On the question of the geometry of curved space*

Alexander Friedmann

Petrograd Polytechnic Institute

15 April 1922

In his well-known book "Raum, Zeit, Materie" Weyl explicates the basics of the curved space geometry using the notion of a parallel transport of a vector, which is more general than the one being developed by Levi-Civita [1], and attaches to this notion the new and extremely original idea of a metric connection of the space. Development of the geometrical ideas lets Weyl generalize Einstein's ideas and obtain the derivation of Maxwell's equation (jointly with Mie's theory) from geometrical properties of the space in question.

While generalizing the idea of a parallel transport, Weyl introduced a number of substantial limitations. For example, he assumed the symmetry of $\Gamma^i_{\lambda\mu}$ in lower indices and also connected the parallel transport of covariant and contravariant vectors. Besides, Weyl's description does not give a clear enough geometrical reason for the change in the norm of a vector under parallel transport to be proportional to the norm of the vector.

Taking into account the startling results of Weyl, it seems interesting to get rid of the constraints mentioned as well as to find out the geometrical reason for the special kind of metric connection of the space used by Weyl [2].

Generalizing the geometrical ideas of Weyl should first of all not constrain the parallel transport of a contravariant vector to be a function of the transport of the covariant. It seems useful while getting rid of this constraint to classify the spaces we got with geometrical objects attached to the spaces. Such a geometrical object will be the notion of a plane, generalized to a curved space of n dimensions accordingly. The classification will allow to choose among such spaces, for which Weyl's will be a special case.

The geometrical reason for the metric connection form used by Weyl, as will be shown later, is in fact that this form is a necessary and sufficient condition for the angle of covariant vectors not to change under the parallel transport of the vectors. Imposing the same condition for contravariant vectors, we will be able to obtain the parameters defining the parallel transport of both

*Unpublished manuscript in Russian, found in the library of Paul Ehrenfest at Leiden University. Translated by *Dmitry Pikulin*. Scans of the original are available at the Instituut-Lorentz, together with related correspondence. In particular, you can find there a letter from Friedmann to Ehrenfest in which he explains the significance of this manuscript and spells out his ideas for further work along these lines.

co- and contravariant vectors with the fundamental metric tensor g_{ik}, two (not one as by Weyl) contravariant scale vectors, and with two special tensors of third rank. Studying the spaces, where the tensors of third rank are zero or depend on both metric tensor and both scale vectors, we will be able to get properties of the space using the metric tensor and the two contravariant scale vectors.

For this kind of generalized space the number of coordinate and scale invariants (in Weyl's notion) will be much larger than for Weyl's space. Some of them, analogous to Weyl's invariants, are straightforward to construct. From this a question arise: could one get Maxwell equations from the properties of the more general space, without relying on Mie's theory? At the end of the current note we will discuss the problem in more details. But we have to remark right away that the existence of one more contravariant scale vector must obligatorily result in a system of equations supplementary to Maxwell's equations. This system seems to be some generalization of Mie's theory.

§1.

1. Let the manifold $\mathscr{M}_n$ of dimension n have the coordinates $x_1, x_2, \dots x_n$. If we apply a point transformation to coordinates to get $\bar{x}_1, \bar{x}_2, \dots \bar{x}_n$, we get the manifold $\bar{\mathscr{M}}_n$,which we define to be obtained with point transformation from $\mathscr{M}_n$. We use the convention to mark with a bar a quantity after the same transformation as $\mathscr{M}_n$ to $\bar{\mathscr{M}}_n$.

If for all of $\mathscr{M}_n$, connected by a point transformation of coordinates we give n^3 functions of the coordinates $\Gamma^i_{\lambda\mu}$, transforming according to:

$$\bar{\Gamma}^i_{\lambda\mu} = \Gamma^\gamma_{\alpha\beta} \frac{\partial x_\alpha}{\partial \bar{x}_\lambda} \frac{\partial x_\beta}{\partial \bar{x}_\mu} \frac{\partial \bar{x}_i}{\partial x_\gamma} + \frac{\partial^2 x_\nu}{\partial \bar{x}_\lambda \partial \bar{x}_\mu} \frac{\partial \bar{x}_i}{\partial x_\nu}, \tag{1}$$

then we call the $\Gamma^i_{\lambda\mu}$ tensorial parameters.

From eq. (1) it directly follows:

$$\frac{\partial^2 x_\nu}{\partial \bar{x}_\lambda \partial \bar{x}_\mu} = \bar{\Gamma}^i_{\lambda\mu} \frac{\partial x_\nu}{\partial \bar{x}_i} - \Gamma^\lambda_{\alpha\beta} \frac{\partial x_\alpha}{\partial \bar{x}_\lambda} \frac{\partial x_\beta}{\partial \bar{x}_\mu}. \tag{2}$$

Using Christoffel brackets, obtained using a symmetric tensor g_{ik}, we get that any tensorial parameters may be represented according to:

$$\Gamma^i_{\lambda\mu} = \left\{ \begin{array}{c} \lambda\mu \\ i \end{array} \right\} + A^i_{\lambda\mu}, \tag{3}$$

where $A^i_{\lambda\mu}$ is an arbitrary mixed tensor of the third rank.

2. Studying the properties of tensorial parameters we notice that a large role is played by the two tensors of third and forth rank, which are defined according to:

$$\gamma^i_{\lambda\mu} = \Gamma^i_{\lambda\mu} - \Gamma^i_{\mu\lambda}, \tag{4}$$

$$F^i_{\kappa\lambda\mu} = \frac{\partial \Gamma^i_{\kappa\lambda}}{\partial x_\mu} - \frac{\partial \Gamma^i_{\kappa\mu}}{\partial x_\lambda} + \Gamma^\sigma_{\kappa\lambda} \Gamma^i_{\sigma\mu} - \gamma^\sigma_{\kappa\mu} \Gamma^i_{\sigma\lambda}, \tag{5}$$

the tensor nature of these expressions is derived from eq. (1) and (2) for tensorial parameters.

The first tensor we introduced is $\gamma^i_{\lambda\mu}$. It is identically zero for Weyl's cases. We call the tensor symmetral. The tensor $F^i_{\kappa\lambda\mu}$ is called the curvature of tensorial parameters. If the symmetral is zero, we call tensorial parameters symmetric.

