Stringing together the quantum phases of matter

Lorentz Lectures, Leiden
May 7, 14, 21, June 4, 2012

Subir Sachdev

See also lecture at the 2011 Solvay conference, Theory of the Quantum World, chair D.J. Gross.
100th anniversary of the first Solvay conference, Radiation and the Quanta, chair H.A. Lorentz.
arXiv:1203.4565

Talk online at sachdev.physics.harvard.edu
Sommerfeld-Bloch theory of metals, insulators, and superconductors: many-electron quantum states are adiabatically connected to independent electron states
Modern phases of quantum matter
Not adiabatically connected to independent electron states:
Modern phases of quantum matter

Not adiabatically connected
to independent electron states:

\textit{many-particle, long-range}

\textit{quantum entanglement}
States of quantum matter with long-range entanglement in $d$ spatial dimensions

Useful classification is provided by nature of excitations with vanishing energy:
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1. Gapped systems without zero energy excitations
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2. “Relativistic” systems with zero energy excitations at isolated points in momentum space
States of quantum matter with long-range entanglement in \(d\) spatial dimensions

Useful classification is provided by nature of excitations with vanishing energy:

1. Gapped systems without zero energy excitations

2. “Relativistic” systems with zero energy excitations at isolated points in momentum space

3. “Compressible” systems with zero energy excitations on \(d-1\) dimensional surfaces in momentum space.
States of quantum matter with long-range entanglement in $d$ spatial dimensions

Gapped quantum matter
Spin liquids, quantum Hall states

Conformal quantum matter
Graphene, ultracold atoms, antiferromagnets

Compressible quantum matter
Graphene, strange metals in high temperature superconductors, spin liquids
States of quantum matter with long-range entanglement in $d$ spatial dimensions

**Gapped quantum matter**

*Spin liquids, quantum Hall states*

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*Graphene, ultracold atoms, antiferromagnets*

**Compressible quantum matter**

*Graphene, strange metals in high temperature superconductors, spin liquids*
Gapped quantum matter
Band insulators

An even number of electrons per unit cell
Metals

An odd number of electrons per unit cell
Mott insulator

Emergent excitations

An odd number of electrons per unit cell but electrons are localized by Coulomb repulsion; state has long-range entanglement
Mott insulator: Triangular lattice antiferromagnet

\[ H = J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j \]

Nearest-neighbor model has non-collinear Neel order
Mott insulator: Triangular lattice antiferromagnet

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Imagine quantum fluctuations are so strong that the Neel order does not have long-range correlations.

Naive “classical” picture: we obtain a quantum disordered state in which all spin-spin correlations decay exponentially over a short length scale.
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Modern “quantum” understanding: the discrete quantum degrees of freedom require a state with long-range entanglement.
Mott insulator: Triangular lattice antiferromagnet

non-collinear Néel state

$Z_2$ spin liquid with neutral $S = 1/2$ spinons and \textit{vison} excitations

Mott insulator: Triangular lattice antiferromagnet

Spin liquid obtained in a generalized spin model with $S=1/2$ per unit cell

\[ \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \]

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Excitations of the $Z_2$ Spin liquid

Spinon: $S=1/2$, charge 0

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A vison

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Visons are the \textit{dark matter} of spin liquids: they likely carry most of the energy, but are very hard to detect because they do not carry charge or spin.

\textbf{A vison}

\begin{itemize}
\item Visons are the \textit{dark matter} of spin liquids: they likely carry most of the energy, but are very hard to detect because they do not carry charge or spin.
\end{itemize}

\textsc{Excitations of the Z}_2 \textsc{Spin liquid}

Topological order in the $\mathbb{Z}_2$ spin liquid ground state

4-fold degeneracy on the torus
Topological order in the $\mathbb{Z}_2$ spin liquid ground state

4-fold degeneracy on the torus
Topological order in the $\mathbb{Z}_2$ spin liquid ground state

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Topological order in the $\mathbb{Z}_2$ spin liquid ground state

$$\rho_A = \text{Tr}_B \rho = \text{density matrix of region } A$$

Entanglement entropy $S_{EE} = -\text{Tr} (\rho_A \ln \rho_A)$
Entanglement entropy of a band insulator:

\[ S_{EE} = aL - \exp(-bL) \]

where \( L \) is the perimeter of the boundary between A and B.
Entanglement entropy of a $Z_2$ spin liquid:

$$S_{EE} = aL - \ln(2)$$

where $L$ is the perimeter of the boundary between A and B. The $\ln(2)$ is a universal characteristic of the $Z_2$ spin liquid, and implies long-range quantum entanglement.

Topological order in the $\mathbb{Z}_2$ spin liquid ground state

These properties of the ground state can be described by effective theories:

- **deconfined phase of a $\mathbb{Z}_2$ gauge theory**
  
  
  

- **topological doubled Chern-Simons gauge theory**
  
  
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- **Good recent numerical evidence of $\mathbb{Z}_2$ spin liquid on kagome and square lattices**
  - J. Hong-Chen Jiang, Hong Yao, and L. Balents, arXiv:1112.2241.
Promising experimental candidate: the kagome antiferromagnet

Young Lee, APS meeting, March 2012

ZnCu$_3$(OH)$_6$Cl$_2$ (also called Herbertsmithite)
Quantum Hall states

Similar topological properties, but no time-reversal symmetry:

- ground state degeneracy on a torus
- universal entanglement entropy
- gapless edge states on spaces with boundaries (can also happen for some spin liquids)
- topological Chern-Simons gauge theories
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A. Field theory: graphene

B. Field theory: superfluid-insulator transition

C. Field theory: antiferromagnets

D. Gauge-gravity duality
Conformal quantum matter

A. Field theory: graphene

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D. Gauge-gravity duality
Honeycomb lattice
(describes graphene after adding long-range Coulomb interactions)

\[ H = -t \sum_{\langle ij \rangle} c^\dagger_{i\alpha} c_{j\alpha} + U \sum_i \left( n_{i\uparrow} - \frac{1}{2} \right) \left( n_{i\downarrow} - \frac{1}{2} \right) \]
Semi-metal with massless Dirac fermions at small $U/t$
We define the Fourier transform of the fermions by

\[ c_A(k) = \sum_r c_A(r) e^{-i k \cdot r} \quad (4) \]

and similarly for \( c_B \). \( A \) and \( B \) are sublattice indices.

The hopping Hamiltonian is

\[
H_0 = -t \sum_{\langle ij \rangle} \left( c_{Ai\alpha}^\dagger c_{Bj\alpha} + c_{Bj\alpha}^\dagger c_{Ai\alpha} \right) \quad (5)
\]

where \( \alpha \) is a spin index. If we introduce Pauli matrices \( \tau^a \) in sublattice space \((a = x, y, z)\), this Hamiltonian can be written as

\[
H_0 = \int \frac{d^2k}{4\pi^2} c^\dagger(k) \left[ -t \left( \cos(k \cdot e_1) + \cos(k \cdot e_2) + \cos(k \cdot e_3) \right) \tau^x \\
+ t \left( \sin(k \cdot e_1) + \sin(k \cdot e_2) + \sin(k \cdot e_3) \right) \tau^y \right] c(k) \quad (6)
\]

The low energy excitations of this Hamiltonian are near \( k \approx \pm Q_1 \).
In terms of the fields near $Q_1$ and $-Q_1$, we define

$$
\begin{align*}
\Psi_{A1\alpha}(k) &= c_{A\alpha}(Q_1 + k) \\
\Psi_{A2\alpha}(k) &= c_{A\alpha}(-Q_1 + k) \\
\Psi_{B1\alpha}(k) &= c_{B\alpha}(Q_1 + k) \\
\Psi_{B2\alpha}(k) &= c_{B\alpha}(-Q_1 + k)
\end{align*}
$$

(7)

We consider $\Psi$ to be an 8 component vector, and introduce Pauli matrices $\rho^a$ which act in the 1, 2 valley space. Then the Hamiltonian is

$$
H_0 = \int \frac{d^2k}{4\pi^2} \bar{\Psi}^{\dagger}(k) \left( \nu \tau^y k_x + \nu \tau^x \rho^z k_y \right) \Psi(k),
$$

