Gravitational Radiation:
The Physics of Gravitational Waves and their Generation

Kip S. Thorne

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PDFs of lecture slides are available at
http://www.cco.caltech.edu/~kip/LorentzLectures/
each Thursday night before the Friday lecture
Introduction

- The Gravitational-Wave window onto the universe is likely to be opened
  - in the next decade
  - in four widely different frequency bands, spanning 22 decades:
Radical New Windows ➤ Great Surprises

- **Radio Window: 1940s & 50s**
  - $10^4$ x lower frequency than optical
  - radio galaxies, quasars, pulsars, cosmic microwave background, ...

- **X-Ray Window: 1960s & 70s**
  - $10^3$ x higher frequency than optical
  - black holes, accreting neutron stars, hot intergalactic gas, ...
Radical New Windows ➞ Great Surprises

- **Gravitational Waves** are far more radical than Radio or X-rays

  *Completely new form of radiation!*

  *Frequencies to be opened span 22 decades*

  $$f_{HF} / f_{ELF} \sim 10^{22}$$

- **What will we learn from Gravitational Waves?**
  - "Warped side of the universe"
    - our first glimpses, then in-depth studies
  - The nonlinear dynamics of curved spacetime
  - Answers to astrophysical & cosmological puzzles:
    - How are supernovae powered?
    - How are gamma-ray bursts powered?
    - What was the energy scale of inflation? ...

- **Surprises**
Growth of GW Community

• 1994: LIGO Approved for Construction: ~ 30 scientists
• Today: ~ 1500 scientists
  – influx from other fields
  – needed for success
  – drawn by expected science payoffs
These Lectures

1. The Physics of Gravitational Waves and their Generation  
   *Today*

2. Astrophysical and Cosmological Sources of Gravitational Waves, and the Information they Carry  
   *Next Friday, Sept 25*

3. Gravitational Wave Detection: Methods, Status, and Plans  
   *Following Friday, Oct 2*
These Lectures

- **Prerequisites for these lectures:**
  - Knowledge of physics at advanced undergraduate level
  - Especially special relativity and Newtonian gravity
  - Helpful to have been exposed to General Relativity; not necessary

- **Goals of these lectures:**
  - Overview of gravitational-wave science
  - Focus on physical insight, viewpoints that are powerful

- **Pedagogical form of these lectures:**
  - Present key ideas, key results, without derivations
  - Give references where derivations can be found
These Lectures

- **Pedagogical references that cover this lecture’s material:**
  
  
  
Resources

• **The best introductory textbook on general relativity:**

• **The best course-length introduction to gravitational-wave science:**
  - *Gravitational Waves, a Web-Based Course* (including videos of lectures, readings, problem sets, problem solutions):
    http://elmer.caltech.edu/ph237/
Outline of This Lecture

1. Gravitational waves (GWs) in the language of tidal gravity
2. GWs in the linearized approximation to general relativity
3. GW generation
   a. Linearized sources
   b. Slow-motion sources
   c. Nonlinear, highly dynamical sources: Numerical relativity
4. GWs in curved spacetime; geometric optics; GW energy
5. Interaction of GWs with matter and EM fields
1. Gravitational Waves in the Language of Tidal Gravity
Relative Motion of Inertial Frames

GW-induced motion of Local Inertial Frames

Non-meshing of local inertial frames ⇒ Curvature of Spacetime

Tidal Gravity

Local Inertial Frame of A

GW field

Riemann Tensor $-R_{jktk}$

$\ddot{x}_j = \frac{1}{2} h_{jk}^{GW} x_k$

like $\ddot{x}_j = \frac{\partial g_j}{\partial x_k} x_k$

$\delta x_j = \frac{1}{2} h_{jk}^{GW} x_k$

$\frac{-\partial^2 \Phi}{\partial x_j \partial x_k} x_k$

analogous to $E_j = -\dot{A}_j^T$

(transverse Lorenz gauge)
The GW field $h_{jk}^{GW}$

- The gravitational-wave field, $h_{jk}^{GW}$

  Symmetric, transverse, traceless (TT);
  two polarizations: $+$, $x$

- $+$ Polarization

  $h_{xx}^{GW} = h_+(t - z/c) = h_+(t - z)$
  $h_{yy}^{GW} = -h_+(t - z)$

  Lines of force
  $\ddot{x}_j = \frac{1}{2} \dddot{h}_{jk}^{GW} x_k$, $\ddot{x} = \dddot{h}_+ x$, $\ddot{y} = -\dddot{h}_+ y$

