

**NOT BY DISCOVERIES ALONE:  
THE CENTENNIAL OF PAUL EHRENFEST**

Martin J. KLEIN

*Yale University, History of Science, Box 2036, Yale Station, New Haven, C.T. 06520, USA*

*This paper is for George Uhlenbeck*

“It is not by discoveries only, and the registration of them by learned societies, that science is advanced. The true seat of science is not in the volume of Transactions, but in the living mind, and the advancement of science consists in the direction of men’s minds into a scientific channel; whether this is done by the announcement of a discovery, the assertion of a paradox, the invention of a scientific phrase, or the exposition of a system of doctrine.”<sup>1)</sup> The words are James Clerk Maxwell’s, and they are particularly appropriate in talking about Paul Ehrenfest, who was born a century ago. Ehrenfest did advance science in all the ways that Maxwell mentions. The adiabatic principle – that pillar of the old quantum theory – was his discovery, a discovery “found with so much joy”<sup>2)</sup>, as he once wrote. His writings propound and analyze many a most ingenious paradox<sup>3)</sup>. It is hard to imagine a scientific phrase more telling than the one Ehrenfest coined to characterize the devastating implications of the distribution law that classical physics requires for blackbody radiation – “the ultraviolet catastrophe”<sup>4)</sup>. There is no exposition of a system of doctrine more incisive, more clarifying than the monograph on the foundations of statistical mechanics written by Paul Ehrenfest and his wife Tatyana for the *Encyklopädie der mathematischen Wissenschaften*<sup>5)</sup>. But what makes Maxwell’s words especially relevant to Ehrenfest is their emphasis on science as living in the minds of men.

At the Seventh Solvay Conference, held only a few weeks after Ehrenfest’s death in 1933, Paul Langevin described him as having been “at the very heart of the drama of contemporary physics”, and even as having personified that drama in his own life<sup>6)</sup>. For Ehrenfest was passionately committed to his science. His way of being alive involved thinking about physics, talking and arguing about physics, working to his utmost to understand physics, and teaching it to anyone who showed an interest in it – students, colleagues, laymen, casual acquaintances, children. Others have been as intensely committed to science, but Ehrenfest was unique in his need to have close human contacts as an essential part of doing physics, in the breadth of human experience and the range of emotions that went into his scientific activity.

Ehrenfest’s way of being devoted to physics contrasts sharply with that of

many theorists. He was no Willard Gibbs, who left New Haven as rarely as possible, who never saw most of his contemporaries in Europe, and whose scientific correspondence consists of answers to letters from others who approached him. Ehrenfest did not share Gibbs's "retiring disposition"<sup>7</sup>). Gibbs was genial and friendly, a sympathetic teacher and helpful adviser, but he did not need or want other people involved in his work. Nor did H.A. Lorentz as a young man. Lorentz, who later travelled widely and became one of the most familiar figures on the international scientific scene, never even tried to meet those colleagues who lived within a day's journey and whose works were constantly in his thoughts, men like Maxwell, Helmholtz, and Hertz<sup>8</sup>). Ehrenfest, who was so close to Albert Einstein, certainly did not share his friend's opinion that a job as lighthouse keeper might be the ideal way for a theoretical physicist to support himself.

Ehrenfest needed other people in his life and in his work. His greatest strength as a physicist was his unmatched critical ability. "His stature lay in his unusually well developed faculty to grasp the essence of a theoretical notion," as Einstein wrote, "to strip a theory of its mathematical accouterments until the simple basic idea emerged with clarity. This capacity made him a peerless teacher. It was on its account that he was invited to scientific congresses; for he always brought clarity and acuteness into any discussion. He fought against fuzziness and circumlocution, when necessary employing his sharp wit and even downright discourtesy"<sup>9</sup>). Ehrenfest used his critical ability to bring out the best in the work of others, to produce "a quickening and sustaining atmosphere"<sup>10</sup>) for doing science, and especially to encourage, excite, and support his students. He was, as Einstein wrote, "passionately preoccupied with the development and destiny of men." This passionate preoccupation cannot be separated from his sense of his own limitations, his ultimately tragic inability to appreciate all that he could do and did in science, all that he actually gave to others.