(8)

where $\nu = 3t/2$; below we set $\nu = 1$. Now define $\bar{\Psi} = \Psi^{\dagger} \rho^z \tau^z$. Then we can write the imaginary time Lagrangian as

$$
\mathcal{L}_0 = -i \bar{\Psi} \left( \omega \gamma_0 + k_x \gamma_1 + k_y \gamma_2 \right) \Psi
$$

(9)

where

$$
\begin{align*}
\gamma_0 &= -\rho^z \tau^z \\
\gamma_1 &= \rho^z \tau^x \\
\gamma_2 &= -\tau^y
\end{align*}
$$

(10)
Exercise: Observe that $\mathcal{L}_0$ is invariant under the scaling transformation $x' = xe^{-\ell}$ and $\tau' = \tau e^{-\ell}$. Write the Hubbard interaction $U$ in terms of the Dirac fermions, and show that it has the tree-level scaling transformation $U' = Ue^{-\ell}$. So argue that all short-range interactions are *irrelevant* in the Dirac semi-metal phase.
The theory of free Dirac fermions is invariant under conformal transformations of spacetime. This is a realization of a simple conformal field theory in 2+1 dimensions: a CFT3.
The Hubbard Model at large $U$

$$H = -\sum_{i,j} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + U \sum_i \left( n_{i\uparrow} - \frac{1}{2} \right) \left( n_{i\downarrow} - \frac{1}{2} \right) - \mu \sum_i c_{i\alpha}^\dagger c_{i\alpha}$$

In the limit of large $U$, and at a density of one particle per site, this maps onto the Heisenberg antiferromagnet

$$H_{AF} = \sum_{i<j} J_{ij} S_i^a S_j^a$$

where $a = x, y, z$,

$$S_i^a = \frac{1}{2} c_{i\alpha}^\dagger \sigma_{\alpha\beta}^a c_{i\beta},$$

with $\sigma^a$ the Pauli matrices and

$$J_{ij} = \frac{4t_{ij}^2}{U}$$
Dirac semi-metal

Insulating antiferromagnet with Neel order

$U/t$
Antiferromagnetism

We use the operator equation (valid on each site $i$):

$$U \left(n_\uparrow - \frac{1}{2}\right) \left(n_\downarrow - \frac{1}{2}\right) = -\frac{2U}{3} S^{a2} + \frac{U}{4}$$  \hspace{1cm} (11)

Then we decouple the interaction via

$$\exp \left(\frac{2U}{3} \sum_i \int d\tau S_i^{a2}\right) = \int D J_i^a(\tau) \exp \left(-\sum_i \int d\tau \left[\frac{3}{8U} J_i^{a2} - J_i^a S_i^a\right]\right)$$  \hspace{1cm} (12)

We now integrate out the fermions, and look for the saddle point of the resulting effective action for $J_i^a$.

Long wavelength fluctuations about this saddle point are described by a field theory of the Néel order parameter, $\varphi^a$, coupled to the Dirac fermions in the Gross-Neveu model.

$$\mathcal{L} = \bar{\Psi} \gamma_\mu \partial_\mu \Psi + \frac{1}{2} \left[(\partial_\mu \varphi^a)^2 + s \varphi^{a2}\right] + \frac{u}{24} (\varphi^{a2})^2 - \lambda \varphi^a \bar{\Psi} \rho_z \sigma^a \Psi$$

Dirac semi-metal

\[ \langle \varphi^a \rangle = 0 \]

Insulating antiferromagnet with Neel order

\[ \langle \varphi^a \rangle \neq 0 \]
At the quantum critical point, the non-linear couplings $\lambda$ and $u$ in the Gross-Neveu model reach non-zero fixed-point values under the renormalization group flow. The critical theory is an *interacting* CFT3.
Dirac semi-metal

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Free CFT3

Interacting CFT3 with long-range entanglement

Monday, May 14, 2012
Long-range entanglement: entanglement entropy obeys \( S_{EE} = aL - \gamma \), where \( \gamma \) is a universal number associated with the CFT3.

An analysis of this quantum critical point requires a RG analysis which goes beyond treexlevely. Such an analysis can be controlled in an expansion in $1/N$ where $N$ is the number of fermion flavors or $s3−d$ where $d$ is the spatial dimensionality. Such analyses show that the couplings $u$ and $λ$ reach a RG fixed point which describes a conformal field theory (CFT). An important result of such an analysis is the following structure in the electron Green’s function:

$$G(k, ω) = \langle ψ(k, ω); ψ^+(k, ω) \rangle \sim \frac{iω + v k_x τ^y + v k_y τ^x ρ^z}{(ω^2 + v^2 k_x^2 + v^2 k_y^2)^{1−η/2}}$$

where $η > 0$ is the anomalous dimension of the fermion. Note that this leads to a fermion spectral density which has no quasiparticle pole: thus the quantum critical point has no well-defined quasiparticle excitations.

Electron Green’s function for the interacting CFT3

Monday, May 14, 2012
\[ \text{Im} G(k, \omega) \]
Quantum phase transition described by a strongly-coupled conformal field theory without well-defined quasiparticles

Dirac semi-metal

\[ \langle \varphi^a \rangle = 0 \]

Insulating antiferromagnet with Neel order

\[ \langle \varphi^a \rangle \neq 0 \]
Electrical transport

The conserved electrical current is

\[ J_\mu = -i \overline{\Psi} \gamma_\mu \Psi. \]  \hspace{1cm} (1)

Let us compute its two-point correlator, \( K_{\mu\nu}(k) \) at a spacetime momentum \( k_\mu \) at \( T = 0 \). At leading order, this is given by a one fermion loop diagram which evaluates to

\[
K_{\mu\nu}(k) = \int \frac{d^3p}{8\pi^3} \text{Tr} \left[ \gamma_\mu (i\gamma_\lambda p_\lambda + m\rho^z\sigma^z) \gamma_\nu (i\gamma_\delta (k_\delta + p_\delta) + m\rho^z\sigma^z) \right] \frac{1}{(p^2 + m^2)((p + k)^2 + m^2)}
\]

\[
= -\frac{2}{\pi} \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \int_0^1 dx \frac{k^2 x (1 - x)}{\sqrt{m^2 + k^2 x (1 - x)}}, \]  \hspace{1cm} (2)

where the mass \( m = 0 \) in the semi-metal and at the quantum critical point, while \( m = |\lambda N_0| \) in the insulator. Note that the current correlation is purely transverse, and this follows from the requirement of current conservation

\[ k_\mu K_{\mu\nu} = 0. \]  \hspace{1cm} (3)
Of particular interest to us is the $K_{00}$ component, after analytic continuation to Minkowski space where the spacetime momentum $k_{\mu}$ is replaced by $(\omega, k)$. The conductivity is obtained from this correlator via the Kubo formula

$$
\sigma(\omega) = \lim_{k \to 0} \frac{-i\omega}{k^2} K_{00}(\omega, k).
$$

(4)

In the insulator, where $m > 0$, analysis of the integrand in Eq. (2) shows that the spectral weight of the density correlator has a gap of $2m$ at $k = 0$, and the conductivity in Eq. (4) vanishes. These properties are as expected in any insulator. In the metal, and at the critical point, where $m = 0$, the fermionic spectrum is gapless, and so is that of the charge correlator. The density correlator in Eq. (2) and the conductivity in Eq. (4) evaluate to the simple universal results

$$
K_{00}(\omega, k) = \frac{1}{4} \frac{k^2}{\sqrt{k^2 - \omega^2}},
$$

$$
\sigma(\omega) = \frac{1}{4}.
$$

(5)

Going beyond one-loop, we find no change in these results in the
semi-metal to all orders in perturbation theory. At the quantum critical point, there are no anomalous dimensions for the conserved current, but the amplitude does change yielding

\[ K_{00}(\omega, k) = \mathcal{K} \frac{k^2}{\sqrt{k^2 - \omega^2}} \]

\[ \sigma(\omega) = \mathcal{K}, \]

where \( \mathcal{K} \) is a universal number dependent only upon the universality class of the quantum critical point. The value of the \( \mathcal{K} \) for the Gross-Neveu model is not known exactly, but can be estimated by computations in the \((3 - d)\) or \(1/N\) expansions.
Dirac semi-metal

\[ \langle \varphi^a \rangle = 0 \]

