Quantum Theory
part b – I: path integrals

• Lagrangian & principle of least action  
• quantum propagator & Feynman path integral  
• stationary phase approximation

KP = Konishi & Paffuti (primary text)  
Ballantine (secondary text): § 4.8
Lagrangian & principle of least action

_classical mechanics_

Hamiltonian: \( H(q, p) = T + V, \quad \dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \)

Lagrangian: \( L(q, \dot{q}) = T - V, \quad p = \frac{\partial L}{\partial \dot{q}}, \quad \dot{p} = \frac{\partial L}{\partial q}. \) Euler-Lagrange equations

why bother? integral formulation of classical equations of motion:

Action: \( S[q(t)] = \int_{t_1}^{t_2} L(q, \dot{q}) \, dt \Rightarrow \delta S[q(t)] = 0 \)

principle of least action — Fermat, Maupertuis, Euler

\[
\delta S = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{\partial L}{\partial q} \delta q \right) \, dt = \int_{t_1}^{t_2} \left( -\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q} \right) \delta q \, dt = 0
\]

• free motion: minimize \( \int_{t_1}^{t_2} v^2(t) \, dt \) at fixed \( \int_{t_1}^{t_2} v(t) \, dt \rightarrow v = \text{const.} \)

• gravitational field: minimize \( \int_{t_1}^{t_2} (mv^2/2 - mgz) \, dt \)
  for \( z_2 = z_1 = 0, \quad x_2 = x_1 + L \rightarrow \text{parabola} \)
Thirty-one years ago, Dick Feynman told me about his “sum over histories” version of quantum mechanics. “The electron does anything it likes,” he said. “It just goes in any direction at any speed, . . . however it likes, and then you add up the amplitudes and it gives you the wavefunction.” I said to him, “You’re crazy.” But he wasn’t. 

**Freeman Dyson**

\[
\text{probability} = |\text{complex amplitude}|^2,
\text{Feynman: amplitude} = \text{sum over all paths of } e^{iS[q(t)]/\hbar}
\]

→ intuitive explanation of the double-slit experiment

in the \(\hbar \to 0\) limit only *classical* paths contribute

(stationary phase approximation → WKB approximation)
derivation of the path integral formula

propagator (Green function)

\[
G(q_1, q_0; T) = \langle q_1 | e^{-iHT/\hbar} | q_0 \rangle = \sqrt{\frac{m}{2\pi i \hbar T}} \int_{q(0)=q_0}^{q(T)=q_1} D[q] e^{iS[q]/\hbar}
\]

time slicing (\(\delta t = T/N\))

\[
G(q_1, q_0; T) = \prod_n \int dq_n \, G(q_{n+1}, q_n; \delta t),
\]

\[
G(q + \delta q, q; \delta t) = \langle q + \delta q | 1 - iH\delta t/\hbar + \mathcal{O}(\delta t)^2 | q \rangle = \int dp \langle q + \delta q | p \rangle \langle p | 1 - iH\delta t/\hbar | q \rangle
\]

\[
= \int \frac{dp}{2\pi \hbar} e^{ip\delta q/\hbar} \left( 1 - i\frac{\delta t}{\hbar} \left[ \frac{p^2}{2m} + V(q) \right] \right) \to \int \frac{dp}{2\pi \hbar} e^{ip\delta q/\hbar} e^{-\left( i\delta t/\hbar \right) \left[ p^2/2m + V(q) \right]}
\]

\[
= \text{constant} \times \exp \left[ i\frac{\delta t}{\hbar} \left( \frac{1}{2} m (\delta q/\delta t)^2 - V(q) \right) \right] = e^{iL(q, q)\delta t/\hbar}
\]

\[
\Rightarrow G(q_1, q_0; T) \sim \prod_n \int dq_n \exp \left[ iL(q_n, \dot{q}_n)\delta t/\hbar \right]
\]

(exponent OK, prefactor requires more work)
stationary phase approximation

see exercise 4