Topological Mechanics

I. Topology and the Wave Equation

II. Topological Phonons and Photons:

- Topological band gaps at finite frequency

III. Classical Mechanical Modes in Isostatic Lattices

- Floppy modes and Maxwell's counting rule
- Twisted Kagome lattice Model
- Topological boundary modes.
- Index theorem
- Analog SSH model

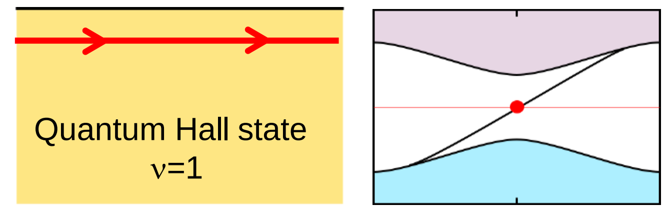
Topological Electronic Phases

Bulk Topological Invariant ⟺ Boundary Topological Modes

2D Integer quantum Hall effect

no symmetry

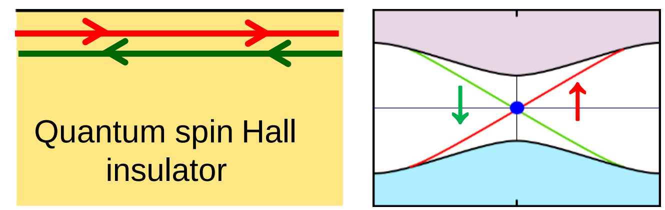
Bulk: Integer Chern invariant
Boundary: Chiral edge states



2D topological insulator

time reversal symmetry

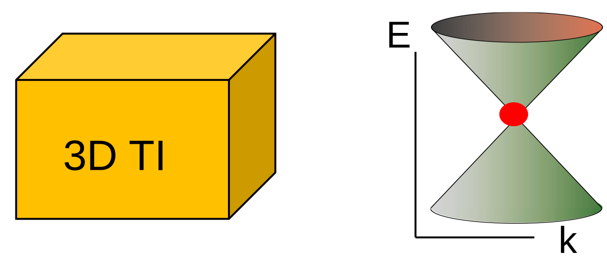
Bulk: Z_2 invariant
Boundary: Helical edge states



3D topological insulator

time reversal symmetry

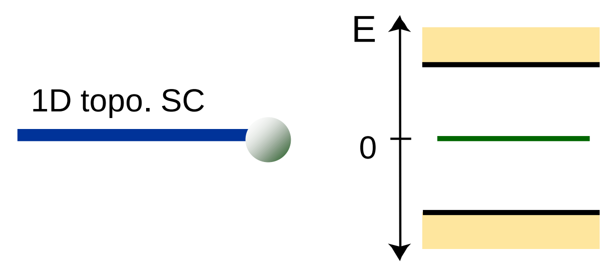
Bulk: Z_2 invariant
Boundary: Helical surface state



1D Topological Superconductor

particle-hole symmetry

Bulk: Z_2 invariant
Boundary: Majorana zero mode



Wave Equations

Quantum:

e.g. Schrodinger Equation $$i\hbar\dot{\psi}_i = H_{ij}\psi_j$$

Classical:

e.g. Newton's Laws $$m\ddot{u}_i = -D_{ij}u_j$$

Common features:

Normal modes define an eigenvalue problem.

Role of symmetry and topology for bulk and boundary modes

Search for periodic "metamaterials" with topological band structures

Periodic Table of Topological Insulators and Superconductors

- Time Reversal : $\Theta H(\mathbf{k})\Theta^{-1} = +H(-\mathbf{k})$; $\Theta^2 = \pm 1$
- Particle - Hole : $\Xi H(\mathbf{k})\Xi^{-1} = -H(-\mathbf{k})$; $\Xi^2 = \pm 1$

Unitary (chiral) symmetry : $\Pi H(\mathbf{k})\Pi^{-1} = -H(\mathbf{k})$; $\Pi \propto \Theta\Xi$

Kitaev, 2008
Schnyder, Ryu, Furusaki, Ludwig 2008

	Symmetry				d							
	AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
Insulator: no symmetry	A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
	AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BdG superconductor	D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
T invariant BdG superconductor	DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
T invariant insulator	AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
	CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Even richer topological classes when accounting for crystalline space group symmetries :
"weak topological insulators", "topological crystalline insulators", ……

Periodic Table of Topological Insulators and Superconductors

- Time Reversal : $\Theta H(\mathbf{k})\Theta^{-1} = +H(-\mathbf{k})$; $\Theta^2 = \pm 1$
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no symmetry	A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
T - invariant	AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
	DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
	AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
	CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Even richer topological classes when accounting for crystalline space group symmetries :
"weak topological insulators", "topological crystalline insulators", ……

Classical Topological Band Phenomena

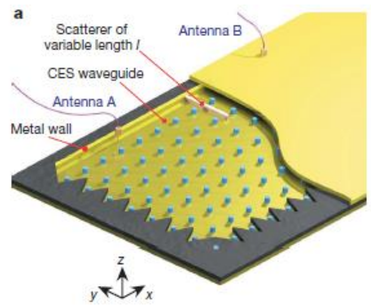
Topological bandgaps and chiral edge modes at finite frequency in classical systems
In two dimensions with broken time reversal symmetry.

Photonic

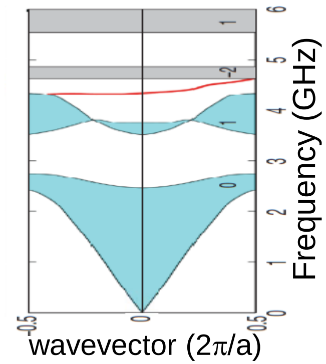
Haldane, Raghu PRL 2008

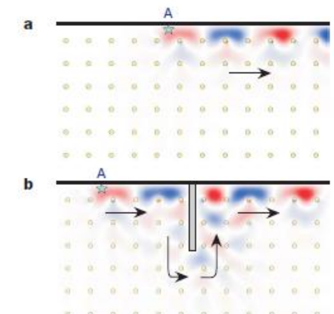
Wang, Chong, Joannopoulos, Soljacic, PRL 2008

…..



