RMT of topological states of matter

- topological insulators/superconductors
- topologically nontrivial RMT ensembles
- conductance of Andreev quantum dot
- fermion parity switches

the three-fold way
(Wigner-Dyson)

time-reversal symmetry
(anti-unitary operator that
commutes with the Hamiltonian)

$$\mathrm{H}\mathcal{T} = \mathcal{T}\mathrm{H}$$

$$\mathcal{T}^2 = \pm 1$$

unitary

orthogonal $\mathcal{T}^2 = +1$

symplectic $\mathcal{T}^2 = -1$

from three-fold way to ten-fold way (1)

$H\mathcal{T} = \mathcal{T}H$ inversion symmetry $H\mathcal{C} = -\mathcal{C}H$

time & *charge*

	time	charge
unitary		
orthogonal	$\mathcal{T}^2 = +1$	
symplectic	$\mathcal{T}^2 = -1$	
superconducting (Altland-Zirnbauer)		$\mathcal{C}^2 = +1$
	$\mathcal{T}^2 = -1$	$\mathcal{C}^2 = +1$
		$\mathcal{C}^2 = -1$
	$\mathcal{T}^2 = +1$	$\mathcal{C}^2 = -1$

$H\mathcal{C}\mathcal{T} = -\mathcal{C}\mathcal{T}H$

+ 3 chiral ensembles

= 10 in total

from three-fold way to ten-fold way (2)

inversion symmetry

	time	*& charge*	
unitary		$S \in U(N)$	*complex unitary*
orthogonal	$S = S^t$	$S \in U(N)$	
symplectic	$S = -S^t$	$S \in U(2N)$	
superconducting		$S \in O(2N)$	*real orthogonal*
	$S = -S^t$	$S \in O(4N)$	
		$S \in Sp(4N)$	*quaternion symplectic*
	$S = S^t$	$S \in Sp(4N)$	

… and beyond …

	inversion symmetry *time*	*& charge*
unitary		$S \in U(N)$
orthogonal	$S = S^t$	$S \in U(N)$
symplectic	$S = -S^t$	$S \in U(2N)$
superconducting		$S \in O(2N)$
	$S = -S^t$	$S \in O(4N)$
		$S \in Sp(4N)$
	$S = S^t$	$S \in Sp(4N)$

topological quantum number Q

$\text{Det}\, S = \pm 1$

$\text{Pf}\, S = \pm 1$

… and beyond …

	inversion symmetry *time*	& *charge*
CUE		$S \in U(N)$
COE	$S = S^t$	$S \in U(N)$
CSE	$S = -S^t$	$S \in U(2N)$
CRE		$S \in O(2N)$
TCRE	$S = -S^t$	$S \in O(4N)$
CQE		$S \in Sp(4N)$
TCQE	$S = S^t$	$S \in Sp(4N)$

topological quantum number Q

$\mathrm{Det}\, S = \pm 1$

$\mathrm{Pf}\, S = \pm 1$

enter topological superconductors

- CRE (aka class D): chiral p-wave superconductor (strontium ruthenate?)
- TCRE (aka class DIII): topological insulator (HgTe quantum well) with s-wave proximity effect

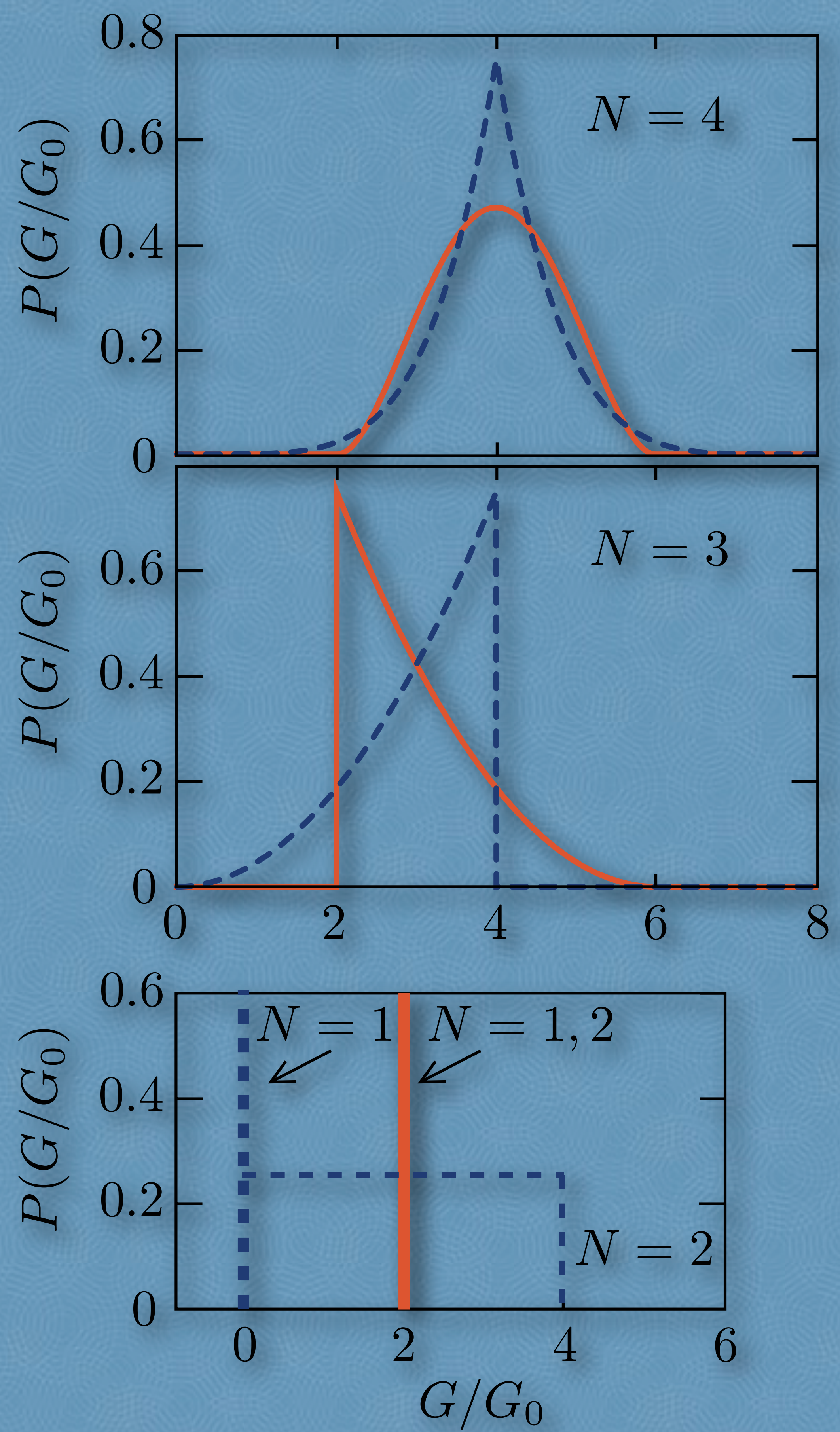
see reviews in Rev.Mod.Phys. by Hasan & Kane and by Qi & Zhang

needed: RMT which knows about topological quantum numbers

conductance distribution in the CRE

Q=−1

Q=+1



what systematics do you notice?