It is easy to check that the curvature of tensorial parameters satisfies:

$$F^i_{\kappa\lambda\mu} = -F^i_{\kappa\mu\lambda}. \tag{6}$$

With the composition of curvature with fundamental tensor g_{ik} and identity tensor δ^k_i we will have following tensors and scalars:

$$\begin{gathered} F_{iklm} = g_{\sigma i} F^{\sigma}_{klm}, \\ F_{ik} = F^{\sigma}_{i\sigma k} = F^{\sigma}_{isk}\delta^s_{\sigma} = g^{\alpha\beta} F_{\alpha i\beta k}, \\ F = g^{ik} F_{ik} = g^{ik} g^{\alpha\beta} F_{\alpha i\beta k}, \end{gathered} \tag{7}$$

we call the first – Riemann tensor, the second – contracted Riemann tensor, and the third – scalar curvature.

When $\Gamma^i_{\lambda\mu} = \left\{ \begin{array}{c} \lambda\mu \\ i \end{array} \right\}$, the tensors and scalars, defined by (5) and (7), become usual Riemann symbols or tensors, derived from them by contraction.

3. Parallel transport of a covariant vector ξ^i is defined by Weyl as follows:

Let us define the curve K via $x_s = x_s(t)$ and the covariant vector ${}^0\xi^i$ at some point ($t = t_0$). We say that this vector is under parallel transport along the curve K, if at any point of the curve it is defined as a solution of the system of linear differential equations:

$$\frac{d\xi^i}{dt} = -\Gamma^i_{rs}\xi^r \frac{dx_s}{dt}, \tag{8}$$

under the following *boundary* conditions:

$$t = t_0,\ \xi^i = {}^0\xi^i,$$

with Γ^i_{rs} being tensorial parameters.

It is easy to see that for $\bar{\mathscr{M}}_n$ the equations (8) read:

$$\frac{d\bar{\xi}^i}{dt} = -\bar{\Gamma}^i_{rs}\bar{\xi}^r \frac{d\bar{x}_s}{dt},$$

i.e. they conserve their form.

Parallel transport of a covariant vector along a closed curve (for arbitrary tensorial parameters Γ^i_{rs}) does not bring the vector to its initial position. In other words, parallel transport of a vector depends on the path, along which the transport is performed.

Weyl has proven a theorem that a necessary and sufficient condition of independence of the parallel transport of a vector on the path is the zero value of the tensorial parameters $\Gamma^i_{\lambda\mu}$.

For symmetric tensorial parameters the condition just mentioned allows us to bring all the tensorial parameters to zero with a coordinate change for all points of $\mathscr{M}_n$. For non-symmetric parameters we cannot state that.

For contravariant vectors we define parallel transport along K as follows. Let at some point ($t = t_0$) of the curve be defined the contravariant vector ${}^0\eta_i$. Then we say that this vector is under parallel transport along K, if at any point of the curve it is defined as a solution of the system of differential equations:

$$\frac{d\eta_i}{dt} = G^r_{is}\eta_r \frac{dx_s}{dt}, \tag{9}$$

under the following boundary conditions:

$$t = t_0,\ \eta_i = {}^0\eta_i$$

where $G^i_{\lambda\mu}$ are tensorial parameters. We will call $\Gamma^i_{\lambda\mu}$ covariant, and $G^i_{\lambda\mu}$ – contravariant tensorial parameters. In Weyl's geometries these two parameters coincide. It is easy to see that the expression $\alpha = \xi^i\eta_i$ does not change as long as ξ^i and η_i are under the parallel transport along the same curve. Indeed a simple calculation shows:

$$\frac{d\alpha}{dt} = \xi^i\eta_r \frac{dx_s}{dt}(G^r_{is} - \Gamma^r_{is}),$$

and our conclusion follows directly.

We do not assume that the two parameters coincide however. Below we find the connection between them using geometrical considerations that are much more general than the ones of Weyl.

One can make the same conclusion about the parallel transport of contravariant vectors as about covariant ones. A necessary and sufficient condition for the independence of parallel transport of a contravariant vector on the path is the zero value of contravariant tensorial parameters.

4. It is absolutely clear that while considering a space from the point of co- and contravariant vectors constituting the space, attaching the notion of parallel transport and describing the space with some manifold $\mathscr{M}_n$ we define its vector properties with co- and contravariant vectorial parameters. This general vector space allows for a simple classification, both of co- and contravariant vectors. We limit ourselves to covariant vectors. The general spaces separate into two classes: non-symmetric, when the symmetral is non-zero and symmetric, when the symmetral is zero. Symmetric spaces in turn fall into two classes: 1. general symmetric, when $\Gamma^i_{\lambda\mu} = \left\{ \begin{matrix} \lambda\mu \\ i \end{matrix} \right\} + A^i_{\lambda\mu}$, where $A^i_{\lambda\mu}$ is a non-zero tensor symmetric in the lower indices; 2. Riemann, when $\Gamma^i_{\lambda\mu} = \left\{ \begin{matrix} \lambda\mu \\ i \end{matrix} \right\}$. From general symmetric spaces we separate a class of spaces, special cases of which are the ones got by Weyl – we call this class Weyl spaces. Among Riemann spaces we separate a special class of spaces, where $\Gamma^i_{\lambda\mu}$ is zero – we call such spaces Euclidean, taking into account the fact that by choosing an appropriate point transformation one can always find such a manifold $\mathscr{M}_n$ defining the space that all g_{ik} will be constant. In general these spaces are pseudo-Euclidean, not Euclidean, because of the inertia law of quadratic forms.

§2.

1. Let us now turn to the definition of a straight line, a plane, and to the comprehension of some basic properties of such geometric objects.

The direction of co- or contravariant vectors is defined by $n-1$ ratios of its components.

Two vectors sharing the same direction obviously have their components different only by a factor, the same for all the components. So if a^i and b^i are two vectors sharing the same direction, then $a^i = \chi b^i$ for all i from 1 to n.

We say that the direction of the vector ξ^i (or η_i) is under parallel transport along the curve K, if one can find a function on the curve λ so that the vector $\lambda\xi^i$ (or $\lambda\eta_i$) is under parallel transport along K.

It is easy to show that a necessary and sufficient condition for the direction of a covariant vector to be under parallel transport along the curve K is the independence on i of the condition:

$$\frac{\frac{d\xi^i}{dt} + \Gamma^i_{rs}\xi^r\frac{dx_s}{dt}}{\xi^i} = \lambda.$$

The same necessary and sufficient condition for the direction of contravariant vectors to be under parallel transport along the curve K is the independence on i of the following condition:

$$\frac{\frac{d\eta_i}{dt} - G^r_{is}\eta_r\frac{dx_s}{dt}}{\xi^i} = \mu.$$

2. We call the direction, tangent to the curve, the direction of the covariant vector $\frac{dx_i}{dt}$.

It is easy to see that the direction of the tangent is independent of the parameter the curve is described with.