Microwave Waveguide





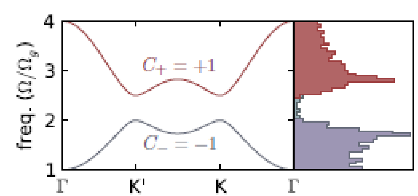
Phononic

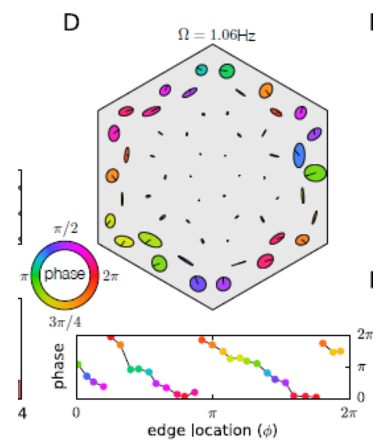
Prodhan, Prodhan, PRL 2009

Nash, Kleckner, Read, Vitelli, Turner, Irvine, PNAS 2015

……..

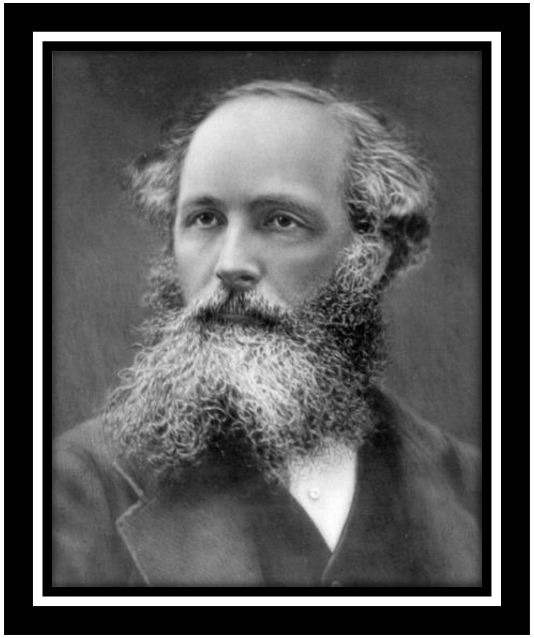
Gyroscopic Metamaterial



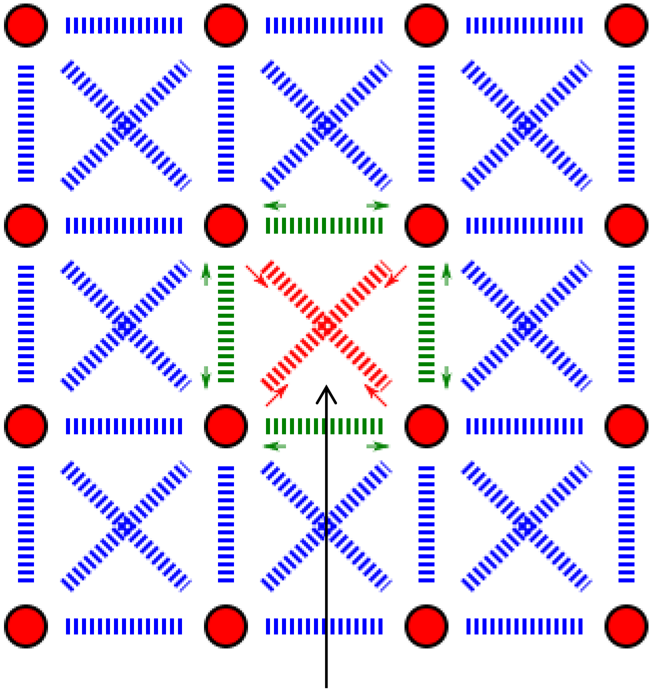


Maxwell Problem

JC Maxwell 1865



Is a “frame” or configuration of masses and springs mechanically stable ?



Maxwell * Counting Rule: (Calladine ’78)

$$N_{fm} - N_{ss} = d\, n_s - n_b$$

n_s = # sites

n_b = # bonds, d = dimension

N_{fm} = # zero frequency “floppy modes”

N_{ss} = # states of self-stress

Proof of Maxwell-Calladine Counting Rule:

elastic energy : $$U = \frac{1}{2} u \cdot D \cdot u = \sum_{m=1}^{n_b} \frac{1}{2} k x_m^2 = \frac{1}{2} u \cdot Q Q^T \cdot u$$

bond extension x_m: $$x_m = \sum_{i=1}^{dn_s} Q_{mi}^T u_i$$

site displacement u_i

dn_s x n_b "Equilibrium Matrix" Q_{im}

site force f_i: $$f_i = -\frac{\partial U}{\partial u_i} = -\sum_{m=1}^{n_b} Q_{im} t_m$$

bond tension $t_m=kx_m$

floppy mode : $$Q^T \cdot u = 0$$

N_{fm} = # zero eigenvectors of Q^T

self-stress state: $$Q \cdot t = 0$$

N_{ss} = # zero eigenvectors of Q

Rank Nullity Theorem of Linear Algebra :

$\nu = N_{fm} - N_{ss}$ = #rows - #columns of Q_{im} = $dn_s - n_b$

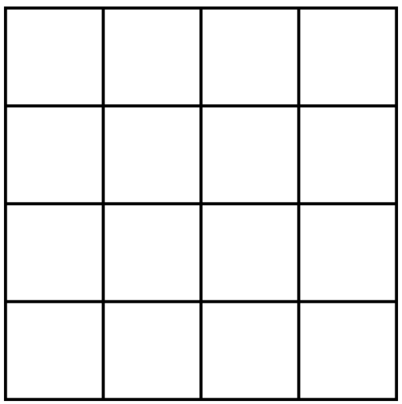
ν = "index" of Q : simplest version of an index theorem.

Periodic Isostatic Lattice

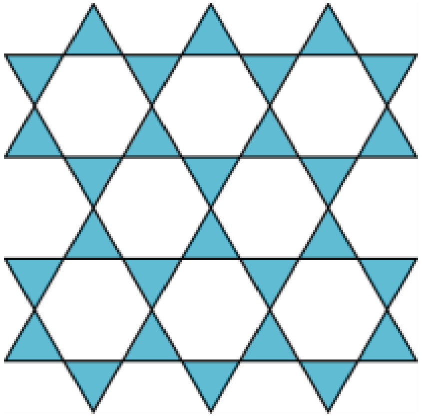
A periodic structure with $dn_s - n_b = 0$

Coordination number (# neighbors): $z = 2d$

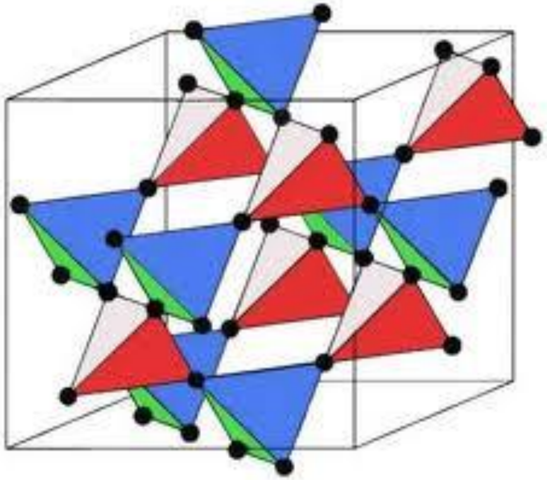
On the verge of mechanical instability



d=2 square lattice (z=4)



d=2 kagome lattice (z=4)



d=3 pyrochlore (z=6)

