|1\rangle \right]. \]

Alice sends both her qubits through a \texttt{cnot} gate (with the unknown qubit as the control) and then sends her unknown qubit through a Hadamard gate. She finally measures both her qubits.

d) Specify for each of the four outcomes 00, 01, 10, 11 of the measurement of Alice, what is the resulting state of Bob's qubit.

Alice communicates to Bob the result of her measurement.

e) Indicate how Bob can use that knowledge to bring his qubit in the state \( \alpha|0\rangle + \beta|1\rangle \) (without actually knowing that state!). This completes the "teleportation".

To communicate the result of her measurement to Bob, Alice must send a message. This takes time and ensures that teleportation does not exceed the speed of light. Suppose that Bob tries to extract some information from his qubit \textit{after} Alice's measurement but \textit{before} her message has arrived.

f) Calculate the reduced density matrix of Bob, \( \rho_B = \text{tr}_A \rho \), and show that \( \rho_B \) contains no information at all about the unknown qubit \( \alpha|0\rangle + \beta|1\rangle \).

Alice must destroy her qubit to transmit its state to Bob. This is an example of the general "no-cloning theorem".
3 Bell inequality

We have encountered the concurrence ℂ as a measure of the degree of entanglement of two qubits A, B in the pure state

$$|Ψ⟩ = \sum_{ij} c_{ij} |i⟩_A |j⟩_B, \quad ℂ = 2|\text{det} c|.$$  

If you dispose of a large number of pairs A, B in the same state, then you can measure the concurrence from the mean correlation of spin A and spin B. The Bell inequality is a way to distinguish classical from quantum mechanical correlations.

The recipe is as follows: Choose two unit vectors ˆa, ˆa′ along which to measure spin A, and two more ˆb, ˆb′ for spin B. Measure for each combination of vectors the spin correlator

$$C_{nm} = ⟨Ψ|( ˆn · ˆσ_A)( ˆm · ˆσ_B)|Ψ⟩,$$

where ˆσ_A acts on spin A and ˆσ_B on spin B. Combine the results to obtain the Bell parameter

$$B = C_{ab} + C_{a'b'} + C_{a'b} - C_{a'b′}.$$  

Maximize B by varying ˆa, ˆb, ˆa′, ˆb′. The maximum B_{max} gives the concurrence via

$$B_{max} = 2\sqrt{1 + ℂ^2}.$$  

Let us prove this.

a) Why can you restrict yourself, without loss of generality, to states of the form |Ψ⟩ = α|00⟩ + β|11⟩ (with real α and β)?

Denote the length of ˆa + ˆa′ as 2cosθ with 0 ≤ θ ≤ π/2. Define two new vectors ˆc, ˆc′ by

$$\hat{a} + \hat{a}' = 2\hat{c}\cos\theta, \quad \hat{a} - \hat{a}' = 2\hat{c}'\sin\theta.$$  

b) Explain why ˆc and ˆc′ have length 1 and why they are orthogonal.

Now you may immediately maximize

$$B = 2\cos\theta \langle Ψ| (\hat{c} · \hat{σ}_A)(\hat{b} · \hat{σ}_B)|Ψ⟩ + 2\sin\theta \langle Ψ| (\hat{c}' · \hat{σ}_A)(\hat{b}' · \hat{σ}_B)|Ψ⟩$$  

over θ.

c) Derive that

$$\max_θ B = 2\left[ \langle Ψ| (\hat{c} · \hat{σ}_A)(\hat{b} · \hat{σ}_B)|Ψ⟩^2 + \langle Ψ| (\hat{c}' · \hat{σ}_A)(\hat{b}' · \hat{σ}_B)|Ψ⟩^2 \right]^{1/2}.$$
In what follows we will take coplanar vectors, say all in the $x$–$z$ plane: $\hat{b} = (\sin \phi, 0, \cos \phi)$, $\hat{b}' = (\sin \phi', 0, \cos \phi')$, $\hat{c} = (\sin \gamma, 0, \cos \gamma)$, $\hat{c}' = (-\cos \gamma, 0, \sin \gamma)$. (We will skip the proof that the coplanar arrangement maximizes $\mathcal{B}$.)

d) Maximize $\mathcal{B}$ over $\phi, \phi', \gamma$ (in that order) to reach the desired result,

$$\mathcal{B}_{\text{max}} = 2 \sqrt{1 + 4(\alpha \beta)^2}.$$  

A non-entangled state has $\mathcal{B}_{\text{max}} = 2$. The Bell inequality

$$\mathcal{B} \leq 2$$  

holds for all classical correlations. John Bell constructed this inequality as a way to demonstrate experimentally that quantum mechanical correlations cannot be described in classical terms.

The argument goes as follows. In a classical description the Bell parameter can be found by measuring the spin polarisation $P_n$ along the axes $\hat{n} = \hat{a}, \hat{a}', \hat{b}, \hat{b}'$. The measurement of spin $A$ is assumed to leave spin $B$ undisturbed (locality), so we can consider each measurement separately. The spin polarisation $P_n$ is $+1$ if the spin is $\uparrow$ along $\hat{n}$ and $-1$ if it is $\downarrow$.

e) Show that

$$P_a P_b + P_a' P_b' + P_a P_b' - P_a' P_b = \pm 2.$$  

f) Explain why $\mathcal{B} \leq 2$.  