Quantum Theory
Fall 2009

Problem Set 2: *Time Evolution; Representations; Heisenberg and interaction picture*

Problem 8: Exercise 4.4.3 LB; The Hellmann-Feynman theorem

Problem 9: Exercise 4.4.4 LB; Time evolution of a two-level system

Problem 10: (Taken from the *Quantum Theory* exam of Jan. 2007)

Consider a spin $\frac{1}{2}$ object. Let the Hamilton operator be: $\hat{H} = -\hbar\omega\hat{\sigma}_y$. The normalised eigenstates of $\hat{\sigma}_y$ can be used as a basis and are given by:

$$|+,y\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + i\,|-\rangle\right) \quad \text{and} \quad |-,y\rangle = \frac{1}{\sqrt{2}}\left(i\,|+\rangle + |-\rangle\right) \ ,$$

in terms of the eigenstates $|+\rangle$ and $|-\rangle$ of $\hat{\sigma}_z$.

(a) Determine the matrix corresponding to $\hat{\sigma}_z$ in the σ_y-representation (i.e. in the basis of eigenstates of $\hat{\sigma}_y$).

(b) Argue that the (unitary) matrix U that transforms a state vector given in the σ_z-representation into the same vector given in the σ_y-representation is:

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \ .$$

Now, the state vector in the σ_z–representation at $t = 0$ is: $\chi(t = 0) = \begin{pmatrix} \cos\gamma \\ \sin\gamma \end{pmatrix}$. Call the state vector in de σ_y–representation:

$$\eta(t) = \begin{pmatrix} e(t) \\ f(t) \end{pmatrix} \ .$$

(c) Determine $\eta(t = 0)$ and solve the Schrödinger-equation $i\hbar\frac{\partial}{\partial t}\eta(t) = \hat{H}\eta(t)$ in the σ_y–representation.

(d) Using the results derived in (a) and (c), show that the expectation value of $\hat{\sigma}_z$ at time t is given by:

$$\langle\sigma_z\rangle(t) = \cos\left(2\omega t - 2\gamma\right) \ .$$

Problem 11: Exercise 4.4.5 LB; Unstable states

Problem 12:

Consider a spin-1/2 system in a magnetic field in the z-direction. The Hamiltonian is given by

$$H = -\left(\frac{qB}{2m}\right) S_z = \omega S_z \tag{1}$$

a) Find the Heisenberg operators $S_x(t)$, $S_y(t)$ and $S_z(t)$.

b) Assume that the system is in either of the Heisenberg states $|\pm, \hat{x}\rangle$. ($S_x |\pm, \hat{x}\rangle = \pm\hbar/2 |\pm, \hat{x}\rangle$). Calculate the expectation values of $S_x(t)$, $S_y(t)$ and $S_z(t)$.

c) Find the *Heisenberg basis kets* $|\pm, \hat{x}; t\rangle$ corresponding to the basis kets $|\pm, \hat{x}\rangle$ at $t = 0$.

d) If the system is in either of the Heisenberg states $|\pm, \hat{x}\rangle$, calculate the probability $|\langle\pm, \hat{x}; t|\pm, \hat{x}\rangle|^2$ that a measurement of $S_x(t)$ at $t > 0$ will give a value equal to the one at $t = 0$.

e) Calculate the Heisenberg basis kets $|\pm, \hat{y}; t\rangle$ of $S_y(t)$ and the probabilities $|\langle\pm, \hat{y}; t|\pm, \hat{x}\rangle|^2$ and $|\langle\mp, \hat{y}; t|\pm, \hat{x}\rangle|^2$ that a Heisenberg state $|\pm, \hat{x}\rangle$ will have eigenvalues $\pm\hbar/2$ of $S_y(t)$ at $t > 0$.

Problem 13:

In the *interaction picture* (or: *Dirac picture*) one considers the case that the Hamiltonian H (Schrödinger picture) is given by the sum of two terms: H_0 and V, where usually V is an interaction term or perturbation of the (non-interacting or solvable) Hamiltonian H_0. The operators (time-dependent) in the interaction picture are given by:

$$A_{\text{IP}}(t) = e^{iH_0t/\hbar} A e^{-iH_0t/\hbar} \tag{2}$$

The state vectors (also time-dependent!) in the interaction picture are given by:

$$|\varphi(t)\rangle_{\text{IP}} = U_{\text{IP}}(t)|\varphi(0)\rangle \ , \tag{3}$$

where

$$U_{\text{IP}}(t) = e^{iH_0t/\hbar} U(t) \tag{4}$$

is the time-evolution operator in the interaction picture.

(a) Show that the time evolution of state vectors in the interaction picture is governed by the perturbation term (and not by the full Hamiltonian as in the Schrödinger picture), as follows:

$$i\hbar \frac{d}{dt}|\varphi(t)\rangle_{\text{IP}} = V_{\text{IP}}(t)|\varphi(t)\rangle_{\text{IP}} \ . \tag{5}$$

(b) Why can the subscript IP with $|\varphi(0)\rangle$ be omitted in the formula for the state vectors?