A model system for problems in soft matter and statistical physics

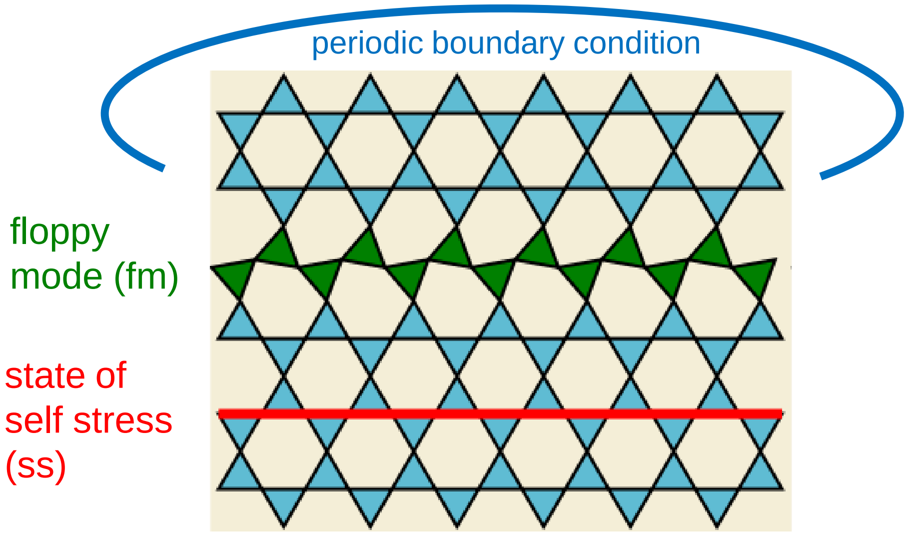
- Rigidity percolation
- Random closed packing, Jamming
- Network glasses

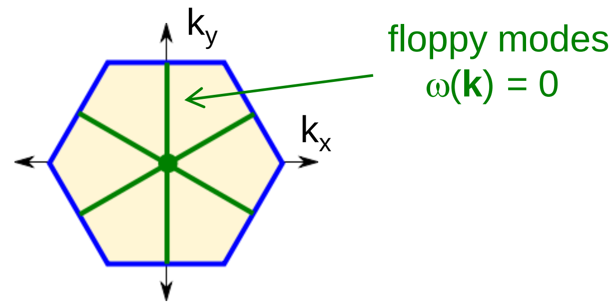
isostatic on the average

Kagome Lattice Model

Sun, Souslov, Mao and Lubensky 2012

Untwisted





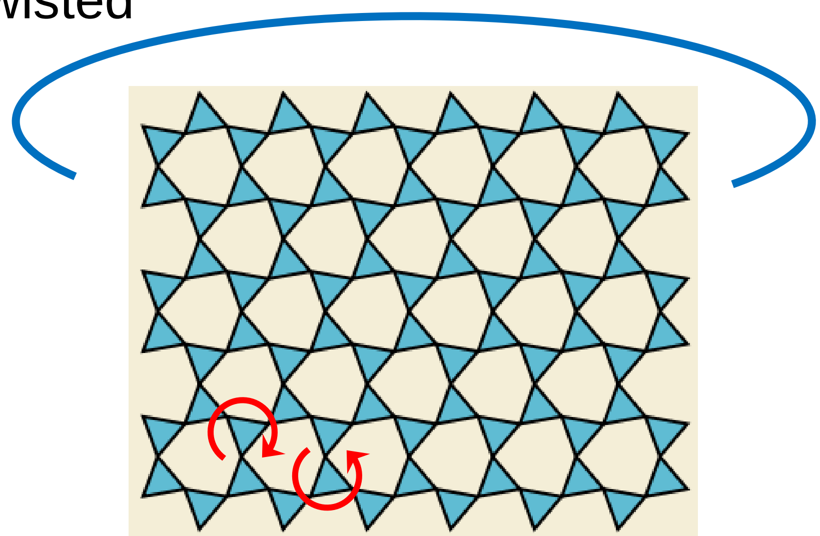
Periodic b.c. : $2\, n_{site} - n_{bond} = 0$

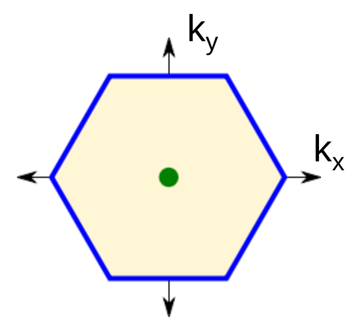
$N_{FM} = N_{SS} \sim L/a$

Open b.c. : $2\, n_{site} - n_{bond} \sim L/a$

$N_{FM} \sim L/a$; $N_{SS} = 0$

Twisted





Periodic b.c. : twisting eliminates both FM and SS

Open b.c. : $N_{FM} \sim L/a$; $N_{SS} = 0$

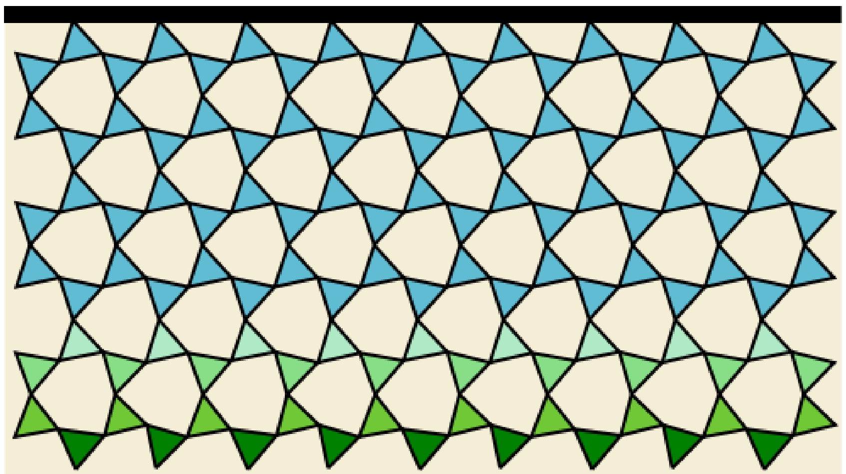
Localized on Boundary

Floppy Modes on a Free Boundary

For twisted Kagome, floppy modes required by Maxwell's count are localized on boundary