Interacting CFT3 with long-range entanglement

Insulating antiferromagnet with Neel order

\[ \langle \varphi^a \rangle \neq 0 \]
Dirac semi-metal

$\langle \varphi^a \rangle = 0$

Insulating antiferromagnet with Neel order

$\langle \varphi^a \rangle \neq 0$

$\sigma(\omega) = \frac{\pi e^2}{2\hbar}$

$\sigma(\omega) = \frac{\kappa e^2}{\hbar}$
Phase diagram at non-zero temperatures

- Quantum critical
- Insulator with thermally excited spin waves
- Semi-metal
- Néel order
Phase diagram at non-zero temperatures

**Insulator with thermally excited spin waves**

**Quantum critical**

**Semi-metal**

**Néel order**
Phase diagram at non-zero temperatures

\[ \sigma(\omega \gg T) = \frac{\pi e^2}{2h} \]

- Insulator with thermally excited spin waves
- Quantum critical
- Semi-metal

Néel order
Phase diagram at non-zero temperatures

\[ \sigma(\omega \gg T) = \frac{\pi e^2}{2h} \]

\[ \sigma(\omega \gg T) = \frac{\kappa e^2}{\hbar} \]
Optical conductivity of graphene

Optical conductivity of graphene

Non-zero temperatures

At the quantum-critical point at one-loop order, we can set \( m = 0 \), and then repeat the computation in Eq. (2) at \( T > 0 \). This only requires replacing the integral over the loop frequency by a summation over the Matsubara frequencies, which are quantized by odd multiples of \( \pi T \). Such a computation, via Eq. (4) leads to the conductivity

\[
\text{Re}[\sigma(\omega)] = (2T \ln 2) \delta(\omega) + \frac{1}{4} \tanh \left( \frac{|\omega|}{4T} \right) ; \tag{7}
\]

the imaginary part of \( \sigma(\omega) \) is the Hilbert transform of \( \text{Re}[\sigma(\omega)] - 1/4 \). Note that this reduces to Eq. (5) in the limit \( \omega \gg T \). However, the most important new feature of Eq. (7) arises for \( \omega \ll T \), where we find a delta function at zero frequency in the real part. Thus the d.c. conductivity is infinite at this order, arising from the collisionless transport of thermally excited carriers.
Electrical transport in a free CFT3 for $T > 0$

$\sigma \sim T \delta(\omega)$
Particle hole symmetry: current carrying state has zero momentum, and collisions can relax current to zero.
Electrical transport for a (weakly) interacting CFT3

\[ \sigma(\omega, T) = \frac{e^2}{\hbar} \sum \left( \frac{\hbar \omega}{k_B T} \right) ; \quad \Sigma \rightarrow \text{a universal function} \]

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\[ \mathcal{O}\left((u^*)^2\right), \]

where \( u^* \) is the fixed point interaction.

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\[ \mathcal{O}\left(\frac{1}{(u^*)^2}\right) \]

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Monday, May 14, 2012
Electrical transport for a (weakly) interacting CFT3

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\[ \mathcal{O}\left( (u^*)^2 \right) \]

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Electrical transport for a (weakly) interacting CFT3

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\[ \text{O}(1/(u^*)^2) \]

Needed: a method for computing the d.c. conductivity of interacting CFT3s (including that of pure graphene!)

Conformal quantum matter

A. Field theory: graphene

B. Field theory: superfluid-insulator transition

C. Field theory: antiferromagnets

D. Gauge-gravity duality
Conformal quantum matter

A. Field theory: graphene

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D. Gauge-gravity duality
Superfluid-insulator transition

Ultracold $^{87}\text{Rb}$ atoms - bosons

The Superfluid-Insulator transition

Boson Hubbard model

Degrees of freedom: Bosons, $b_j^\dagger$, hopping between the sites, $j$, of a lattice, with short-range repulsive interactions.

$$H = -t \sum_{\langle ij \rangle} b_i^\dagger b_j - \mu \sum_j n_j + \frac{U}{2} \sum_j n_j (n_j - 1) + \cdots$$

$$n_j \equiv b_j^\dagger b_j$$

$$[b_j, b_k^\dagger] = \delta_{jk}$$

Excitations of the insulator:

\[ S = \int d^2 r d\tau \left[ |\partial_\tau \psi|^2 + v^2 |\vec{\nabla} \psi|^2 + (g - g_c) |\psi|^2 + \frac{u}{2} |\psi|^4 \right] \]

$\langle \psi \rangle \neq 0$ 

Superfluid 

$\langle \psi \rangle = 0$ 

Insulator 

$0 \quad g_c \quad g$
\[ \langle \psi \rangle \neq 0 \quad \text{Superfluid} \]

\[ \langle \psi \rangle = 0 \quad \text{Insulator} \]

\[ g_c \quad \text{CFT3} \]
CFT3 at $T>0$

- **Superfluid**
- **Insulator**
- **Quantum critical**

Diagram showing the phase transition between superfluid and insulator phases with temperature ($T$) and coupling constant ($g$). The critical point is marked as $g_c$. The $T_{KT}$ line separates the superfluid and quantum critical phases.
Quantum critical transport

Quantum “nearly perfect fluid” with shortest possible equilibration time, $\tau_{eq}$

$$\tau_{eq} = \mathcal{C} \frac{\hbar}{k_B T}$$

where $\mathcal{C}$ is a universal constant

Quantum “nearly perfect fluid” with shortest possible equilibration time, $\tau_{eq}$

$$\tau_{eq} = C \frac{\hbar}{k_B T}$$

where $C$ is a universal constant

Quantum critical transport

Quantum “nearly perfect fluid” with shortest possible equilibration time, $\tau_{eq}$

$$\tau_{eq} = C \frac{\hbar}{k_B T}$$

where $C$ is a universal constant

Zaanen: Planckian dissipation

Quantum critical transport

Transport coefficients not determined by collision rate, but by universal constants of nature

\[ \sigma = \frac{Q^2}{\hbar} \times [\text{Universal constant } \mathcal{O}(1)] \]

(Q is the “charge” of one boson)

Quantum critical transport

Transport co-efficient not determined by collision rate, but by universal constants of nature

Momentum transport

\[ \eta \equiv \frac{\text{viscosity}}{s} \times \text{entropy density} \]

\[ \eta = \frac{\hbar}{k_B} \times [\text{Universal constant } O(1)] \]

Quantum critical transport

Describe charge transport using Boltzmann theory of interacting bosons:

\[
\frac{dv}{dt} + \frac{v}{\tau_c} = F.
\]

This gives a frequency (\(\omega\)) dependent conductivity

\[
\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau_c}
\]

where \(\tau_c \sim \hbar/(k_B T)\) is the time between boson collisions.
Quantum critical transport

Describe charge transport using Boltzmann theory of interacting bosons:

$$\frac{dv}{dt} + \frac{v}{\tau_c} = F.$$ 

This gives a frequency ($\omega$) dependent conductivity

$$\sigma(\omega) = \frac{\sigma_0}{1 - i \omega \tau_c}$$

where $\tau_c \sim \hbar/(k_B T)$ is the time between boson collisions.

Also, we have $\sigma(\omega \to \infty) = \sigma_\infty$, associated with the density of states for particle-hole creation (the “optical conductivity”) in the CFT3.
Electrical transport in a free-field theory for $T > 0$

\[ \sigma \sim T \delta(\omega) \]
Boltzmann theory of bosons

\[ \text{Re}[\sigma(\omega)] \]

\[ \sigma_0 \]

\[ \frac{1}{\tau_c} \]

\[ \sigma_\infty \]
So far, we have described the quantum critical point using the boson particle and hole excitations of the insulator.

\[ \langle \psi \rangle \neq 0 \quad \text{Superfluid} \]

\[ \langle \psi \rangle = 0 \quad \text{Insulator} \]

\[ g_c \]

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However, we could equally well describe the conductivity using the excitations of the superfluid, which are \textit{vortices}. 

\begin{align*}
\langle \psi \rangle \neq 0 & \quad \text{Superfluid} \\
\langle \psi \rangle = 0 & \quad \text{Insulator}
\end{align*}
However, we could equally well describe the conductivity using the excitations of the superfluid, which are *vortices*.