cumulants of conductance

$$G/G_0 = \tfrac{1}{2}\mathrm{Tr}\,(1 + JSJS^\dagger)$$

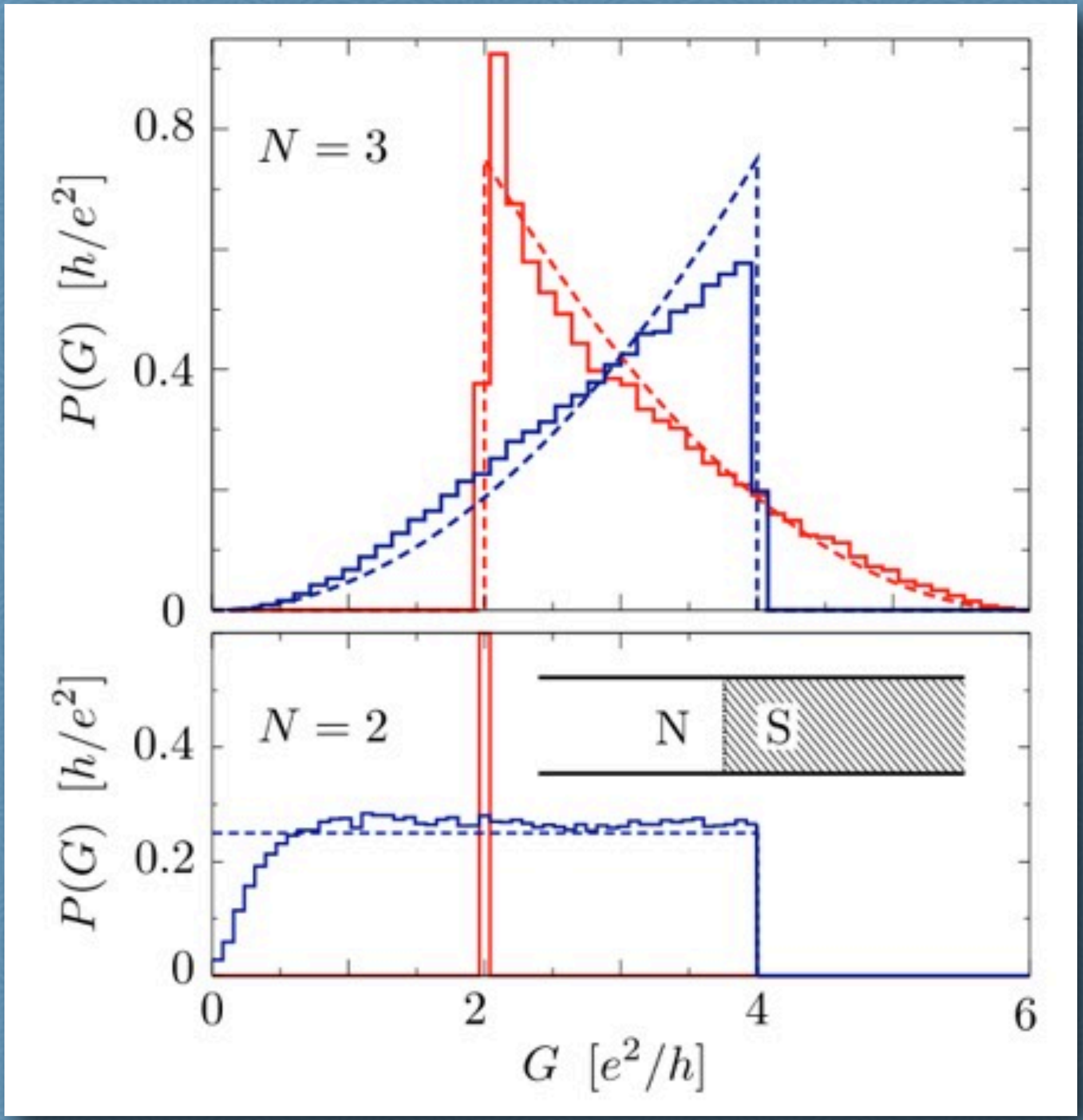
$$G_0 = e^2/h \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\langle (G/G_0)^p \rangle_Q = \int_{O(2N)} d\mu(S)\, \left[\tfrac{1}{2}\mathrm{Tr}(1 + JSJS^\dagger)\right]^p \tfrac{1}{2}(1 + Q\,\mathrm{Det}\,S)$$

lemma: $\int_{O(2N)} d\mu(S)\, \left(\mathrm{Tr}\,JSJS^\dagger\right)^p \mathrm{Det}\,S = 0$ if $p < N$

hence the *p*-th moment or cumulant of the conductance is independent of the topological quantum number if *p*<*N*

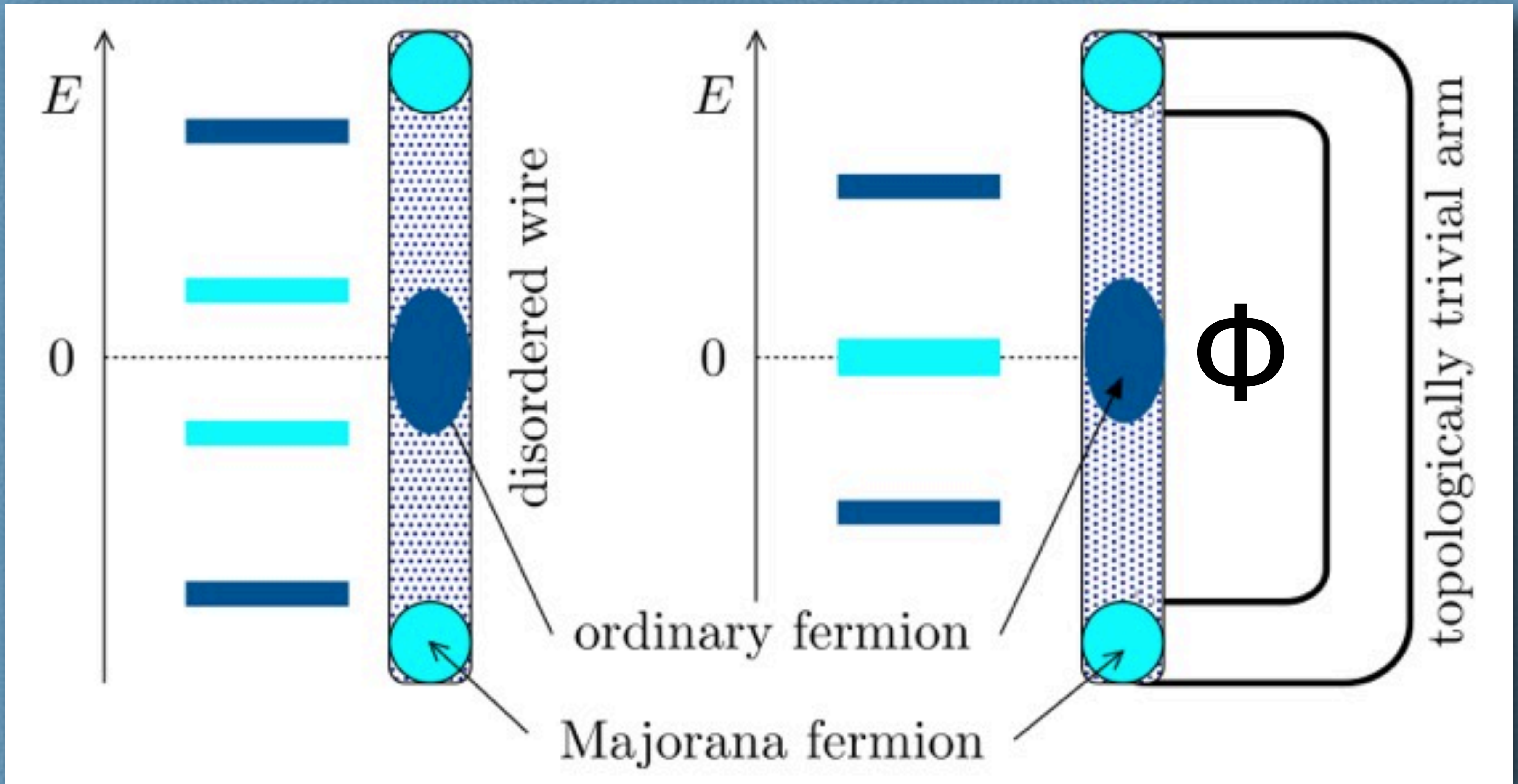
no Q-dependence of weak localization or UCF *comparison of RMT with a microscopic model calculation*



$$H_R = \frac{\mathbf{p}^2}{2m_{\mathrm{eff}}} + U(\mathbf{r}) + \frac{\alpha_{\mathrm{so}}}{\hbar}(\sigma_x p_y - \sigma_y p_x) + \frac{1}{2} g_{\mathrm{eff}} \mu_B B \sigma_x$$