Paul Ehrenfest was born in Vienna on 18 January 1880, youngest of the five sons of a Jewish grocer in a working class quarter of the city<sup>11</sup>). His early fascination with mechanical gadgets and mathematical problems was encouraged by his brothers, particularly by Arthur, the oldest brother, a student of engineering, who delighted in giving lively and graphic answers to Paul's many questions. Although Ehrenfest grew up in Vienna, he did not love the city. The anti-Semitism he experienced was an intrinsic part of his Vienna; he could never feel at home there. Ehrenfest studied theoretical physics at the University of Vienna under Ludwig Boltzmann, whose lectures and writings had a lasting influence on him. Although Boltzmann appreciated his student's abilities and knowledge – "If only I knew my own works that

well," he is said to have remarked, after Ehrenfest had referred to a point in one of Boltzmann's papers – no close relationship developed between them. Ehrenfest had come to Boltzmann too late, and knew him only during his last, troubled years.

After receiving his doctorate in 1904 Ehrenfest spent some time at Göttingen, which he had visited during his student days and where he had met Tatyana Alexeyevna Afanassjewa, a Russian student of mathematics who became his wife and sometime scientific collaborator. A talk Ehrenfest gave to Felix Klein's Mathematical Colloquium late in 1906 made a significant difference in Ehrenfest's career. Ehrenfest discussed the urn model (or dog-flea model) that he and his wife had developed for simply elucidating the consistency of Boltzmann's statistical mechanical explanation of the second law, despite the objections to it based on both the reversibility of mechanics and the Poincaré recurrence theorem. Klein was impressed enough by Ehrenfest's lecture to invite him to write the article on the foundations of statistical mechanics for the *Encyklopädie der mathematischen Wissenschaften*, an article originally assigned to Boltzmann but not yet written at the time of his death.

The Ehrenfests planned that review of the basic problems of statistical mechanics together, and Paul wrote it during the next few years when they lived in Russia, at St. Petersburg. The article is a masterly analysis of just what Boltzmann achieved in his long series of memoirs, something not easily unearthed from the papers themselves, but it is also a clear statement of what Boltzmann did not achieve. Ehrenfest made crystal clear what had been asserted, what had been proved, and what was still to be proved. For all its subtlety the Ehrenfests' article contains a minimum of mathematical machinery; for example, although it plays an important part in the discussion, the Boltzmann integro-differential equation is never written down explicitly in the article.

The five years Ehrenfest spent in Russia were probably his happiest. Had he been able to obtain an academic position there he would have stayed, but there was no chance of that. He felt more at home in Russia than he ever did anywhere else. Ehrenfest especially loved the wide ranging, open discussions he could have with his Russian friends: "Conversations over tea in a tightly packed room until late into the night, and then out with one or two friends through the endless winter streets of Petersburg in the piercing cold, bundled up to the eyes."

His search for a job at a university in western Europe was long and frustrating, but it ended with a success that exceeded his wildest hopes. In 1912 Paul Ehrenfest was appointed to the chair of theoretical physics at Leyden, as the successor of H.A. Lorentz. Leyden would be his home for the

rest of his life, though he travelled often. Ehrenfest became a familiar figure at the physics colloquia of Berlin, Göttingen, and Copenhagen, and his trips took him as far afield as Kharkov and Pasadena.

Lorentz chose Ehrenfest to follow him at Leyden on the strength of the *Encyklopädie* article on statistical mechanics, as well as Ehrenfest's penetrating studies of the very early quantum theory and his many other contributions to the continuing discussion that constitutes much of the life of science. Lorentz would have preferred Einstein, but was happy with his second choice. Ehrenfest's arrival at Leyden was "like a bolt of lightning" to one young observer, as he threw himself into the creation of a genuine scientific community. The colloquium he immediately organized, the reading room that served as an intellectual center for physics and mathematics students, the newly revived Christiaan Huygens Society for student discussions of physics, all these along with Ehrenfest's lively lectures and his strong presence and constant personal concern brought fresh life to the quiet Dutch university. Lorentz was pleased, if perhaps a bit amused, by the whirlwind of activity Ehrenfest generated, but he very much appreciated that Ehrenfest "had gotten the students talking", something that Lorentz himself had never been able to do.

