The Kutta-Jukowski Lift formula and some Hydrodynamical Wisdom

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1 Introduction

When I wanted to study Theoretical Physics I knocked on the door of the then only professor of Theoretical Physics in Amsterdam, Jan de Boer, and I asked him whether I could continue my study as a theoretician. He kindly advised me not to do so, since a former student of his, Hayo Meyer, then working at Philips, had calculated the number of theoreticians needed in the Netherlands outside the universities. He arrived at five: three at Philips and two at the Shell company. You know what to do with the advice

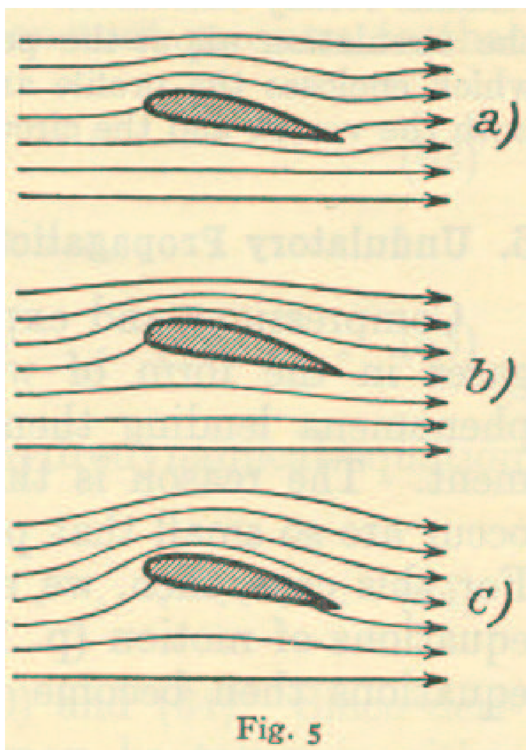

Figure 1: Streamlines around an Airplane Wing

of an older and wise professor: just ignore it! So I asked de Boer what I had to do in order to become a student in Theoretical Physics. He suggested that I read some good book on Theoretical Physics, e.g. the book by Georg Joos. So did I and after 3 months struggling through the 800 pages, I knocked again at his door and told that I had read Joos and still wanted to do Theoretical Physics. He accepted me right away and did not examine me on Joos.

I remembered this story during the discussions on hydrodynamics at the Institute-Lorentz. I wanted to refresh my knowledge on hydrodynamics and having only older books, I turned to Joos; *ieder nadeel heb zijn voordeel*[1]. There is a picture in Joos that had intrigued me very much: streamlines of air flow around an airplane wing. At the time, still in the wake of the war, everything concerning airplanes was interesting, next in line to nuclear energy. The explanation of these pictures was provided by the Kutta-Jukowski lift formula. So I felt impelled to study this again and see whether I could make more sense out of it than 50 years ago. The Hydrodynamics Discussionclub seems the ideal forum to tell the implications of this remarkable formula.

But first some hydrodynamical wisdom. On the basis of the foregoing story you will likely ignore it, but if you are still interested, here it is.

2 Material and Spatial Coordinates

When matter can be described as a continuous medium, one can associate material coordinates $\mathbf{R}$ to the points of the medium. These points change position $\mathbf{r}$ in a space fixed frame. The coordinates $\mathbf{R}$ and $\mathbf{r}$ are equivalent descriptions of the medium which can be transformed into each other. For instance, one could define the material coordinates $\mathbf{R}$ such that at time 0, they are equal to their spatial positions $\mathbf{r}$. The velocity is defined as

$$\mathbf{v} = \frac{d\mathbf{r}(t)}{dt}, \tag{1}$$

where $\mathbf{r}(t)$ is the position of $\mathbf{R}$ as function of time.

I prefer to use the spatial coordinates for the description of the phenomena in fluids. Material coordinates are fine for solids, but for fluids the material coordinates are too political: they flow with the mass. So velocity, density, entropy, etc are taken as function of $\mathbf{r}, t$. There is, however, a useful relation between the time derivatives

$$\left(\frac{\partial}{\partial t}\right)_{\mathbf{R}} = \left(\frac{\partial}{\partial t}\right)_{\mathbf{r}} + \left(\frac{d\mathbf{r}}{dt}\right)_{\mathbf{R}} \frac{\partial}{\partial \mathbf{r}} \tag{2}$$

It tells the relation between the comoving time derivative and the space and time derivatives in the space-fixed frames. We abbreviate this relation as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla, \tag{3}$$

where from now on all partial derivatives are with respect to the frame $(\mathbf{r}, t)$. We will refer to the left hand side of (3) as the material (time) derivative.

