EXAM QUANTUM THEORY January 15, 2009
(Course Dr. P.J.H. Denteneer, Fall 2008) 10.00 - 13.00h

Write NAME, INITIALS and STUDENT NUMBER on every sheet you hand in. Start each new problem on a new page.
All problems count for the same number of points (25) in the grading.

1) Consider a spin-$\frac{1}{2}$ object. The eigenvectors of $S_{\hat{n}} = \frac{1}{2}\hbar\vec{\sigma}\cdot\hat{n}$ for a general direction $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ in the basis of the eigenstates $|+\rangle$ and $|-\rangle$ of $\hat{\sigma}_z$ are given by:

$$|+,\hat{n}\rangle = \begin{pmatrix} e^{-i\phi/2}\cos\theta/2 \\ e^{i\phi/2}\sin\theta/2 \end{pmatrix} \quad \text{and} \quad |-,\hat{n}\rangle = \begin{pmatrix} -e^{-i\phi/2}\sin\theta/2 \\ e^{i\phi/2}\cos\theta/2 \end{pmatrix}$$

(a) Determine the matrix corresponding to $\hat{\sigma}_z$ in the σ_x-representation (i.e. in the basis of eigenstates of $\hat{\sigma}_x$).
(Hint: First write down the *spectral decomposition* of $\hat{\sigma}_z$)

One can deduce a *rotation matrix for spin-$\frac{1}{2}$*:

$$D^{(1/2)}(\theta,\phi) = \begin{pmatrix} e^{-i\phi/2}\cos\theta/2 & -e^{-i\phi/2}\sin\theta/2 \\ e^{i\phi/2}\sin\theta/2 & e^{i\phi/2}\cos\theta/2 \end{pmatrix}$$

(b) Describe qualitatively what is meant here by *rotation*.

Consider now the *EPR-pair* state of a system of two spin-$\frac{1}{2}$ objects (EPR: Einstein, Podolsky, Rosen):

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|++\rangle + |--\rangle\right)$$

(c) Is $|\Psi\rangle$ invariant under rotations as given by $D^{(1/2)}(\theta,\phi)$?

(d) Consider the two-spin system in the pure state $|\Psi\rangle$. Compute $\rho^{(1)} = \mathrm{Tr}_2\,\rho$, where ρ is the state operator. Does $\rho^{(1)}$ describe a pure state or a mixture?

(e) In an EPR-type of experiment, Alexia measures the spin of particle 1 and Bernhard that of particle 2 of the two-spin system in state $|\Psi\rangle$. Calculate the probability amplitude that Alexia measures spin up and Bernhard spin down, if Alexia's analyzer is oriented in the $z-$direction and Bernhard's analyzer is oriented in a direction $\hat{b}$ making an angle θ with the $z-$direction.

2) Consider a simple harmonic oscillator with raising- and lowering operators $\hat{a}^\dagger$ and $\hat{a}$, respectively; $[\hat{a}, \hat{a}^\dagger] = \hat{1}$, $\hat{n} \equiv \hat{a}^\dagger \hat{a}$, $\hat{n}$-basis $\{|n\rangle\}$ with $n = 0, 1, 2, \ldots,$ $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$, $\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$. The position- and momentum operators are (in dimensionless form) given by:

$$\hat{Q} = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger + \hat{a}\right) \qquad \hat{P} = \frac{i}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right)$$

A *coherent state* $|z\rangle$ is given by:

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \ ,$$

where z is a complex number.

(a) What is the characterizing property of a coherent state? Show that it holds for the state as given above.

(b) Which state (or: density) operator $\hat{\rho}$ describes a *pure* coherent state? Show that for your answer holds: $\mathrm{Tr}\,\hat{\rho} = 1$.

One defines a *squeezing operator*:

$$S(\lambda) = \exp\left[\tfrac{1}{2}\lambda^* \left(\hat{a}^\dagger\right)^2 - \tfrac{1}{2}\lambda \hat{a}^2\right]$$

(c) Compute $\langle Q\rangle$ and $\langle P\rangle$ for the state $S(\lambda)|z\rangle$, where $|z\rangle$ is a coherent state and λ is real. Hint: Use the following operator identity:

$$e^{\lambda A}\, B\, e^{-\lambda A} = B + \lambda\,[A, B] + \frac{\lambda^2}{2!}\,[A, [A, B]] + \cdots$$

(d) How do $\langle Q\rangle$ and $\langle P\rangle$ in the state $S(\lambda)|z\rangle$ evolve in time? Answer this question without doing an explicit calculation, but knowing that a squeezed state is a generalized coherent state.

