Assignment 5. (Due Nov. 23, 2004)

1. A specialty of coherent states as introduced in question 1, assignment 4, is that the expectation value of the position operator oscillates with the classical frequency of the harmonic oscillator: another manifestation of its 'optimally classical' nature. Let us investigate the time dependences in more detail. Use the Heisenberg picture (i.e., the operators are time dependent) and be aware that a and $a^\dagger$ save you a lot of work,

   a. Calculate the time dependent dispersions $\langle\lambda|(\Delta x(t))^2|\lambda\rangle$ and $\langle\lambda|(\Delta p(t))^2|\lambda\rangle$. What is happening with the uncertainty relation $\langle\lambda|(\Delta x(t))^2|\lambda\rangle\langle\lambda|(\Delta p(t))^2|\lambda\rangle$?

   b. Calculate the energy expectation value $\langle\lambda|H|\lambda\rangle$ directly and compare the outcome with what can be deduced from intermediary results obtained in (a.).

   c. Create the coherent state according to Sakurai 2.18-d by displacing the ground state over a distance l. Evaluate the time dependent expectation values of position, momentum and energy. What does this mean physically?

2. The following problem has to do with the interactions of a localized electron with the quantized vibrations of an atomic lattice (phonons). The effect is that a polaron is formed: the electron gets 'dressed' with a deformation of the lattice. This is a problem with a long history: it turns out to become rather intractable at the moment that the electron is allowed to delocalize. This kept Richard Feynman busy for many years and his approximate solution of the 1950's is still the best on the market. Consider the Hamiltonian

$$H = \epsilon c^\dagger c + \sum_k \epsilon_k a_k^\dagger a_k + c^\dagger c \sum_k M_k (a_k + a_k^\dagger) \tag{1}$$

where $c^\dagger$ creates a fermion in a state with energy ϵ, $a_k^\dagger$ creates a boson in a state with energy ϵ_k and M_k are k dependent coupling constants.

   a. Explain why this Hamiltonian describes the localized polaron problem.

This problem can actually be exactly solved. We need the Baker-Hausdorff lemma (Sakurai, Eq. 2.3.47), where S and A are operators while $\bar{A}$ is the transformed operator,

$$\bar{A} = e^S A e^{-S} = A + [S, A] + \frac{1}{2!}[S,[S,A]] + \cdots + \frac{1}{n!}[S,[S,[S,\cdots,[S,A]]]] \tag{2}$$

   b. Proof this lemma.

Take $S = -c^\dagger c \sum_k \frac{M_k}{\epsilon_k}(a_k^\dagger - a_k)$.

   c. Show that $\bar{c} = cX$, $\bar{c}^\dagger = c^\dagger X^\dagger$, $\bar{a}_k = a_k + \frac{M_k}{\epsilon_k} c^\dagger c$ and $\bar{a}_k^\dagger = a_k^\dagger + \frac{M_k}{\epsilon_k} c^\dagger c$, where $X = \exp\left[\sum_k \frac{M_k}{\epsilon_k}(a_k^\dagger - a_k)\right]$ (note that $X^\dagger = X^{-1}$).

d. Show that $\bar{H} = \bar{c}^\dagger \bar{c}(\epsilon - \Delta) + \sum_k \epsilon_k \bar{a}_k^\dagger \bar{a}_k$, where $\Delta = \sum_k \frac{M_k^2}{\epsilon_k}$. What is the physical meaning of Δ when this Hamiltonian is interpreted as describing localized polarons?

3. One of the remarkable trics in second quantization is the Bogoliubov transformation: as long as the Hamiltonian is bilinear in creation- and anihilation operators, one can still diagonalize the Hamiltonian even when it contains terms mixing states characterized by different particle numbers. Among others, for bosons it is at the heart of Hawking's demonstration that Black holes act like thermal sources of radiation.

   For one species of Bosons annihilated by a, the Bogoliubov transformation reads

$$\begin{aligned} b &= ua + va^\dagger \\ b^\dagger &= ua^\dagger + va \end{aligned}$$

   where u and v are c-numbers which can be taken real in this excercise.

   a. Show that this transformation preserves the canonical commutation relations provided $|u|^2 - |v|^2 = 1$.

   b. Using the results of (a), diagonalize the Hamiltonian

$$H = \epsilon(a^\dagger a + \frac{1}{2}) + \frac{1}{2}\Delta(a^\dagger a^\dagger + aa) \tag{3}$$

   by transforming it in the form $H = \omega(b^\dagger b + \frac{1}{2})$. Find ω, u and v in terms of ϵ and Δ. Imagine that a and $a^\dagger$ have to do with the elementary harmonic oscillator. What happens when $\Delta = \epsilon$?

   c. The ground state is the one annihilated by b: $b|0\rangle = 0$. Evaluate the expectation values of $a^\dagger a$ and $a^\dagger, a$ in the ground state and discuss the physical meaning of these outcomes.

4. The fermionic version of the Bogoliubov transformation is at the heart of another highlight of twentieth century physics: the Bardeen-Cooper-Schrieffer theory of superconductivity. Consider two fermions a_1 and a_2 (in the BCS theory these are electrons with opposite momentum and spin).

   a. Show that the Bogoliubov transformation

$$\begin{aligned} c_1 &= ua_1 + va_2^\dagger \\ c_2^\dagger &= -v^* a_1 + u^* a_2^\dagger \end{aligned}$$

   preserves the canonical anti-commutation relations if $|u|^2 + |v|^2 = 1$.

   b. Use this result to show that the Hamiltonian

$$H = \epsilon(a_1^\dagger a_1 - a_2 a_2^\dagger) + \Delta(a_1^\dagger a_2^\dagger + a_2 a_1) \tag{4}$$

   can be diagonalized in the form,

$$H = \sqrt{\epsilon^2 + \Delta^2}\,(c_1^\dagger c_1 + c_2^\dagger c_2 - 1) \tag{5}$$

   c. Determine the ground state energy and wave function. Interpreted in the BCS context this implies that the electrons form ('Cooper') pairs. Why?