Topological Properties of Quantum States of Condensed Matter: some recent surprises.

F. D. M. Haldane
Princeton University
and
Instituut Lorentz

I. Berry phases, zero-field Hall effect, and “one-way light”

II. Anomalous and Spin Hall effect, Topological insulators

III. Non-abelian FQHE states

Laughlin FQHE state

$$\Psi = \Phi(z_1, z_2, \ldots, z_N) \prod_{i=1}^{N} e^{-\varphi(\boldsymbol{r}_i)}$$

lowest Landau level

N-variable (anti)symmetric polynomial

$\nabla^2 \varphi(\boldsymbol{r}) = 2\pi B(\boldsymbol{r})/\Phi_0$

- ν = $1/\mathrm{m}$ Laughlin state

$$\Phi(z_1, z_2, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^m$$

- "occupation number"-like representation in orbitals $\mathrm{z^m}$, $\mathrm{m} = 0,1,\ldots, \mathrm{N}_\Phi = \mathrm{m(N\text{-}1)}$ orbitals

m=0 orbital $\rightarrow$ $1001001001001001001...1001$ **(m=3)**

This is the "dominant" configuration of the Laughlin state

“Dominance”

- convert occupation pattern to a **partition** λ, “padded” with zeroes to length N:
- $1001001 \rightarrow \lambda = \{\lambda_1, \lambda_2, \lambda_3\} = \{6,3,0\}$
- λ dominates λ' if
  - $|\lambda| \equiv (\sum_i \lambda_i) = |\lambda'| = M$
  - $(\sum_{j \leq i} \lambda'_j) \leq (\sum_{j \leq i} \lambda_j)$ for all $i = 1,2,..N-1$

“dominance” and “squeezing”

- (pairwise) squeezing: move a particle from orbital m_1-1 to m_1 and another from m_2+1 to m_2 where $m_1 \leq m_2$.

$$1001001001001001001\ldots1001 \qquad \text{A}$$

$$1000101001001010001\ldots1001 \qquad \text{B}$$

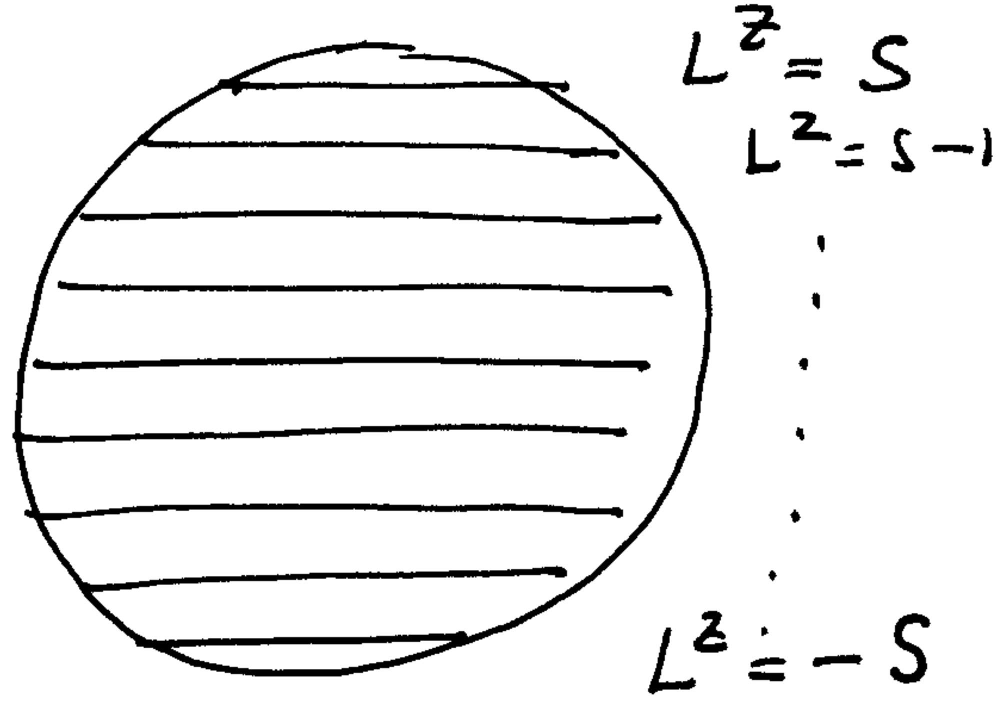
A dominates B (A > B)

- dominance is a partial ordering: if A > B and B > C, then A > C.

- When expanded in occupation number states, the (polynomial) $1/\mathrm{m}$ Laughlin state only contains configurations dominated by the most compressed (minimum M) "(1,m)-admissible configuration" where no group of m consecutive orbitals contains more than 1 particle.
- "admissibility" can be thought of as a generalized Pauli principle.

Compactification of the Lowest Landau level on the Riemann sphere.

Identify orbitals m = 0,1,...,N_Φ with orbitals Lz = S,S-1,...,-S on a sphere enclosing magnetic monopole charge N_Φ= 2S



Uniform QHE states are rotationally-invariant, L_{tot} = 0.

Beyond "standard" occupation number formalism

- $\boldsymbol{k}$-particle $1/\boldsymbol{m}$ Laughlin droplet creation operator (circular droplet centered at $\boldsymbol{R}$):

Gaussian centered at $\boldsymbol{R}$

$$\eta_{km}(\boldsymbol{R})^\dagger|vac\rangle \propto \prod_{i>j}(z_i - z_j)^m \prod_{i=1}^{k} \psi_{\boldsymbol{R}}(\boldsymbol{r}_i)$$

- For $k = 1$, $(m\ has\ no\ meaning\ in\ this\ case)$, this is just the standard lowest Landau-level single-particle creation operator

$$c(\boldsymbol{R})^\dagger|vac\rangle \propto \psi_{\boldsymbol{R}}(\boldsymbol{r}_1)$$

- Read-Rezayi (includes Laughlin, Moore-Read) FQHE states are defined by

$$\nu = \frac{k}{km+2}$$

(k+1)-particle destruction

$$\eta_{k+1,m}(\boldsymbol{R})|\Psi\rangle = 0$$

$$\eta_{2,m'}(\boldsymbol{R})|\Psi\rangle = 0, m' < m$$

for all $\boldsymbol{R}$

k-particle destruction

$$\eta_{k,m}(\boldsymbol{R}_i)|\Psi\rangle = 0$$

at locations $\boldsymbol{R}_i$ of pinned elementary quasiholes

"Admissible" configurations:

Not more than k particles in km+2 consecutive orbitals

For m >0, not more than one particle in m consecutive orbitals

- On the sphere, the number of charge -e/(km+2) elementary quasiholes for a given N, N_Φ is

$$N_{qh} = k(N_\Phi - \tfrac{1}{2}mk(k-1)) - (km+2)(N-k)$$

- The size of the basis set of quantum states (with unpinned quasi holes) is equal to the number of admissible configurations.
- The states can be completely constructed out of configurations dominated by the dominant admissible configuration ("top" configuration).
- These are a very small subset of lowest Landau level states!

Jack Polynomials.

- For m = 0 (bosonic case) the $\nu = k/2$ Read–Rezayi multi–quasihole states are spanned by the set of **Jack symmetric polynomials** $J_\lambda^{-(k+1)}(z_1,\ldots z_N)$ with an admissible partition λ, which form a complete but non–orthogonal basis. (See Feigin, Miwa, Jimbo and Mukhin 2002, and Bernevig and Haldane, cond–mat/0707.3637)

- $J^\alpha_\lambda(z)$ with parameter α real positive and unrestricted are orthogonal polynomials; here α is in general **negative rational**, and λ are restricted to "admissible" partitions.

Fermionic 2/4=1/2 Moore-Read state

uniform vacuum state on sphere:

1100110011001100110011001100110011

even fermion number -e/2 double quasihole (h/e vortex) at North Pole:

01100110011001100110011001100110011

odd fermion number -e/2 double quasihole (h/e vortex) at North Pole:

10011001100110011001100110011001 1

fractionalization: one -e/4 quasihole (h/2e vortex) at North Pole, one near equator.

10101010101010100110011001100110011001 1

These translate into explicit wavefunctions that can be calculated in finite-size systems

3/5 (fibonacci) Read-Rezayi state primary configurations

111001110011100111001110011100 . . .

elementary -e/5 vortex at North pole

11010110101101011010110101101 . . .

110011100111001110011100111001 . . .

10101101011010110101101011010 . . .

01110011100111001110011100111 . . .

0101101011010110101101011010 . . .

1001110011100111001110011100111 . . .