Ehrenfest looked up to Lorentz with a veneration that went far beyond even what one might expect on the basis of Lorentz's age, his position as the acknowledged leader among the world's theoretical physicists, and his status as Ehrenfest's benefactor. At the close of his Leyden inaugural speech Ehrenfest addressed his predecessor directly, following the tradition for such occasions. "When one of us younger men comes in contact with you he must feel one thing above all else: you read our souls like an open book, smiling quietly. You see not only our scientific plans and thoughts spread out before you, with all their merits and defects which we ourselves will only recognize much later; you also see clearly and penetratingly into our purely human feelings, desires, and abilities." Ehrenfest would turn to Lorentz with his problems and his joys as one turns to one's father, and he never lost those feelings of love and respect that he expressed in his Leyden inaugural. Ehrenfest had not been able to come close to Boltzmann, but his need for such a relationship, for such a second father in physics, or rather through physics, was fulfilled with Lorentz.

Ehrenfest also needed and formed close personal ties with many of his physicist contemporaries. Each of these friendships and collaborations had its own special characteristics, but perhaps a sketch of some aspects of his friendship with Niels Bohr will suggest the richness of these varied relationships. Bohr, five years younger than Ehrenfest, made an immediate impact on

physics in 1913 with the radically new and remarkably successful ideas in his series of three papers, "On the Constitution of Atoms and Molecules"<sup>12</sup>). His

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feeling of real urgency about it." Ehrenfest responded to Bohr's wish to visit Leyden when the war ended, and travel became possible once again, by writing: "I have continually heard you spoken of so much from a number of independent sides that I am impatiently awaiting the day when you will enter our home as our guest. I hope that Einstein can also come to us then!"<sup>18)</sup>

That eagerly anticipated meeting took place early in 1919 when Bohr came to Leyden to lecture. The visit was a great success and the two men became instant friends, much as Ehrenfest and Einstein had done when they met in Prague in 1912. For Bohr it was "a most wonderful time of the greatest intellectual enrichment." He went on to say: "I am sitting and thinking of all that you have told me about so many different things, and whatever I think of I feel that I have learned so much from you which will be of great importance for me; but, at the same time, I miss so much to express my feeling of happiness over your friendship and of thankfulness for the confidence and sympathy you have shown me. I find myself so utterly incapable of finding words for it."<sup>19)</sup>

Ehrenfest's response to Bohr can best be seen this way. In October 1919 Einstein spent two weeks in Leyden. He brought a unique gift to his friends—the great news, not yet publicly announced, that Eddington's measurements of the bending of starlight made during the recent solar eclipse provided brilliant confirmation of the general theory of relativity. Ehrenfest also had a gift for his friend, namely, the good news of Niels Bohr, whom Einstein had not yet met and whose work he had not studied very much. "Ehrenfest told me about many of the details of what comes out of Niels Bohr's intellectual kitchen," Einstein wrote soon after to Planck, "he must be a mind of the very first rank, extremely critical and farseeing, who never loses sight of the totality of things."<sup>20)</sup> To Ehrenfest himself Einstein wrote that he was now going to "plunge deeply" into Bohr's papers: "You have shown me that there is a man of profound vision behind them, one in whom great connections come alive."<sup>21)</sup>

Later on Ehrenfest felt free to advise Bohr on many matters, treating him at times as an older brother might treat his more innocent, less practical junior. When Bohr was considering a proposal that a collection of his pre-1918 papers be published in German translation, Ehrenfest wrote:

"Please, please translate each paper word for word including the lines which are 'incorrect'. (Add supplementary footnotes at those places *the way Rayleigh did* when editing his papers.) It is already so difficult to follow your train of thought, *even for an Einstein*, because you condense everything so."<sup>22)</sup>

When Bohr was terribly busy trying to prepare his report for the 1921 Solvay Conference and at the same time overseeing every aspect of his new Institute of Theoretical Physics at Copenhagen, Ehrenfest wrote, at Lorentz's

suggestion, to offer his help, “*in case you wish it.*” Ehrenfest reported that he had told Lorentz “that from everything I know about your way of writing I consider it completely impossible, n.b. especially that I myself find writing so terribly hard.” He could offer no more than “a quite short manuscript, composed of a few theses on your ideas (distilled out of your two big papers and your Berlin lecture.)” But Ehrenfest recognized that nothing written by anyone else could ever seem to Bohr to be an adequate expression of his ideas, since Bohr expressed himself in such a particular way. “I would adapt myself to your genetic mode of thinking with pleasure, but ultimately I would still have to assert, or at least suggest, a thesis; inevitably a few dogmatic, solidified bones would be stuck in the genetic mollusc.”<sup>23)</sup>

Bohr was, as it happens, unable to attend that Solvay Conference, but the problems of writing a report for the published proceedings continued to trouble him anyway. Ehrenfest intervened to try to save Bohr from agonizing too much over his task. After discussions with Lorentz, Ehrenfest wrote the following explicit advice.