3) Consider the *Hubbard model* for identical electrons on a lattice in the discrete position-representation. The Hamilton operator for this quantum many-body system is:

$$\hat{H} = \hat{T} + \hat{V} - \mu\hat{N} - h\hat{M} \ , \tag{1}$$

where

$$\hat{T} = -t \sum_{<ij>\sigma} c^{\dagger}_{i\sigma} c_{j\sigma} \qquad \hat{V} = U \sum_{j} n_{j+} n_{j-} \tag{2}$$

$$\hat{N} = \sum_{j\sigma} n_{j\sigma} \qquad \hat{M} = \sum_{j\sigma} \sigma n_{j\sigma} \ . \tag{3}$$

$c^{\dagger}_{j\sigma}$ and $c_{j\sigma}$ are the creation- and annihilation-operators of an electron at site j with spin σ, respectively. $< ij >$ denotes a sum over pairs of nearest-neighbor sites i and j; the spin-index σ as a subscript takes values $+$ and $-$ and elsewhere takes values $+1$ and -1, corresponding to spin-up and spin-down, respectively; $n_{j\sigma} = c^{\dagger}_{j\sigma} c_{j\sigma}$ and t and U are constants, μ is a chemical potential and h a (Zeeman) magnetic field

(a) Give the complete set of algebraic relations of the annihilation- and creation-operators $c_{j\sigma}$ and $c^{\dagger}_{j\sigma}$.

(b) Compute the commutator $\left[n_{\ell\sigma}, c^{\dagger}_{i\sigma} c_{j\sigma}\right]$.

(c) From the result of (b) compute the commutator $\left[\hat{N}, \hat{T}\right]$. What is the physical meaning of the result for $\left[\hat{N}, \hat{T}\right]$?

(d) The operator $\hat{V}$ gives the interaction between the electrons. Give a qualitative description of how the electrons interact in this model.

A *bipartite lattice* is a lattice that can be seen as made up of two separate sublattices such that nearest-neighbor sites are always on different sublattices. An example is a two-dimensional square lattice. Consider the following *staggered particle-hole transformation* (transformed operators are denoted by an over-bar):

$$\bar{c}_{j\sigma} = (-1)^{j}\, c^{\dagger}_{j\sigma} \ .$$

The numbering of the sites by site-index j is such that the sign-factor $(-1)^j$ is different on the two sublattices and equal on a sublattice.

(e) Show that, on a bipartite lattice, the staggered particle-hole transformation is a *canonical* transformation.

(f) For which values of μ and h is the Hamiltonian invariant under the staggered particle-hole transformation?

4) The quantummechanical *propagator* for a particle that at time t_0 is at position x_0 and at a later time t' is at position x' is given by the path integral:

$$<x', t'|x_0, t_0>= \int_{x_0}^{x'} \mathfrak{D}[x(t)] e^{i\mathcal{S}[x(t)]/\hbar} \ ,$$

in which the action $\mathcal{S}[x(t)]$ in terms of the Lagrangian $L(\dot{x}, x)$ is given by:

$$\mathcal{S}[x(t)] = \int_{t_0}^{t'} \mathrm{d}t \, L(\dot{x}, x) \quad .$$

(the dot above x means differentiation with respect to time)

In classical mechanics a harmonic oscillator with mass m and (angular) frequency ω in one spatial dimension is given by the following Lagrangian:

$$L(\dot{x}, x) = \tfrac{1}{2} m \dot{x}^2 - \tfrac{1}{2} m \omega^2 x^2 \quad .$$

The equation of motion follows from the Euler-Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad .$$

(a) Compute the classical path $x_{\mathrm{cl}}(t)$ for the harmonic oscillator in terms of x_0, t_0, x', t' and ω.

(b) Show, by splitting off the classical path, $x(t) = x_{\mathrm{cl}}(t) + y(t)$, that the propagator for the harmonic oscillator can be written as:

$$<x', t'|x_0, t_0>= e^{\frac{i}{\hbar}\mathcal{S}[x_{\mathrm{cl}}(t)]} \int_0^0 \mathfrak{D}[y(t)] \exp\left\{ \frac{i}{\hbar} \mathcal{S}[y(t)] \right\} .$$

(c) Argue that the propagator for the harmonic oscillator for the case $t_0 = 0$ is of the form:

$$<x', t'|x_0, 0>= A(t') F(x_0, x', t') \ ,$$

where A and F are functions of the variables indicated.

(d) Compute $F(x_0, x', t')$ in case $x_0 = 0$.

Useful formulas: $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha \qquad \sin 2\alpha = 2 \sin \alpha \cos \alpha$