01101011010110101101011010110 . . .

vortex moves by hopping 5 orbitals at a time

For charge -ne/5, n > 1 there are always 2 orthogonal primary states.

explicit numerical calculations

- Strategy: obtain full set of highest-weight states by solving

$$L^{+}_{tot}|\Psi\rangle = 0$$

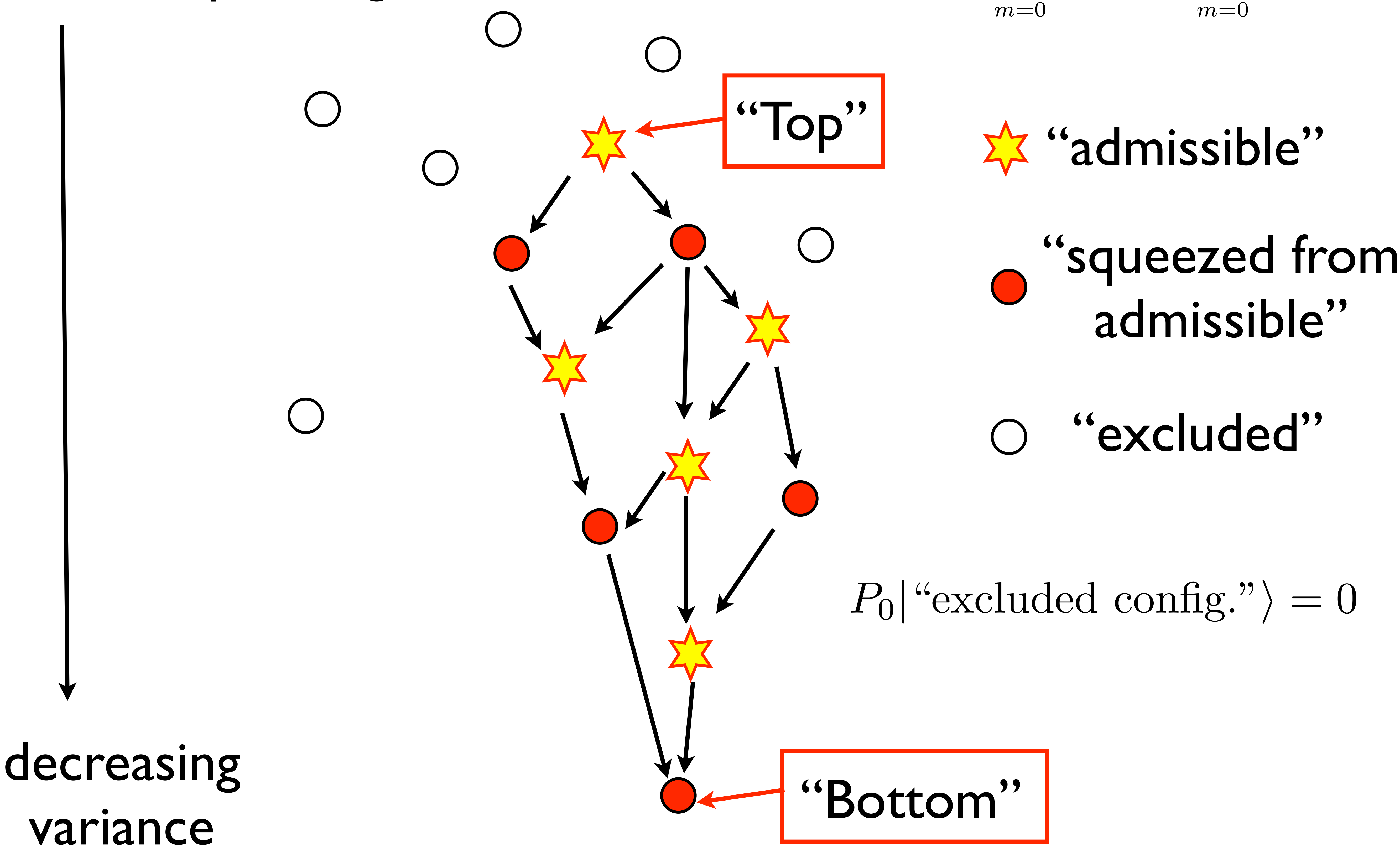
- The number of admissible configs at each Lz tells us how many we need. <u>We exclude from the basis set configs not dominated by the dominant admissible config</u>. This gives a highly overdetermined system of equations!

- Within the full basis set thus obtained, impose the condition that pins the quasiholes at the desired locations.

Partial ordering of occupation number configurations with fixed L_z

- squeezing decreases the variance

$$\sum_{m=0}^{m_{max}} m^2 n_m - \left(\sum_{m=0}^{m_{max}} m n_m\right)^2$$



$$P_0 | \text{“excluded config.”} \rangle = 0$$

key point:

Null space is invariant under the Euclidean group

- Disk: $[P_0, a] = 0$
- Sphere: $[P_0, L^+] = 0$

- Use Wigner-Eckert: need to (simultaneously) solve

$$L^+|\Psi\rangle = 0 \text{ and } P_0|\Psi\rangle = 0$$

highest weight null modes

- In the full basis this is an undetermined problem (more columns than rows)
- After "excluded" states are removed, it is overdetermined (more rows than columns)!
- (can efficiently solve with a variant Lanczos-type technique to full floating-point accuracy.)

example:

16 electrons on sphere, maximum ν=1/2 Moore-Read density, plus 2h/e extra flux (single qubit when vortices are fixed)

```
  16 spinless fermions on the sphere with   32 orbitals:
 full basis: 601080390 projected basis:       825 (summed over LZ)
==========================================================================================
Ltot=     0.0     1.0     2.0     3.0     4.0     5.0     6.0     7.0     8.0     9.0    10.0    11.0
all:     6235   17625   30017   41207   53324   64172   75813   86131   97177  106789  117059  125864
nul:        3       0       6       2       7       4       7       4       7       3       5       2
==========================================================================================
Ltot=    12.0    13.0    14.0    15.0    16.0    17.0    18.0    19.0    20.0    21.0    22.0    23.0
all:   135218  143078  151432  158231  165486  171188  177258  181794  186683  190016  193686  195853
nul:        3       1       2       0       1       0       0       0       0       0       0       0
==========================================================================================
```

L_z = 2: find 6 zero modes of a sparse 5,800,384 x 6,170,810 overdetermined matrix (52×10^6 non-zero matrix elements)

```
PURGE: nstate =   8884686 LZ=     2.0 root =   11001100110011001001001100110011
nstate before purge=   8884686 after purge =   5800384
 BINARY: registered binary code total size =     2:
    1 components with sizes:      2
PURGE: nstate =   8854669 LZ=     3.0 root =   11001100110011001010001100110011
L+ has   52060614 +  21167057 non-zero elements
   6170810 constraints,   52060614 nonzero matrix elements, and    370432 linear dependencies
 second representation of L+:   16 distinct values  48727308 elements
```

601,080,390 lowest LL states
825 MR null-mode states, of which
57 are highest weight

```
zero mode #    1: maximum error       3.0D-18
zero mode #    2: maximum error       5.2D-18
zero mode #    3: maximum error       4.8D-18
zero mode #    4: maximum error       6.2D-18
zero mode #    5: maximum error       3.8D-18
zero mode #    6: maximum error       5.5D-18
final overlap matrix eigenvalues:
    1.00000000     1.00000000     1.00000000     1.00000000     1.00000000
    1.00000000
two-body interaction energies:
     -7.4892956319233095      -7.4689790893071946      -7.4496977681804388
     -7.4039194994824538      -7.3871308896580405      -7.3668521272231633
   6 highest weight zero modes found
```

- Laughlin 1/3 state is represented by "occupation number" pattern

$$1001001001001001001001\ldots$$

any 3 consecutive "orbitals" contain exactly 1 particle

- Moore-Read "Pfaffian" 2/4 (= 1/2) state has the occupation pattern

$$1100110011001100110011 00\ldots$$

any 4 consecutive "orbitals" contain exactly 2 particles

(These are not simple Slater determinant states, but (related to) Jack polynomials with specific negative integer Jack parameters)

- The 3 Laughlin 1/3 configurations

$$\begin{aligned} &\ldots 100100100100100 \ldots \\ &\ldots 010010010010010 \ldots \\ &\ldots 001001001001001 \ldots \end{aligned}$$

repeat period 3

- The 6 Moore-Read 2/4 configurations

$$\begin{aligned} &\ldots 1100110011001100 11 \ldots \\ &\ldots 011001100110011001 \ldots \\ &\ldots 001100110011001100 \ldots \\ &\ldots 100110011001100110 \ldots \end{aligned}$$

repeat period 4

$$\begin{aligned} &\ldots 010101010101010101 \ldots \\ &\ldots 101010101010101010 \ldots \end{aligned}$$

repeat period 2

These counts give the "topological degeneracies" of these states when constructed on a 2-torus

- operator that creates a circular"filled Landau level droplet" of k particles, centered at position R: $\eta_k^\dagger(\boldsymbol{R})$

$$\langle \boldsymbol{r}_1, \ldots \boldsymbol{r}_k | \eta_k^\dagger(\boldsymbol{R}) | \mathrm{vac} \rangle \propto \prod_{i<j} (z_i - z_j) \prod_i \psi_{\boldsymbol{R}}(\boldsymbol{r}_i)$$

Gaussian coherent state centered at R

$$|\psi_{\boldsymbol{R}}(\boldsymbol{r})|^2 \propto \exp -|\boldsymbol{r} - \boldsymbol{R}|^2/2\ell^2$$

"magnetic length"

- (Abelian) Laughlin 1/3 state with elementary charge -e/3 quasi holes at positions $\boldsymbol{R}_i^h$

$$\Psi_L(\{\boldsymbol{r}_i\};\{\boldsymbol{r}_j^h\}) \propto \prod_{ij}(z_i - z_j^h)\prod_{i<j}(z_i - z_j)^3\prod_i \psi_0(\boldsymbol{r}_i)$$

$$\eta_2(\boldsymbol{R})|\Psi_L(\{\boldsymbol{r}_j^h\})\rangle = 0, \text{all } \boldsymbol{R}$$ can't destroy a m=1 pair anywhere

$$\eta_1(\boldsymbol{R}_i^h)|\Psi_L(\{\boldsymbol{r}_j^h\})\rangle = 0$$ can't destroy a single electron at positions of holes