Strip Geometry

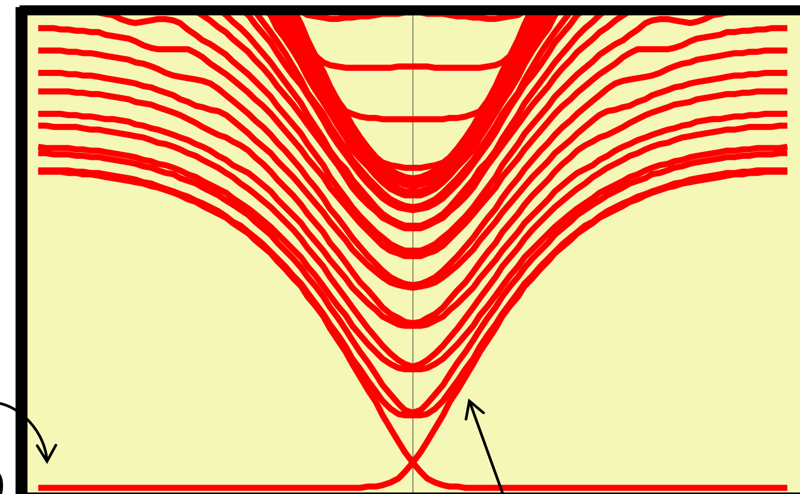
Fixed Boundary



Free Boundary

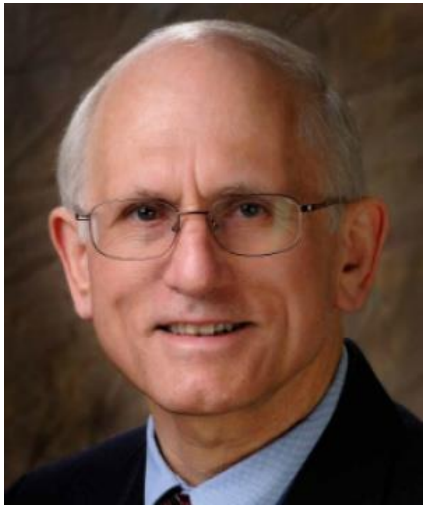
x

Normal Mode Spectrum



zero frequency mode
localized at boundary

bulk acoustic
modes



Tom Lubensky

2012 Tom: Are my boundary modes related to your boundary modes?

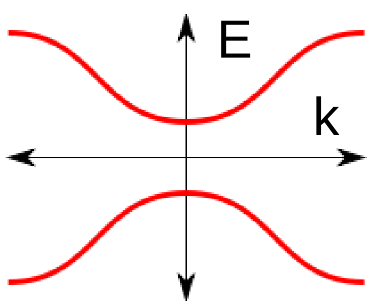
CLK: I don't think so

2013 Tom: Are you sure ?

Schrodinger Equation

$$i\hbar\dot{\psi}_i = H_{ij}\psi_j$$

1st order in time
Hamiltonian H has positive
or negative eigenvalues E

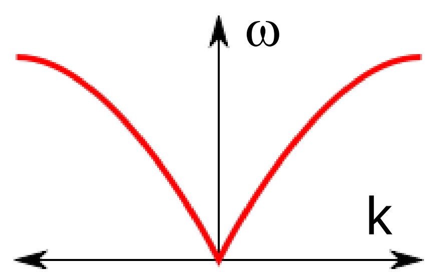


Topologically classify valence band

Newton's Laws

$$m\ddot{u}_i = -D_{ij}u_j$$

2nd order in time
Dynamical matrix D has only
positive eigenvalues $m\omega^2$



No "valence band"

Dirac's Problem :

Klein Gordon Equation $(\vec{p} = -i\vec{\partial})$

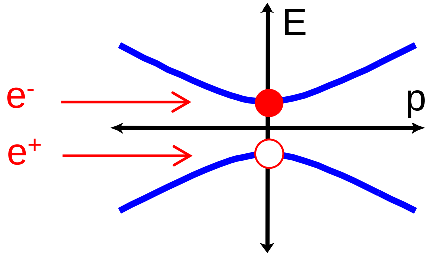
$$-\partial_t^2 \psi = (p_x^2 + p_y^2 + m^2)\psi$$

$$\begin{pmatrix} m & p_x - ip_y \\ p_x + ip_y & -m \end{pmatrix}\begin{pmatrix} m & p_x - ip_y \\ p_x + ip_y & -m \end{pmatrix} = \begin{pmatrix} p_x^2 + p_y^2 + m^2 & 0 \\ 0 & p_x^2 + p_y^2 + m^2 \end{pmatrix}$$

$$\sqrt{(p_x^2 + p_y^2 + m^2)I} = p_x\sigma_x + p_y\sigma_y + m\sigma_z$$

Paul Dirac: "I'm trying to take the *square root* of something"

Dirac's Square Root predicted the anti-electron (= positron)



Our Problem : $-\partial_t^2 u = D \cdot u$

$$D = QQ^T$$

$$U = \frac{1}{2} u \cdot D \cdot u = \frac{1}{2} k \sum_n x_n^2 = \frac{1}{2} u \cdot QQ^T \cdot u$$

"Supersymmetric partners"

$$D = QQ^T \qquad \tilde{D} = Q^T Q$$

D and $\tilde{D}$ have same eigenvalues: ω_n^2

$$QQ^T u_n = \omega_n^2 u_n \quad \Rightarrow \quad Q^T Q (Q^T u_n) = \omega_n^2 (Q^T u_n)$$

Except for zero modes

$QQ^T \cdot u = 0$ floppy mode

$Q^T Q \cdot t = 0$ state of self stress

Equivalent "Quantum Hamiltonian"

$$H = \begin{bmatrix} 0 & Q \\ Q^T & 0 \end{bmatrix} \quad ; \quad H^2 = \begin{bmatrix} QQ^T & 0 \\ 0 & Q^T Q \end{bmatrix}$$

eigenvalues of H :

$$E_n = \pm \omega_n$$

+ both kinds of zero modes

Symmetries

$\left.\begin{array}{l}\text{Time reversal (H=H*)} \\ \text{Particle – Hole (H } \tau^z = -\tau^z \text{ H)}\end{array}\right\}$ Class "BDI" (same as SSH model)

	Symmetry				d							
	AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
	A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
no symmetry	AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
	AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
T - invariant	BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
	D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
	DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
	AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
	CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