Rashba-Zeeman Hamiltonian + *s*-wave proximity effect

what is counted by Q?

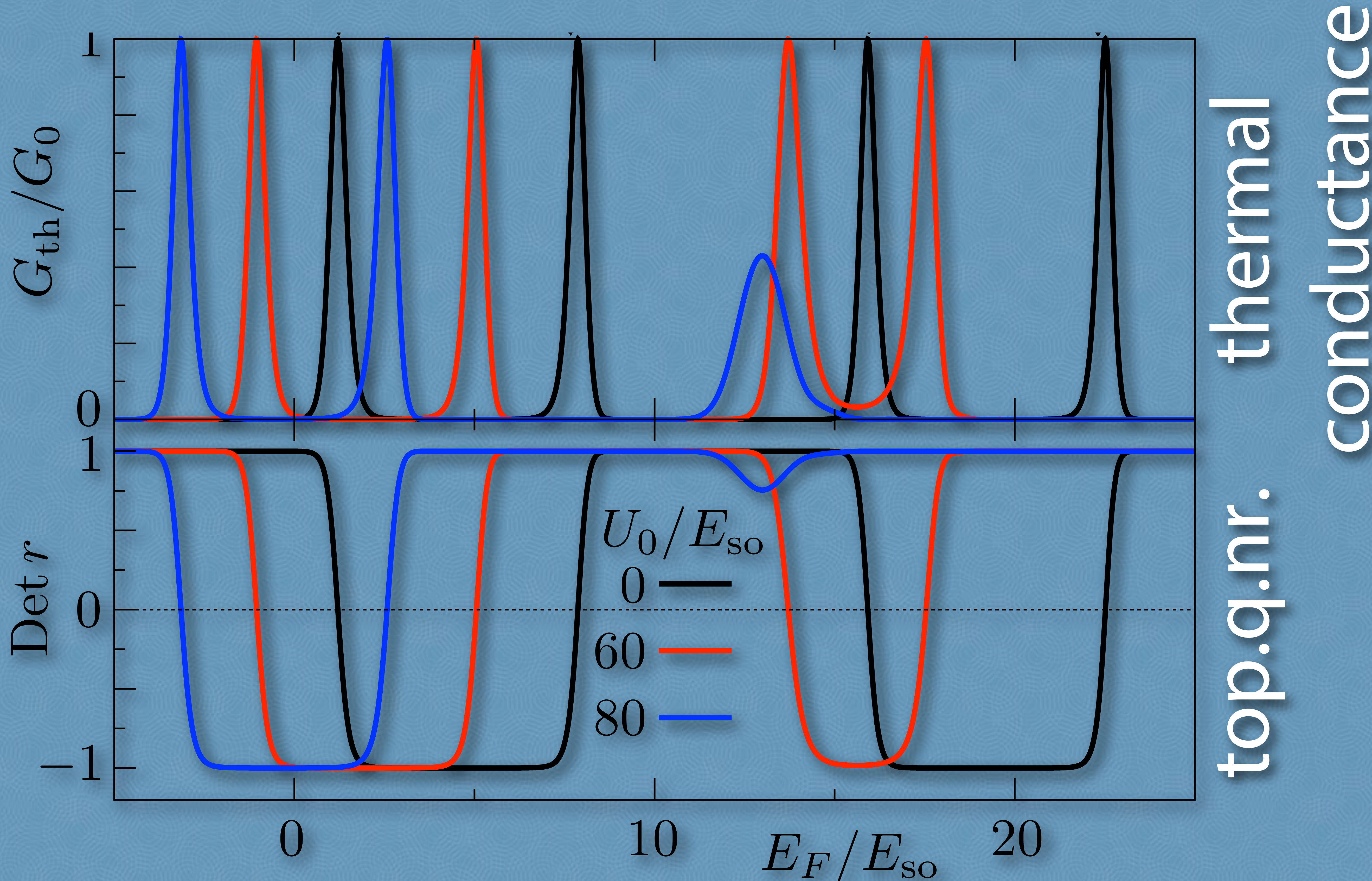


$$\gamma(E) = \gamma^\dagger(-E) \Rightarrow \gamma = \gamma^\dagger \text{ for } E = 0$$