We call straight the line, to which the tangent is under parallel transport along the curve.

This means that the equation for a straight line boils down to the following system of $n-1$ differential equations of second order:

$$\frac{\frac{d^2x_i}{dt^2} + \Gamma^i_{rs}\frac{dx_r}{dt}\frac{dx_s}{dt}}{\frac{dx_i}{dt}} = \lambda,$$

where λ is an undefined function of t.

It is obvious that the parameter t can always be chosen so that the equation of the straight line is written in the form:

$$\frac{d^2x_i}{dt^2} + \Gamma^i_{rs}\frac{dx_r}{dt}\frac{dx_s}{dt} = 0. \tag{10}$$

This choice of the parameter t we call principal, because for the straight line definition it has the same meaning as the arc length for the equation of a geodesic curve.

Upon a closer look on eq. (10), one can see that there is a curve through the point with definite tangent at the point. If our space is not Riemann, then the straight is not geodesic and only for Riemann spaces the geodesic and straight lines are equivalent.

3. The straight line definition was made by looking at covariant vectors and their parallel transport; a definition of a plane hypersurface may be given by utilizing the parallel transport of a contravariant vector.

We define the contravariant vector normal to the hypersurface S: $f(x_1, x_2, \dots, x_n) = 0$ at a given point, the vector:

$$f_i = \frac{\partial f}{\partial x_i}.$$

For any curve of S we have:

$$f_i \frac{dx_i}{dt} = 0,$$

and conversely, if the equation

$$a_i = \frac{dx_i}{dt} = 0$$

is true for any curve on S, then the direction of the vector a_i and the direction of the contravariant normal to S coincides at the given point.

A plane in the usual Euclidean space has the property that the normal to it is under parallel transport along any curve on the plane.

Searching for hypersurfaces in a general space such that the direction of their contravariant normal is under parallel transport over any curve on the hypersurface, we come to quite restrictive constraints for the curvature of covariant tensorial parameters, making the space quite similar to Euclidean, where the curvature is identically zero. We do not go into details here.

A plane in the usual Euclidean space has also a more convenient property for generalization.

If one considers a pencil of lines through a given point of the Euclidean space that lie on a plane, then the normal to the plane is under parallel transport along each of the curves.

If one considers a plane and a normal for any given point then for the usual space the plane, perpendicular to the normal in any of its points is the same plane, while for the generalization of the definition of the plane we consider this is not true.

<u>Let us define a straight hypersurface, perpendicular to a contravariant vector ${}^0 f_i$ at a given point $\mathscr{P}$, as the hypersurface formed by the pencil of straight lines through the point and having at $\mathscr{P}$ a definite normal, coinciding with the direction of ${}^0 f_i$.</u>

It is easy to see that there exists a single straight hyperspace through the given point $\mathscr{P}$ and perpendicular to the vector ${}^0 f_i$.

<u>We define a hypersurface through a point $\mathscr{P}$ that is perpendicular to the vector ${}^0 f_i$ to be a plane hypersurface, corresponding to the point $\mathscr{P}$ and normal to the vector ${}^0 f_i$, if a contravariant normal to the hypersurface is under parallel transport along any straight line going through $\mathscr{P}$.</u>

It is clear that a plane hypersurface, corresponding to the point $\mathscr{P}$, is not the plane hypersurface corresponding to any other point on it; the opposite is valid only in a limited number of cases.

A space, where any hypersurface is a plane we call a <u>vectorially perfect space</u>.

3. Let us obtain the restrictions on co- and contravariant tensorial parameters for the space to be vectorially perfect.

Theorem. Necessary and sufficient conditions for the tensorial space to be perfect are: 1. The contravariant tensorial parameters are symmetric. 2. The following is satisfied:

$$\begin{gathered} \gamma^i_{rs} + \gamma^i_{sr} = 0, \quad (i \neq r,\, i \neq s), \\ \gamma^i_{ri} + \gamma^i_{ir} = \omega_r, \quad (\text{independent of } i,\, i \neq r), \\ \gamma^i_{ii} = \omega_i, \quad (\text{no summing over } i) \end{gathered} \tag{11}$$

where $\gamma^i_{\lambda\mu} = G^i_{\lambda\mu} - \Gamma^i_{\lambda\mu}$.

It is obvious that $\gamma^i_{\lambda\mu}$ is a mixed tensor of third rank and ω_i is a covariant vector. Equations (11) may be rewritten as follows:

$$\gamma^i_{rs} + \gamma^i_{sr} = \delta^i_r \omega_s + \delta^i_s \omega_r. \tag{12}$$

Before proving the theorem let us prove two lemmas.

Lemma 1. If a_{rs} does not depend on ξ^i, and if furthermore $a_{rs}\xi^r\xi^s = 0$ for all ξ^i such that $f_i\xi^i = 0$, then:

$$a_{rs} + a_{sr} = f_s \frac{a_{rr}}{f_r} + f_r \frac{a_{ss}}{f_s}. \tag{a}$$

The conditions of the lemma 1 result in the fact that $a_{rs}\xi^r\xi^s$ is dividable by a linear form $f_i\xi^i$, i.e.:

$$a_{rs}\xi^r\xi^s = (f_i\xi^i)(\lambda_j\xi^j),$$

then making the coefficients before $\xi^r\xi^s$ on both sides of the equality equal, we prove the lemma.

Lemma 2. If a^i_{rs} does not depend on f_i and ξ^i, and if $a^i_{rs} f_i \xi^r\xi^s = 0$ for all f_i and ξ^i satisfying $f_i\xi^i = 0$, then a^i_{rs} satisfy:

$$a^i_{rs} + a^i_{sr} = \delta^i_r \omega_s + \delta^i_s \omega_r, \tag{b}$$

where $\omega_r = a^r_{rr}$.

Using formula (a) from the lemma 1 we have for all f_i:

$$a^i_{rs} f_i + a^i_{sr} f_i = f_s \frac{a^i_{rr} f_i}{f_r} + f_r \frac{a^i_{ss} f_i}{f_s},$$

then eliminating the denominator we obtain:

$$(a^i_{rs} + a^i_{sr}) f_i f_s f_r = f_s^2 f_i a^i_{rr} + f_r^2 f_i a^i_{ss},$$

where the summation goes only over i. Then, making the coefficient before different products $f_i f_s f_r$ the same on both sides, we prove the lemma.