- This state is completely defined (up to a phase factor) by the positions of the quasi holes

- (Non-Abelian) Moore-Read 2/4 = 1/2 state with elementary charge -e/4 quasi holes at positions $\boldsymbol{R}_i^h$

$$\eta_3(\boldsymbol{R})|\Psi_{MR}(\{\boldsymbol{r}_j^h\})\rangle = 0, \text{all } \boldsymbol{R}$$ can't destroy a 3-particle droplet anywhere

$$\eta_2(\boldsymbol{R}_i^h)|\Psi_{MR}(\{\boldsymbol{r}_j^h\})\rangle = 0$$ can't destroy a 2-particle droplet at positions of non-abelian quasiholes

- This state is NOT fully defined by the positions of the quasi holes
- residual degeneracy $2^{N_{qh}/2-1}, N_{qh}$ even

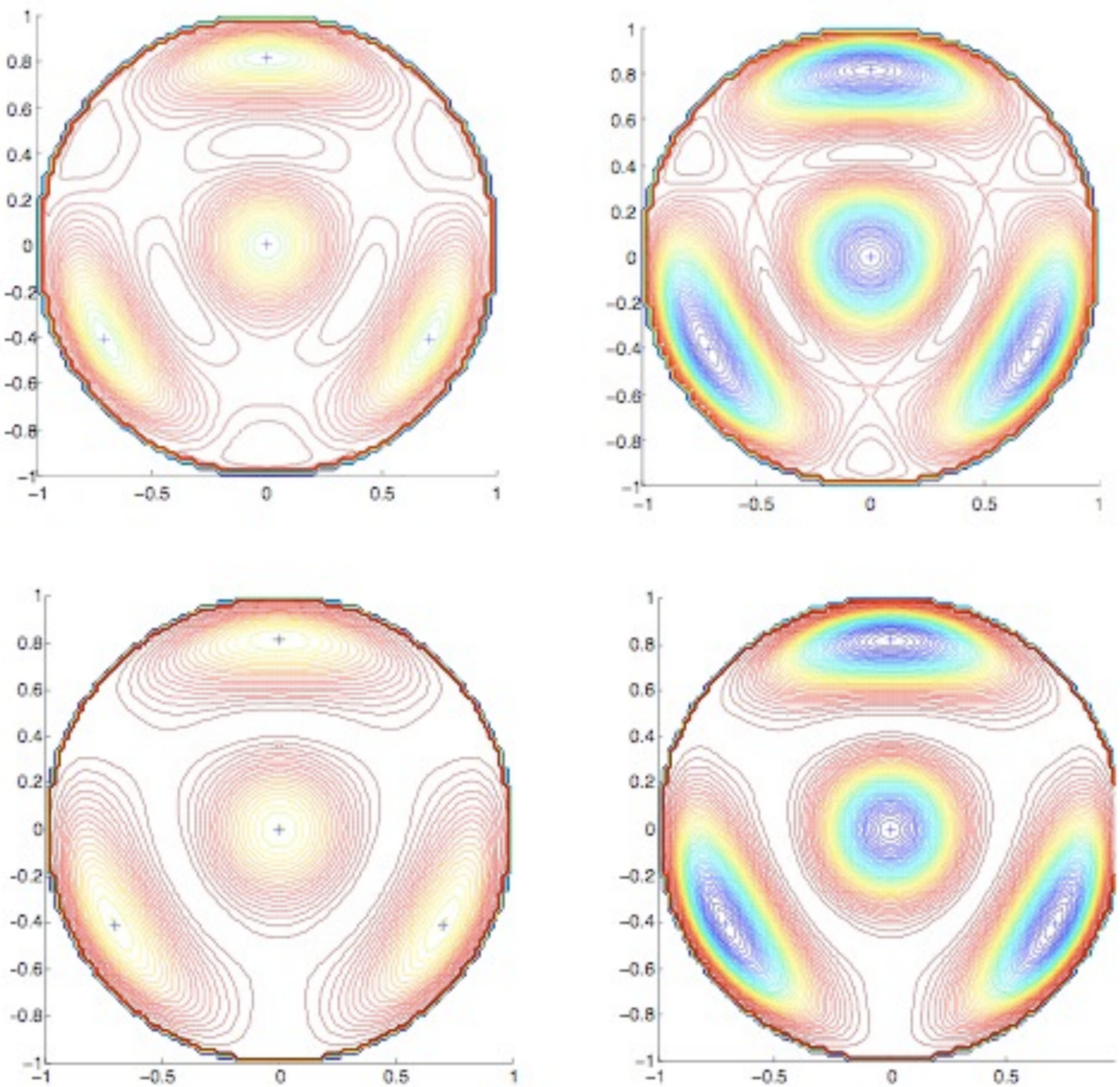
- The only local operations possible are
  - (i) add/remove an electron orbital (h/e flux)
  - (ii) add/remove an electron
- isolated non-Abelian Moore-Read -e/4 (h/2e) quasiholes cannot be created by local operations, only PAIRS of quasiholes can be locally created, then split apart.

- Topological quantum computing dream: encode and process quantum information in the residual degeneracy that is left after the positions of the non-Abelian quasiholes have been specified
- When the quasiholes are widely separated, local measurements dont distinguish the states, so they are immune to decoherence by local enviromental perturbations

What distinguishes the “internal states” physically?

single-particle density

$m=1$ two-particle density



Tetrahedral arrangement of 4 MR h/2e vortices, (14 electrons, 28 orbitals)

One qubit is left after positions of vortices are fixed.

Sphere is mapped to unit disk.

the qubit doublet is split by the Coulomb interaction, both states are shown. THE SPLITTING AND LOCAL DIFFERENCE BETWEEN THE TWO STATES IS EXPECTED TO DISAPPEAR AS THE SYSTEM SIZE INCREASES.

- states are distinguished by small oscillations of charge density (like interference fringes) around a common background density pattern

- the amplitude of these oscillations becomes exponentially small as the separation becomes large on the magnetic length scale.

- a pair of quasiholes has TWO states, distinguished by local fermion number parity (even/odd):

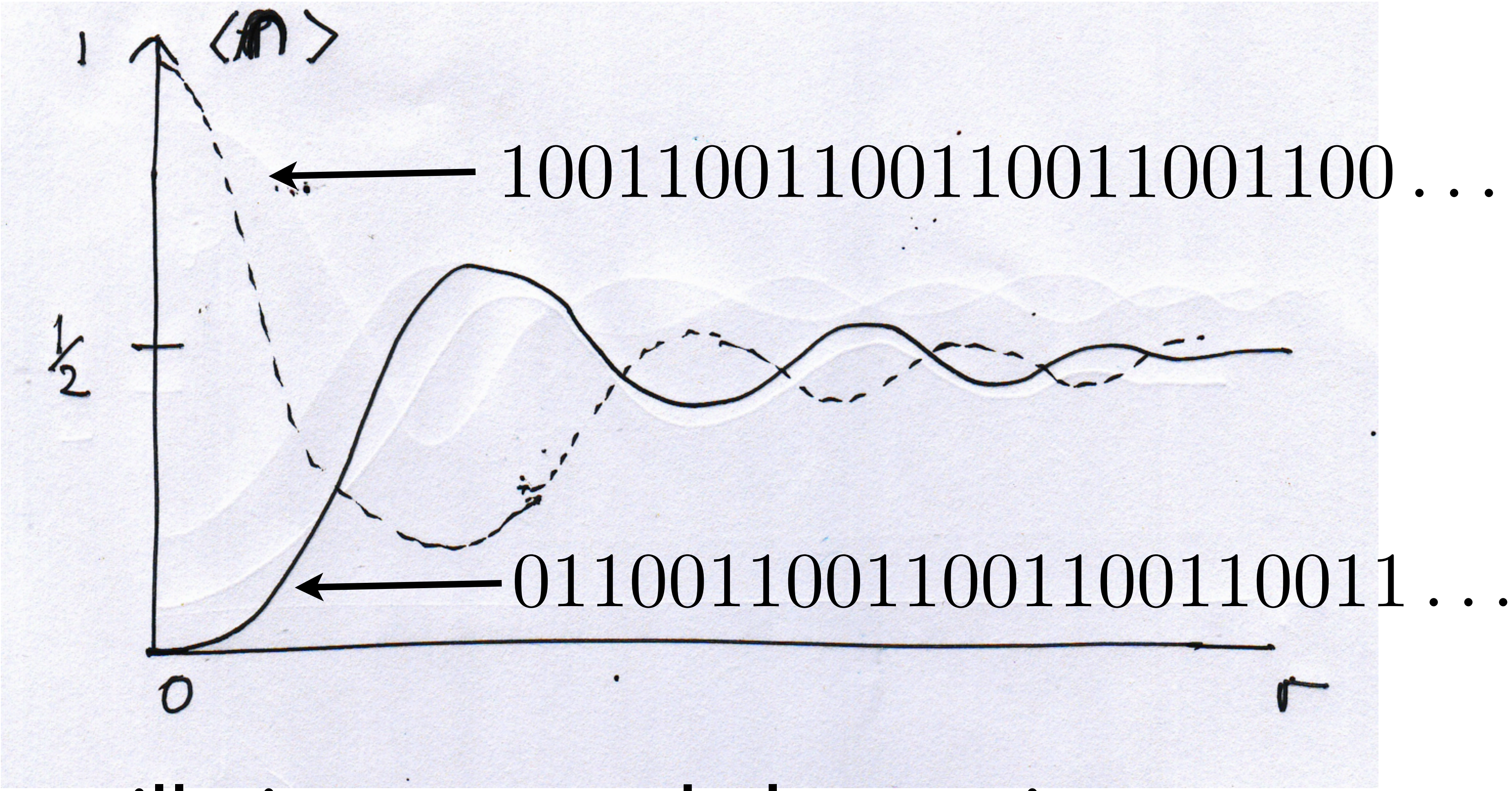
uniform state: $11001100110011001100110\ldots$

two quasiholes at north pole:

$^{*}\,{}^{*}01100110011001100110011\ldots$

$^{*}1^{*}0011001100110011001100\ldots$

unpaired electron:



- The oscillations around close pairs or quasiholes clearly distinguish even and odd fermion number states of the pair

two Moore-Read vortices (fused)

7

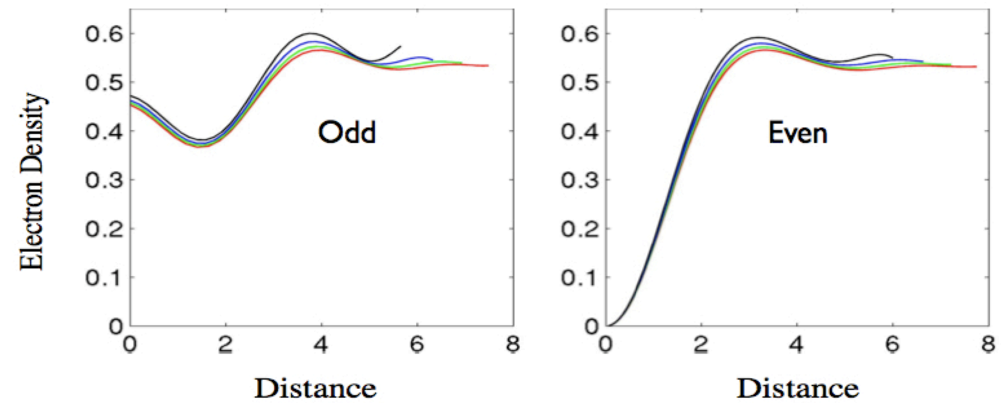


FIG. 4: The particle density for fused probes as function of distance from the fusing point. Left/right panel refers to odd/even number of electrons. On the left, the different curves correspond to N/N_ϕ=9/16, 11/20, 13/24, 15/28 and, on the right, the different curves correspond to N/N_ϕ=10/18, 12/22, 14/26, 16/30.

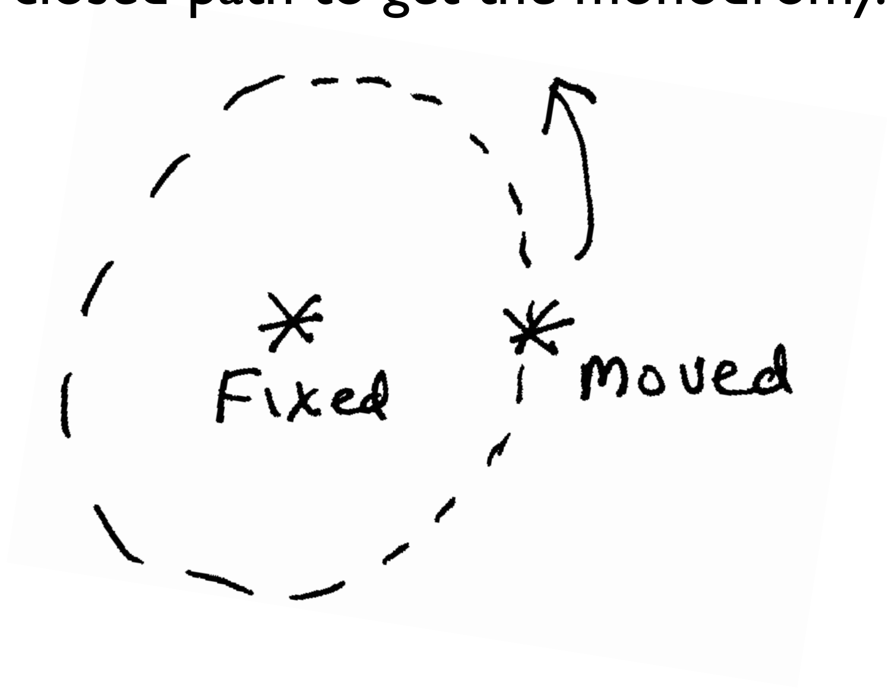
$1001100110011001 1 0011 \ldots$ $\quad$ $0110011001100110 01100 \ldots$

unpaired electron at North pole

Monodromy

- Hold one vortex at the north pole, and move the other in infinitesimal loops to map out the Berry curvature, in the two cases of even and odd fermion number.

- Integrate the berry curvature inside a closed path to get the monodromy.



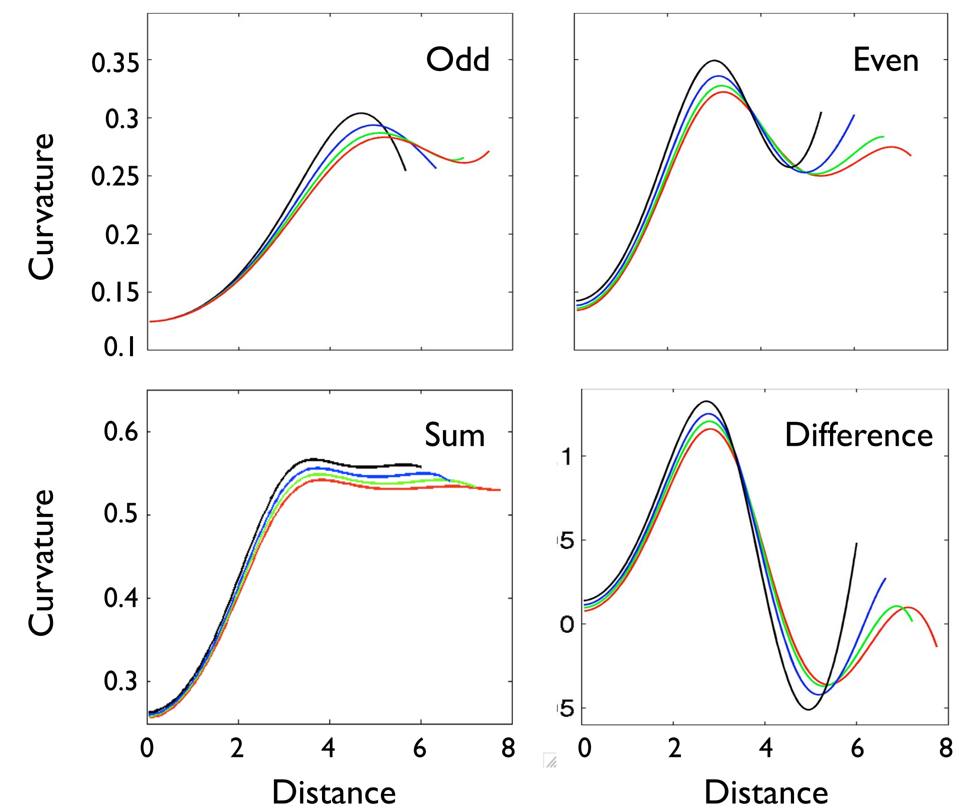


FIG. 10: (Color online.) The Berry curvature obtained by moving one anyon while keeping the other fixed. Left-upper panel shows the results for odd number of electrons: N/N_ϕ=9/16, 11/20, 13/24, 15/28 and the right-upper panel shows the results for even number of electrons: N/N_ϕ=10/18, 12/22, 14/26, 16/30. The lower-left and lower-right panels show the sum and the difference between the odd and even results, respectively. For example, we added and subtracted the result for N/N_ϕ=10/18 and N/N_ϕ=9/16, and then the results for N/N_ϕ=12/22 and N/N_ϕ=11/20, etc..

for a path with
a large radius, the
relative Berry phase
factor between
the even and odd
fermion number
cases approaches -1
(as predicted!)