These are quantum particles (in 2+1 dimensions) which described by a (mirror/e.m.) “dual” CFT3 with an emergent U(1) gauge field. Their $T > 0$ dynamics can also be described by a Boltzmann equation:

\[
\langle \psi \rangle \neq 0 \quad \text{Superfluid}
\]

\[
\langle \psi \rangle = 0 \quad \text{Insulator}
\]
Boltzmann theory of bosons

\[ \text{Re}[\sigma(\omega)] \]

\( \sigma_0 \)

\( 1/\tau_c \)

\( \omega \)

\( \sigma_\infty \)
Boltzmann theory of vortices

\[ \text{Re}[\sigma(\omega)] \]

\[ \frac{1}{\sigma_{0v}} \]

\[ \frac{1}{\tau_{cv}} \]

\[ \sigma_{\infty} \]
Boltzmann theory of bosons
Boltzmann theory of bosons

\[ \text{Re}[\sigma(\omega)] \]

\[ \sigma_0 \]

\[ 1/\tau_c \]

\[ \sigma_\infty \]
Boltzmann theory of bosons

\[
\text{Re}[\sigma(\omega)] \quad \sigma_0 \quad \omega
\]

Needed:

a method for computing the d.c. conductivity of interacting CFT3s (including that of the boson Hubbard model !)
Conformal quantum matter

A. Field theory: graphene

B. Field theory: superfluid-insulator transition

C. Field theory: antiferromagnets

D. Gauge-gravity duality
Conformal quantum matter

A. Field theory: graphene

B. Field theory: superfluid-insulator transition

C. Field theory: antiferromagnets

D. Gauge-gravity duality
Quantum critical point in a frustrated square lattice antiferromagnet

Néel state

Valence bond solid (VBS) state with a nearly gapless, emergent “photon”

Long-range entanglement described by a CFT3 with an emergent U(1) “photon”

Quantum critical point in a frustrated square lattice antiferromagnet

Néel state

Valence bond solid (VBS) state

with a nearly gapless, emergent “photon”

Critical theory for photons and deconfined spinons:

\[
S_z = \int d^2 r d\tau \left[ |(\partial_\mu - iA_\mu)z_\alpha|^2 + s|z_\alpha|^2 + u(|z_\alpha|^2)^2 + \frac{1}{2\epsilon_0^2}(\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)^2 \right]
\]

Circular symmetry is evidence for emergent $U(1)$ photon.
Conformal quantum matter

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D. Gauge-gravity duality
Conformal quantum matter

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C. Field theory: antiferromagnets

D. Gauge-gravity duality
Field theories in $d+1$ spacetime dimensions are characterized by couplings $g$ which obey the renormalization group equation

$$u \frac{dg}{du} = \beta(g)$$

where $u$ is the energy scale. The RG equation is *local* in energy scale, *i.e.* the RHS does not depend upon $u$. 
Key idea: ⇒ Implement $r$ as an extra dimension, and map to a local theory in $d + 2$ spacetime dimensions.
For a relativistic CFT in $d$ spatial dimensions, the metric in the holographic space is uniquely fixed by demanding the following scale transformation \((i = 1 \ldots d)\)

\[ x_i \rightarrow \zeta x_i \ , \quad t \rightarrow \zeta t \ , \quad ds \rightarrow ds \]
For a relativistic CFT in $d$ spatial dimensions, the metric in the holographic space is uniquely fixed by demanding the following scale transformation $(i = 1 \ldots d)$

$$x_i \rightarrow \zeta x_i \ , \ t \rightarrow \zeta t \ , \ ds \rightarrow ds$$

This gives the unique metric

$$ds^2 = \frac{1}{r^2} (-dt^2 + dr^2 + dx_i^2)$$

Reparametrization invariance in $r$ has been used to the prefactor of $dx_i^2$ equal to $1/r^2$. This fixes $r \rightarrow \zeta r$ under the scale transformation. This is the metric of the space $\text{AdS}_{d+2}$. 
AdS/CFT correspondence

$\text{AdS}_{d+2}$

$\mathbb{R}^{d,1}$

Minkowski

$CFT_{d+1}$
AdS/CFT correspondence

$AdS_4$

$R^{2,1}$

Minkowski

CFT3

$r$
This emergent spacetime is a solution of Einstein gravity with a negative cosmological constant

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \left( R + \frac{6}{L^2} \right) \right] \]
There is a family of solutions of Einstein gravity which describe non-zero temperatures.

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \left( R + \frac{6}{L^2} \right) \right] \]
There is a family of solutions of Einstein gravity which describe non-zero temperatures. A 2+1 dimensional system at its quantum critical point:

\[ k_B T = \frac{3\hbar}{4\pi R}. \]

The metric is given by:

\[ ds^2 = \left( \frac{L}{r} \right)^2 \left[ \frac{dr^2}{f(r)} - f(r) dt^2 + dx^2 + dy^2 \right] \]

with \[ f(r) = 1 - \left( \frac{r}{R} \right)^3 \]
AdS/CFT correspondence at non-zero temperatures

AdS\(_4\)-Schwarzschild black-brane

\[ ds^2 = \left( \frac{L}{r} \right)^2 \left[ \frac{dr^2}{f(r)} - f(r)dt^2 + dx^2 + dy^2 \right] \]

with \( f(r) = 1 - (r/R)^3 \)

Black-brane at temperature of 2+1 dimensional quantum critical system

\[ k_B T = \frac{3\hbar}{4\pi R} \]
AdS/CFT correspondence at non-zero temperatures

AdS$_4$-Schwarzschild black-brane

A 2+1 dimensional system at its quantum critical point:

\[ k_B T = \frac{3\hbar}{4\pi R}. \]

Black-brane at temperature of 2+1 dimensional quantum critical system

Beckenstein-Hawking entropy of black brane = entropy of CFT3

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AdS/CFT correspondence at non-zero temperatures

A 2+1 dimensional system at its quantum critical point:

\[ k_B T = \frac{3\hbar}{4\pi R}. \]

AdS\(_4\)-Schwarzschild black-brane

Black-brane at temperature of 2+1 dimensional quantum critical system

Friction of quantum criticality = waves falling into black brane
AdS$_4$ theory of “nearly perfect fluids”

To leading order in a gradient expansion, charge transport in an infinite set of strongly-interacting CFT3s can be described by Einstein-Maxwell gravity/electrodynamics on AdS$_4$-Schwarzschild

$$S_{EM} = \int d^4 x \sqrt{-g} \left[ -\frac{1}{4g_4^2} F_{ab} F^{ab} \right].$$

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\[ S = S_{CFT} + i \int dxdydt A_\mu J_\mu \]

AdS4 theory of electrical transport in a strongly interacting CFT3 for $T > 0$.

Conductivity is independent of $\omega/T$. 
AdS4 theory of electrical transport in a strongly interacting CFT3 for $T > 0$

Consequence of self-duality of Maxwell theory in 3+1 dimensions

Electrical transport in a free CFT3 for $T > 0$

$\sigma \approx T \delta(\omega)$

Complementary $\omega$-dependent conductivity in the free theory
Improving the AdS$_4$ theory of “nearly perfect fluids”

To leading order in a gradient expansion, charge transport in an infinite set of strongly-interacting CFT3s can be described by Einstein-Maxwell gravity/electrodynamics on AdS$_4$-Schwarzschild.

To leading order in a gradient expansion, charge transport in an infinite set of strongly-interacting CFT3s can be described by Einstein-Maxwell gravity/electrodynamics on AdS$_4$-Schwarzschild

We include all possible 4-derivative terms: after suitable field redefinitions, the required theory has only one dimensionless constant $\gamma$ ($L$ is the radius of AdS$_4$):

$$S_{EM} = \int d^4x \sqrt{-g} \left[ -\frac{1}{4g_4^2} F_{ab} F^{ab} + \frac{\gamma L^2}{g_4^2} C_{abcd} F^{ab} F^{cd} \right],$$

where $C_{abcd}$ is the Weyl curvature tensor. Stability and causality constraints restrict $|\gamma| < 1/12$.

Improving the AdS$_4$ theory of “nearly perfect fluids”

The $\gamma > 0$ result has similarities to the quantum-Boltzmann result for transport of particle-like excitations.

The $\gamma < 0$ result can be interpreted as the transport of vortex-like excitations.
The $\gamma = 0$ case is the exact result for the large $N$ limit of SU($N$) gauge theory with $\mathcal{N} = 8$ supersymmetry (the ABJM model). The $\omega$-independence is a consequence of self-duality under particle-vortex duality ($S$-duality).