- $x$ Polarization

  $h_{xy}^{GW} = h_{yx}^{GW} = h_+(t - z)$
Gravitons

- Quantum spin and rest mass: imprint on classical waves

\[
\text{spin} = \frac{180^\circ}{\text{return angle}}
\]

- Photon
  - return angle = 360°
  - spin = 1

- Graviton
  - return angle = 180°
  - spin = 2

propagation speed = \( c \equiv 1 \) \( \Rightarrow \) rest mass = 0
Behavior of $h_{jk}^{GW}$ Under boosts in z direction

GW Field

- $h_{jk}^{GW}$, $h_+$, $h_x$
- Transform as scalar fields
- $h'_{jk}^{GW} (t' - z') = h_{jk}^{GW} (t - z)$
- $= h_{jk}^{GW} (\mathcal{D} (t' - z'))$

Riemann amplified

- $R'_{jktk} = \frac{1}{2} \dot{h}_{jk}^{GW}$
- $= \mathcal{D}^2 R_{jktk}$

Boost weight 2

EM waves in Transverse Lorenz gauge

- $A_j^T$ is transverse:
- $A_x^T (t - z)$, $A_y^T (t - z)$
- Transform as scalar fields
- $A_{j'}^T (t' - z')$
- $= A_j^T (\mathcal{D} (t' - z'))$

Electric field amplified

- $E'_j = -\dot{A}_j^T$
- $= -\mathcal{D} E_j$

Boost weight 1
GWs as Seen in Laboratory on Earth

- Proper Reference Frame: analog of local Lorentz frame
  \[ ds^2 = -(1 + 2g \cdot x) dt^2 + dx^2 + dy^2 + dz^2 \]

- GWs unaffected by earth’s gravity
  - except for a very tiny, unimportant gravitational blue shift

- Total gravitational force
  \[ \ddot{x}_j = \frac{1}{2} \ddot{h}^{GW}_{jk} x_k + g_j \]
2. GWs in Linearized Approximation to General Relativity
Metric Perturbation, Lorenz Gauge, Einstein Field Equation

- **Metric:** \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \)

- **Field theory in flat spacetime** \( h_{\mu\nu} \) analogous to \( A_\mu \)

- **Gauge freedom (ripple coordinates)**
  \[ x'^{\mu}_{\text{new}} = x^{\mu}_{\text{old}} - \xi^\mu \]
  \[ h^{\text{new}}_{\mu\nu} = h^{\text{old}}_{\mu\nu} + \frac{\partial \xi_\mu}{\partial x^\nu} + \frac{\partial \xi_\nu}{\partial x^\mu} \text{ analogous to } A^{\text{new}}_\mu = A^{\text{old}}_\mu + \frac{\partial \phi}{\partial x^\mu} \]

- **Lorenz gauge**
  \[ \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h^{\alpha\beta} \eta_{\alpha\mu} \eta_{\beta\nu} \]
  \[ \frac{\partial \bar{h}_{\mu\nu}}{\partial x^\nu} = 0 \text{ analogous to } \frac{\partial A^{\nu}}{\partial x^\nu} = 0 \]

- **Einstein field equation in Lorenz gauge**
  \[ \Box \bar{h}^{\mu\nu} \equiv \eta^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} \bar{h}^{\mu\nu} = -16\pi G T^{\mu\nu} \] analogous to \( \Box A^\mu = -4\pi J^\mu \)

- **Gravitational-wave field**

  In wave zone, gauge change with \( \Box \xi_\alpha = 0 \) (analogous to \( \Box \phi = 0 \)) \( \rightarrow \)

  Project out TT piece; get GW field:
  \[ h^{\text{new}}_{tt} = h^{\text{new}}_{jt} = 0, \quad h^{\text{new}}_{jk} = (h^{\text{old}}_{jk})^{\text{TT}} = h^{\text{GW}}_{jk} \]

  where \( (h^{\text{old}}_{\theta\phi})^{\text{TT}} = h^{\text{old}}_{\theta\phi} = h_\times, \quad (h^{\text{old}}_{\theta\theta})^{\text{TT}} = h^{\text{old}}_{\theta\theta} - \frac{1}{2}(h_{\theta\theta} + h_{\phi\phi}) = \frac{1}{2}(h_{\theta\theta} - h_{\phi\phi}) = h_+ \),

  analogous to obtaining Transverse Lorenz gauge by projecting:
  \[ A^T_j = (A^\text{old}_j)^T \]
3. Gravitational Wave Generation
Example: Linearized, Point Particles in Lorenz Gauge