- “1. Have no fear of writing at too great length, because everybody’s hunger for your ideas is so great that the more there is by you in it, the more the book will be demanded – that is a dry fact which *I* can judge better than *you*.
2. Leave what you have written unabridged and as it is.
3. Break off the report at any point you wish and say: explanations about such and such should come here, but I shall publish these in such and such a place.

For you see how the situation is:

1. The Solvay book *must* finally appear – that it contain a bit of Bohr is necessary, but that it contain all of Bohr is superfluous.
2. It is *much* more important for the development of physics that you are not only completely healthy but also *happy and free of cares* than that any of your publications remains a fragment or a little awry. . . .
3. Lorentz is fond of you, as is every acute person who meets you. And so the worst that could perhaps happen is that he would react to you as a father does when his little boy, who is *very* dear to him, tears his socks while climbing. Then (suppressing a laugh!!!) he makes an ‘an angry face’. And you are still very far even from that with Lorentz.

So make a *sketchy* conclusion and period, and send off the manuscript.”<sup>24)</sup>

Ehrenfest first visited Bohr in Copenhagen in December 1921. He found it “a marvelously beautiful and rich experience” for many reasons, but especially for “the possibility of being able to talk repeatedly with Bohr about his works.” This was not so easy, however, partly because Bohr was so busy

reading proofs in his own “completely grotesque way,” which involved making “fundamental changes again and again, even in the galley proofs.” There was also the difficulty of getting Bohr to answer a particular question. “He reacts to that like a very rapidly rotating top – completely transversely.” Ehrenfest was consoled when Harald Bohr, Niels’s younger brother and a brilliant mathematician, said to him: “When Niels tells me something, I absolutely don’t understand what he is talking about and what he is driving at for 59 minutes; but in the 60th minute a light suddenly dawns and I see that everything he had said previously was absolutely necessary.” Ehrenfest followed the same procedure: “Ask and then listen patiently and attentively to what follows, and just don’t come back to the question. In this way I learned an enormous amount, but I had hardly a tenth of my questions ‘answered’.”

What appealed so much to Ehrenfest was Bohr’s “conviction that we have just touched the beginning of an essentially new physics.” Bohr somehow knew just which features of the current atomic model must be taken “inexorably seriously” and which could be ignored or taken very lightly. But most of all: “What I find so liberating in Bohr is that one can *think* again instead of just calculating.”<sup>25)</sup>

Ehrenfest acquired a great respect for Bohr’s style of working and writing. Bohr’s manuscripts, and even his individual sentences, tended to grow longer with each revision as Bohr struggled to incorporate every shade of meaning, every subtle distinction, into his formulations. In a conversation with Robert Oppenheimer, Ehrenfest once compared his two great friends, so different from each other, to two correspondingly great painters. “Einstein is like Holbein, in which everything within the frame is luminously clear and harmonious. With Bohr, as with Rembrandt, there is an intense patch of light, whose intensity is enhanced by the surrounding darkness.”<sup>26)</sup> When Ehrenfest himself wrote his short sharp sentences, he sketched like a great caricaturist, capturing the essential, the defining features of his subject in a few seemingly effortless strokes.

James Joyce argued that Ulysses was the complete hero because we see him in a full range of human relationships: “Ulysses is son to Laertes, but he is father to Telemachus, husband to Penelope, lover of Calypso, companion in arms of the Greek warriors around Troy and King of Ithaca.”<sup>27)</sup> Ehrenfest’s relationships with physicists had some of this Odyssean sweep, but no account of them, however brief, can omit mention of his students. Here is an excerpt from the Foreword to the doctoral dissertation one of them wrote: “It is difficult to express how intense and how many-sided was the influence on me of Professor Ehrenfest’s daily guidance. His stimulating lectures, the



really lively discussions at the colloquium, and in addition . . . the international atmosphere had a powerful effect on me, and taught me above all how necessary it is always to seek out what is essential in problems. It would be hard to exaggerate the joy in work that pervaded those surroundings.”<sup>28</sup>) Jan Tinbergen’s words are echoed in similar remarks by Ehrenfest’s other doctoral students. Tinbergen’s situation as a student was a special one, however, since he decided along the way that his vocation was not for physics but rather for social questions. And it was Ehrenfest who encouraged him, helping him to overcome the difficulties associated with such a change of fields and to make a smooth transition from theoretical physics to mathematical economics and statistics<sup>29</sup>).