A somewhat related issue is the difference between the densities based on volume and based on mass. The extensive thermodynamic quantities, like the internal energy U or the entropy S, can be made intensive by dividing them by the volume V or by the mass M. We will generally use greek symbols for the ratio $\sigma = S/V$ and latin for $s = S/M$. Thermodynamically one converts the two into each other with $\sigma = \rho s$ where $\rho = M/V$. For the space fixed coordinate system the densities based on the volume

[1] Johan Cruyff, public communication

are most useful, while for the material coordinates the ratios with the mass are more natural. In hydrodynamics the difference is important for compressible fluids where the mass density ρ varies in space and time.

3 Conservation Laws

Hydrodynamics hinges on conservation laws. They are the origin of slow motion. We have 3 basic conservation laws.

- Conservation of mass:
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \tag{4}$$

- Conservation of momentum:
$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \mathbf{T} = 0 \tag{5}$$

- Conservation of energy:
$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot \mathbf{J}_\epsilon = 0 \tag{6}$$

If there are sources in the system, the right hand side has to be replaced by the source density. For instance, the gravitional field is a source of momentum. The momentum current density is by definition the stress tensor $\mathbf{T}$. Nevertheless equation (5) is not empty, since the stress tensor, as it is codetermined by (5), can be used to calculate forces on e.g. an airplane wing. To make the conservation laws meaningful one has to supply them with constitutive relations for the currents. Only for the mass density this is trivial, as the current density has only the convective contribution. The whole game of hydrodynamics is the question what to put for the current densities of the momentum and energy.

The expressions we use here, are based on the concept of local equilibrium. Local equilibrium copies the equilibrium expression for the microscopic weight of a configuration, but allows the parameters, density $\rho(\mathbf{r}, t)$ and temperature $T(\mathbf{r}, t)$ to vary in space and time. In addition the system has a mean local velocity $\mathbf{v}(\mathbf{r}, t)$. This is an approximation to the reality since gradients in the density, velocity and temperature will automatically cause deviations from local equilibrium. We will see that the approximation implies the neglect of dissipative processes. For local equilibrium one can work out the expression for the stress tensor from the microscopic Liouville equation.

$$\mathbf{T} = \rho \, \mathbf{v} \, \mathbf{v} + p \, \mathbf{I}, \tag{7}$$

where p is the hydrostatic pressure. Clearly this expression only holds for an isotropic fluid. Local equilibrium gives for the energy current density

$$\mathbf{J}_\epsilon = \epsilon \, \mathbf{v} + \mathbf{T} \cdot \mathbf{v}. \tag{8}$$

The first term is the convective term and the second is the result of the work done a volume element.

Local equilibrium implies that the standard thermodynamic relations remain valid. Thus we can use the equation of state to relate the pressure p to the density ρ and the temperature or rather to the energy density and so close the system (4)–(8). The densities, based on the mass, are more convenient than those on the volume for the use of thermodynamics. In particular we make the energy density explicit as

$$\epsilon = \rho(v^2/2 + u), \tag{9}$$

where the first term is the kinetic energy of the fluid and the second, u, is the internal energy (per mass), which contains the potential and kinetic energy of the molecules (in the comoving system). The equation for u is neater in terms of comoving derivative D/Dt. Using this derivative the conservation laws become for the mass

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{v}, \tag{10}$$

for the velocity

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p, \tag{11}$$

and for the internal energy

$$\rho \frac{Du}{Dt} = -p \nabla \cdot \mathbf{v}. \tag{12}$$

The advantage of this formulation is, that it is easy to show that the entropy per mass s is a constant along the flow. Thermodynamics

$$TdS = dU + pdV \tag{13}$$

and (10) and (12) tell us that

$$T\frac{Ds}{Dt} = \frac{Du}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt} = 0. \tag{14}$$

This confirms that the ansatz of local equilibrium ignores dissipative processes. As a consequence, the energy equation, although still valid, decouples from the other equations. Although (14) only implies that s is constant along the flowlines, in practice s will often be constant through all space, since usually the flow conditions are such that asymptotically the flow is uniform and therefore s is also constant in a plane perpendicular to the flowlines. Now with s constant, the pressure p is effectively only coupled to the density ρ. Thus the adiabatic compressibility enters in e.g. the sound velocity.