- **Electromagnetic**
  \( \Box A^\alpha = -4\pi J^\alpha \Rightarrow \text{at } O, \quad A^\alpha = \frac{q p^\alpha}{k_\mu p^\mu} \)
  in rest frame of particle, reduces to \( A^t = \frac{q}{r} \)

  In wave zone \( E_j = -(\dot{A}_j)^T \) (Liénard-Wiechart)

- **Gravitational**
  \( \Box \bar{h}^{\alpha\beta} = -16\pi GT^{\alpha\beta} \Rightarrow \text{at } O, \quad \bar{h}^{\alpha\beta} = G \frac{p^\alpha p^\beta}{k_\mu p^\mu} \)
  in rest frame of particle, reduces to \( \bar{h}^{tt} = \frac{4Gm}{r} \)

  In wave zone \( h_{jk}^{GW} = (\bar{h}_{jk})^{TT} = G \left( \frac{p^{j} p^{k}}{k_\mu p^\mu} \right)^{TT} \)

- **Gravitational-Wave Memory**
  \( \Delta h_{jk}^{GW} = G \left( \Delta \sum_A \frac{4 p_{A}^{j} p_{A}^{k}}{k_A \mu p_{A}^{\mu}} \right)^{TT} \)
Slow-Motion GW Sources

**Slow Motion:** speeds $<< c = 1$; wavelength $= \lambda >>$ (source size) $= L$

examples: me waving arms; pulsar (spinning neutron star); binary made from two black holes

- **Weak-field, near zone:** *Newtonian Potential*
  \[
  \Phi = -G \frac{m}{r} \quad \text{and} \quad G \frac{\text{mass dipole}}{r^2} \quad \text{and} \quad G \frac{\text{mass quadrupole}}{r^3} \quad \text{and} \quad \ldots
  \]

- **Wave zone:** $h_{jk}^{\text{GW}} \sim \frac{1}{r}$ [by energy conservation] and dimensionless $\Rightarrow$
  \[
  h_{jk}^{\text{GW}} \sim G \frac{m}{r} \quad \text{and} \quad G \frac{\partial \text{(mass dipole)}/\partial t}{r} \quad \text{and} \quad G \frac{\partial^2 \text{(mass quadrupole)}/\partial t^2}{r} \quad \text{and} \quad \ldots
  \]

mass; cannot oscillate

canonical field theory $\Rightarrow$ radiation field carried by
quanta with spin $s$ has multipoles confined to $\ell \geq s$

- **Mass quadrupole dominates**

\[
  h_{jk}^{\text{GW}} = 2G \left( \frac{\ddot{I}_{jk}}{r} \right)^{TT} \quad \text{for Newtonian source} \quad I_{jk} = \int \rho (x^j x^k - \frac{1}{3} r^2) d^3 x
  \]
Common Textbook Derivation

1. Linearized Approximation to General Relativity \((\text{set } G=1)\)

\[ \square \bar{h}^{jk} = -16\pi \bar{h}^{jk} \Rightarrow \bar{h}^{jk}(t, x) = 4 \int \frac{T^{jk}(t - |x - x'|, x')}{|x - x'|} d^3 x' \]

slow motion \(\Rightarrow\) \[ \bar{h}^{jk}(t, x) = \frac{4 \int T^{jk}(t - r, |x'|) d^3 x'}{r} \]

2. Conservation of 4-momentum \(T^{\alpha\beta},_{\beta} = 0 \Rightarrow\)

\[ 2T^{jk} = (T^{tt} x^j x^k),_{tt} - (T^{ab} x^j x^k),_{ab} - 2(T^{aj} x^k + T^{ak} x^j),_a \]

3. Insert 2 into 1; integral of divergence vanishes

\[ \bar{h}^{jk}(t, x) = \frac{2 \bar{I}^{jk}(t - r)}{r}, \text{ where } \bar{I}^{jk} = \int T^{00} x^j x^k d^3 x \]