Ehrenfest was unusually responsive to such career problems of young people, whether or not they were his own students. He helped to persuade Richard Courant, Professor of Mathematics at Göttingen, that his student, Otto Neugebauer, should be encouraged to pursue his studies of ancient mathematics. Ehrenfest recognized at once that Neugebauer’s concern with Babylonian and Greek mathematics was a calling, not a distracting hobby, and that one ought not try to keep him from his true interests<sup>30</sup>).

Even these brief interactions Ehrenfest had with students during his visits to other universities could have major effects. Victor Weisskopf reports that Ehrenfest made a greater impression on him as a visitor than any of his regular teachers at Göttingen, teaching him “to distrust the complicated mathematics and formalisms that were then very popular in Göttingen.” Ehrenfest never hesitated to ask “stupid questions,” and encouraged students to do the same. “He showed me how to get at the real physics,” Weisskopf wrote, adding, “The older I get, the more aware I am of his influence.”<sup>31</sup>)

As Tinbergen’s remarks suggest, working with Ehrenfest demanded all the energy and intellectual effort a student could muster. Ehrenfest was prepared to put all his own resources into working with his students, and he expected no less from them. “He worked essentially always only with one student, and that practically every afternoon during the week,” George Uhlenbeck wrote. “He discussed with him either the problem on which he was working or recent papers in the literature which he wanted to understand in detail. It went fast. . . . I can personally testify that in the beginning, since one understood things so to say only with the tips of one’s fingers, at the end of the afternoon one was dead tired. Especially because one *had* to follow in detail; the greatest sin was to say that one had understood the point if it was not the case. And it was always found out! The wonder was that after a while the tiredness disappeared, and after a year one worked almost as equals. In fact, as a student you often had the sneaking suspicion that you really knew the things much better. At that point one stood on one’s own legs and one had become a physicist!”<sup>32</sup>)

Since Ehrenfest demanded this complete commitment to science from his students, his relationships with them were intense and strong. For some these demands were too great, and the process did not work. When the teacher–student relationship did not develop to Ehrenfest’s satisfaction, he would find some reason or some pretext for sending the student away, if necessary. These failed relationships sometimes took a heavy emotional toll on the student, and also on Ehrenfest himself. Even with some of his most successful and best-loved students, Ehrenfest could become an interfering and overbearing “mother-in-law,” as he put it, and the younger man might find the need to break out of a situation that made too many emotional demands on him<sup>33</sup>). But when things worked out in the way Ehrenfest hoped for, and tried his hardest to achieve, the result was a joyful common pursuit of understanding.

Some of that feeling comes through in a letter Ehrenfest wrote to Oppenheimer in the summer of 1928. Oppenheimer, who had received his doctorate at Göttingen early the previous year, was still only twenty-four. He had already demonstrated his brilliance on two continents. Oppenheimer had met Ehrenfest when he visited Leyden in 1927, and now wanted to return for a more extended stay during the tenure of his National Research Fellowship. Ehrenfest liked the young man and was willing to have him, but laid down some conditions. He did not want Oppenheimer to “shoot immediately at a problem with gigantic calculating cannon, instead of first thinking through an example – as simplified as possible – eagerly and joyfully.” “If you intend to mount heavy mathematical artillery again during your coming year in Europe,” he wrote, “I would ask you not only not to come to Leyden, but if possible not even to Holland, and just because I am really so fond of you and want to keep it that way. But if, on the contrary, you want to spend at least your first few months patiently, comfortably, and joyfully in discussions that keep coming back to the same few points, chatting about a few basic questions with me and our young people – and without thinking much about publishing (!!!) – why then I welcome you with open arms!!”<sup>34</sup>)

For Ehrenfest the “real essence” of theoretical physics was “the wish to *understand* physics better through calm reflection and discussions with friends.”<sup>35</sup>) He never lost the conviction he once expressed to Lorentz that it was “a magnificent thing to make clearer and clearer to oneself something that seemed at first to be full of internal contradictions or totally confused,” and that it was “also magnificent to make something clear to others, if they can take pleasure in it!”<sup>36</sup>)

This approach to physics – a joyful fascination with understanding that could be “caught like a contagion from the lips of a living teacher” and “become the guiding principle of a life”<sup>37</sup>) – was Ehrenfest’s greatest legacy to his science.

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