Stability constraints on the effective theory ($|\gamma| < 1/12$) allow only a limited $\omega$-dependence in the conductivity.
Theory for transport of conserved quantities in CFT3s:

\[ S_{EM} = \int d^4 x \sqrt{-g} \left[ -\frac{1}{4g_4^2} F_{ab} F^{ab} + \frac{\gamma L^2}{g_4^2} C_{abcd} F^{ab} F^{cd} \right], \]

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where \( C_{abcd} \) is the Weyl curvature tensor.

**General approach:**

- Theory has 2 free dimensionless parameters: \( g_4^2 \) and \( \gamma \). We match these to correlators of the CFT3 of interest at \( \omega \gg T \): \( g_4^2 \) is determines the current correlator \( \langle J_\mu J_\nu \rangle \), while \( \gamma \) determines the 3-point function \( \langle T_{\mu\nu} J_\rho J_\sigma \rangle \), where \( T_{\mu\nu} \) is the stress-energy tensor.

Improving the AdS$_4$ theory of “nearly perfect fluids”

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Improving the AdS$_4$ theory of “nearly perfect fluids”

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- We determine these $\omega \gg T$ correlators of the CFT3 by other methods (e.g. vector large $N$ expansion), and so obtain values of $g_4^2$ and $\gamma$.

- We use $S_{EM}$ to extrapolate to transport properties for $\omega \ll T$. This step is traditionally carried out by descendants of the Boltzmann equation.
Conclusions

Conformal quantum matter
Conclusions

Conformal quantum matter

New insights and solvable models for diffusion and transport of strongly interacting systems near quantum critical points
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The description is far removed from, and complementary to, that of the quantum Boltzmann equation which builds on the quasiparticle/vortex picture.
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Conformal quantum matter

New insights and solvable models for diffusion and transport of strongly interacting systems near quantum critical points

The description is far removed from, and complementary to, that of the quantum Boltzmann equation which builds on the quasiparticle/vortex picture.

Prospects for experimental tests of frequency-dependent, non-linear, and non-equilibrium transport
States of quantum matter with long-range entanglement in $d$ spatial dimensions

Gapped quantum matter

Spin liquids, quantum Hall states

Conformal quantum matter

Graphene, ultracold atoms, antiferromagnets

Compressible quantum matter

Graphene, strange metals in high temperature superconductors, spin liquids
States of quantum matter with long-range entanglement in $d$ spatial dimensions

**Gapped quantum matter**

*Spin liquids, quantum Hall states*

**Conformal quantum matter**

*Graphene, ultracold atoms, antiferromagnets*

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Conformal quantum matter

A. Fermi liquids: graphene

B. Holography: Reissner - Nördstrom solution

C. Non-Fermi liquids: nematic critical point (and U(1) spin liquids)

D. Holography: scaling arguments for entropy and entanglement entropy
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Dirac semi-metal

\[ \langle \varphi^a \rangle = 0 \]

Insulating antiferromagnet with Neel order

\[ \langle \varphi^a \rangle \neq 0 \]

Free CFT3

Interacting CFT3 with long-range entanglement

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Quantum phase transition in graphene tuned by a chemical potential (gate voltage)

Dirac semi-metal
Quantum phase transition in graphene tuned by a chemical potential (gate voltage)
Quantum phase transition in graphene tuned by a chemical potential (gate voltage)

Hole Fermi surface

Electron Fermi surface

$\mu < 0$

$\mu > 0$
Dirac semi-metal
\[ \langle \varphi^a \rangle = 0 \]

Insulating antiferromagnet with Neel order
\[ \langle \varphi^a \rangle \neq 0 \]

Free CFT3

Interacting CFT3 with long-range entanglement

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Insulating antiferromagnet

Free CFT3

Interacting CFT3
Free CFT3

Electron metal

Interacting CFT3

Hole metal

Insulating antiferromagnet

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Transport in graphene at non-zero $\mu$

From the Kubo formula

$$\sigma(\omega) = 2 (e v_F)^2 \frac{\hbar}{i} \sum_{s s'} \int \frac{d^2 k}{4\pi^2} \frac{f(\varepsilon_s(k)) - f(\varepsilon_{s'}(k))}{(\varepsilon_s(k) - \varepsilon_{s'}(k))(\varepsilon_s(k) - \varepsilon_{s'}(k) + \hbar \omega + i\eta)}$$

where $\varepsilon_s(k) = s\hbar v_F |k|$ and $s, s' = \pm 1$ for the valence and conduction bands.

A is inversely proportional to disorder. In the clean limit $A \to \infty$, at $T = 0$

$$\text{Re}[\sigma(\omega)] = \frac{e^2}{\hbar} \left[ \frac{\varepsilon_F}{\hbar} \delta(\omega) + \frac{1}{4} \theta(|\omega| - 2\varepsilon_F) \right]$$

Notice delta function is present even at $T = 0$ at non-zero density: this is a generic consequence of the conservation of momentum in any clean interacting Fermi liquid. Only “umklapp” scattering can broaden this delta function.
Current carrying state has non-zero momentum, and collisions cannot relax current to zero.
Optical conductivity of graphene

Conformal quantum matter

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Begin with a CFT
Holographic representation: AdS$_4$

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \left( R + \frac{6}{L^2} \right) \right] \]
Holographic representation: AdS$_4$

A 2+1 dimensional CFT at $T=0$

\[ ds^2 = \left( \frac{L}{r} \right)^2 \left[ \frac{dr^2}{f(r)} - f(r)dt^2 + dx^2 + dy^2 \right] \]

with $f(r) = 1$
Apply a chemical potential

$\mu > 0$
AdS$_4$ theory of “nearly perfect fluids”

To leading order in a gradient expansion, charge transport in an infinite set of strongly-interacting CFT3s can be described by Einstein-Maxwell gravity/electrodynamics on AdS$_4$-Schwarzschild

$$S_{EM} = \int d^4 x \sqrt{-g} \left[ -\frac{1}{4g_4^2} F_{ab} F^{ab} \right].$$

This is to be solved subject to the constraint

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where $A_\mu$ is a source coupling to a conserved U(1) current $J_\mu$ of the CFT3

$$S = S_{CFT} + i \int dx dy dt A_\mu J_\mu$$
**AdS$_4$ theory of “nearly perfect fluids”**

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where $A_\mu$ is a source coupling to a conserved U(1) current $J_\mu$ of the CFT3

\[
S = S_{CFT} + i \int dxdydt A_\mu J_\mu
\]

At non-zero chemical potential we simply require $A_\tau = \mu$. 

---
The Maxwell-Einstein theory of the applied chemical potential yields a AdS$_4$-Reissner-Nordström black-brane

\[ \mathcal{E}_r = \langle Q \rangle \]

\[ S = \int d^4 x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \left( R + \frac{6}{L^2} \right) - \frac{1}{4g_4^2} F_{ab} F^{ab} \right] \]

The Maxwell-Einstein theory of the applied chemical potential yields a AdS$_4$-Reissner-Nordström black-brane

\[ dS^2 = \left( \frac{L}{r} \right)^2 \left[ \frac{dr^2}{f(r)} - f(r)dt^2 + dx^2 + dy^2 \right] \]

with \( f(r) = \left( 1 - \frac{r}{R} \right)^2 \left( 1 + \frac{2r}{R} + \frac{3r^2}{R^2} \right) \) and \( R = \frac{\sqrt{6Lg_4}}{\kappa \mu} \), and \( A_\tau = \mu \left( 1 - \frac{r}{R} \right) \).
The Maxwell-Einstein theory of the applied chemical potential yields a $\text{AdS}_4$-Reissner-Nordström black-brane.

At $T = 0$, we obtain an extremal black-brane, with a near-horizon (IR) metric of $\text{AdS}_2 \times R^2$

$$ds^2 = \frac{L^2}{6} \left( -dt^2 + \frac{dr^2}{r^2} \right) + dx^2 + dy^2$$

Compute conductivity using response to a time-dependent vector potential as a function of $\omega/T$ and $\mu/T$. 

Compute conductivity using response to a time-dependent vector potential as a function of $\omega/T$ and $\mu/T$.

$$\sigma(\omega) = \frac{e^2 \pi \rho^2}{\hbar (\varepsilon + P)} \delta(\omega)$$

where $\rho$ is the number density, $\varepsilon$ is the energy density, and $P$ is the pressure.