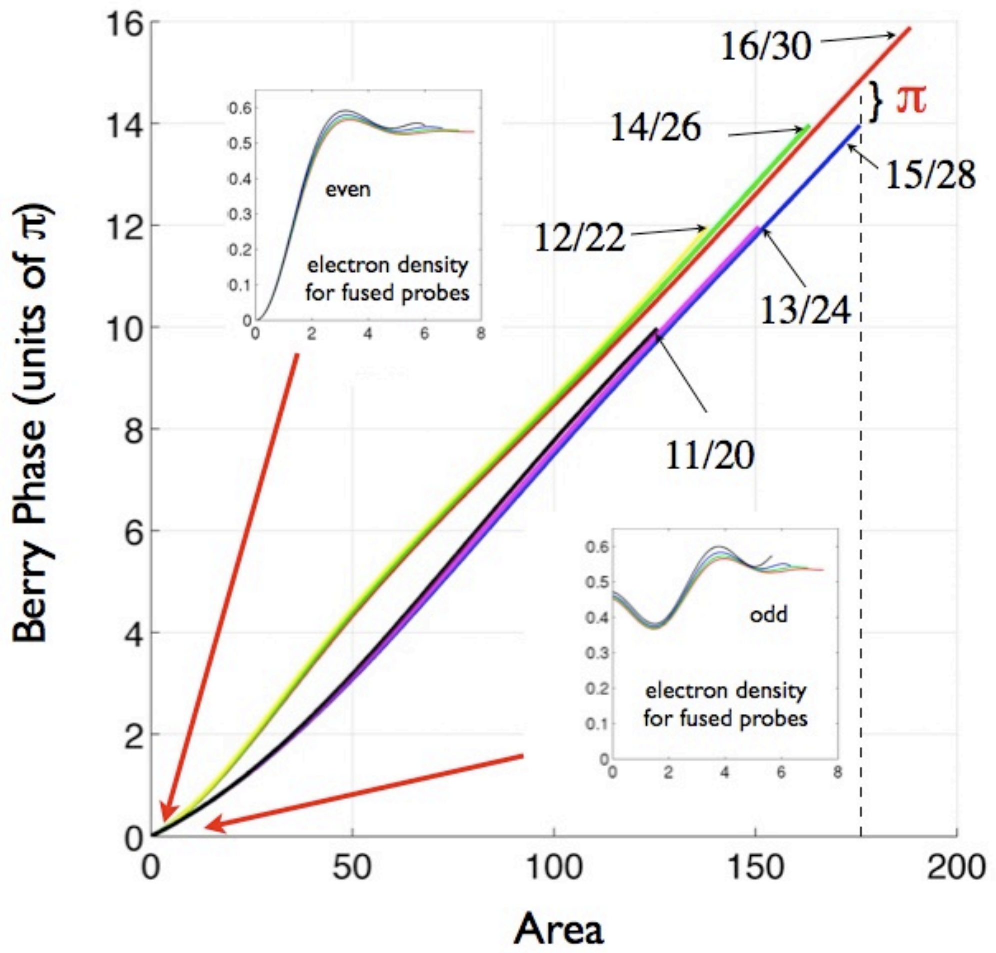


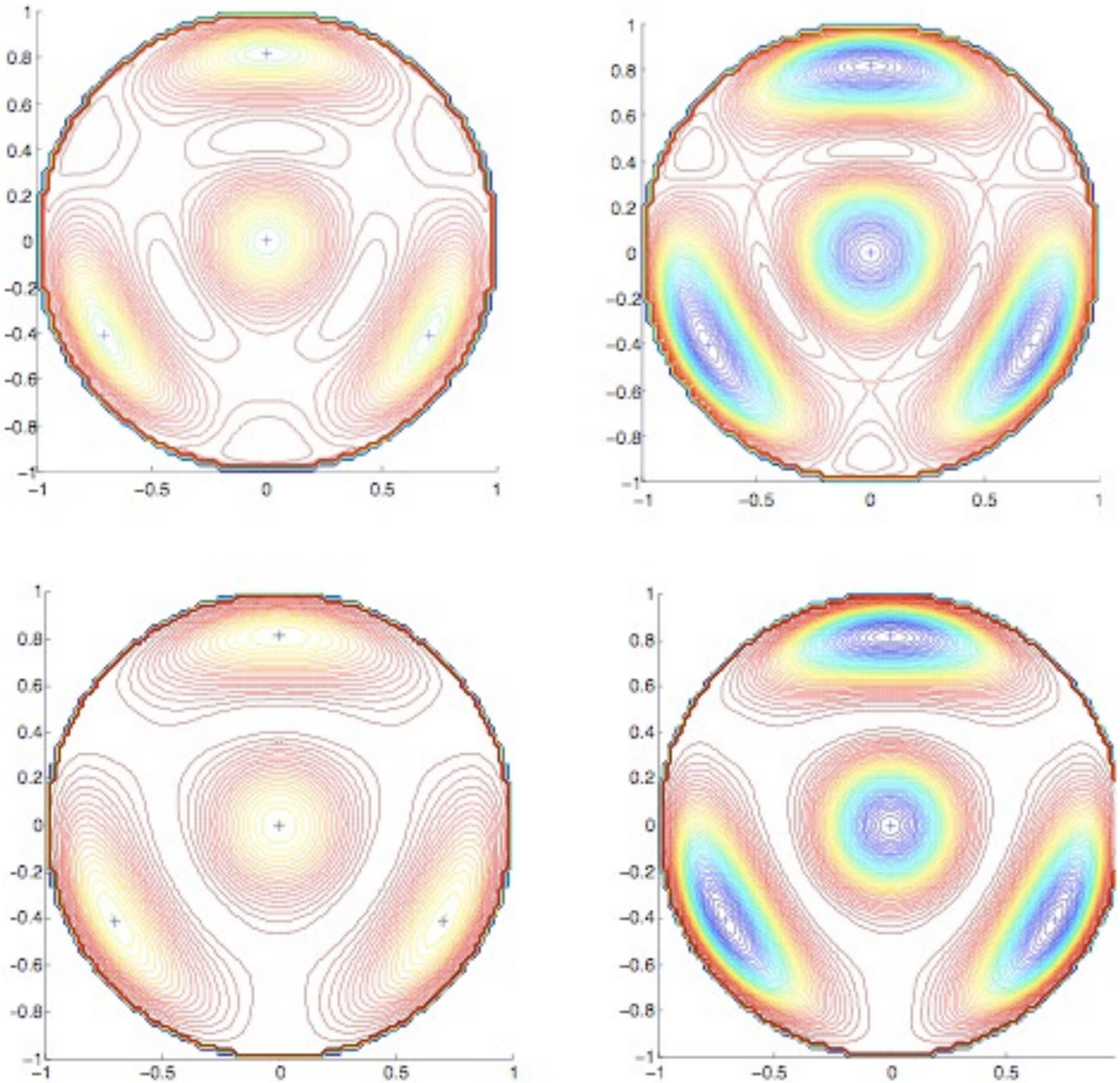
FIG. 11: (Color online.) The Berry phase accumulated by an anyon when moved along a path θ=const, with the other anyon fixed at the North pole. The Berry phase is plotted against the area enclosed by the paths. Each curve is marked with the corresponding N/N_ϕ numbers. The insets show the electron density for the fused anyons, computed in Fig. 4, which one can use, experimentally, to distinguish between even/odd cases.

4 well-separated vortices (a qubit)

Note that the two state have slightly different "interference ripple" patterns in the electron density that will be exponentially small as the distance between the vortices increases, but which is a residual local physical difference between the states.

single-particle density

$m=1$ two-particle density



Tetrahedral arrangement of 4 MR h/2e vortices, (14 electrons, 28 orbitals)

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Sphere is mapped to unit disk.