Now let us come to the proof of the necessity conditions of our theorem. Let $f(x_1, x_2, \dots, x_n) = 0$ be a plane hypersurface for point $\mathscr{P}$, let $f_i = \frac{\partial f}{\partial x_i}$ and $\xi^s = \frac{dx_s}{dt}$, where $x_s = x_s(t)$ is the equation of straight lines going through $\mathscr{P}$ and lying in our plane hypersurface. For each of the straight lines we will have:

$$f_i\xi^i = 0,$$

taking t as the principal parameter of some straight line, differentiating the equality by t and taking into account that for a straight line:

$$\frac{d\xi^i}{dt} = -\Gamma^i_{rs}\xi^r\xi^s,$$

we get the following equation:

$$\xi^r\left(\frac{df_r}{dt} - \Gamma^i_{rs}f_i\xi^s\right) = 0. \tag{c}$$

At this point let us remember that according to the property of the plane hypersurface, the normal is under parallel transport over a straight plane of the generatrix. Then we get the following equation:

$$\frac{df_r}{dt} - G^i_{rs}f_i\xi^s = \omega' f_r, \tag{d}$$

Multiplying the equation by ξ^r and summing over r from 1 to n, and substituting the result from (c), we get:

$$\gamma^i_{rs}f_i\xi^r\xi^s = 0,$$

which holds for a vectorial perfect space for all f_i, ξ^i satisfying $f_i\xi^i = 0$. Using the lemma 2 we arrive at the second equation of the theorem.

To get the first equation let us notice that $\frac{df_r}{dt} = \frac{\partial f}{\partial x_s}\xi^s$, then straightforwardly from (d) we get:

$$\left(\frac{\partial f_r}{\partial x_s} - G^i_{rs}f_i\right)\xi^s = \omega' f_r,$$

multiplying this equation by ξ^r and summing over r from 1 to n and replacing the expression $\frac{df_r}{dx_s} - G^i_{rs}f_i$ by a_{rs} we find:

$$a_{rs}\xi^r\xi^s = 0,$$

for $f_i\xi^i = 0$. Using the lemma 1 we get:

$$a_{rs} + a_{sr} = f_s\frac{a_{rr}}{f_r} + f_r\frac{a_{ss}}{f_s},$$

then we multiply it by ξ^s and sum over s from 1 to n to get:

$$a_{rs}\xi^s + a_{sr}\xi^s = f_r\sum_{s=1}^{n}\frac{a_{ss}}{f_s}\xi^s = \omega'' f_r,$$

where ω'' does not depend on r. Then obtaining a_{sr} we get:

$$a_{sr} = a_{rs} + \left(G^i_{rs} - G^i_{sr}\right)f_i,$$

then the previous equation is written as:

$$2a_{rs}\xi^s + \left(G^i_{rs} - G^i_{sr}\right)f_i\xi^s = \omega'' f_r,$$

but $a_{rs}\xi^s = \omega' f_r$, therefore:

$$(G^i_{rs} - G^i_{sr})\xi^s f_i = \omega f_r,\ \omega = \omega'' - 2\omega',$$

from which we get:

$$\frac{(G^i_{rs} - G^i_{sr})\xi^s f_i}{f_r} = \frac{(G^i_{\alpha s} - G^i_{s\alpha})\xi^s f_i}{f_\alpha}.$$

We eliminate the denominator and equalize the coefficients before some of the products $f_i f_\beta$ and get:

$$(G^i_{rs} - G^i_{sr})\xi^s = 0,\ i \neq r.$$

As this equation is valid for all ξ^i such that $f_i\xi^i = 0$ and as $G^i_{rs} - G^i_{sr}$ does not depend on f_i:

$$G^i_{rs} - G^i_{sr} = 0,\ i \neq r,$$

then taking $i = s$ we get:

$$G^s_{rs} - G^s_{sr} = 0,$$

which proves the symmetry of the contravariant tensorial parameters.

Let us now prove the sufficiency condition of the theorem.

From the condition $f_i\xi^i = 0$ for $\xi^i = \frac{dx_i}{dt}$, where t is the principal parameter of the straight line lying in the plane hypersurface $f = 0$ and going through point $\mathscr{P}$, we have:

$$\frac{d f_i \xi^i}{dt} = 0,$$

or

$$\xi^r\left(\frac{df_r}{dt} - \Gamma^i_{rs} f_i \xi^s\right) = 0,$$

remembering that $\Gamma^i_{rs} = G^i_{rs} - \gamma^i_{rs}$, we find:

$$\xi^r\left(\frac{df_r}{dt} - G^i_{rs} f_i \xi^s\right) + \gamma^i_{rs} f_i \xi^r \xi^s = 0,$$

but owing to the conditions of the theorem we find:

$$\gamma^i_{rs} f_i \xi^r \xi^s = \frac{1}{2}(\gamma^i_{rs} + \gamma^i_{sr}) f_i \xi^r \xi^s = \frac{1}{2}(\delta^i_r \omega_s + \delta^i_s \omega_r) f_i \xi^r \xi^s,$$

because $f_r\xi^r = 0$; then:

$$\xi^r\left(\frac{df_r}{dt} - G^i_{rs} f_i \xi^s\right) - a_{rs}\xi^r\xi^s = 0, \tag{e}$$

with $a_{rs} = \frac{\partial f_r}{\partial x_s} - G^i_{rs} f_i$ being symmetrical in the lower indices due to the conditions of the theorem:

$$a_{rs} = a_{sr}.$$

Then let us remember that (e) holds for all ξ^i satisfying $f_i\xi^i = 0$. Then using lemma 1 we find:

$$a_{rs} + a_{sr} = 2a_{rs} = f_s \frac{a_{rr}}{f_r} + f_r \frac{a_{ss}}{f_s}.$$

We multiply the above equation by ξ^s and sum over s from 1 to n to get:

$$a_{rs}\xi^s = f_r \frac{1}{2}\sum_{s=1}^{n}\frac{a_{ss}}{f_s}\xi^s,$$

or

$$a_{rs}\xi^s = \frac{df_r}{dt} = G^i_{rs} f_i \xi^s = \mu f_r,$$

i.e. the contravariant normal to our hypersurface does experience parallel transport along the straight lines going through the point $\mathscr{P}$ and lying on the hypersurface. Consequently the hypersurface in question is a plane and the sufficiency conditions of the theorem are proven.