Optical conductivity of graphene

Features of AdS$_2 \times R^2$

- Has non-zero entropy density at $T = 0$, and “volume” law for entanglement entropy.

- Green’s function of a probe fermion (a *mesino*) can have a Fermi surface, but self energies are momentum independent, and the singular behavior is the same on and off the Fermi surface.

- Deficit of order $\sim N^2$ in the volume enclosed by the mesino Fermi surfaces: presumably associated with “hidden Fermi surfaces” of gauge-charged particles (the *quarks*).

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High temperature superconductors

\[ \text{YBa}_2\text{Cu}_3\text{O}_{6+x} \]
Evolution of the (ARPES) Fermi surface on the cuprate phase diagram

Smaller hole Fermi-pockets

Large hole Fermi surface
Evolution of the (ARPES) Fermi surface on the cuprate phase diagram

Small hole Fermi-pockets

Large hole Fermi surface

K.M. Shen et al., Science 2005

M. Platé et al., PRL 2005
Evolution of the (ARPES) Fermi surface on the cuprate phase diagram

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Smaller hole Fermi-pockets

Large hole Fermi surface

Temperature [K]

AF insulator

$T_N$

$T^*$

Superconductor

$T_c$

Hole doping, $\rho$

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Broken rotational symmetry in the pseudogap phase of a high-Tc superconductor

R. Daou, J. Chang, David LeBoeuf, Olivier Cyr-Choiniere, Francis Laliberte, Nicolas Doiron-Leyraud, B. J. Ramshaw, Ruixing Liang, D. A. Bonn, W. N. Hardy, and Louis Taillefer

STM measurements of $Z(r)$, the energy asymmetry in density of states in Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$.

STM measurements of $Z(r)$, the energy asymmetry in density of states in Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$.


$O_N = Z_A + Z_B - Z_C - Z_D$
STM measurements of $Z(r)$, the energy asymmetry in density of states in $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$.

Strong anisotropy of electronic states between $x$ and $y$ directions: Electronic “Ising-nematic” order

$O_N = Z_A + Z_B - Z_C - Z_D$

Quantum criticality of Ising-nematic ordering

Fermi surface with full square lattice symmetry
Quantum criticality of Ising-nematic ordering

Spontaneous elongation along $x$ direction:
Quantum criticality of Ising-nematic ordering

Spontaneous elongation along $y$ direction:
Ising-nematic order parameter

\[ \phi \sim \int d^2 k \left( \cos k_x - \cos k_y \right) c_{k\sigma}^\dagger c_{k\sigma} \]

Measures spontaneous breaking of square lattice point-group symmetry of underlying Hamiltonian
Quantum criticality of Ising-nematic ordering

Spontaneous elongation along $x$ direction:
Ising order parameter $\phi > 0$. 
Quantum criticality of Ising-nematic ordering

Spontaneous elongation along $y$ direction:
Ising order parameter $\phi < 0$. 

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Quantum criticality of Ising-nematic ordering

Pomeranchuk instability as a function of coupling $r$

or

$\langle \phi \rangle \neq 0$

$\langle \phi \rangle = 0$
Quantum criticality of Ising-nematic ordering

Phase diagram as a function of $T$ and $r$
Quantum criticality of Ising-nematic ordering

Phase diagram as a function of $T$ and $r$
Quantum criticality of Ising-nematic ordering

Phase diagram as a function of $T$ and $r$
Quantum criticality of Ising-nematic ordering

Phase diagram as a function of $T$ and $r$
Quantum criticality of Ising-nematic ordering

Phase diagram as a function of $T$ and $r$
Quantum criticality of Ising-nematic ordering

Phase diagram as a function of $T$ and $r$
Quantum criticality of Ising-nematic ordering

Effective action for Ising order parameter

\[ S_\phi = \int d^2r d\tau \left[ (\partial_\tau \phi)^2 + c^2(\nabla \phi)^2 + (\lambda - \lambda_c)\phi^2 + u\phi^4 \right] \]
Quantum criticality of Ising-nematic ordering

Effective action for Ising order parameter

\[ S_\phi = \int d^2r d\tau \left[ (\partial_\tau \phi)^2 + c^2(\nabla \phi)^2 + (\lambda - \lambda_c)\phi^2 + u\phi^4 \right] \]

Effective action for electrons:

\[ S_c = \int d\tau \sum_{\alpha=1}^{N_f} \left[ \sum_i c_{i\alpha}^\dagger \partial_\tau c_{i\alpha} - \sum_{i<j} t_{ij} c_{i\alpha}^\dagger c_{i\alpha} \right] \]

\[ \equiv \sum_{\alpha=1}^{N_f} \sum_k \int d\tau c_{k\alpha}^\dagger \left( \partial_\tau + \varepsilon_k \right) c_{k\alpha} \]
Quantum criticality of Ising-nematic ordering

Coupling between Ising order and electrons

\[ S_{\phi_c} = -\gamma \int d\tau \sum_{\alpha=1}^{N_f} \sum_{k,q} \phi_q (\cos k_x - \cos k_y) c^\dagger_{k+q/2,\alpha} c_{k-q/2,\alpha} \]

for spatially dependent \( \phi \)

\[ \langle \phi \rangle > 0 \]

\[ \langle \phi \rangle < 0 \]
Quantum criticality of Ising-nematic ordering

\[ S_\phi = \int d^2r d\tau \left[ (\partial_\tau \phi)^2 + c^2(\nabla \phi)^2 + (\lambda - \lambda_c)\phi^2 + u\phi^4 \right] \]

\[ S_c = \sum_{\alpha=1}^{N_f} \sum_k \int d\tau c_{k\alpha}^\dagger (\partial_\tau + \varepsilon_k) c_{k\alpha} \]

\[ S_{\phi c} = -\gamma \int d\tau \sum_{\alpha=1}^{N_f} \sum_{k,q} \phi_q (\cos k_x - \cos k_y) c_{k+q/2,\alpha}^\dagger c_{k-q/2,\alpha} \]
- $\phi$ fluctuation at wavevector $\vec{q}$ couples most efficiently to fermions near $\pm \vec{k}_0$. 

- Expand fermion kinetic energy at wavevectors about $\pm \vec{k}_0$ and boson ($\phi$) kinetic energy about $\vec{q} = 0$. 

Quantum criticality of Ising-nematic ordering
• $\phi$ fluctuation at wavevector $\vec{q}$ couples most efficiently to fermions near $\pm \vec{k}_0$.

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\[ \mathcal{L}[\psi_\pm, \phi] = \]

\[ \psi_+^\dagger (\partial_\tau - i\partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\partial_\tau + i\partial_x - \partial_y^2) \psi_- \]

\[ -\phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} \left( \partial_y \phi \right)^2 \]

Model of a spin liquid ("Bose metal"): couple fermions to a dynamical gauge field $A_\mu$.

$$
\mathcal{L} = f_\sigma^\dagger \left( \partial_\tau - i A_\tau - \frac{(\nabla - i A)^2}{2m} - \mu \right) f_\sigma
$$

$$
= \frac{1}{\sqrt{2}} (\langle \uparrow \downarrow \rangle - \langle \downarrow \uparrow \rangle)
$$
Quantum criticality of Ising-nematic ordering

\[ \mathcal{L}[\psi_\pm, \phi] = \]

\[ \psi_+^\dagger \left( \partial_\tau - i \partial_x - \partial_y^2 \right) \psi_+ + \psi_-^\dagger \left( \partial_\tau + i \partial_x - \partial_y^2 \right) \psi_- \]

\[ - \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} \left( \partial_y \phi \right)^2 \]

Field theory of U(1) spin liquid

\[ \mathcal{L}[\psi_{\pm}, a] = \]

\[ \psi_+^{\dagger} \left( \partial_\tau - i \partial_x - \partial_y^2 \right) \psi_+ + \psi_-^{\dagger} \left( \partial_\tau + i \partial_x - \partial_y^2 \right) \psi_- \]

\[ -a \left( \psi_+^{\dagger} \psi_+ - \psi_-^{\dagger} \psi_- \right) + \frac{1}{2g^2} (\partial_y a)^2 \]