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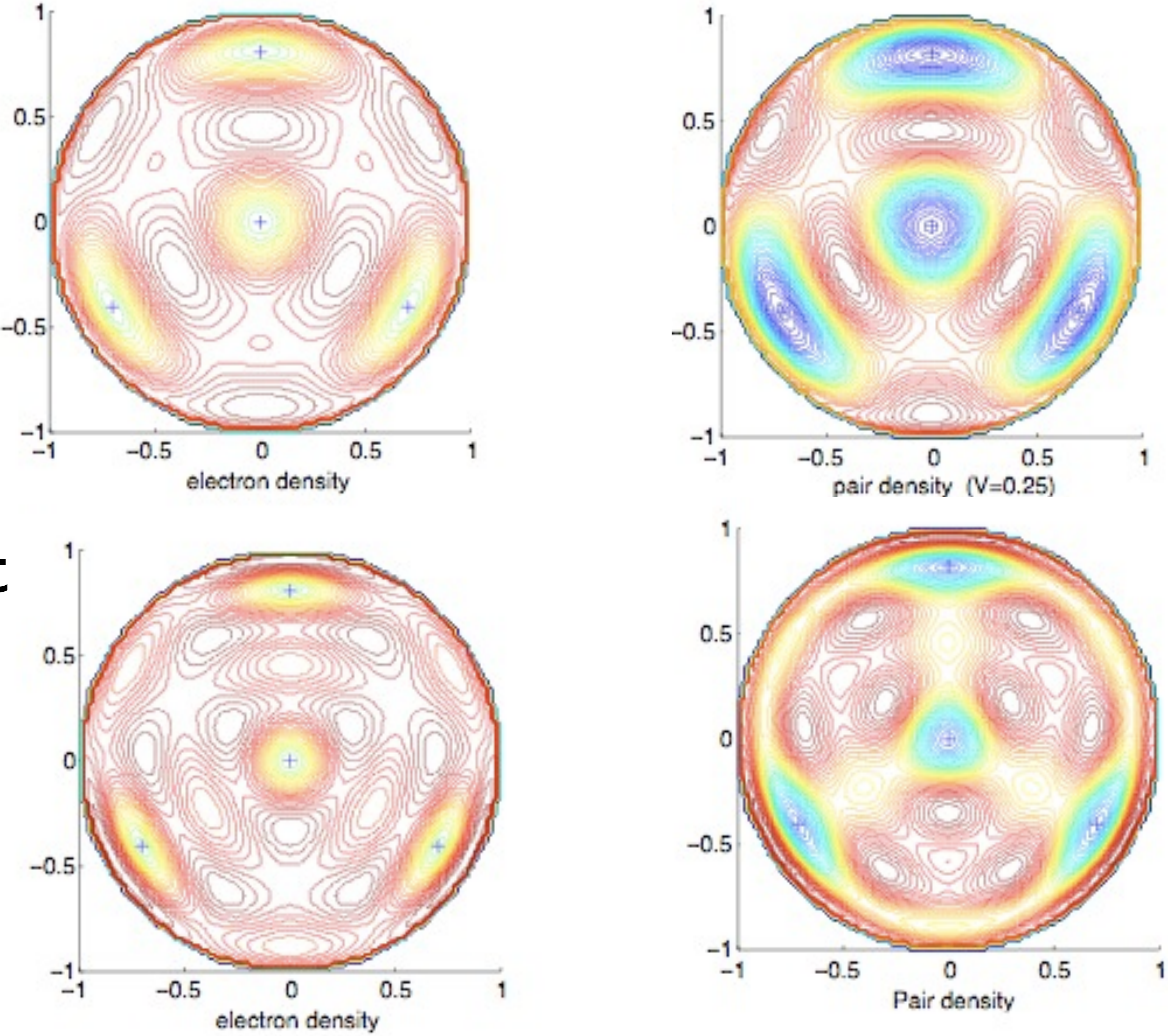
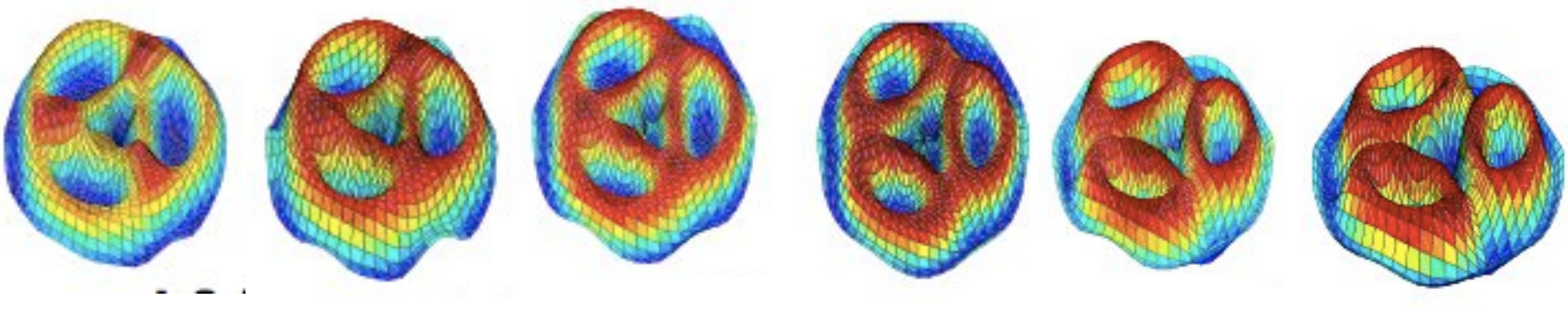
four probes, tetrahedral pattern: candidate qubit pair, 14/28

zero-point motion of vortex positions



These are made with "STM + coulomb repulsion": very close to the "exact" states!

non-Abelian Berry curvature , for increasing size (10-15 electrons)



as size increases, the (magnititude) of the non-abelian curvature field is seen to be concentrated near the quasiparticle cores, consistent with braiding. (For widely separated vortices, there should be vanishing non-abelian curvature in the regions in between the vortices, so the monodromy becomes purely topological)

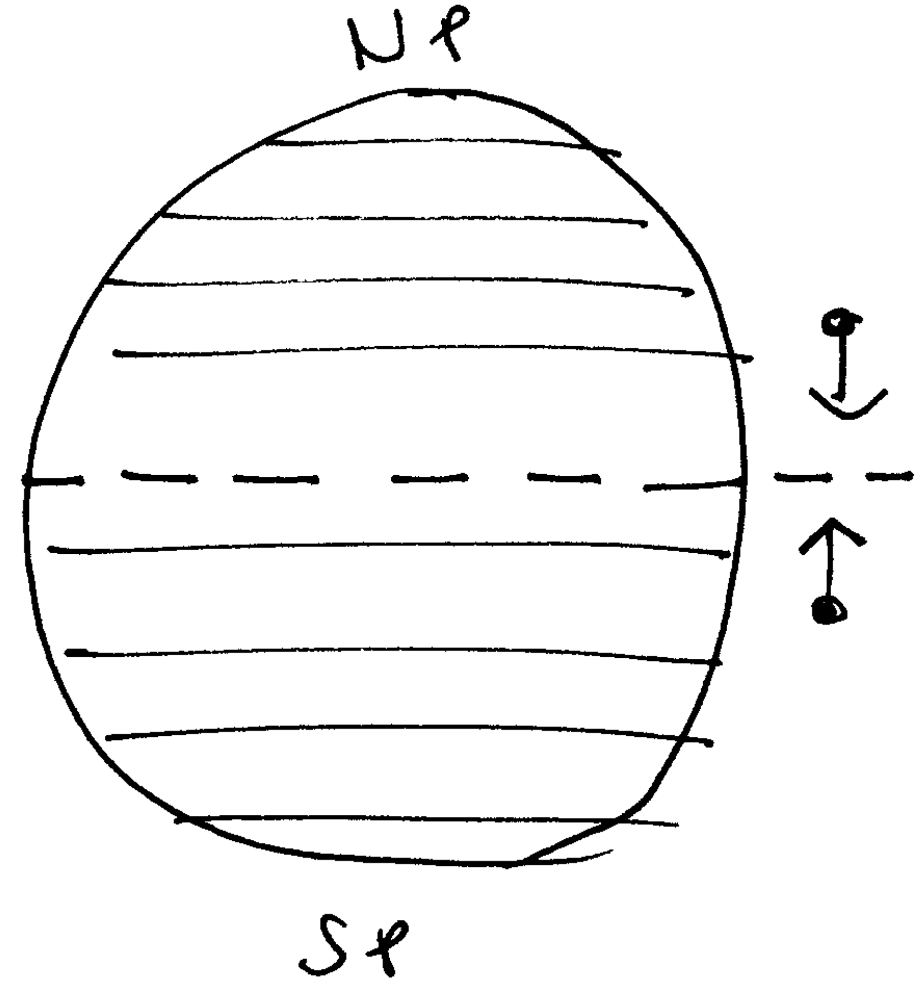
10/20

11/22

12/24

Entanglement spectra and "dominance"

- Schmidt decomposition of Fock space into N and S hemispheres.
- Classify states by Lz and N in northern hemisphere, relative to dominant configuration. Lz always decreases relative to this (squeezing)



Represent bipartite Schmidt decomposition like an excitation spectrum (with Hui Li)

$$|\Psi\rangle = \sum_\alpha e^{-\beta_\alpha/2}|\Psi_{N\alpha}\rangle \otimes |\Psi_{S\alpha}\rangle$$

- like CFT of edge states.
- A lot more information than single number (entropy)