4 Incompressible Fluids and Bernoulli's Law

Fluids like water are rather incompressible. This means that the pressure varies very rapidly with the density or reversely that for modest pressure variation the density is constant. That means that the pressure is uncoupled to the density. In that case we do not need an equation of state to relate pressure to the other variables and simply take ρ as a constant. For incompressible fluids the flow field obeys

$$\nabla \cdot \mathbf{v} = 0. \tag{15}$$

As we shall see that extra condition on the flow field determines the flow and the flow field determines the pressure.

We now look to stationary states and rewrite (10) using conservation of mass and the identity

$$\mathbf{v} \times (\nabla \times \mathbf{v}) = \frac{1}{2}\nabla v^2 - \mathbf{v} \cdot \nabla\, \mathbf{v}. \tag{16}$$

This gives the stationary case of (11) the form

$$\mathbf{v} \times (\nabla \times \mathbf{v}) = \frac{1}{2}\nabla v^2 + \frac{1}{\rho}\nabla p \tag{17}$$

We would like to get the right hand side as a gradient. For incompressible fluids one can simply pull ρ under the gradient. So we follow the textbooks and comment on it later

$$\mathbf{v} \times (\nabla \times \mathbf{v}) = \nabla \left(\frac{1}{2}\nabla v^2 + p/\rho\right) \equiv \nabla Q. \tag{18}$$

A streamline is a line where the velocity $\mathbf{v}$ is tangential in each point of the line. So if we integrate (18) along a streamline from point $\mathbf{r}_0$ to $\mathbf{r}$ we find

$$Q(\mathbf{r}_0) = Q(\mathbf{r}), \tag{19}$$

because the left hand side is perpendicular to the streamlines. So $Q(\mathbf{r})$ is constant along a streamline. This is Bernoulli's law for incompressible fluids. It holds also if $\nabla \times \mathbf{v}$ would differ from 0.

One derives a stronger statement, if we can find a cross section of the streamlines where the combination $Q(\mathbf{r})$ is a constant for other reasons. For instance when flows are asymptotically uniform. Then we can find a plane crossing the streamlines in which Q is also constant. The flow around a airplane wing is uniform far away from the wing and in such case Q is independent of $\mathbf{r}$, for all $\mathbf{r}$. Then the right hand of (18) is zero and as a consequence

$$\nabla \times \mathbf{v} = \mathbf{0}. \tag{20}$$

Thus the flow pattern is irrotational, not by a physical principle but as the solution of the hydrodynamic equations.

The conventional derivation of Bernoulli's law is based on an energy consideration. The objection to these considerations, which give

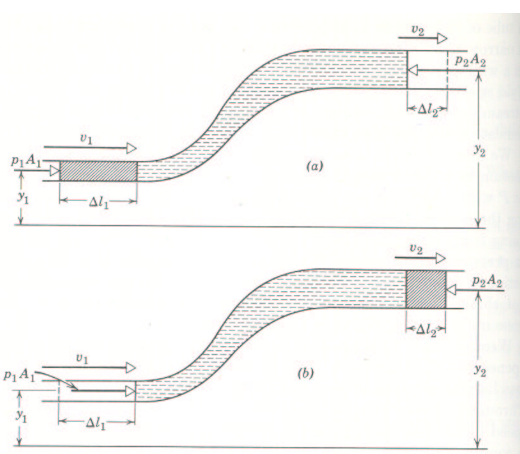

Figure 2: Derivation of Bernoulli's law using energy.

$$\frac{1}{2}\rho v^2 + p + \rho g z = \text{constant} \tag{21}$$

along a flow line, is that possible changes of the internal energy are not considered. This is only correct for incompressible fluids.

Now Bernoulli's law is often used for airflow and air is compressible. Is it still valid? The essence of the derivation is that we can write the right hand side of (17) as a gradient. For compressible fluids in local equilibrium this can be achieved for isentropic processes using the thermodynamic relation for the enthalpy H

$$dH = TdS + Vdp. \tag{22}$$

Dividing this through the mass M and using (14), that $ds = dS/M = 0$ along a flowline, we get

$$dh = \frac{1}{\rho}dp. \tag{23}$$

So we get for (17)

$$\mathbf{v} \times (\nabla \times \mathbf{v}) = \nabla \left(\frac{1}{2}\nabla v^2 + h \right) \equiv \nabla \tilde{Q}, \tag{24}$$

with the consequence that "total enthalpy" $\tilde{Q}$ is constant. Again it follows from the same arguments that the flow is irrotational (20). Now for ideal gases like air the relation between enthalpy and pressure is simple (air has 3 translational and 2 rotational degrees of freedom)

$$h = u + p/\rho = (5/2)k_BT/m + k_BT/m = (7/2)k_BT/m = (7/2)(p/\rho). \tag{25}$$

So Bernouilli is recovered with a factor 7/2 as correction. This is a non-neglible difference and it is amazing that not much attention is paid to this factor in the textbooks. The aerodynamicists, however, are well aware of this factor. They also claim that the flow around an airplane wing is to a good approximation incompressible, for low Mach numbers ($<$ 0.3 Mach).[2] The argument is based on some thermodynamics.