From equation (12) and the symmetry of the contravariant parameters we get the following expression of them through covariant ones:

$$G^i_{\lambda\mu} = \frac{\Gamma\lambda\mu^i + \Gamma^i_{\mu\lambda}}{2} + \frac{1}{2}\delta^i_\lambda \omega_\mu + \frac{1}{2}\delta^i_\mu \omega_\lambda, \tag{13}$$

where ω_λ is any contravariant vector.

In the case of the symmetry of covariant tensorial parameters the equation (13) gives the following:

$$G^i_{\lambda\mu} = \Gamma^i_{\lambda\mu} + \frac{1}{2}\delta^i_\lambda \omega_\mu + \frac{1}{2}\delta^i_\mu \omega_\lambda. \tag{14}$$

The spaces considered by Weyl ($G^i_{\lambda\mu} = \Gamma^i_{\lambda\mu}$) are obtained for the case when $\omega_\lambda = 0$. These kind of spaces where $\omega_\lambda = 0$ and $\Gamma^i_{\lambda\mu}$ are symmetric we call <u>principal vectorially perfect spaces</u>. The vector ω_i we call <u>characteristic vector</u>.

The considerations about the plane hypersurfaces brings us to a natural definition of a contravariant vector ω_i different from the scale vector of Weyl. So we get that the space is characterized not only by the metric tensor g_{ik} and scale vector ϕ_i, but also by ω_i. If the properties of the space are in the equations of *gravity* and electrodynamics, then this new vector is possibly the one entering the electrodynamic equations as well.

§3.

1. To obtain the metric properties of the space we introduce the following definitions:

<u>The norm of a covariant vector ξ^i</u> we call, in agreement with Weyl, the following scalar:

$$l = l(\xi^i) = g_{ik}\xi^i\xi^k, \tag{15}$$

where g_{ik} is a fundamental tensor.

<u>The norm of a contravariant vector η_i</u> we call the scalar obtained in a similar fashion:

$$L = L(\eta_i) = g^{ik}\eta_i\eta_k, \tag{16}$$

where g^{ik} is a conjugate tensor to the fundamental one g_{ik}.

We will now study the change of the norm of the co- and contravariant vectors when they are under parallel transport along some curve $\mathscr{K}$.

As we noticed before, Weyl considers only such types of the metrical spaces, where the norm of the covariant vector under parallel transport of it along the curve $x_i = x_i(t)$ changes according to:

$$\frac{dl}{dt} = -\varphi_s l \frac{dx_s}{dt},$$

where φ_s is a definite contravariant vector. As we show now the above equation has a close connection to the law of the angle change between two vectors under parallel transport.

The angle ω between two covariant vectors ξ^i and η^i is defined according to:

$$\cos\omega = \frac{\Delta(\xi^i,\eta^i)}{\sqrt{l(\xi^i)}\sqrt{l(\eta^i)}}, \tag{17}$$

where

$$\Delta(\xi^i,\eta^i) = g_{ik}\xi^i\eta^k. \tag{18}$$

The angle Ω between two contravariant vectors ξ_i and η_i is defined according to:

$$\cos\Omega = \frac{\nabla(\xi_i,\eta_i)}{\sqrt{L(\xi_i)}\sqrt{L(\eta_i)}}, \tag{19}$$

where

$$\nabla(\xi_i,\eta_i) = g^{ik}\xi_i\eta_k. \tag{20}$$

We will not go into details to make the angles obtained from (17) and (19) explicit.

We will also not discuss the internal connection between (15) and (17), discussed in [3].

2. Now let us study the change of the vector norms and angles between them under parallel transport along a curve. After a straightforward calculation we come to:

$$\begin{aligned}
\frac{d\Delta}{dt} &= A_{iks}\frac{dx_s}{dt}\xi^i\eta^k,\\
\frac{dl}{dt} &= A_{iks}\frac{dx_s}{dt}\xi^i\xi^k,\\
\frac{d\nabla}{dt} &= B^{ik}_s\frac{dx_s}{dt}\xi_i\eta_k,\\
\frac{dL}{dt} &= B^{ik}_s\frac{dx_s}{dt}\xi_i\xi_k,
\end{aligned} \tag{21}$$

where the tensors A_{iks} and B^{ik}_s are defined according to:

$$A_{iks} = \frac{\partial g_{ik}}{\partial x_s} - g_{\alpha k}\Gamma^{\alpha}_{is} - g_{i\alpha}\Gamma^{\alpha}_{ks}, \tag{22}$$

$$B^{ik}_s = \frac{\partial g^{ik}}{\partial x_s} + g^{\alpha k}G^{i}_{\alpha s} + g^{i\alpha}G^{k}_{\alpha s}, \tag{23}$$

in other words A_{iks} is the tensorial derivative of g_{ik} obtained with the tensorial parameters $\Gamma^i_{\lambda\mu}$, while B^{ik}_s is the tensorial derivative of g^{ik} obtained with the tensorial parameters $G^i_{\lambda\mu}$.

Using the previous equation we get:

$$\begin{aligned}\frac{d\cos\omega}{dt} &= \frac{1}{\sqrt{l_1}\sqrt{l_2}}\left(A_{iks}\xi^i\eta^k - \frac{1}{2}\frac{\Delta}{l_1}A_{pqs}\xi^d\xi^q - \frac{1}{2}\frac{\Delta}{l_2}A_{pqs}\eta^p\eta^q\right)\frac{dx_s}{dt},\\ \frac{d\cos\Omega}{dt} &= \frac{1}{\sqrt{L_1}\sqrt{L_2}}\left(B_s^{ik}\xi_i\eta_k - \frac{1}{2}\frac{\nabla}{L_1}B_s^{pq}\xi_p\xi_q - \frac{1}{2}\frac{\nabla}{L_2}B_s^{pq}\eta_p\eta_q\right)\frac{dx_s}{dt},\end{aligned} \tag{24}$$

where for brevity we used:

$$\begin{aligned}&l_1 = l(\xi^i),\ l_2 = l(\eta^i),\ L_1 = L(\xi_i),\ L_2 = L(\eta_i),\\ &\Delta = \Delta(\xi^i,\eta^i),\ \nabla = \nabla(\xi_i,\eta_i).\end{aligned}$$

Let us also note that ξ^i,η^i and ξ_i,η_i are not connected.

3. The space, where the angles of both co- and contravariant vectors stay unchanged under the parallel transport we call Weyl's space. It is possible to study spaces, having this property only for co- or contravariant vectors, but we do not go into detail there.

The main property of Weyl's space is:

Theorem. For a space to be Weyl's it is necessary and sufficient for the ratios $\frac{A_{iks}}{g_{ik}}$ and $\frac{b_s^{ik}}{g^{ik}}$ to be independent of the indices i and k.