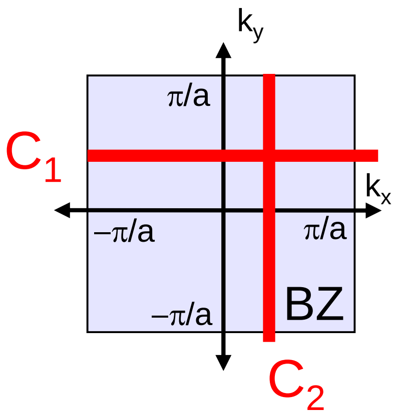
Integer Topological Invariant:

$$Q(k) \in GL(n,C)$$ invertable complex n x n matrix $$0 \neq \det[Q] \in C$$

d=1 : $$n = \frac{1}{2\pi i}\oint_{BZ} \mathrm{Tr}[Q^{-1}(k)dQ(k)] = \frac{1}{2\pi i}\oint_{BZ} dk\ \partial_k \log\left(\det[Q(k)]\right)$$ = winding number of phase of det[Q]

d=3 : $$n = \frac{1}{24\pi^2}\oint_{BZ} \mathrm{Tr}\left[\left(Q^{-1}(k)dQ(k)\right)^{\wedge 3}\right]$$

D=2: "Weak Topological Invariants"



Two independent (1D) winding numbers

$$n_{j=1,2} = \frac{1}{2\pi i}\oint_{C_j} \mathrm{Tr}[Q^{-1}(k)dQ(k)]$$

The two invariants define a *lattice vector*

$$\mathbf{R}_T = n_1\mathbf{a}_1 + n_2\mathbf{a}_2$$

where C_j is along reciprocal lattice generator $\mathbf{b}_j$ with $\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi\delta_{ij}$

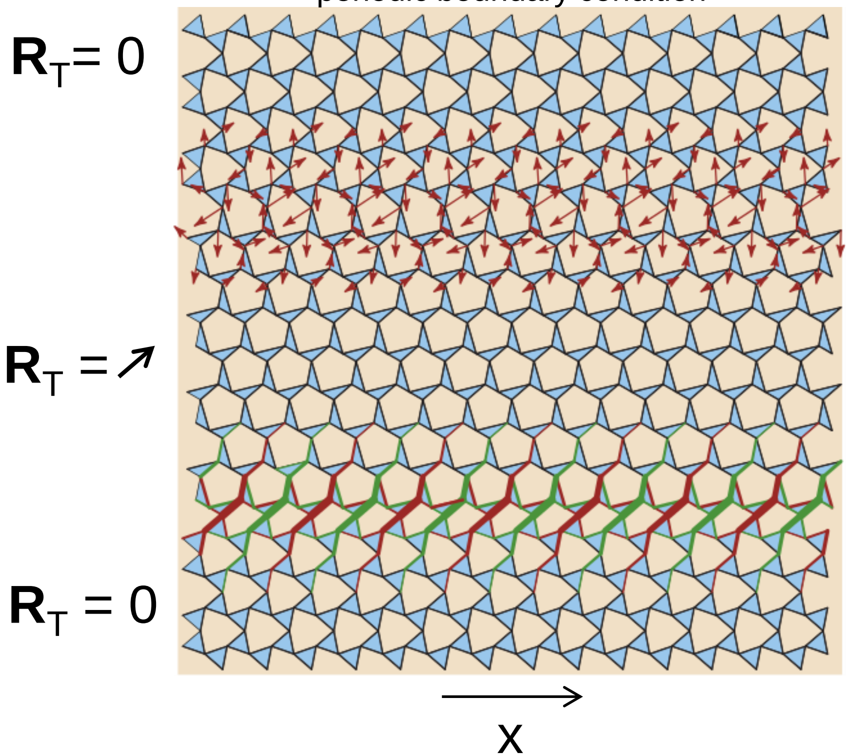
Twisted Kagome lattice model : $\mathbf{R}_T = 0$

New Topological Phases and Domain Walls

Z x Z topological invariant: $\mathbf{R}_T = n_1\, \mathbf{a}_1 + n_2\, \mathbf{a}_2$ (lattice vector)

"Deformed" Kagome lattice model can have : $\mathbf{R}_T \neq 0$

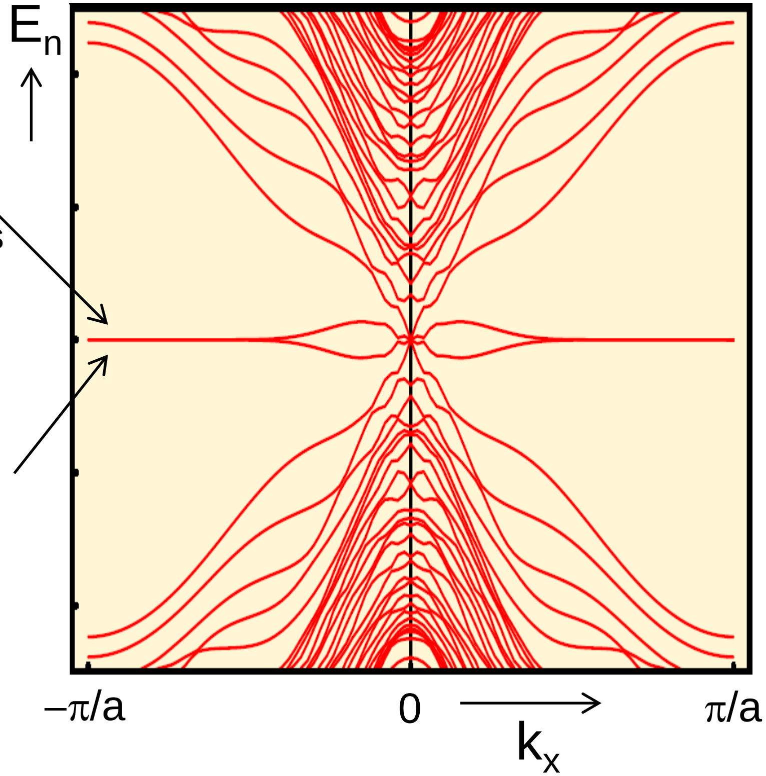
periodic boundary condition



floppy modes

self-stress

Spectrum of $H \sim \sqrt{D}$

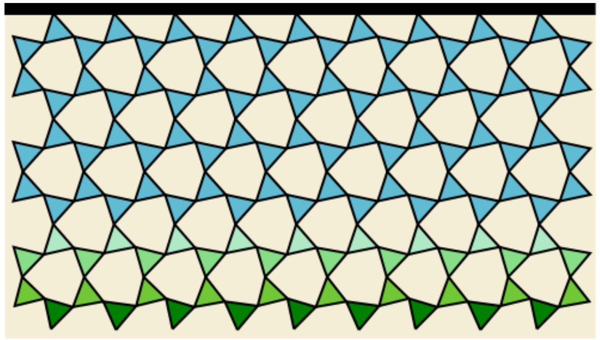


Two Kinds of Zero Modes?