4. Take transverse traceless part

\[ \bar{h}^{GW}_{jk}(t, x) = \frac{2 \left( \bar{I}^{jk}(t - r) \right)^{TT}}{\bar{T}^{jk}}, \text{ where } \bar{T}^{jk} = \int T^{00}(x^j x^k - r^2 \delta^{jk}) d^3 x \]

**PROBLEM:** Derivation Not valid when self gravity influences source’s dynamics!!
Derivation via Propagation from Weak-Field Near Zone to Wave Zone

- **Weak-gravity regions**: Linearized approximation to GR
  \[ \Box h^{\alpha \beta} = 0 \]
  \[ \bar{h}^{\alpha \beta} \,_{,\beta} = 0 \]
  i.e. \[ \bar{h}^{tt} \,_{,t} = -\bar{h}^{tj} \,_{,j} \]
  \[ \bar{h}^{tj} \,_{,t} = -\bar{h}^{jk} \,_{,k} \]

In weak-field near zone (wfnz) \( \Phi = -\frac{M}{r} - \frac{3 I_{jk} n^j n^k}{2 r^3} \), \( \bar{h}^{tt} = -4\Phi \)

- **Quadrupolar solution in Induction Zone**: 
  \[ \bar{h}^{tt} = 2 \left[ \frac{1}{r} I_{jk} (t - r) \right] \,_{,jk} \simeq \frac{6}{r^3} I_{jk} n^j n^k \text{ in wfnz, } \simeq \frac{2}{r} \bar{I}_{jk} n^j n^k \text{ in lwz} \]
  \[ \bar{h}^{tj} = 2 \left[ \frac{1}{r} \dot{I}_{jk} (t - r) \right] \,_{,k} \simeq \frac{-2}{r^2} \dot{I}_{jk} n^k \text{ in wfnz, } \simeq \frac{-2}{r^2} \ddot{I}_{jk} (t - r) n^l \text{ in lwz} \]
  \[ \bar{h}^{jk} = \frac{2}{r} \dddot{I}_{jk} (t - r) \text{ tiny } \]

Take TT part to get GW field in wfnz:
\[ h_{jk}^{GW}(t, x) = \frac{2 \left( \dddot{I}_{jk} (t - r) \right)^{TT}}{r} \]
Order of Magnitude

- **Source parameters:**
  
  mass $\sim M$, size $\sim L$, rate of quadrupolar oscillations $\sim \omega$, distance $\sim r$,
  
  internal kinetic energy of quadrupolar oscillations $\sim E_{\text{kin}} \sim M(\omega L)^2$

- **GW strength:**

  $h^{\text{GW}}_{jk} = 2G\frac{\ddot{T}_{jk}}{r} \sim G\frac{\omega^2(ML^2)}{r} \sim G\frac{E_{\text{kin}}/c^2}{r}$
  
  $\sim \Phi$ produced by kinetic energy of shape changes.

\[
\begin{align*}
h^{\text{GW}}_{jk} &\sim h_+ \sim h_\times \sim 10^{-21} \left( \frac{E_{\text{kin}}}{M_\odot c^2} \right) \left( \frac{100\text{Mpc}}{r} \right) \\
100\text{Mpc} &= 300\text{ million light years} \sim \frac{1}{30}(\text{Hubble distance})
\end{align*}
\]
Slow-Motion Sources: Higher-Order Corrections

- **Source Dynamics: Post-Newtonian Expansion**
  \[
  \text{in } \frac{v}{c} \sim \sqrt{\frac{\Phi}{c^2}} \sim \sqrt{\frac{P}{\rho c^2}}
  \]

- **GW Field: Higher-Order Moments (octopole, ...)**
  - computed in same manner as quadrupolar waves: by analyzing the transition from weak-field near zone, through induction zone, to local wave zone
  - actually Two families of moments (like electric and magnetic) -
    - moments of mass distribution, moments of angular-momentum distribution
  - Use symmetric, trace-free (STF) tensors to describe the moments and the GW field ... (19th century approach; Great Computational Power)
STF Tensors [an aside]

- **Multipole moments of Newtonian gravitational potential**
  
  - Spherical-harmonic description: \( \Phi \sim \sum_{m=-\ell}^{+\ell} \frac{M_{\ell m} Y_{\ell m}(\theta \phi)}{r^{\ell+1}} \)
  
  \( M_{\ell m} \) has \( 2\ell + 1 \) components: \( m = -\ell, \ell + 1, \ldots, +\ell \)