To prove the necessity condition let us consider the parallel transport of orthogonal covariant vectors, i.e. such vectors that $\cos\omega = 0$ or $\Delta(\xi^i,\eta^i) = 0$. For Weyl's space we have $\frac{d\Delta}{dt} = 0$, or, using (24) and $\Delta = 0$ we get:

$$A_{iks}\frac{dx_s}{dt}\xi^i\eta^k = 0,$$

for ξ^i, η^i satisfying:

$$g_{ik}\xi^i\eta^k = 0.$$

From these equations follows:

$$\frac{A_{iks}}{g_{ik}}\frac{dx_s}{dt} = \frac{A_{lms}}{g_{lm}}\frac{dx_s}{dt},$$

which being satisfied for any $\frac{dx_s}{dt}$ proves the necessity condition of the theorem for covariant vectors.

The proof for contravariant vectors goes in exactly the same way.

The sufficiency of the conditions of the theorem is checked straightforwardly using (24).

We call $\frac{A_{iks}}{g_{ik}}$ and $\frac{B_s^{ik}}{g^{ik}}$ respectively $-\varphi_s$ and $-f_s$. Then:

$$A_{iks} = -\varphi_s g_{ik},\ \ B_s^{ik} = -f_s g^{ik}, \tag{25}$$

with φ_s and f_s being contravariant vectors.

Using formulas (21) and (25) we get:

$$\frac{dl}{dt} = -\varphi_s\frac{dx_s}{dt}l,\ \ \frac{dL}{dt} = -f_x\frac{dx_s}{dt}L, \tag{26}$$

so that φ_s and f_s are the contravariant vectors characterizing the change of the norm of vectors as long as they are under parallel transport along some curve. We call them first and second scale vectors. There is a connection between them for vectorially perfect spaces, which we discuss below.

4. The tensors φ_{ik} and f_{ik} are defined according to:

$$
\begin{aligned}
\varphi_{ik} &= \frac{\partial \varphi_i}{\partial x_k} - \frac{\partial \varphi_k}{\partial x_i}, \\
f_{ik} &= \frac{\partial f_i}{\partial x_k} - \frac{\partial f_k}{\partial x_i},
\end{aligned} \tag{27}
$$

we call them the first and second metric curvature of a space. It is easy to see that a necessary and sufficient condition for the independence of the change of the norm of a co- or contravariant vector of the path, along which it is under parallel transport, is the zero value of respectively the first or second metric curvature of the space.

We say that we change scale, if all g_{ik} are multiplied by $\lambda = \lambda(x_1, x_2, \ldots, x_n)$. In this case the norm of a covariant vector is multiplied by λ and the norm of a contravariant vector is multiplied by $\frac{1}{\lambda}$, as all g^{ik} will also be multiplied by $\frac{1}{\lambda}$. Then the first scale vector φ_s becomes $\varphi_s - \frac{\partial \ln \lambda}{\partial x_s}$, and the second scale vector f_s becomes $f_s + \frac{\partial \ln \lambda}{\partial x_s}$. As to the first and second metric curvature, they do not change.

We denote the expression, to which a value $\mathscr{A}$ changes under the scale change with $\tilde{\mathscr{A}}$. The values that do not change upon the scale change we call (in agreement with Weyl) scale invariants, keeping for the usual invariant the name coordinate invariant.

Since in what follows we will have to construct scale invariants, we have to find out how different expressions change upon changing the scale. It is easy to get the following formulas:

$$
\begin{aligned}
&\tilde{g}_{ik} = \lambda g_{ik},\ \tilde{g}^{ik} = \frac{1}{\lambda} g^{ik},\ \tilde{g} = \lambda^n g, \\
&\tilde{R} = \frac{1}{\lambda} R + \frac{n-1}{\lambda} \mathscr{D}\psi + \frac{(n-1)(n-2)}{4\lambda} \Delta\psi,
\end{aligned} \tag{28}
$$

where $\mathscr{D}\psi$ and $\Delta\psi$ are defined according to:

$$
\begin{aligned}
&\mathscr{D}\psi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_\beta} \left(\sqrt{g} g^{\alpha\beta} \frac{\partial \psi}{\partial x_\alpha} \right), \\
&\Delta\psi = g^{\alpha\beta} \frac{\partial \psi}{\partial x_\alpha} \frac{\partial \psi}{\partial x_\beta}, \\
&\psi = \ln \lambda,
\end{aligned} \tag{29}
$$

and R is the scalar curvature, obtained for the tensorial parameters $\Gamma^i_{\lambda\mu} = \left\{ \begin{array}{c} \lambda\mu \\ i \end{array} \right\}$: $R = g^{ik} g^{\alpha\beta} (i\alpha, \beta k)$. It is also easy to see that the following holds:

$$
\begin{aligned}
&\tilde{\varphi}_i = \varphi_i - \frac{d\psi}{dx_i},\ \tilde{f}_i = f_i + \frac{d\psi}{dx_i}, \\
&\tilde{\varphi}_{ik} = \tilde{\varphi}_{ik},\ \tilde{f}_{ik} = f_{ik}.
\end{aligned} \tag{30}
$$

§4.

1. Now let us define the tensorial parameters for Weyl's spaces. We start from the covariant tensorial parameters $\Gamma^i_{\lambda\mu}$; they must be defined from eq. (25), i.e. from the equation:

$$\frac{\partial g_{ik}}{\partial x_s} + \varphi_s g_{ik} = g_{\alpha k}\Gamma^\alpha_{is} + g_{\alpha i}\Gamma^\alpha_{ks}. \tag{31}$$

If $\Gamma^i_{\lambda\mu}$ are symmetric parameters, then from eq. (31) Weyl finds the following expressions for $\Gamma^i_{\lambda\mu}$:

$$\Gamma^i_{\lambda\mu} = \mathscr{L}^i_{\lambda\mu} = \left\{ \begin{array}{c} \lambda\mu \\ i \end{array} \right\} + \frac{1}{2}\varphi_\lambda \delta^i_\mu + \frac{1}{2}\varphi_\mu \delta^i_\lambda - \frac{1}{2} g_{\lambda\mu}\varphi^i, \tag{32}$$

where φ^i is defined according to:

$$\varphi^i = g^{\sigma i}\varphi_\sigma. \tag{33}$$

Let us show that the general solution of the equation (31) is defined according to:

$$\Gamma^i_{\lambda\mu} = \mathscr{L}^i_{\lambda\mu} + a^i_{\lambda\mu} - g_{\alpha\lambda}g^{\sigma i}a^\alpha_{\sigma\mu} - g_{\alpha\mu}g^{\sigma i}a^\alpha_{\sigma\lambda}, \tag{34}$$

where $a^i_{\lambda\mu}$ is an arbitrary tensor skew-symmetric in the lower indices. From eq. (34) it follows that if the tensorial parameters are symmetric, $a^i_{\lambda\mu} = a^i_{\mu\lambda}$. Consequently it is identically zero: $a^i_{\lambda\mu} = 0$.