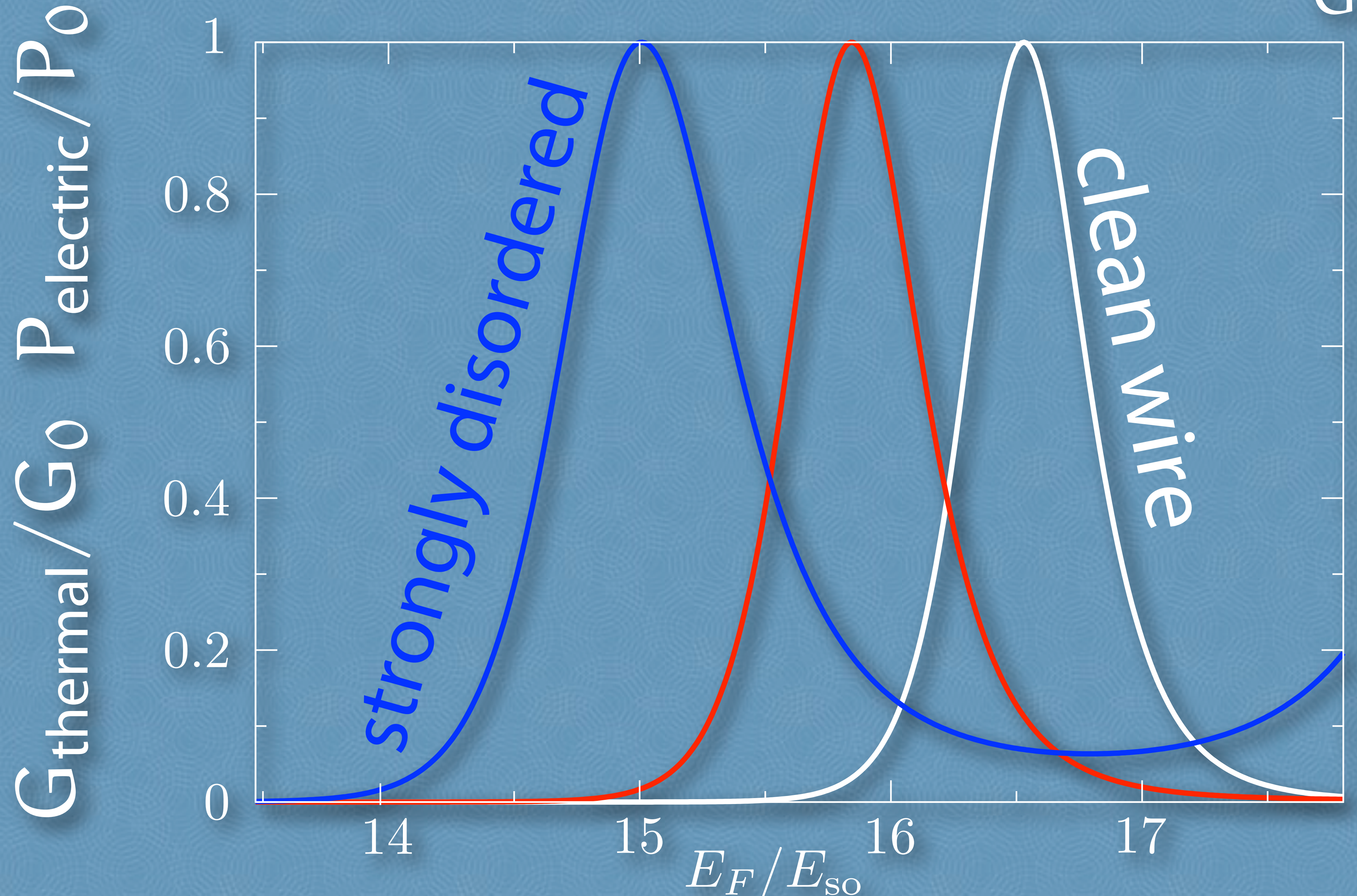
m pairs of Majorana fermions

$$Q = (-1)^m$$

topological phase transitions



quantized conductance & shot noise at the topological phase transition



$G_0 = \pi^2 k_B^2 T_0/6h$

$P_0 = e^3 V/2h$

$E_{\text{so}} \simeq 0.1\,\text{meV}$

$U_{\text{disorder}}/E_{\text{so}} =$

0 60 80

$\Delta = 1\,\text{meV}$

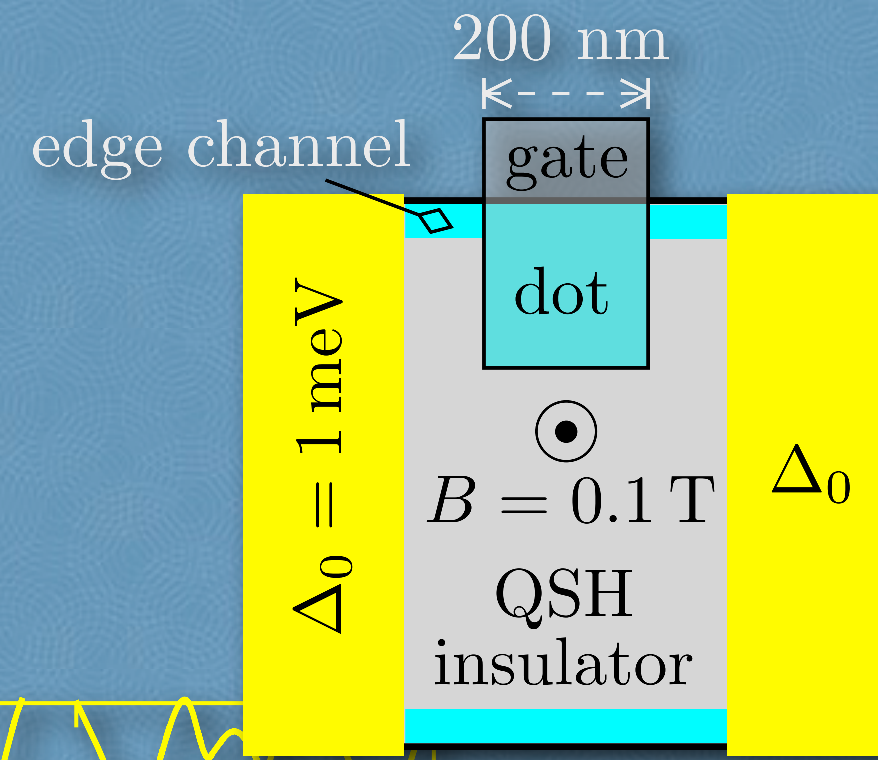
$B = 1\,\text{T}$

no disorder corrections to quantization

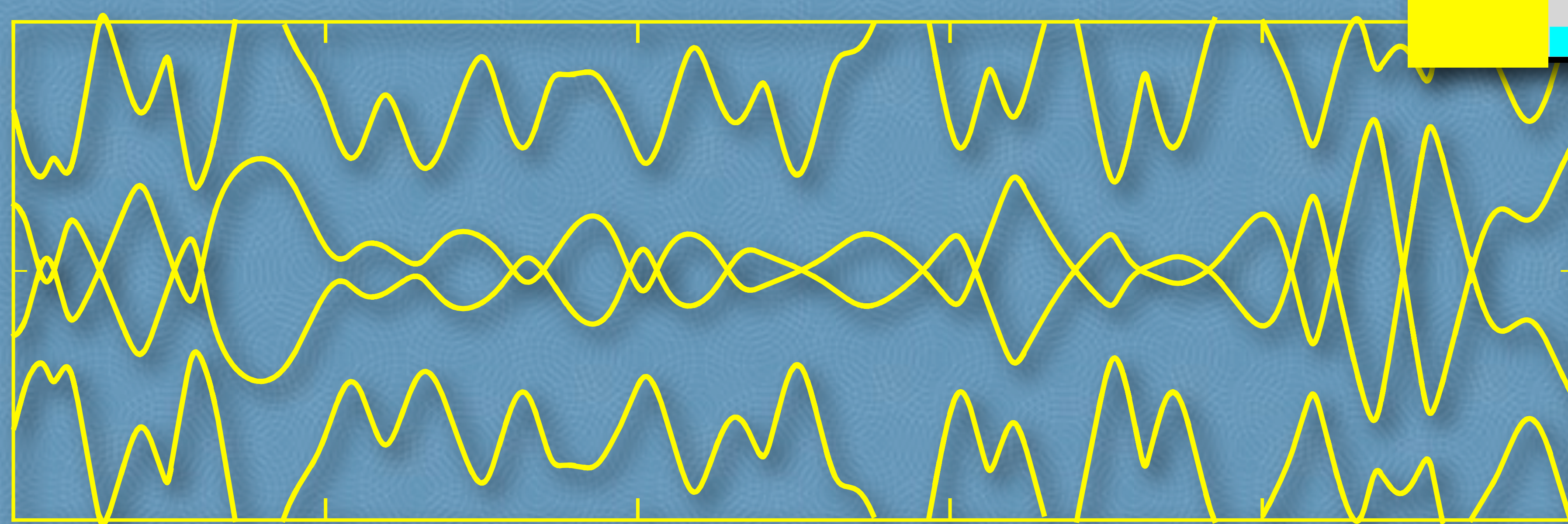
Wigner-Poisson statistics of fermion parity transitions