5 Complex Velocity Potential

An irrotational flow pattern can be obtained from a velocity potential

$$\mathbf{v} = \nabla\Phi. \tag{26}$$

Generally this leads to a coupled set of equations for the scalars ρ and Φ. An airplane wing may be idealized as an infinitely long cylinder with the axis along the z axis. The cross section of the wing can be any closed contour in the $x-y$ plane. For such set-up, all the functions will be independent of the coordinate z. This setting leads to the overwhelming temptation to try the function Φ as the real part of an analytic function $\Omega(z)$ of the complex variable $z = x + iy$. This has only one drawback. For analytic functions

$$\Omega = \Phi + i\Psi \tag{27}$$

there are the Cauchy relations between the real and imaginary parts

$$\frac{\partial\Phi}{\partial x} = \frac{\partial\Psi}{\partial y}, \qquad \frac{\partial\Phi}{\partial y} = -\frac{\partial\Psi}{\partial x}. \tag{28}$$

[2]private communication, W. J. Bannink, TU Delft.

If Φ is the velocity potential this implies that

$$\frac{\partial v_x}{\partial x} = \frac{\partial^2 \Psi}{\partial y \partial x} = -\frac{\partial v_y}{\partial y}, \tag{29}$$

or that the flow is incompressible

$$\nabla \cdot \mathbf{v} = 0. \tag{30}$$

Let us not be deterred by this restriction and assume that compressibility effects are not of great importance. We know that the lines of constant Φ are perpendicular to the lines of constant Ψ. So if Φ is the flow potential the curves of constant Ψ are the streamlines. Note that the velocity field follows also from Ψ as

$$v_x = \frac{\partial \Psi}{\partial y}, \qquad v_y = -\frac{\partial \Psi}{\partial x} \tag{31}$$

One may interchange the role of real and imaginary parts, e.g. by multiplying Ω by i. So one can obtain a flow pattern in two ways from a complex flow potential. E.g. consider the function

$$\Omega(z) = A \log(z) = (A/2) \log(x^2 + y^2) + iA \arctan(y/x). \tag{32}$$

Taking the real part as flow potential gives the pattern

$$v_x = A \frac{x}{x^2 + y^2}, \qquad v_y = A \frac{y}{x^2 + y^2}. \tag{33}$$

which is a radially outflowing pattern with the source in the middle. If we take the imaginary part as flow potential we get

$$v_x = -A \frac{y}{x^2 + y^2}, \qquad v_y = A \frac{x}{x^2 + y^2}. \tag{34}$$

which is a vortex pattern: a concentric circulation with a singularity in the origin.

For complex flow potentials the derivative

$$w = \frac{d\Omega}{dz} = \frac{\partial \Omega}{\partial x} = \frac{\partial \Phi}{\partial x} + i \frac{\partial \Psi}{\partial x} = v_x - i v_y \tag{35}$$

is called the complex velocity. Note that in the example (32) the expressions (33) can be obtained easily in this way.

Flows derived from a potential are irrotational. It does not necessarily mean that the circulation

$$\Gamma = \oint \mathbf{v} \cdot \mathbf{ds} = \oint (v_x dx + v_y dy) = -\mathrm{Re} \oint w dz \tag{36}$$

vanishes. The minus sign arises from the convention that in aerodynamics the contour is taken clockwise, while in mathematics it is taken contourclockwise. The integral is independent of the contour. When the contour can be contracted to a point inside the domain of the flow, the circulation vanishes.