  - STF description: \( \Phi \sim \frac{\mathcal{I}_{a_1 a_2 \ldots a_\ell} n^{a_1} n^{a_2} \ldots n^{a_\ell}}{r^{\ell+1}} \)

  \( \mathcal{I}_{a_1 a_2 \ldots a_\ell} \) has \( 2\ell + 1 \) independent components

- **Multipolar Expansion of gravitational-wave field**

  \[
  h_{jk}^{GW} = \left\{ \sum_{\ell=2}^{\infty} \frac{4}{\ell!} \frac{\partial^\ell}{\partial t^\ell} \frac{\mathcal{I}_{jka_1 \ldots a_{\ell-2}} (t-r)}{r} n^{a_1} \ldots n^{a_{\ell-2}} \right\}_{TT} \\
  + \left\{ \sum_{\ell=2}^{\infty} \frac{8\ell}{(\ell+1)!} \epsilon_{pq(j} \frac{\partial^\ell}{\partial t^\ell} S_k p a_1 \ldots a_{\ell-1} (t-r) \frac{n^q n^{a_1} \ldots n^{a_{\ell-1}}}{r} \right\}_{TT}
  \]

  - Indices carry directional, multipolar and tensor information, all at once
Strong-Gravity \((GM/c^2 L \sim 1)\), Fast-Motion \((v \sim c)\) Sources

- Most important examples [next week]
  - Black-hole binaries: late inspiral, collision, merger, ringdown
  - Black-hole / neutron-star binaries: late inspiral, tidal disruption and swallowing
  - neutron star / neutron-star binaries: late inspiral, collision and merger
  - supernovae

- These are the strongest and most interesting of all sources
- Slow-motion approximation fails
- Only way to compute waves: \textit{Numerical Relativity}
4. Gravitational Waves in curved spacetime; geometric optics; GW energy
GWs Propagating Through Curved Spacetime (distant wave zone)

- **Definition of gravitational wave:** the rapidly varying part of the metric and of the curvature
  
  \[ \lambda = \lambda/2\pi \ll \mathcal{L} = \text{(lengthscale on which background metric varies)} \lesssim \mathcal{R} \]

\[ g_{\alpha\beta} = g^B_{\alpha\beta} + h_{\alpha\beta} \equiv \langle g_{\alpha\beta} \rangle \equiv g_{\alpha\beta} - \langle g_{\alpha\beta} \rangle \]

Same definition used for waves in plasmas, fluids, solids

- **In local Lorentz frame of background:** GW theory same as in flat spacetime (above)

  \[ \Box h_{\alpha\beta} = 0 \text{ in vacuum. Same propagation equation as for EM waves: } \Box A_\alpha = 0 \]

\[ \implies \text{GWs exhibit same geometric-optics behavior as EM waves} \]
Geometric-Optics Propagation

- GWs and EM waves propagate along the same rays: null geodesics in the background spacetime

- Label each ray by its direction $\left(\theta, \varphi\right)$ in source’s local wave zone, and the retarded time it has in the local wave zone,
  \[ t_{\text{ret}} \equiv (t - r)_{\text{local wave zone}} \]

- Wave’s amplitude dies out as $1/r$ in local wave zone. Along the ray, in distant wave zone, define $A = \left(\text{cross sectional area of a bundle of rays}\right)$, and $r \equiv r_o \sqrt{A/A_o}$ where $r_o$ and $A_o$ are values at some location in local wave zone. Then amplitude continues to die out as $1/r$ in distant wave zone.

- Transport the unit basis vectors $\vec{e}_\theta$ and $\vec{e}_\varphi$ parallel to themselves along the ray, from local wave zone into and through distant wave zone. Use them to define $+$ and $\times$

- Then in distant wave zone:
  \[ A_\theta = \frac{Q_\theta(t_{\text{ret}}, \theta, \varphi)}{r}, \quad A_\varphi = \frac{Q_\varphi(t_{\text{ret}}, \theta, \varphi)}{r} \]

\[ h_+ = \frac{Q_+(t_{\text{ret}}, \theta, \varphi)}{r}, \quad h_\times = \frac{Q_\times(t_{\text{ret}}, \theta, \varphi)}{r} \]
Geometric-Optics Propagation

• Form of waves:

\[ A_\theta = \frac{Q_\theta(t_{ret}, \theta, \varphi)}{r}, \quad A_\varphi = \frac{Q_\varphi(t_{ret}, \theta, \varphi)}{r} \]

\[ h_+ = \frac{Q_+(t_{ret}, \theta, \varphi)}{r}, \quad h_\times = \frac{Q_\times(t_{ret}, \theta, \varphi)}{r} \]

• ⇒ GWs experience identically the same geometric-optics effects as EM waves:

  - gravitational redshift,
  - cosmological redshift,
  - gravitational lensing, ...