Majorana meets a mermaid

level statistics of a Josephson junction



energy

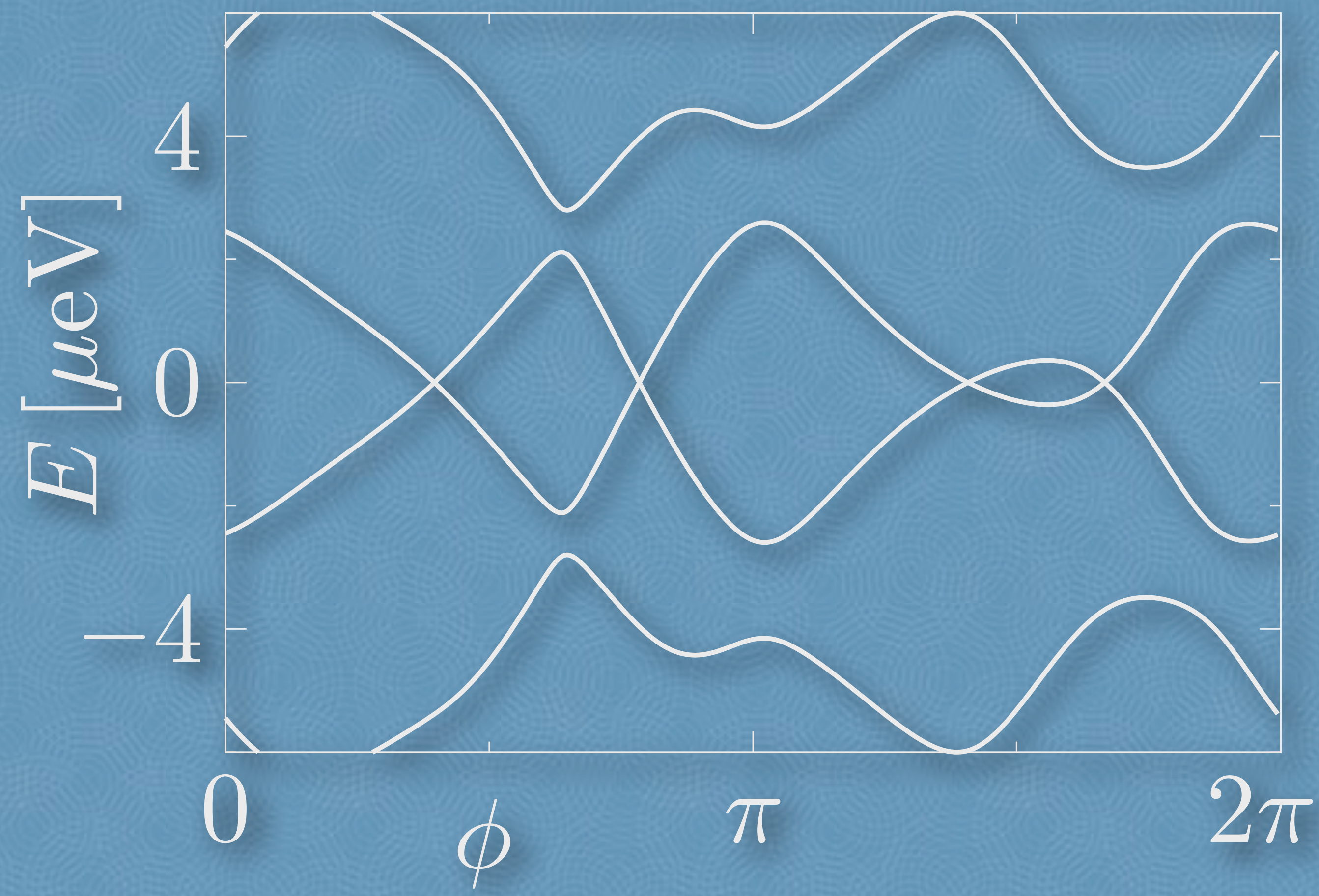


Fermi level

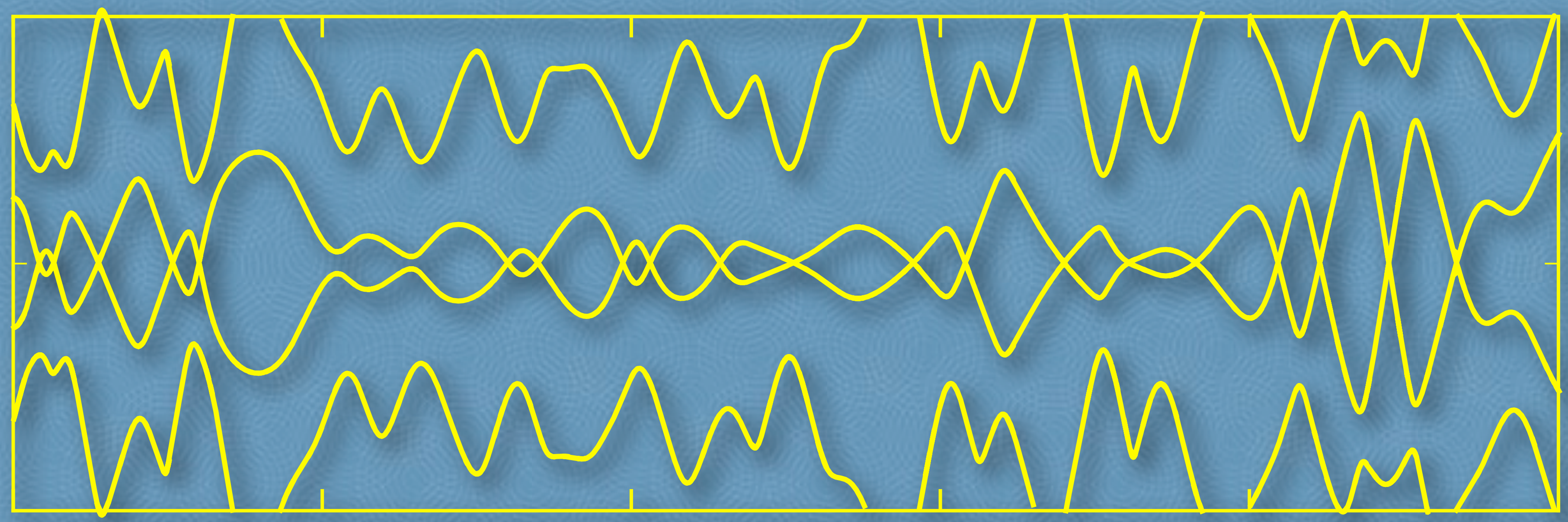
external parameter (V_{gate})