It is straightforward to check that $\Gamma^i_{\lambda\mu}$ defined by eq. (34) satisfies (31), if one takes into account $a^i_{\lambda\mu} = -a^i_{\mu\lambda}$. Now let us prove that any solution of (31) may be represented in the form of (34). From (31) we find:

$$\left[\begin{array}{c} \lambda\mu \\ i \end{array} \right] + \frac{1}{2}\varphi_\mu g_{\lambda i} + \frac{1}{2}\varphi_\lambda g_{\mu i} - \frac{1}{2}\varphi_i g_{\lambda\mu} = g_{\alpha i}\Gamma^\alpha_{\lambda\mu} - \frac{1}{2}g_{\alpha i}\gamma^\alpha_{\lambda\mu} - \frac{1}{2}g_{\alpha\lambda}\gamma^\alpha_{\mu i} - \frac{1}{2}g_{\alpha\mu}\gamma^\alpha_{\lambda i},$$

where $\gamma^i_{\lambda\mu} = \Gamma^i_{\lambda\mu} - \Gamma^i_{\mu\lambda}$. Replacing in the equation i by σ, multiplying both sides with $g^{i\sigma}$, and summing over σ from 1 to n, we get:

$$\Gamma^i_{\lambda\mu} = \mathscr{L}^i_{\lambda\mu} + \frac{1}{2}\gamma^i_{\lambda\mu} - \frac{1}{2}g_{\alpha\lambda}g^{\sigma i}\gamma^\alpha_{\sigma\lambda} - \frac{1}{2}g_{\alpha\mu}g^{\sigma i}\gamma^\alpha_{\sigma\lambda},$$

in other words the equation (34), where $a^i_{\lambda\mu}$ is replaced by the tensor $\frac{1}{2}\gamma^i_{\lambda\mu}$, is skew-symmetric by definition.

From eq. (34) it is easy to conclude that the covariant tensorial parameters of Weyl's space are determined by the metric tensor g_{ik}, the scale vector φ_i and the lower-indices skew-symmetric tensor $a^i_{\lambda\mu}$, i.e. it requires $\frac{n(n+1)}{2} + n + \frac{n^2(n-1)}{2}$ coordinate functions. If the tensorial parameters are symmetric, then one needs $\frac{n(n+1)}{2} + n$ functions to determine them.

2. Now let us get the contravariant tensorial parameters for Weyl's space. According to eq. (25) the definition may be obtained from the solution of the equations:

$$\frac{\partial g^{ik}}{\partial x_s} + f_s g^{ik} = -g^{\alpha k}G^i_{\alpha s} - g^{\alpha i}G^k_{\alpha s}. \tag{35}$$

The general solution of equation (35) is given by:

$$G^i_{\lambda\mu} = \Gamma^i_{\lambda\mu} + \delta^i_\lambda \psi_\mu + g^{i\sigma} b_{\sigma\lambda\mu}, \tag{36}$$

where the contravariant vector ψ_μ is given by:

$$\psi_\mu = -\frac{\varphi_\mu + f_\mu}{2},$$

and the tensor $b_{\sigma\lambda\mu}$ is skew-symmetric in the first two indices.

We multiply both sides of (35) by $g_{kl}g_{im}$, sum over i and k from 1 to n and add the result to eq. (31), where i and k are replaced with l and m. Then we get:

$$g_{im}g_{kl}\frac{\partial g^{ik}}{\partial x_s} + \frac{\partial g_{lm}}{\partial x_s} + (\varphi_s + f_s)g_{lm} = -g_{\alpha m}\gamma^\alpha_{ls} - g_{\alpha l}\gamma^\alpha_{ms},$$

where $\gamma^i_{\lambda\mu} = G^i_{\lambda\mu} - \Gamma^i_{\lambda\mu}$. Direct computation gives:

$$g_{im}g_{kl}\frac{\partial g^{ik}}{\partial x_s} + \frac{\partial g_{lm}}{\partial x_s} = 0,$$

which means that the previous equation may be rewritten as:

$$2\psi_s g_{lm} = g_{\alpha m}\gamma^\alpha_{ls} + g_{\alpha l}\gamma^\alpha_{ms},$$

then replacing:

$$\gamma^i_{\lambda\mu} = \delta^i_\lambda \psi_\mu + v^i_{\lambda\mu},$$

we find:

$$g_{\alpha m}v^\alpha_{ls} + g_{\alpha l}v^\alpha_{ms} = 0,$$

then we introduce the tensor:

$$g_{\alpha m}v^\alpha_{ls} = b_{mls}, \quad v^i_{ls} = g^{\sigma i}b_{\sigma ls}.$$

From the previous equation we find that it is skew-symmetric in the first two indices, and as $\gamma^i_{\lambda\mu} = \delta^i_\lambda \psi_\mu + g^{\sigma i}b_{\sigma\lambda\mu}$, we get that any solution of eq. (35) is given by (36).

Direct simple computation shows that $G^i_{\lambda\mu}$ given by (36) is the solution of eq. (35).

For the definition of contravariant tensorial parameters in the general Weyl's space one needs not only the quantities defining the covariant parameters, but also the second scale vector f_i and the first-two-indices skew-symmetric tensor $b_{\sigma\lambda\mu}$, so one needs additionally $n + \frac{n^2(n-1)}{2}$ more coordinate functions.

3. Now let us study the additional constraints placed on the vectors and tensors defining the tensorial parameters in the Weyl's space by the vectorial perfectness of the space.