- many zero eigenvalues

$$e^{-\beta_\alpha} = 0$$

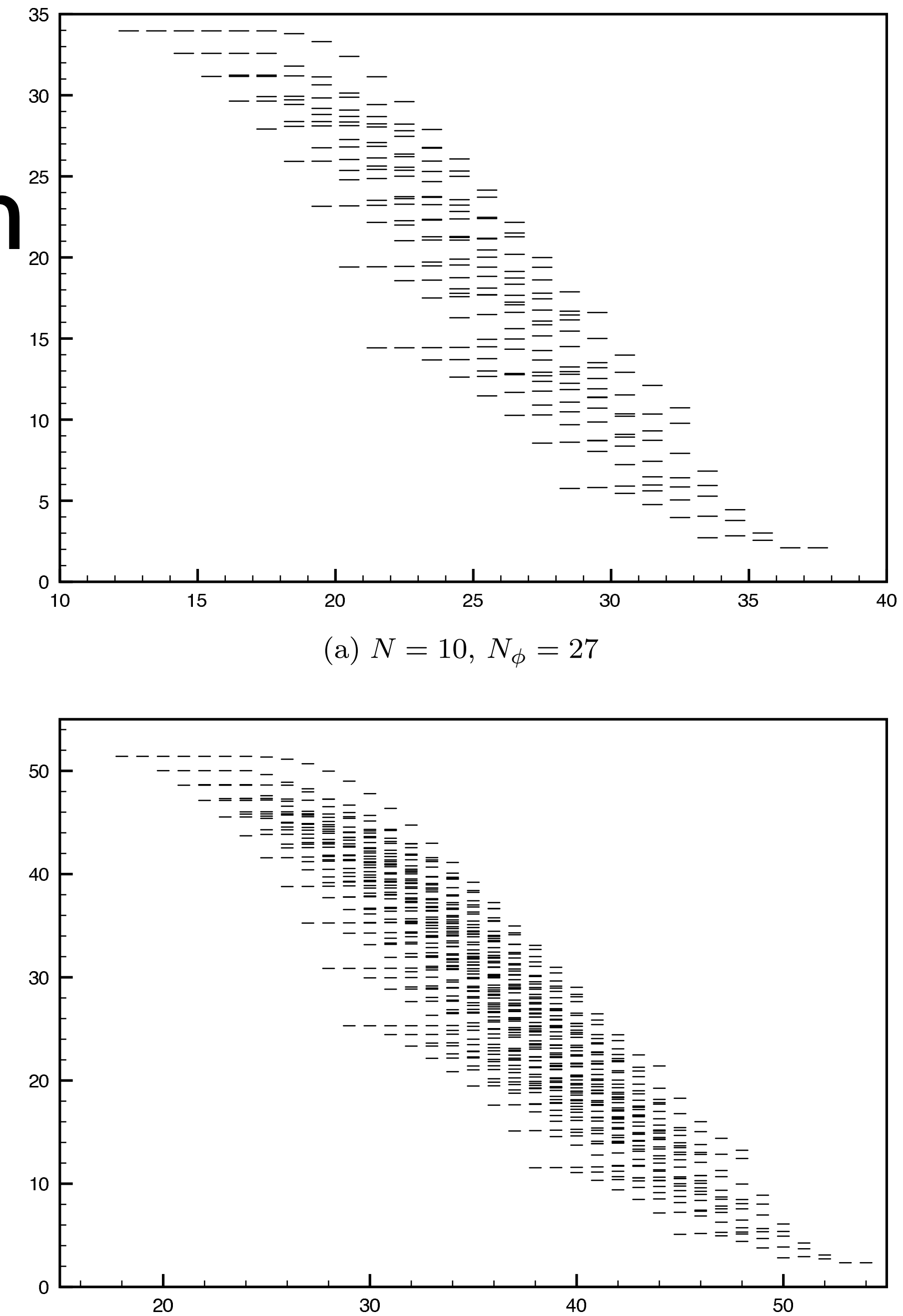
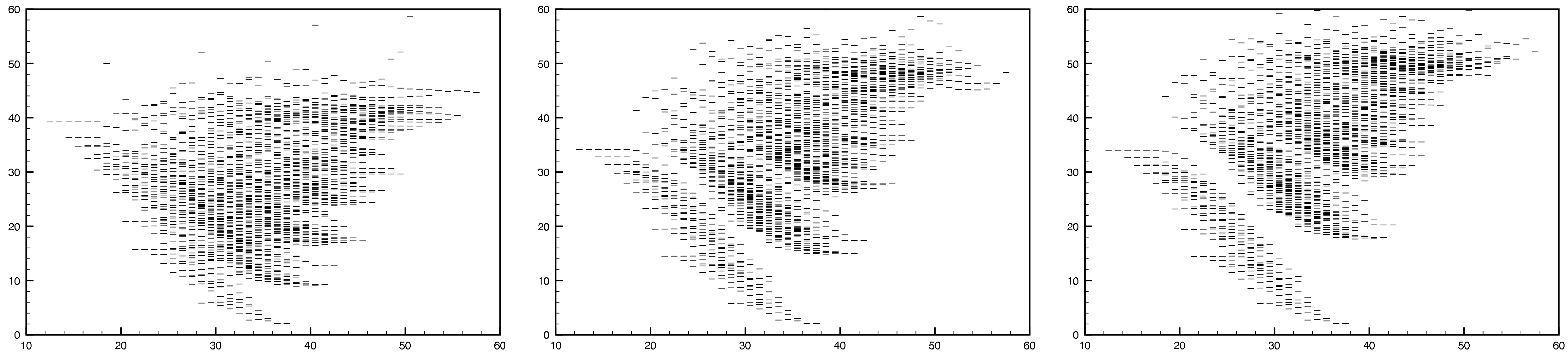


FIG. 1: Entanglement spectrum for the 1/3-filling Laughlin states, for $N = 10, m = 3, N_\phi = 27$ and $N = 12, m = 3, N_\phi = 33$. Only sectors of $N_A = N_B = N/2$ are shown.

Look at difference between Laughlin state,entanglement spectrum and state that interpolates to Coulomb ground state.

3



(a) $x = 1$ (b) $x = 1/3$ (c) $x = 1/10$

FIG. 2: Entanglement spectrum for the ground state, for a system of $N = 10$ electrons in the lowest Landau level on a sphere enclosing $N_\phi = 27$ flux quanta, of the Hamiltonian in Eq. (12) for various values of x.

$$H = xH_c + (1-x)V_1$$

x=0 is pure Laughlin

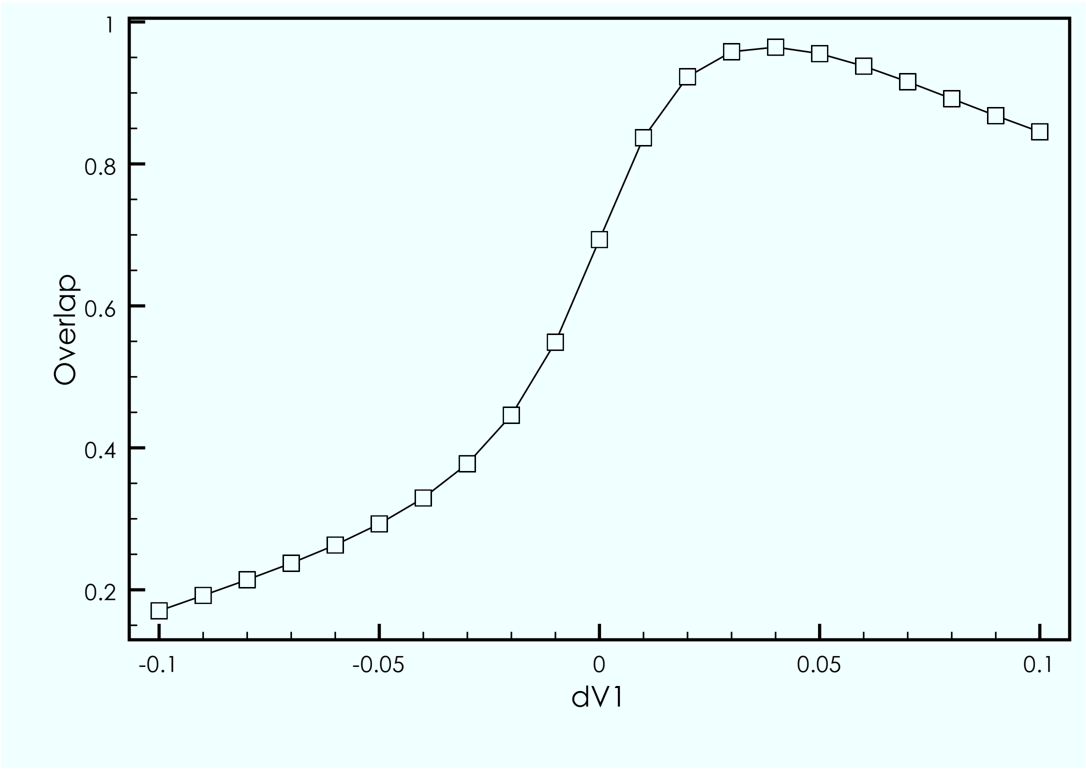
Can we identify topological order in "physical as opposed to model wavefunctions from low-energy entanglement spectra?

Latest results showing change in entangle spectrum of half-filled second-landau-level coulomb interaction with additional V(1) pseudopotential (with Hui Li)

The interaction potential is Coulomb in LL=1 (spherical geometry) plus δV_1. System size is $N_e = 14$, $N_{orb} = 26$.

I. VARYING δV_1

The overlap with model Moore-Read state.



V(1) modifies the m=1 pair energy, and drives a transition between a Moore-Read-like state and a gapless state