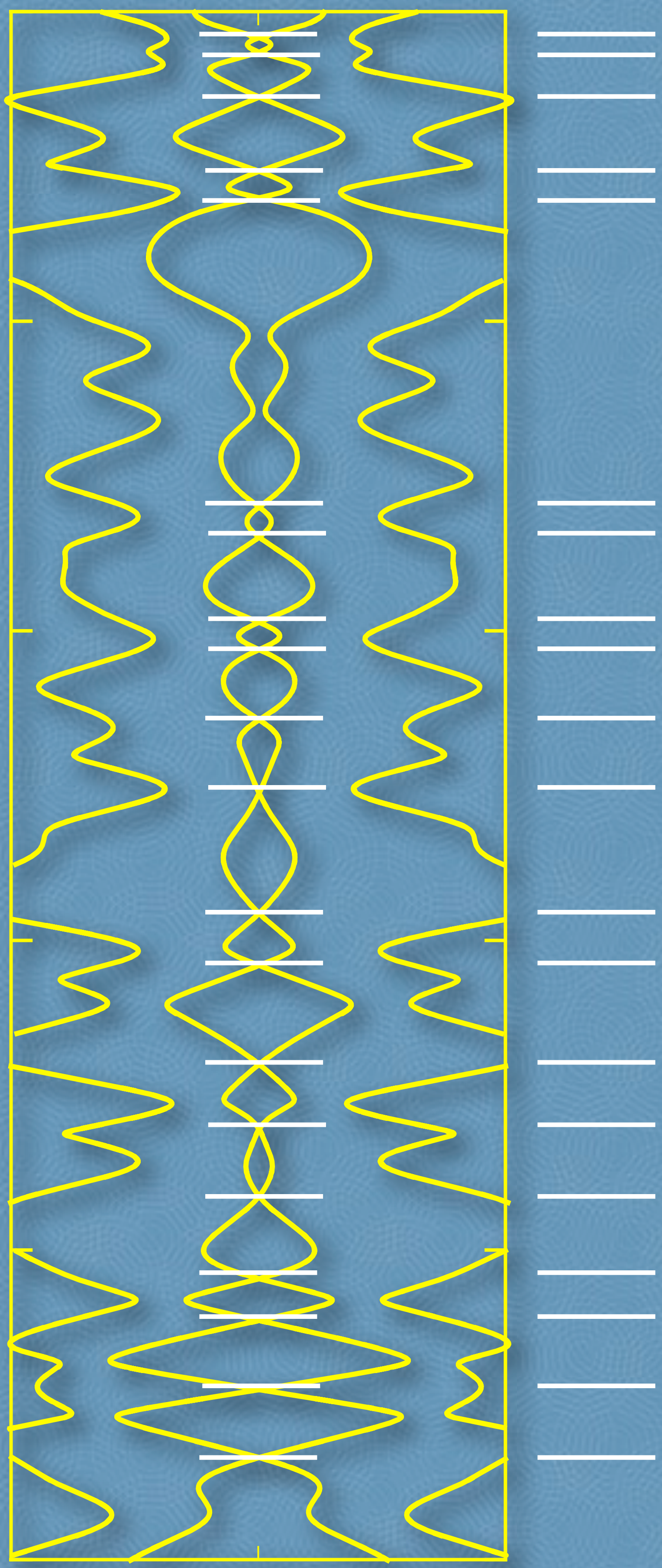
absence of level repulsion at E_F – signature of Majorana zero-modes

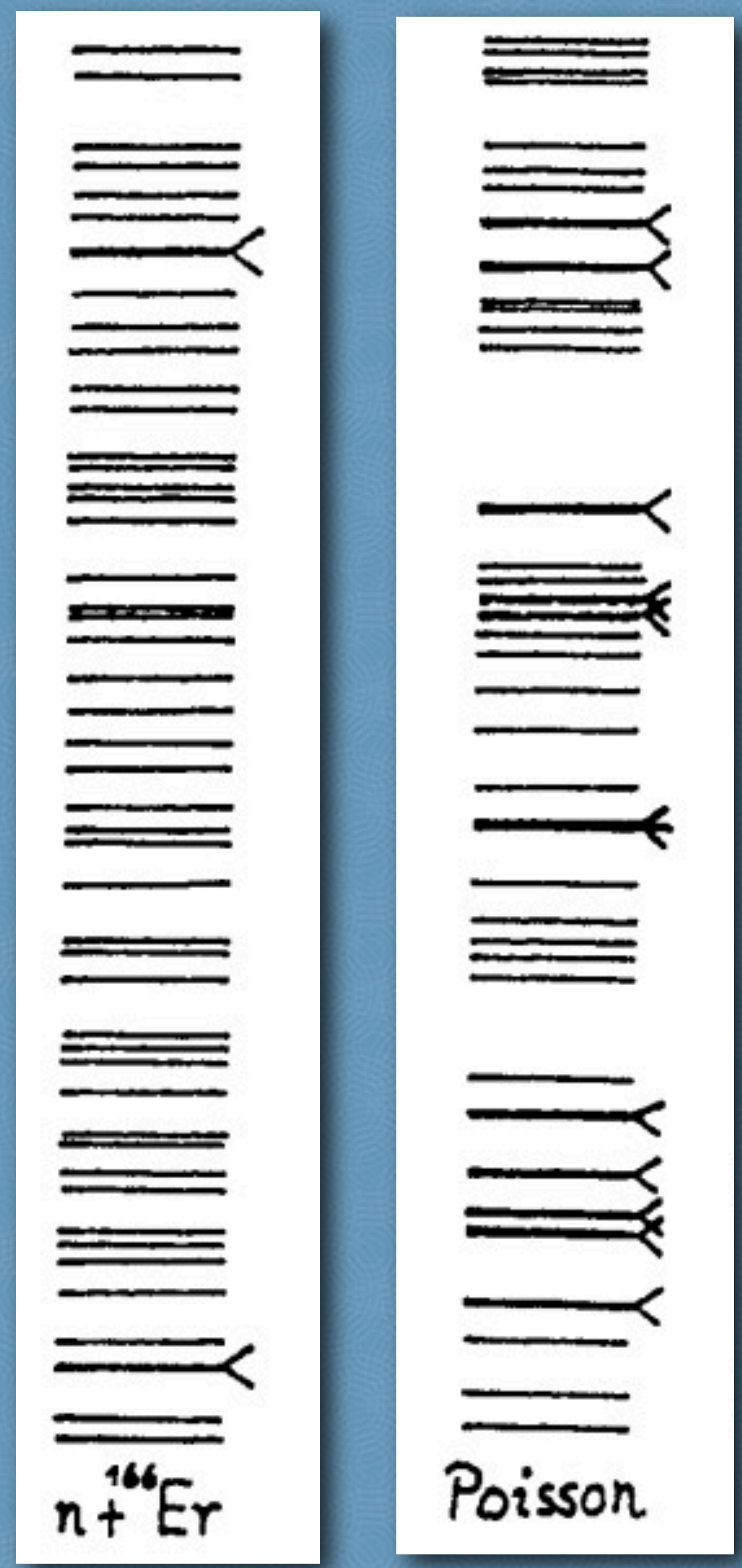
level crossings are *fermion-parity transitions* (Kitaev)



- ground state fermion parity is a Z_2 topological quantum number
- switches *even* | *odd* number of times for a topologically *trivial* | *nontrivial* superconductor
  (→4π-periodic Josephson effect)
- level crossing is a topological transition
  (zero-dimensional class D)





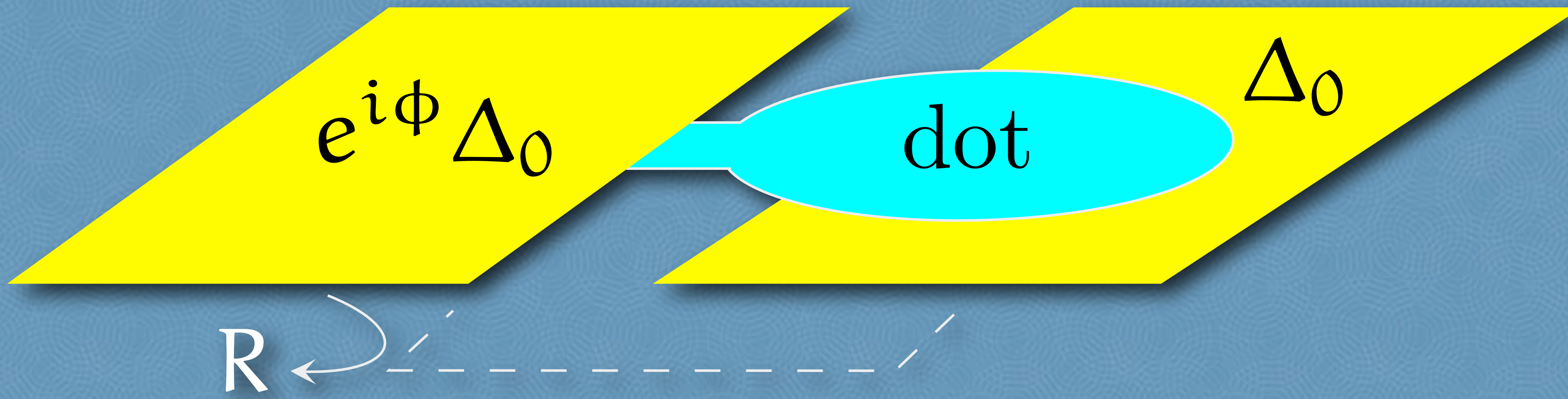


GOE (Wigner)

repulsion of level *crossings*

problem: find a random matrix that has the level crossings as eigenvalues

level crossings as an eigenvalue problem



$$\mathrm{Det}\left[1 - R\begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}\right] = 0$$