6 The Kutta-Jukowski Lift Formula

Let us turn again towards the flow around the idealized wing of an airplane. The surface of the wing is a streamline. We skip the frictional forces on the wing, i.e. those which are tangential to the surface and focus on the normal forces. Then the convective part of the stress tensor does not contribute. The normal force follows from the pressure integral along the surface

$$\mathbf{F} = \oint p\,\mathbf{n}ds \tag{37}$$

Here $\mathbf{n}$ is the normal to the surface, pointing inwardly. Then rewrite the pressure with Bernoulli in terms of of v^2 and a constant k. For neatness we have to assume an incompressible fluid, which we have to do anyway, because we are going to use the complex formulation.

$$\mathbf{F} = \oint \left(-\frac{\rho}{2}v^2 + k\right)\mathbf{n}ds \tag{38}$$

The constant does not contribute. Along the surface we have

$$\frac{dx}{ds} = n_y, \qquad \frac{dy}{ds} = -n_x, \qquad n_x dx + n_y dy = 0. \tag{39}$$

Then consider the complex number

$$Z = F_x + iF_y = -\frac{\rho}{2}\oint v^2(n_x + in_y)ds = -\frac{\rho i}{2}\oint v^2(dx - idy) \tag{40}$$

The velocity is along the surface, so we have

$$v_x dy = v_y dx \tag{41}$$

One easily verifies with (41) that

$$v^2(dx - idy) = (v_x - iv_y)^2(dx + idy) = w^2 dz \tag{42}$$

So one finds for Z

$$Z = \frac{\rho i}{2}\oint w^2 dz \tag{43}$$

(remember the sign convention, complex integrations are counterclockwise!).

Now suppose that the flow pattern can be derived from a complex potential Ω, for which we make the asymptotic expansion

$$\left\{\begin{array}{lcl} \Omega & = & w(\infty)z - \dfrac{\Gamma}{2\pi i}\log(z) + \omega_0 + \dfrac{a_{-1}}{z}\cdots \\ w & = & w(\infty) - \dfrac{\Gamma}{2\pi i z} - \dfrac{a_{-1}}{z^2}\cdots \end{array}\right. \tag{44}$$

From (36) one observes that Γ is indeed the circulation around the wing (clockwise for $\Gamma > 0$). For (43) one has to square the second relation (44) and one finds for the contour integral far from the wing

$$Z = F_x + iF_y = -\rho i w(\infty)\,\Gamma = -i\rho\Gamma[v_x(\infty) - iv_y(\infty)] \tag{45}$$

This is the Kutta-Jukowski lift formula. The main message is that the lift is determined by the circulation! Amongst others this implies the Magnus force around a circulating cylinder.

Let us now comment on the approximation that the flow can be taken as incompressible.[3] Since the flow is isentropic one has the relation

$$\nabla p = \left(\frac{\partial p}{\partial \rho}\right)_s \nabla \rho = c^2 \nabla \rho. \tag{46}$$

with c the sound velocity. We find from the adiabatic equation of state for an ideal gas that

$$\frac{p}{p_\infty} = \left(\frac{\rho}{\rho_\infty}\right)^\gamma \qquad c^2 = \gamma \frac{p}{\rho}, \tag{47}$$

where $\gamma = C_p/C_V$ is the ratio of the specific heats, which equals 7/5 for air. We have taken the values far from the wing as reference values for pressure and density. The expression (25) shows that the local temperature T is the relevant quantity. We can convert this into a local sound velocity $c^2 = k_B T/m$. The total enthalpy then can be expressed as

$$\tilde{Q} = \frac{c^2}{2(\gamma - 1)} + \frac{v^2}{2} = \frac{c_\infty^2}{2(\gamma - 1)} + \frac{v_\infty^2}{2}. \tag{48}$$

For the relation between the density and c, we use

$$\frac{c^2}{c_\infty^2} = \frac{T}{T_\infty} = \frac{p/\rho}{p_\infty/\rho_\infty} = \left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1}. \tag{49}$$

With the last two equations we can express ρ in terms of c as

$$\rho = \rho_\infty \left(1 + (\gamma - 1)\frac{v_\infty^2 - v^2}{c_\infty^2}\right)^{1/(\gamma-1)}. \tag{50}$$

Now v will in general be less than the asymptotic speed v_∞ and let us take the worst case $v = 0$ which occurs at a frontpoint of the wing (the stagnation point, see later). Then we expand in terms of the Mach number $M = v_\infty/c_\infty$ and find

$$\rho = \rho_\infty (1 + M^2 + (1 - \gamma/2)M^4 + \cdots) \tag{51}$$

For $M = 0.3$ the second term is only 11 % of the first. So the changes in the density are still small. Expression (50) shows however, that as soon as the velocity changes in absolute value, the density changes. Moreover the adiabatic equation of state shows that density differences are reflected in pressure differences and vice versa. Thus assuming that the flow is incompressible can only be a first approximation. The whole point of Bernoulli's formula is that the pressure (and therefore the density) is not constant.

