

Assignment 1. (Due Sept. 28, 2005)

1. A particle in a one dimensional box, and Dirac brackets.
- a. Consider the problem of a quantum mechanical particle confined in an one dimensional box. Consider a Schrödinger equation,

$$H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \quad (1)$$

with a potential $V(x) = 0$ in the interval $-a \leq x \leq a$, and infinite elsewhere. Derive the complete spectrum of energy eigenstates $|n\rangle$ with wavefunctions in position representation $\Psi_n(x)$ and eigenvalues E_n .

- b. Using 24-th century quantum technology an experimentalist manages to prepare a state $|\Phi\rangle$ in this box corresponding in x representation with a wavefunction $\Phi(x) = N(a - |x|)$, where N is a normalization constant which is yet to be determined. Evaluate the overlap matrix elements $\langle n|\Phi\rangle$ of this state with the energy eigenstates determined in question (1.a).
- c. Calculate the overlap $\langle x|\Phi\rangle$ where $|x\rangle$ refers to the eigenstates of the position operator x of the particle in the box.
- d. Calculate the overlap matrix elements $\langle x|x'\rangle$ where $|x\rangle, |x'\rangle$ both refer to position eigenstates.

2. Using the rules of bra-ket algebra, prove or evaluate the following:

- a) $(XY)^\dagger = Y^\dagger X^\dagger$, where X and Y are operators.
- b) $\text{tr}(XY) = \text{tr}(YX)$,
where $\text{tr}(XY) = \sum_{a'} \langle a'| XY |a'\rangle$ with $\{|a'\rangle\}$ a complete basis set, and X and Y are operators.
- c) $\sum_{a'} \Psi_{a'}^*(\mathbf{x}') \Psi_{a'}(\mathbf{x}'')$, where $\Psi_{a'}(\mathbf{x}') = \langle \mathbf{x}' | a'\rangle$

3. Using the orthonormality of $|+\rangle$ and $|-\rangle$, prove

$$[S_i, S_j] = i\varepsilon_{ijk} \hbar S_k,$$

$$\{S_i, S_j\} = \frac{\hbar^2}{2} \delta_{ij},$$

where

$$\begin{aligned}
S_x &= \frac{\hbar}{2} (|+\rangle\langle-| + |- \rangle\langle+|), \\
S_y &= \frac{i\hbar}{2} (-|+\rangle\langle-| + |- \rangle\langle+|), \\
S_z &= \frac{\hbar}{2} (|+\rangle\langle+| - |- \rangle\langle-|).
\end{aligned}$$

4. Construct $|\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle$ such that

$$\mathbf{S} \cdot \hat{\mathbf{n}}|\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle = \frac{\hbar}{2}|\mathbf{S} \cdot \hat{\mathbf{n}}; +\rangle \quad (2)$$

in terms of the S_z up- and down $S = 1/2$ states $|\pm\rangle$, while $\hat{\mathbf{n}}$ is parametrized in terms of the Euler angles θ, ϕ as $\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. N.B.: this is an elementary example of a generalized coherent state for $S = 1/2$.

5. A certain observable in quantum mechanics has a 3×3 matrix representation as follows:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (3)$$

- a. Find the normalized eigenvectors of this observable and the corresponding eigenvalues. Is there any degeneracy?
- b. Give a physical example where all of this is relevant.

6. Consider the following Hamiltonian for a 2-state system:

$$H = a(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|),$$

where a is a number with the dimension of energy. Find the energy eigenvalues and the corresponding energy eigenkets (as linear combinations of $|1\rangle$ and $|2\rangle$). $|1\rangle$ and $|2\rangle$ form a complete orthonormal basis set.