Cayley transform $\quad R = \dfrac{1-A}{1+A} \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$M = AJ \qquad \varepsilon = \tan(\phi/2)$$

$$\mathrm{Det}\,(M - \varepsilon) = 0$$

skew-Hamiltonian ensemble

$$M^* = M, \quad M^{\mathsf{T}} = -JMJ$$

→ all eigenvalues twofold degenerate and symmetric around the real axis

level crossing = real eigenvalue

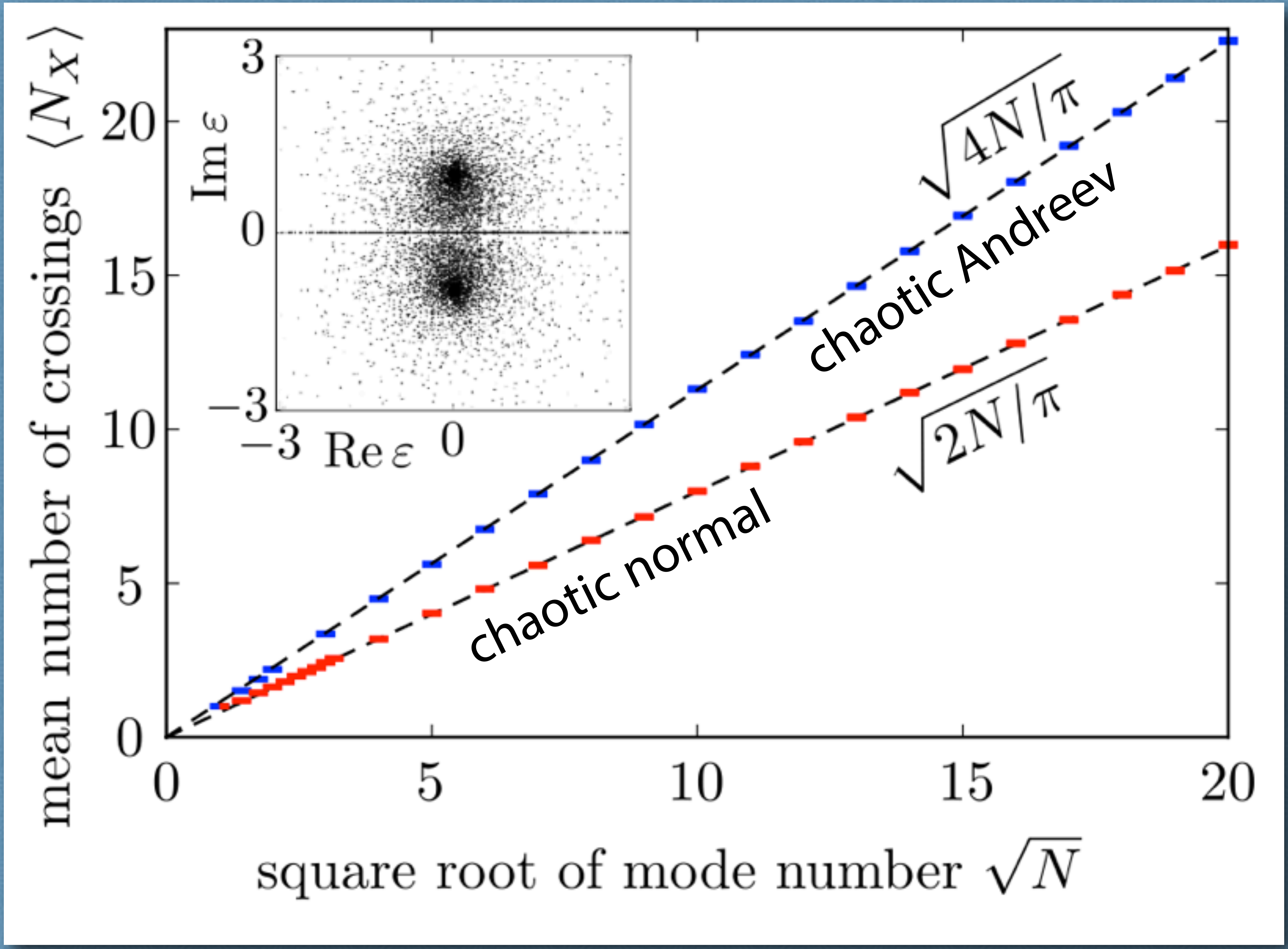
HOW MANY EIGENVALUES OF A RANDOM MATRIX ARE REAL?