  ▶ and at the focus of a gravitational lense: the same diffraction effects as EM waves
Energy and Momentum in GWs

- Einstein’s general relativity field equations say
  \[ G_{\alpha\beta} = 8\pi \ G \ T_{\alpha\beta} \]

- Break metric into background plus GW:
  \[ g_{\alpha\beta} = g_{\alpha\beta}^B + h_{\alpha\beta} \]

- Expand Einstein tensor in powers of \( h_{\alpha\beta} \)
  \[ G_{\alpha\beta} = G_{\alpha\beta}^B + G_{\alpha\beta}^{(1)} + G_{\alpha\beta}^{(2)} \]

- Average over a few wavelengths to get quantities that vary on background scale \( \mathcal{L} \), not wavelength scale \( \lambda \)
  \[ \langle G_{\alpha\beta} \rangle = G_{\alpha\beta}^B + \langle G_{\alpha\beta}^{(2)} \rangle = 8\pi \langle T_{\alpha\beta} \rangle \]

- Rearrange:
  \[ G_{\alpha\beta}^B = 8\pi (\langle T_{\alpha\beta} \rangle + T_{\alpha\beta}^{GW}) \quad \text{where} \quad T_{\alpha\beta}^{GW} \equiv -\frac{\langle G_{\alpha\beta}^{(2)} \rangle}{8\pi} \]

- Evaluate the average:

- In Local Lorentz Frame

\[ T_{\alpha\beta}^{GW} = \frac{1}{16\pi} \langle h_{+,\alpha} h_{+,\beta} + h_{x,\alpha} h_{x,\beta} \rangle \]

\[ T_{tt}^{GW} = T_{tz}^{GW} = T_{zz}^{GW} = \frac{1}{16\pi} \langle \dot{h}_+^2 + \dot{h}_x^2 \rangle. \]
Energy and Momentum Conservation

- Einstein’s field equations \( G^{B}_{\alpha\beta} = 8\pi(\langle T_{\alpha\beta} \rangle + T^{GW}_{\alpha\beta}) \) guarantee energy and momentum conservation, e.g.

- Source loses mass (energy) at a rate
  \[
  \frac{dM}{dt} = - \int_{S} T^{tr}_{GW} dA = - \frac{1}{16\pi} \int_{S} (\dot{h}^2_{+} + \dot{h}^2_{\times}) dA
  \]

- Source loses linear momentum at a rate
  \[
  \frac{dp^{j}}{dt} = - \int_{S} T^{j}_{GW} dA = - \frac{1}{16\pi} \int_{S} (\vec{e}_{j} \cdot \vec{e}_{r}) (\dot{h}^2_{+} + \dot{h}^2_{\times}) dA
  \]

- Angular momentum is a little more delicate
5. Interaction of Gravitational Waves with matter and EM fields
Plane GW Traveling Through Homogeneous Matter

- **Fluid:**
  - GW shears the fluid, (rate of shear) = \( \sigma_{jk} = \frac{1}{2} \dot{h}_{jk}^{GW} \)
  - no resistance to shear, so no action back on wave
  - Viscosity \( \eta \sim \rho vs \) = (density)(mean speed of particles)(mean free path) produces stress \( T_{jk} = -2\eta \sigma_{jk} = -\eta \dot{h}_{jk}^{GW} \) \[\text{NOTE: } s \text{ must be } < \lambda \]
  - Linearized Einstein field equation: \( \Box h_{jk}^{GW} = -16\pi (T_{jk})^{TT} = 16\pi \eta \dot{h}_{jk}^{GW} \)
  - Wave attenuates: \( h_{jk}^{GW} \sim \exp(-z/\ell_{att}) \) where \( \ell_{att} = \frac{1}{8\pi \eta} = \frac{1}{8\pi \rho vs} \)
  - Fluid’s density curves spacetime (background Einstein equations)
    \( \frac{1}{R^2} \sim G_{00}^B = 8\pi \rho \)
  - Therefore \( \ell_{att} \sim \frac{R^2}{vs} = R \frac{R}{s} \frac{c}{v} \geq R \frac{R}{\lambda} \frac{c}{v} \gg R \)