7 Conformal Invariance

We now try to answer the question: why should there be circulation around an airplane wing? The nice property of analytic functions is that one can apply a conformal transformation and again obtain an analytic function. So suppose that we change variables

$$z = z(u), \qquad u = u(z) \tag{52}$$

[3] W. J. Bannink TU Delft, private communication.

The circle $z = R\exp i\theta$ will transform to a curve $u = u(R\exp i\theta)$ in the complex u plane. Now we might manage to give this curve the shape of an airplane wing. Then the function

$$\Gamma(u) = \Omega(z(u)) \tag{53}$$

provides a flow potential in the complex u plane. Since streamlines transform into streamlines the contour of the wing is a streamline of the flow in the complex u plane and $\Gamma(u)$ gives the flow pattern around the wing. We take the potential

$$\Omega(z) = v\left(z + \frac{R^2}{z}\right) + \frac{\Gamma}{2\pi i}\log(z/R) \tag{54}$$

as starting point. The complex velocity w is given by

$$w = v\left(1 - \frac{R^2}{z^2}\right) + \frac{\Gamma}{2\pi i z} \tag{55}$$

The first part corresponds to an asymptotically uniform flow in the x direction with velocity v. The second part is a circular flow around the cylinder with circulation Γ. The circle $|z| = R$ is a streamline of the flow, since the imaginary part of Ω vanishes on the circle ($\Psi = 0$). Thus (55) gives the flow around a circular cylinder of radius R. For $\Gamma = 0$ the real axis $y = 0$ is a flow line up to the points $z_\pm = \pm R$ where the flow

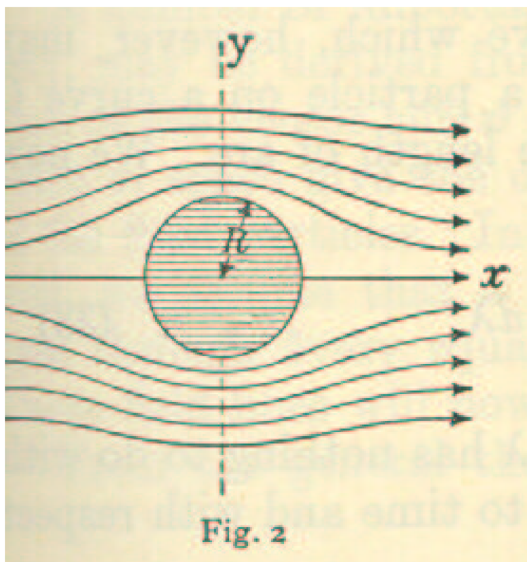

Figure 3: Streamlines around a Cylinder

vanishes. Points where w vanish are called stagnation points (see Fig. 2). For $\Gamma \neq 0$ the stagnation points change position to

$$z_\pm = R\left[i\frac{\Gamma}{4\pi v R} \pm \left(1 - \left(\frac{\Gamma}{4\pi v R}\right)^2\right)^{1/2}\right] \tag{56}$$

They are still located on the circle $|z| = R$. Note that we count the circulation positive when it is counterclockwise. The shape of the airplanes induce a negative circulation. So the stagnation points are located in the lower complex halfplane.

Now assume that our transformation (52) reduces asymptotically to the identity. Then the flow in the complex u plane will have the same asymptotic velocity and the same circulation around the wing. By varying the circulation Γ we can vary the location of the stagnation points. They go to the positions $u_\pm = u(z_\pm)$. In Fig. 1 we see that in case a) they are on top of the wing while in case c) they are underneath the wing. Case b) has a stagnation point at the tip of the wing. This will be the flow pattern that settles. The argument is that for another location there has to be a flow around the sharp tip of the wing and that empirically such flows will cause turbulence, which we have excluded. This is the Kutta condition (Wilhelm Kutta 1902).

A few final comments.

- We have only assumed that the profile of the wing is a flowline. A more precise treatment introduces a boundary layer, in which viscosity plays a role, but where the flow is still laminar. Then the drag on the wing can also be calculated. Outside the boundary layer the present approximation is still useful.

- The wing is not an infinitely long cylinder, but ends in a tip. The circulation around the wing produces a vortex at the tip. Tip fans help to use the vortex for the lift. Nevertheless, large airplanes create a lot of turbulence in particular at the tip of the plane.

- Remains the question how the transformation (52) has to be designed in order to obtain a reasonable wing.