1. Edge modes due to mismatch of # sites and # bonds

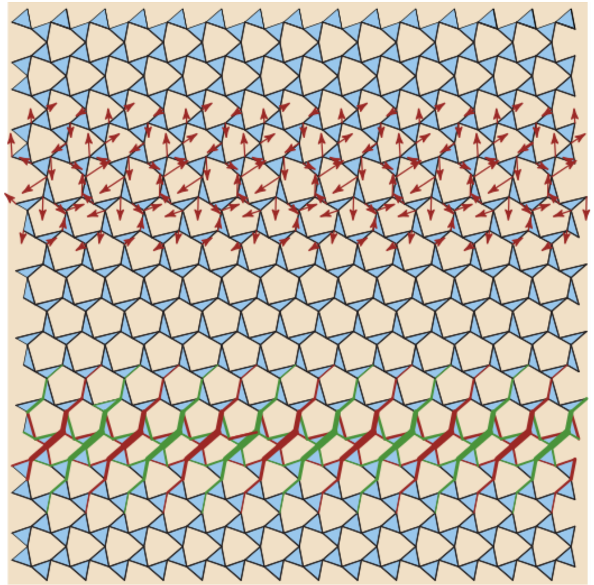
Global count of zero modes:
Maxwell/Calladine rule

$$N_{fm} – N_{ss} = d\ n_s - n_b$$



2. Topological boundary modes

No mismatch in sites
and bonds



Are they related?

Index Theorem

A "local" generalization of the Maxwell-Caladine counting rule

Variant of a famous theorem in mathematics

Attiyah and Singer '63
Callias, Bott and Seeley '78

floppy modes and states of self stress in region S

$$N_{\text{fm}}^S - N_{\text{ss}}^S = \nu_L^S + \nu_T^S$$

"Local count" of sites and bonds in S

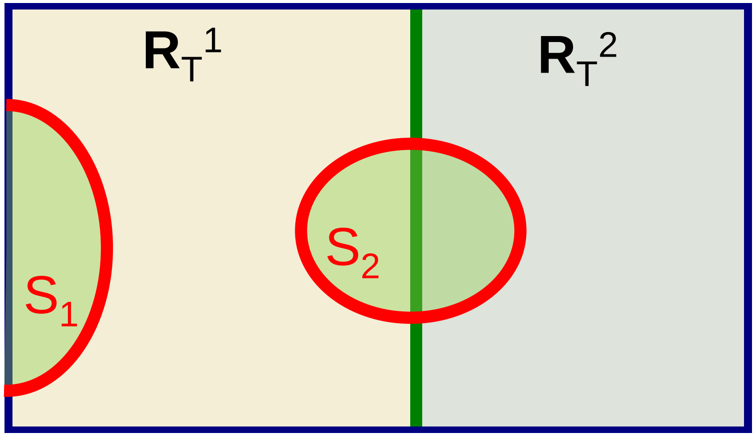
$$\nu_L^S = dn_s^S - n_b^S$$

Depends on edge termination

"Topological count" on boundary of S

$$\nu_T^S = \int_{\partial S} \frac{d^{d-1}S}{V_{\text{cell}}} \hat{n} \cdot \mathbf{R}_T$$

Depends on topological class(es) of bulk



Sketch of proof of Index Theorem

$$H = \begin{pmatrix} 0 & Q^T \\ Q & 0 \end{pmatrix} \qquad \tau^z = \begin{pmatrix} 1_{dn_s} & 0 \\ 0 & -1_{n_b} \end{pmatrix} \qquad \left\{H, \tau^z\right\} = 0 \qquad \hat{\mathbf{r}} = \begin{pmatrix} \mathbf{r}_i \delta_{ii'} & 0 \\ 0 & \mathbf{r}_m \delta_{mm'} \end{pmatrix}$$

Region S defined by “support function”: $\rho_S(\mathbf{r}) = \begin{cases} 1 \text{ for } \mathbf{r} \in S \\ 0 \text{ for } \mathbf{r} \notin S \end{cases}$ $\qquad \hat{\rho}_S = \rho_S(\hat{\mathbf{r}})$

$$\upsilon_S \equiv N_{fm}^S - N_{ss}^S = \mathrm{Tr}\left[\rho_S(\hat{\mathbf{r}})\tau^z \lim_{\varepsilon \to 0} \frac{i\varepsilon}{i\varepsilon + H}\right] = \nu_S^L + \nu_S^T$$