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The viscous attenuation length is always far larger than the background radius of curvature. Attenuation is never significant!
Plane GW Traveling Through Homogeneous Matter

- **Elastic Medium:**
  - GW shears the medium, (rate of shear) = \( \sigma_{jk} = \frac{1}{2} h_{jk}^{GW} \), (shear) = \( \Sigma_{jk} = \frac{1}{2} \dot{h}_{jk}^{GW} \)
  - Medium resists with stress \( T_{jk} = -2\mu \Sigma_{jk} - 2\eta \sigma_{jk} = -\mu h_{jk}^{GW} - \eta \dot{h}_{jk}^{GW} \)
  - Einstein equation becomes \( \Box h_{jk}^{GW} = -16\pi (T_{jk})^{TT} = 16\pi (\mu h_{jk}^{GW} + \eta \dot{h}_{jk}^{GW}) \)
  - Insert \( h_{jk}^{GW} \propto \exp(-i\omega t + ikz) \). Obtain dispersion relation
    \[ \omega^2 - k^2 = 16\pi (\mu - i\omega \eta) \]; i.e. \( \omega = k(1 + 8\pi \lambda^2 \mu) - i8\pi \eta \), where \( \lambda = 1/k \)
  - Same attenuation length as for fluid: \( \ell_{att} = \frac{1}{8\pi \eta} \gg \mathcal{R} \)
  - Phase and group velocities (dispersion):
    \[ v_{phase} = \frac{\omega}{k} = 1 + 8\pi \lambda^2 \mu, \quad v_{group} = \frac{d\omega}{dk} = 1 - 8\pi \lambda^2 \mu \]
  - Dispersion length (one radian phase slippage) \( \ell = \frac{\lambda}{\delta v_{phase}} = \frac{1}{8\pi \lambda \mu} \gg \frac{\mathcal{R}^2}{\lambda} \gg \mathcal{R} \)

The dispersion length is always far larger than the background radius of curvature. Dispersion is never significant!
GW Scattering

- **Strongest scattering medium is a swarm of black holes:** hole mass $M$, number density of holes $n$
  - Scattering cross section $\sigma \lesssim M^2$
  - Graviton mean free path for scattering
    \[ \ell = \frac{1}{n\sigma} \gtrsim \frac{1}{nM^2} = \frac{1}{\rho M} \sim \frac{R^2}{M} \gg R \]

The scattering mean free path is always far larger than the background radius of curvature. Scattering is never significant!
Interaction with an Electric or Magnetic Field

• Consider a plane EM wave propagating through a DC magnetic field  
  \( B_{\text{wave}} = B_o \sin[\omega(t - z)]e_y, \quad B_{\text{DC}} = B_{\text{DC}} e_y \)

• Beating produces a TT stress  
  \( T_{xx} = -T_{yy} = \frac{B_o B_{\text{DC}}}{4\pi} \)

• TT stress resonantly generates a GW  
  \( \Box h_{jk}^{\text{GW}} = -16\pi (T_{jk})^{\text{TT}} \)

\( h_+ = h_{xx}^{\text{GW}} = -h_{yy}^{\text{GW}} = \frac{2B_{\text{DC}}B_o}{\omega} z \cos[\omega(t - z)] \)  The “Gertsenshtein effect”

• Ratio of GW energy to EM wave energy:
  \[
  \frac{T_{\text{GW}}^{tt}}{T_{\text{EMwave}}^{tt}} = \frac{\langle \dot{h}_+^2 \rangle/16\pi}{B_0^2/8\pi} = B_{\text{DC}}^2 z^2 = \frac{z^2}{R^2}
  \]

The lengthscale for significant conversion of EM wave energy into GW energy is equal to the radius of curvature of spacetime produced by the catalyzing DC magnetic field.

• The lengthscale for the inverse process is the same

There can never be significant conversion in the astrophysical universe.
Conclusion

• Gravitational Waves propagate through the astrophysical universe without significant attenuation, scattering, dispersion, or conversion into EM waves

• Next Friday: Astrophysical and Cosmological Sources of Gravitational Waves, and the Information they Carry
  
  slides will be available Thursday night at http://www.cco.caltech.edu/~kip/LorentzLectures/