Quantum criticality of Ising-nematic ordering

\[ \mathcal{L}[\psi_{\pm}, \phi] = \]

\[ \psi_{+}^\dagger \left( \partial_{\tau} - i \partial_{x} - \partial_{y}^{2} \right) \psi_{+} + \psi_{-}^\dagger \left( \partial_{\tau} + i \partial_{x} - \partial_{y}^{2} \right) \psi_{-} \]

\[ - \phi \left( \psi_{+}^\dagger \psi_{+} + \psi_{-}^\dagger \psi_{-} \right) + \frac{1}{2g^{2}} \left( \partial_{y} \phi \right)^{2} \]

Quantum criticality of Ising-nematic ordering

\[ \mathcal{L} = \psi_+^\dagger (\partial_\tau - i \partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\partial_\tau + i \partial_x - \partial_y^2) \psi_- \\
- \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2 \]

One loop \( \phi \) self-energy with \( N_f \) fermion flavors:

\[ D(\vec{q}, \omega) = N_f \int \frac{d^2 k}{4\pi^2} \frac{d\Omega}{2\pi} \frac{1}{[-i(\Omega + \omega) + k_x + q_x + (k_y + q_y)^2] [-i\Omega - k_x + k_y^2]} \]

\[ = N_f \frac{\lvert \omega \rvert}{4\pi \lvert q_y \rvert} \]

Landau-damping
Quantum criticality of Ising-nematic ordering

\[ \mathcal{L} = \psi_+^\dagger (\partial_\tau - i\partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\partial_\tau + i\partial_x - \partial_y^2) \psi_- 
- \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2 \]

Electron self-energy at order $1/N_f$:

\[
\Sigma(\vec{k}, \Omega) = -\frac{1}{N_f} \int \frac{d^2q}{4\pi^2} \frac{d\omega}{2\pi} \frac{1}{\left[ -i(\omega + \Omega) + k_x + q_x + (k_y + q_y)^2 \right] \left[ \frac{q_y^2}{g^2} + \frac{|\omega|}{|q_y|} \right]}
\]

\[
= -i \frac{2}{\sqrt{3} N_f} \left( \frac{g^2}{4\pi} \right)^{2/3} \text{sgn}(\Omega)|\Omega|^{2/3}
\]
Quantum criticality of Ising-nematic ordering

\[ \mathcal{L} = \psi_+^\dagger (\partial_\tau - i \partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (\partial_\tau + i \partial_x - \partial_y^2) \psi_- \\
- \phi \left( \psi_+^\dagger \psi_+ + \psi_-^\dagger \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2 \]

Schematic form of \( \phi \) and fermion Green’s functions

\[ D(\vec{q}, \omega) = \frac{1/N_f}{q_x^2 + \frac{|\omega|}{|q_y|}}, \quad G_f(\vec{q}, \omega) = \frac{1}{q_x + q_y^2 - i \text{sgn}(\omega) |\omega|^{2/3} / N_f} \]

In both cases \( q_x \sim q_y^2 \sim \omega^{1/z} \), with \( z = 3/2 \). Note that the bare term \( \sim \omega \) in \( G_f^{-1} \) is irrelevant.

Strongly-coupled theory without quasiparticles.
Quantum criticality of Ising-nematic ordering

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Simple scaling argument for \( z = 3/2 \).
Quantum criticality of Ising-nematic ordering

\[ \mathcal{L}_{\text{scaling}} = \psi_+^\dagger (-i \partial_x - \partial_y^2) \psi_+ + \psi_-^\dagger (+i \partial_x - \partial_y^2) \psi_- - g \phi \left( \psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_- \right) + (\partial_y \phi)^2 \]

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\[ \mathcal{L}_{\text{scaling}} = \psi^\dagger_+ \left( -i \partial_x - \partial_y^2 \right) \psi_+ + \psi^\dagger_- \left( +i \partial_x - \partial_y^2 \right) \psi_- - g \phi \left( \psi^\dagger_+ \psi_+ - \psi^\dagger_- \psi_- \right) + (\partial_y \phi)^2 \]

Simple scaling argument for \( z = 3/2 \).

Under the rescaling \( x \to x/s \), \( y \to y/s^{1/2} \), and \( \tau \to \tau/s^z \), we find invariance provided

\[
\begin{align*}
a & \to a \, s^{(2z+1)/4} \\
\psi & \to \psi \, s^{(2z+1)/4} \\
g & \to g \, s^{(3-2z)/4}
\end{align*}
\]

So the action is invariant provided \( z = 3/2 \).
Quantum criticality of Ising-nematic ordering

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Schematic form of \( \phi \) and fermion Green's functions

\[
D(\vec{q}, \omega) = \frac{1/N_f}{q_x^2 + \frac{\sqrt{\omega}}{|q_y|}} \quad , \quad G_f(\vec{q}, \omega) = \frac{1}{q_x + q_y^2 - i\text{sgn}(\omega)|\omega|^{2/3}/N_f}
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- \phi \left( \psi^\dagger_+ \psi_+ + \psi^\dagger_- \psi_- \right) + \frac{1}{2g^2} (\partial_y \phi)^2 \]

The $1/N_f$ expansion is *not* determined by counting fermion loops, because of infrared singularities created by the Fermi surface. The $|\omega|^{2/3}/N_f$ fermion self-energy leads to additional powers of $N_f$, and a breakdown in the loop expansion.
All planar graphs of $\psi_+$ alone are as important as the leading term

Graph mixing $\psi_+$ and $\psi_-$ is $O\left(N^{3/2}\right)$ (instead of $O\left(N\right)$), violating genus expansion

Sung-Sik Lee, Physical Review B 80, 165102 (2009)

• There is a sharp Fermi surface defined by the fermion Green’s function: $G_f^{-1}(|k| = k_F, \omega = 0) = 0$.  

Properties of the strange metal at the Ising-nematic critical point

- There is a sharp Fermi surface defined by the fermion Green’s function: $G_f^{-1}(|k| = k_F, \omega = 0) = 0$.

- Area enclosed by the Fermi surface $A = Q$, the fermion density

There is a sharp Fermi surface defined by the fermion Green’s function: $G^{-1}_f(|k| = k_F, \omega = 0) = 0$.

Area enclosed by the Fermi surface $A = Q$, the fermion density

Critical continuum of excitations near the Fermi surface with energy $\omega \sim |q|^z$, where $q = |k| - k_F$ is the distance from the Fermi surface and $z$ is the dynamic critical exponent.
Properties of the strange metal at the Ising-nematic critical point

- Fermion Green’s function $G_f^{-1} = q^{1-\eta} F(\omega/q^z)$. Three-loop computation shows $\eta \neq 0$ and $z = 3/2$. 

● Fermion Green’s function $G_f^{-1} = q^{1-\eta} F(\omega/q^z)$. Three-loop computation shows $\eta \neq 0$ and $z = 3/2$.

● The phase space density of fermions is effectively one-dimensional, so the entropy density $S \sim T^{d_{\text{eff}}/z}$ with $d_{\text{eff}} = 1$. 

Entanglement entropy

Measure strength of quantum entanglement of region A with region B.

\[ \rho_A = \text{Tr}_B \rho = \text{density matrix of region } A \]

Entanglement entropy \( S_{EE} = -\text{Tr} (\rho_A \ln \rho_A) \)
Logarithmic violation of “area law”: \[ S_{EE} = \frac{1}{12} (k_F P) \ln(k_F P) \]

for a circular Fermi surface with Fermi momentum \( k_F \), where \( P \) is the perimeter of region A with an arbitrary smooth shape.

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Non-Fermi liquids have, at most, the "1/12" prefactor modified.

Conformal quantum matter

A. Fermi liquids: graphene

B. Holography: Reissner - Nördstrom solution

C. Non-Fermi liquids: nematic critical point (and U(1) spin liquids)

D. Holography: scaling arguments for entropy and entanglement entropy
Conformal quantum matter

A. Fermi liquids: graphene

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C. Non-Fermi liquids: nematic critical point (and $U(1)$ spin liquids)

D. Holography: scaling arguments for entropy and entanglement entropy
Consider the metric which transforms under rescaling as

\[ x_i \rightarrow \zeta x_i \]
\[ t \rightarrow \zeta^z t \]
\[ ds \rightarrow \zeta^{\theta/d} ds. \]

This identifies \( z \) as the dynamic critical exponent (\( z = 1 \) for “relativistic” quantum critical points).