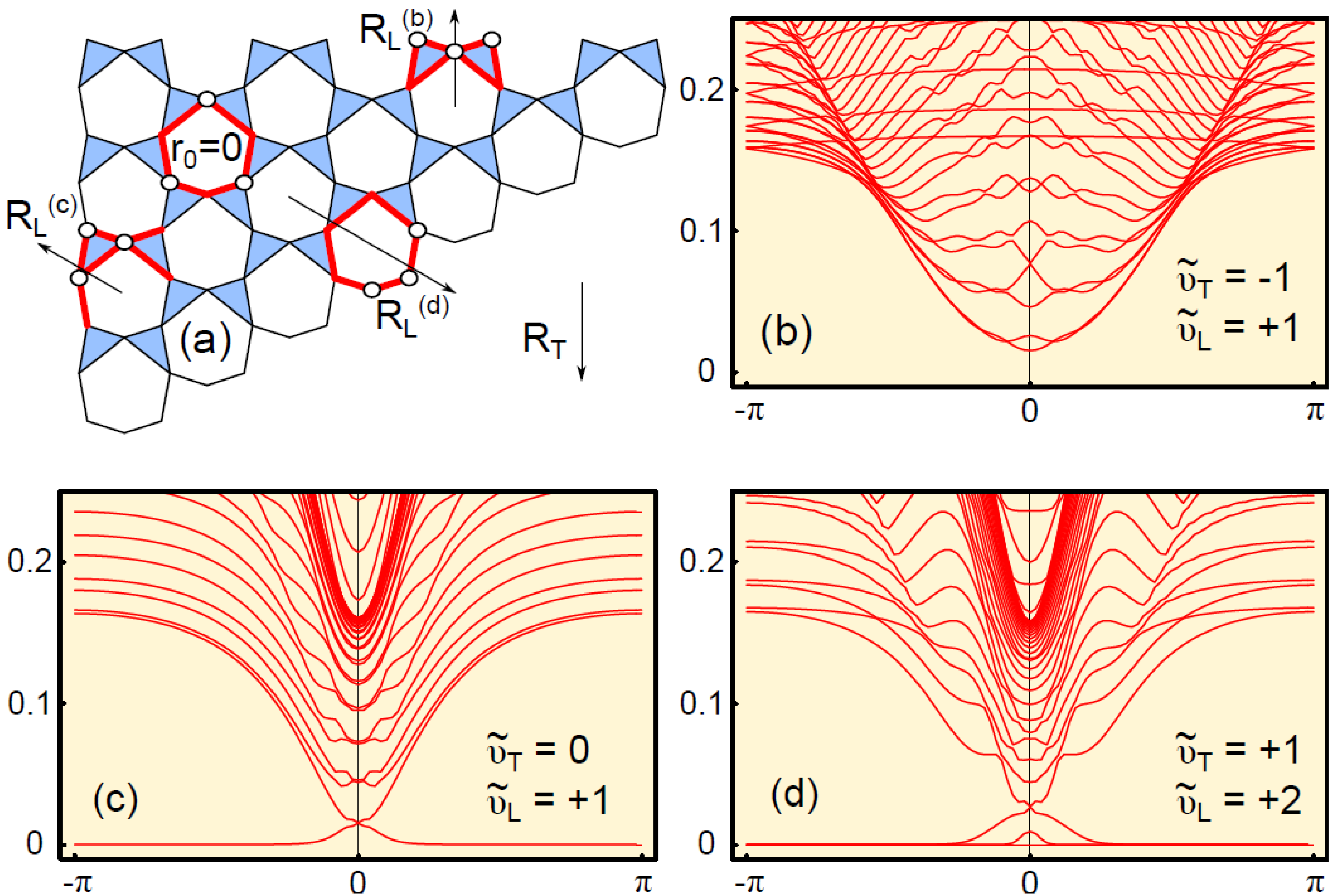
projection onto zero modes

$$\nu_S^L = \mathrm{Tr}\left[\rho_S(\hat{\mathbf{r}})\tau^z\right] = dn_s^S - n_b^S$$

$$\nu_S^T = \nu_S - \nu_S^L = \lim_{\varepsilon \to 0} \mathrm{Tr}\left[\rho_S(\hat{\mathbf{r}})\tau^z \frac{-H}{i\varepsilon + H}\right] = \frac{1}{2}\lim_{\varepsilon \to 0} \mathrm{Tr}\left[[\rho_S(\hat{\mathbf{r}}), H]\tau^z \frac{1}{i\varepsilon + H}\right]$$

Since H is local, contribution comes only from boundary of S. Further manipulation relates the result to the topological polarization integrated around boundary.

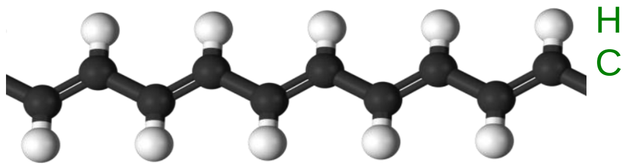
Boundary modes for different edge terminations

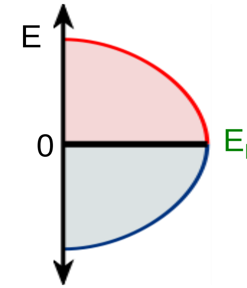


Recall the Su Schrieffer Heeger Model

Polyacetylene: A 1D conducting polymer

- **Undimerized** :

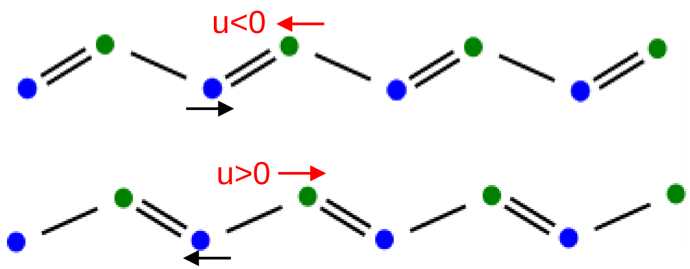


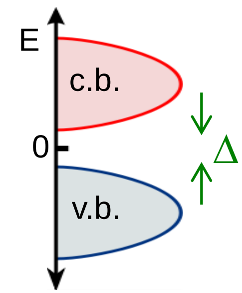


Conductor

Gap $\Delta=0$

- **Dimerized** : $u = \pm u_0$



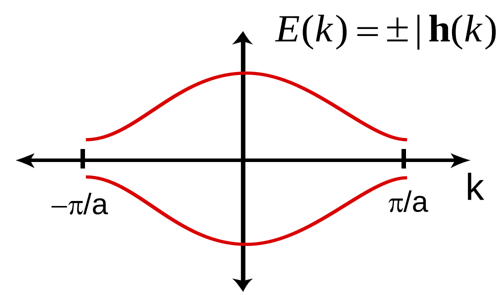


Insulator

Gap $\Delta \sim |u| \neq 0$

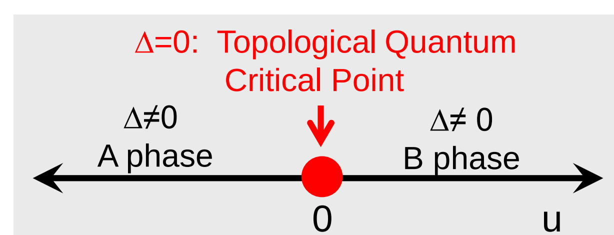
2 band model:

$$H(k) = \begin{pmatrix} 0 & h_x - i h_y \\ h_x + i h_y & 0 \end{pmatrix} = \mathbf{h}(k)\cdot\vec{\sigma}$$