ALAN EDELMAN, ERIC KOSTLAN, AND MICHAEL SHUB

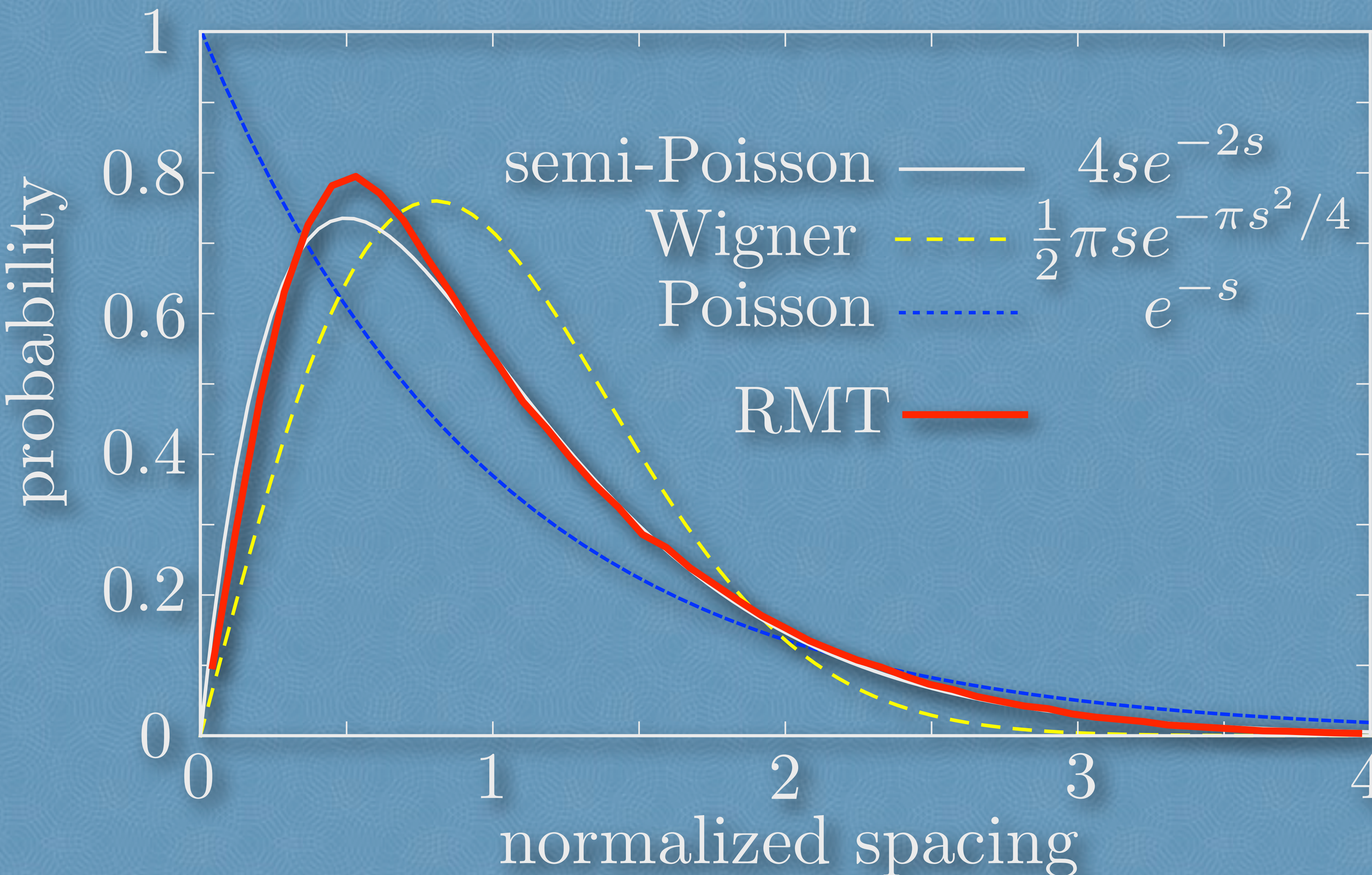
a classic problem in RMT

Ginibre, Forrester, Kanzieper, Sommers, Akemann….

square-root law

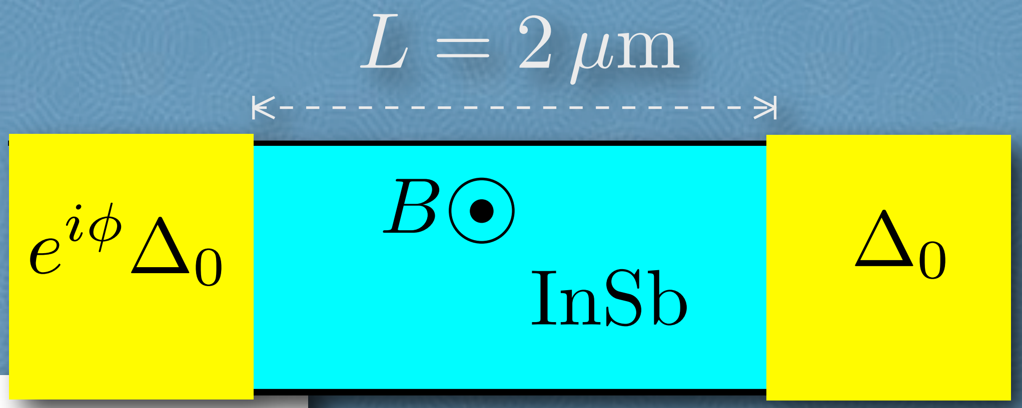


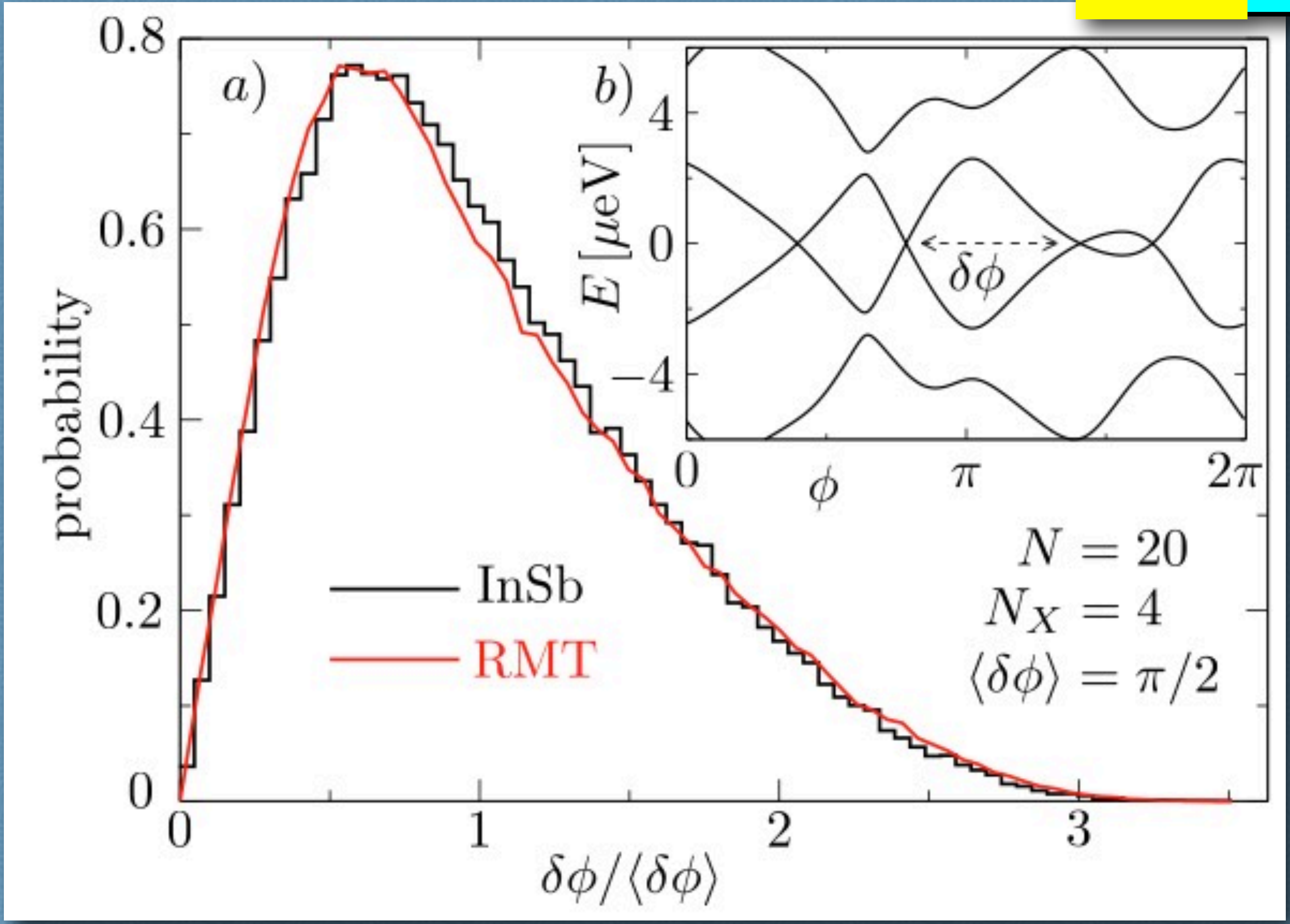
hybrid Wigner-Poisson statistics



same as level statistics at the metal-insulator transition (Shklovskii's *mermaid*)

disordered InSb channel





no adjustable parameters!

Conclusion

- topologically nontrivial RMT ensemble (CRE)
- Q-dependence not accessible by large-N perturbation theory
- Q counts Majorana fermions
- cumulant theorem for conductance
- topological phase transition with quantized conductance

experimental realization in 2012?!