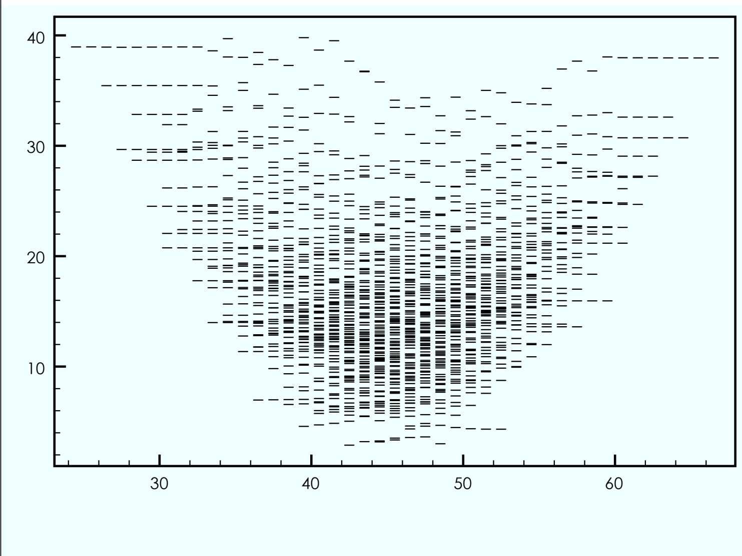


FIG. 1: $\delta V_1 = -0.05$

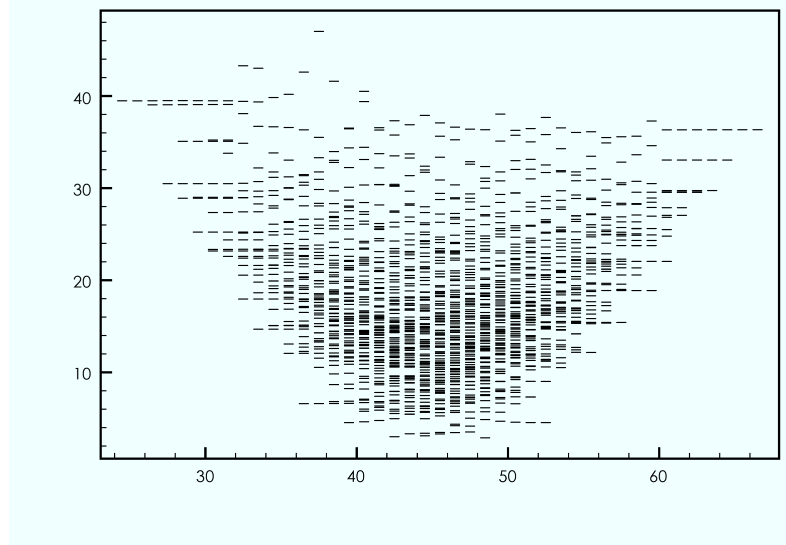


FIG. 2: $\delta V_1 = -0.02$

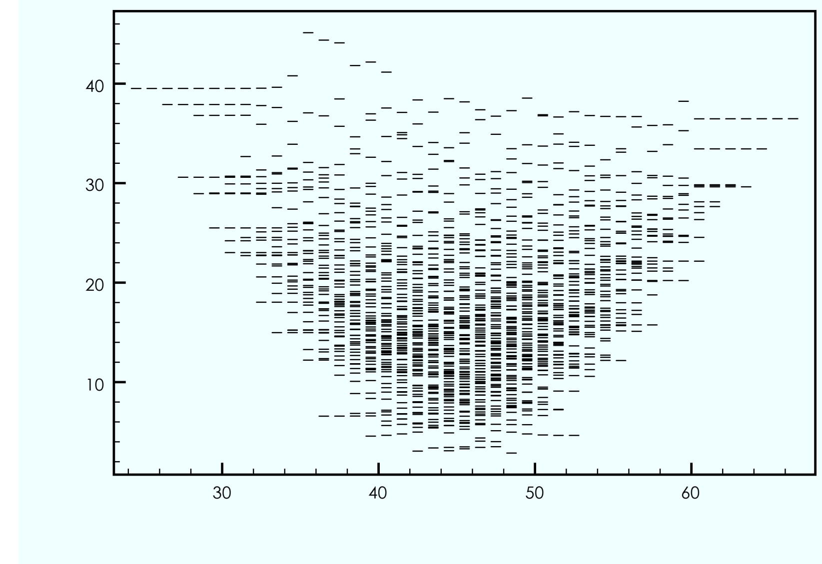


FIG. 3: $\delta V_1 = -0.015$

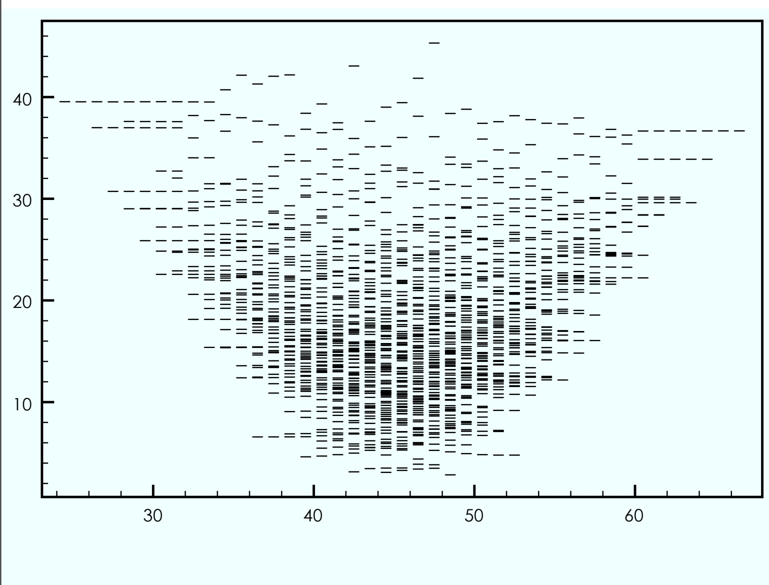


FIG. 4: $\delta V_1 = -0.01$

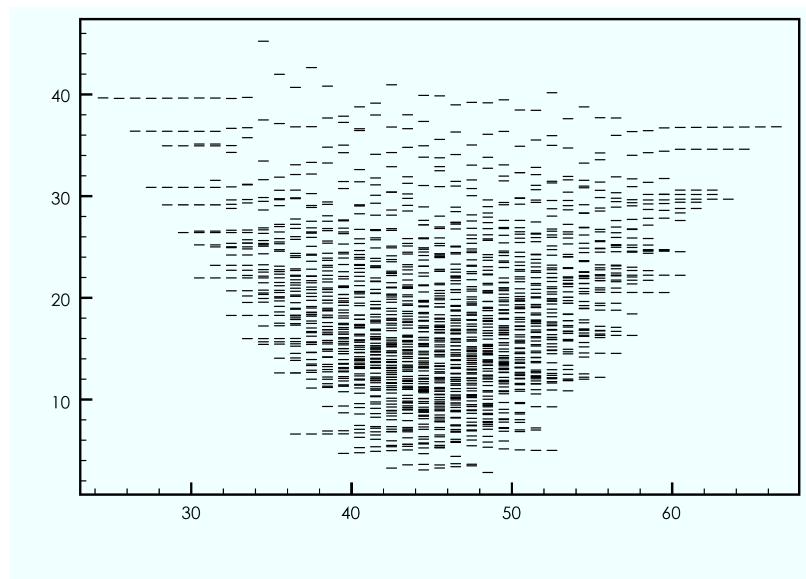


FIG. 5: $\delta V_1 = -0.005$

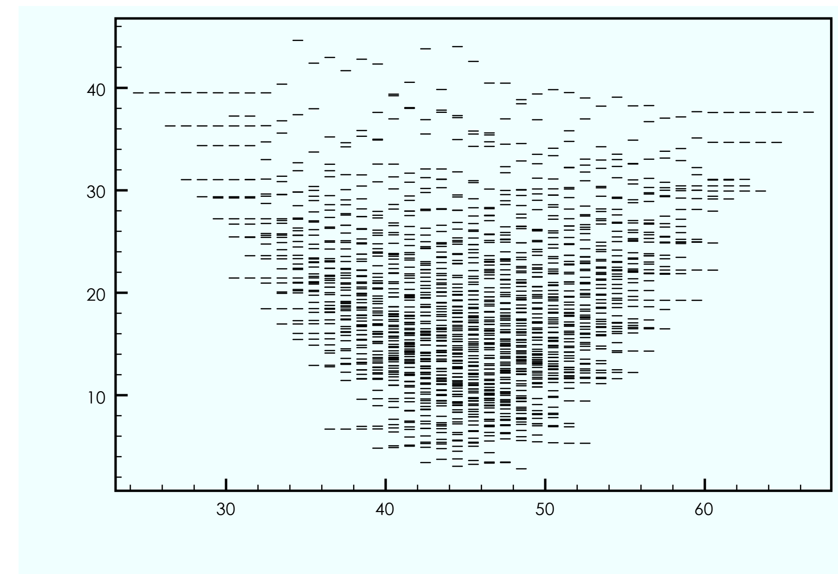


FIG. 6: $\delta V_1 = 0$

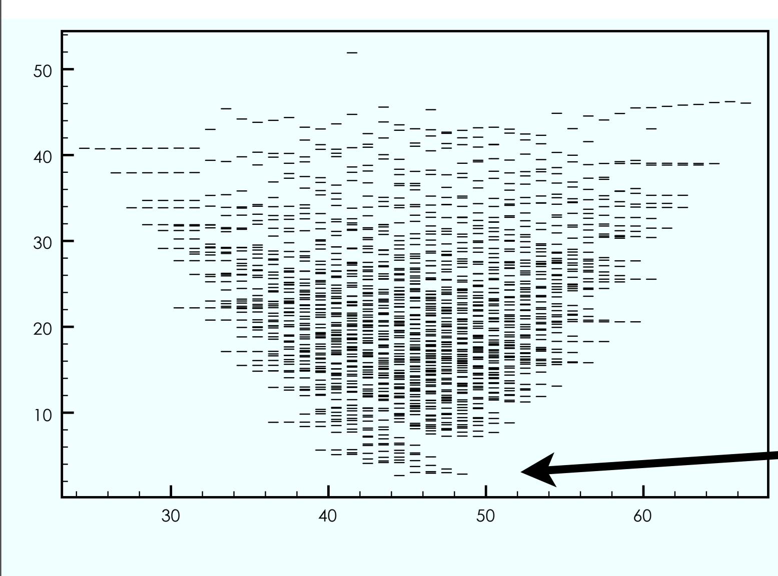


FIG. 7: $\delta V_1 = 0.04$

low-lying entanglement spectrum
matches that of pure MR state