$$\{H(k), \sigma^z\} = 0$$

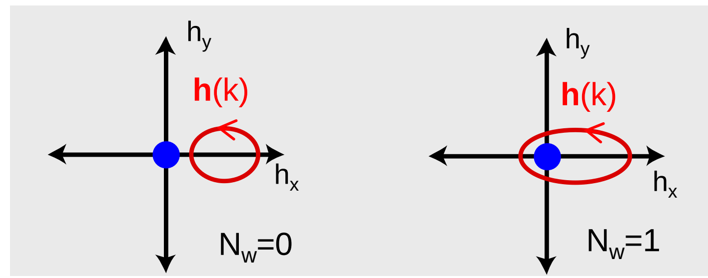
Class BDI



A and B phases are *topologically distinct*

Distinguished by integer* topological invariant

N_w = winding number characterizing $\mathbf{h}(k)$



* Assuming 'particle-hole' symmetry

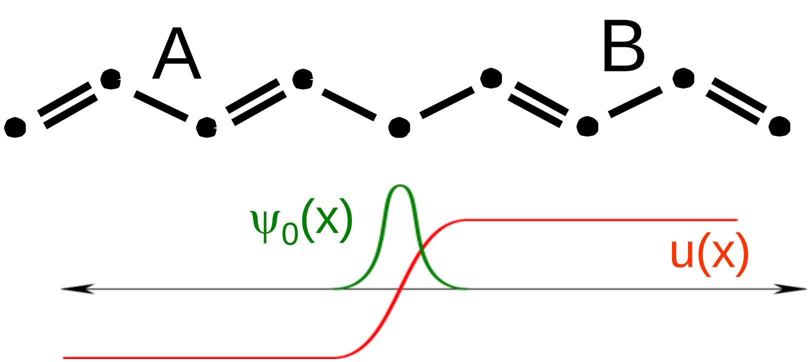
$$H\sigma^z = -\sigma^z H \quad \rightarrow \quad h_z = 0$$

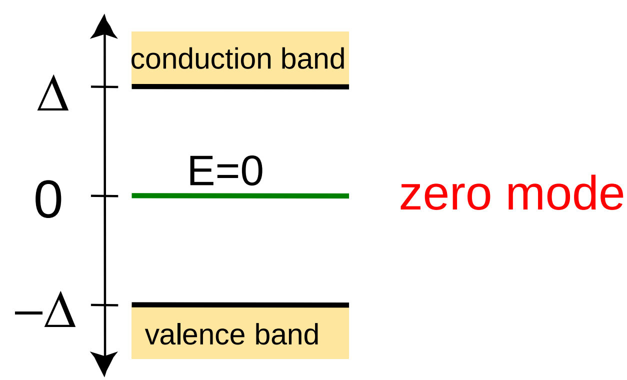
$$\sigma^z |\psi_E\rangle = |\psi_{-E}\rangle \quad \rightarrow$$ spectrum symmetric under E → -E

Bulk Boundary Correspondence :

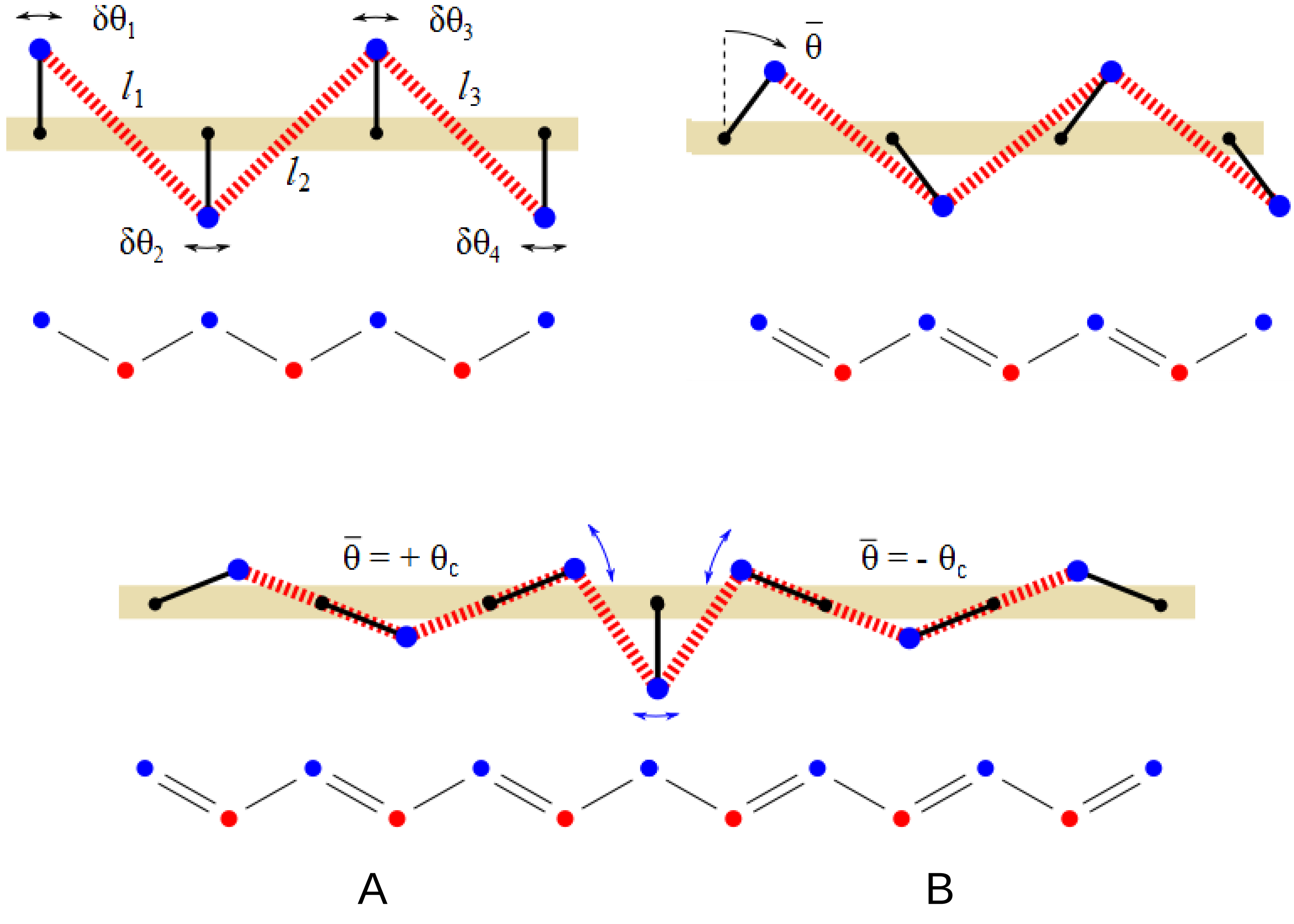
At the boundary between topologically distinct insulating phases, there exist topologically protected low energy states.

Jackiw and Rebbi 76,
Su Schrieffer, Heeger 79





Mechanical Analog of SSH Model



A *model* of the model

B. Chen, N. Upadhyaya, V. Vitelli, PNAS 111, 13004 (2014).

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Conclusion

Topological boundary modes are an elegant consequence of a mathematical structure that has applications in diverse areas

- Topological Electronic Phases
- Mechanical Modes of isostatic systems

Much more to do:

- New materials and experiments on electronic systems
- Experiments on metamaterials?
  - mechanical systems
  - optical, electronic, plasmonic systems?
- Role of interactions and nonlinearities