\( \theta \) is the violation of hyperscaling exponent.
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\( \theta \) is the violation of hyperscaling exponent.

The most general choice of such a metric is

\[
ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)} dr^2 + dx_i^2 \right)\
\]

We have used reparametrization invariance in \( r \) to choose so that it scales as \( r \rightarrow \zeta^{(d-\theta)/d} r \).
At $T > 0$, there is a “black-brane” at $r = r_h$.

The Beckenstein-Hawking entropy of the black-brane is the thermal entropy of the quantum system $r = 0$.

The entropy density, $S$, is proportional to the “area” of the horizon, and so $S \sim r_h^{-d}$.
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Under rescaling $r \to \zeta^{(d-\theta)/d}r$, and the temperature $T \sim t^{-1}$, and so

$$S \sim T^{(d-\theta)/z} = T^{d_{\text{eff}}/z}$$

where $\theta = d - d_{\text{eff}}$ measures "dimension deficit" in the phase space of low energy degrees of a freedom.
The thermal entropy density scales as

\[ S \sim T^{(d-\theta)/z}. \]

The third law of thermodynamics requires \( \theta < d \).

\[
d s^2 = \frac{1}{r^2} \left( -\frac{d t^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)} d r^2 + d x_i^2 \right)
\]
Holographic entanglement entropy

Emergent holographic direction
Holographic entanglement entropy

Area of minimal surface equals entanglement entropy

Emergent holographic direction

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The entanglement entropy, \( S_E \), of an entangling region with boundary surface ‘area’ \( \Sigma \) scales as

\[
S_E \sim \begin{cases} 
\Sigma & \text{for } \theta < d - 1 \\
\Sigma \ln \Sigma & \text{for } \theta = d - 1 \\
\Sigma^{\theta/(d-1)} & \text{for } \theta > d - 1 
\end{cases}
\]

All local quantum field theories obey the “area law” (upto log violations) and so \( \theta \leq d - 1 \).
\[ ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^2 d (z-1)/(d-\theta)} + r^{2\theta/(d-\theta)} dr^2 + dx_i^2 \right) \]

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- The null energy condition implies \( z \geq 1 + \frac{\theta}{d} \).
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The null energy condition implies \( z \geq 1 + \frac{\theta}{d} \).
The value of $\theta$ is fixed by requiring that the thermal entropy density $S \sim T^{1/z}$ for general $d$.
Conjecture: this metric then describes a compressible state with a hidden Fermi surface of quarks coupled to gauge fields.

Holography of non-Fermi liquids

\[ ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^{2d(z-1)/(d-\theta)}} + r^{2\theta/(d-\theta)} dr^2 + dx_i^2 \right) \]

\[ \theta = d - 1 \]

- The value of \( \theta \) is fixed by requiring that the thermal entropy density \( S \sim T^{1/z} \) for general \( d \).
  Conjecture: this metric then describes a compressible state with a hidden Fermi surface of quarks coupled to gauge fields.

- The null energy condition yields the inequality \( z \geq 1 + \theta/d \). For \( d = 2 \) and \( \theta = 1 \) this yields \( z \geq 3/2 \). The field theory analysis gave \( z = 3/2 \) to three loops!
The entanglement entropy exhibits logarithmic violation of the area law only for this value of $\theta$!!

The logarithmic violation is of the form $P \ln P$, where $P$ is the perimeter of the entangling region. This form is independent of the shape of the entangling region, just as is expected for a (hidden) Fermi surface!!

$$\theta = d - 1$$

$ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^2d(z-1)/(d-\theta)} + r^{2\theta/(d-\theta)} dr^2 + dx_i^2 \right)$

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$\theta = d - 1$

Holographic theory of a non-Fermi liquid (NFL)

Add a relevant “dilaton” field

\[ \mathcal{E}_r = \langle Q \rangle \]

\[ S = \int d^{d+2}x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \left( R - 2(\nabla \Phi)^2 - \frac{V(\Phi)}{L^2} \right) - \frac{Z(\Phi)}{4e^2} F_{ab} F^{ab} \right] \]

with \( Z(\Phi) = Z_0 e^{\alpha \Phi} \), \( V(\Phi) = -V_0 e^{-\beta \Phi} \), as \( \Phi \to \infty \).


Monday, May 14, 2012
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This is a “bosonization” of the Fermi surface

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Add a relevant “dilaton” field

\[ \mathcal{E}_r = \langle Q \rangle \]

\[ r \leftarrow \]

Electric flux

\[ \mathcal{E}_r = \langle Q \rangle \neq 0 \]

Leads to metric

\[ ds^2 = L^2 \left( -f(r)dt^2 + g(r)dr^2 + \frac{dx^2 + dy^2}{r^2} \right) \]

with

\[ f(r) \sim r^{-\gamma}, \quad g(r) \sim r^{\delta}, \quad \Phi(r) \sim \ln(r) \quad \text{as} \quad r \to \infty. \]

Holographic theory of a non-Fermi liquid (NFL)

\[ ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^2 d(z-1)/(d-\theta)} + \frac{r^2 \theta/(d-\theta)}{d^2 (d-\theta)} \, dr^2 + dx_i^2 \right) \]

The \( r \to \infty \) metric has the above form with

\[ \theta = \frac{d^2 \beta}{\alpha + (d - 1) \beta} \]

\[ z = 1 + \frac{\theta}{d} + \frac{8(d(d - \theta) + \theta)^2}{d^2 (d - \theta) \alpha^2}. \]

Note \( z \geq 1 + \theta/d \).
Holographic theory of a non-Fermi liquid (NFL)

\[ ds^2 = \frac{1}{r^2} \left( -\frac{dt^2}{r^2d(z-1)/(d-\theta)} + \frac{r^{2\theta}/(d-\theta)}{dr^2} + dx_i^2 \right) \]

The solution also specifies the missing numerical prefactors in the metric. In general, these depend upon the details on the UV boundary condition as \( r \to 0 \). However, the coefficient of \( dx_i^2/r^2 \) turns out to be \textit{independent} of the UV boundary conditions, and proportional to \( Q^{2\theta}/(d(d-\theta)) \).

The square-root of this coefficient is the prefactor of the log divergence in the entanglement entropy for \( \theta = d - 1 \).
The entanglement entropy has log-violation of the area law

\[ S_E = \Xi Q^{(d-1)/d} \Sigma \ln\left( Q^{(d-1)/d} \Sigma \right). \]

where \( \Sigma \) is surface area of the entangling region, and \( \Xi \) is a dimensionless constant which is independent of all UV details, of \( Q \), and of any property of the entangling region. Note \( Q^{(d-1)/d} \sim k_F^{d-1} \) via the Luttinger relation, and then \( S_E \) is just as expected for a Fermi surface !!!!

Gauss Law and the “attractor” mechanism ⇔ Luttinger theorem on the boundary
Holographic theory of a fractionalized-Fermi liquid (FL*)

Hidden Fermi surfaces of “quarks”

Visible Fermi surfaces of “mesinos”

\[ \mathcal{E}_r = \mathcal{Q} - \mathcal{Q}_{\text{mesino}} \]

\[ \mathcal{E}_r = \mathcal{Q} \]

A state with partial confinement

Now the entanglement entropy implies that the Fermi momentum of the hidden Fermi surface is given by $k^d_F \sim Q - Q_{\text{mesino}}$, just as expected by the extended Luttinger relation. Also the probe fermion quasiparticles are sharp for $\theta = d - 1$, as expected for a FL* state.
Confining geometry leads to a state which has all the properties of a Landau Fermi liquid.

S. Sachdev, Physical Review D 84, 066009 (2011)
Conclusions

Compressible quantum matter

Evidence for *hidden Fermi surfaces* in compressible states obtained for a class of holographic Einstein-Maxwell-dilaton theories. These theories describe a *non-Fermi liquid* (NFL) state of gauge theories at non-zero density.
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Conclusions

Compressible quantum matter

Evidence for *hidden Fermi surfaces* in compressible states obtained for a class of holographic Einstein-Maxwell-dilaton theories. These theories describe a *non-Fermi liquid* (NFL) state of gauge theories at non-zero density.

Fermi liquid (FL) state described by a confining holographic geometry

Hidden Fermi surfaces can co-exist with Fermi surfaces of mesinos, leading to a state with *partial confinement*: the fractionalized Fermi liquid (FL*)