Let us prove the following theorem:

Theorem. If Weyl's space is a vectorially perfect space, then the half-sum of the first and the second scale vectors is a characteristic vector with opposite sign:

$$\omega + i = \psi_i = -(\varphi_i + f_i)/2.$$

According to the formulas of the previous section we get:

$$\gamma^i_{\lambda\mu} = \delta^i_\lambda \psi_\mu + g^{i\sigma} b_{\sigma\lambda\mu},$$

but due to the vectorial perfectness condition:

$$\gamma^i_{\lambda\mu} + \gamma^i_{\mu\lambda} = \delta^i_\lambda \omega_\mu + \delta^i_\mu \omega_\lambda,$$

from which follows:

$$\begin{aligned} b_{\kappa\lambda\mu} + b_{\kappa\mu\lambda} &= g_{\lambda\kappa}\chi_\mu + g_{\mu\kappa}\chi_\lambda, \\ b_{\lambda\kappa\mu} + b_{\lambda\mu\kappa} &= g_{\kappa\lambda}\chi_\mu + g_{\mu\lambda}\chi_\kappa, \\ b_{\mu\kappa\lambda} + b_{\mu\lambda\kappa} &= g_{\kappa\mu}\chi_\lambda + g_{\lambda\mu}\chi_\kappa, \end{aligned}$$

where $\chi_i = \omega_i - \psi_i$.

Summing the obtained formulas and remembering that $b_{\sigma\lambda\mu}$ is skew-symmetric in the first two indices, we get:

$$g_{\kappa\lambda}\chi_\mu + g_{\mu\kappa}\chi_\lambda + g_{\lambda\mu}\chi_\kappa = 0.$$

We multiply the equation by $g^{\mu s} g^{\lambda\sigma}$ and sum over μ, λ from 1 to n. Then considering s, σ different from κ we get:

$$\chi_\kappa = 0,$$

QED.

Theorem. If a vectorially perfect Weyl's space has symmetric covariant parameters, then it is a principal vectorially perfect space.

In other words, for such a space $\omega_i = \psi_i = 0$, so $\varphi_i = -f_i$. As $\Gamma^i_{\lambda\mu} = \Gamma^i_{\mu\lambda}$ and the space is vectorially perfect, $\gamma^i_{\lambda\mu} = \gamma^i_{\mu\lambda}$. Then:

$$\gamma^i_{\lambda\mu} = \frac{1}{2}\delta^i_\mu \omega_\lambda + \frac{1}{2}\delta^i_\lambda \omega_\mu.$$

From the above equation we obtain $v^i_{\lambda\mu}$ as well as b_{mls} and get the following relation:

$$b_{mls} = \frac{1}{2} g_{sm}\omega_l - \frac{1}{2} g_{lm}\omega_s.$$

Then we use that b_{mls} is skew-symmetric in the first two indices and prove the theorem.

One may think that Weyl's space is a rather particular case once the tensorial parameters of the tensor $b_{\sigma\lambda\mu}$ are non-symmetric. We do not discuss it at this point, but this is not the case.

In other words, if a Weyl's space is vectorially perfect, its tensorial parameters are defined with a fundamental metric tensor, first and second scale vectors, and the tensor $a^i_{\lambda\mu}$, skew-symmetric in the lower indices. So for defining the vectorially perfect Weyl's space one needs $\frac{n(n+1)}{2} + 2n + \frac{n^2(n-1)}{2}$ coordinate functions. Below we consider only such vectorially perfect Weyl's spaces, for which the tensors $a^i_{\lambda\mu}$ can be expressed through scale vectors. In other words, we assume that a space is defined by a metric tensor and two scale vectors.

§5.

1. From a metric tensor and both scale vectors one can construct several coordinate invariants. From the components of the metric tensor, with its first and second derivatives ,it is easy to obtain the scalar curvature in the Riemann sense $\mathscr{R}$. From the scale vectors one can get the following invariants:

$$\varphi = g^{ik}\varphi_i\varphi_k,\ f = g^{ik}f_i f_k,\ \mu = g^{ik}\varphi_i f_k. \tag{37}$$

Using the Christoffel brackets one gets the tensorial derivatives of the vectors φ_i and f_i. We denote them φ'_{ik} and f'_{ik} and find the equations:

$$\varphi'_{ik} = \frac{\partial \varphi_i}{\partial x_k} - \varphi_\sigma \left\{ \begin{array}{c} ik \\ \sigma \end{array} \right\},\ f'_{ik} = \frac{\partial f_i}{\partial x_k} - f_\sigma \left\{ \begin{array}{c} ik \\ \sigma \end{array} \right\},$$

then using these equations as well as the metric curvatures φ_{ik} and f_{ik} we can get the following invariants, involving the scale vectors, their first derivatives, as well as the metric tensor and its first derivatives:

$$\begin{aligned} \Phi &= g^{ik}\varphi'_{ik} = \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}g^{\sigma k}\varphi_\sigma)}{\partial x_k}, \\ \mathscr{F} &= g^{ik}f'_{ik} = \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}g^{\sigma k}f_\sigma)}{\partial x_k}, \end{aligned} \tag{38}$$

$$\begin{aligned} S_1 &= \varphi_{ik}\varphi^{ik} = g^{i\alpha}g^{k\beta}\varphi_{\alpha\beta}\varphi_{ik}, \\ S_2 &= f_{ik}f^{ik} = g^{i\alpha}g^{x\beta}f_{\alpha\beta}f_{ik}, \\ S_3 &= \varphi_{ik}f^{ik} = f_{ik}\varphi^{ik} = g^{i\alpha}g^{k\beta}\varphi_{\alpha\beta}f_{ik}. \end{aligned} \tag{39}$$

We get therefore nine coordinate invariants. They may give several scale invariants. We discuss them below.

2. If the multiplication of a metric tensor by an arbitrary coordinate function λ results in a coordinate invariant multiplied by λ^c, i.e. if we have the condition:

$$\tilde{\mathscr{A}} = \lambda^l \mathscr{A},$$

then we call the coordinate invariant $\mathscr{A}$ <u>relative scale invariant of the order l</u>.

It is straightforward to see that the coordinate invariants S_i, defined by (39) are scale invariants of the order -2.

From the coordinate invariants (37), (38) and $\mathscr{R}$ it is easy to get linear combinations, which are scale invariants. After simple calculations we arrive at:

$$
\begin{aligned}
\tilde{\varphi} &= \frac{1}{\lambda}(\varphi + \Delta\psi - 2Z_1\psi), \\
\tilde{\Phi} &= \frac{1}{\lambda}\left(\Phi - \mathscr{D}\psi - \frac{n-2}{2}\Delta\psi + \frac{n-2}{2}Z_1\psi\right), \\
\tilde{f} &= \frac{1}{\lambda}(f + \delta\psi + 2Z_2\psi), \\
\tilde{\mathscr{F}} &= \frac{1}{\lambda}\left(\mathscr{F} + \mathscr{D}\psi - \frac{n-2}{2}\Delta\psi - \frac{n-2}{2}Z_2\psi\right), \\
\tilde{\mu} &= \frac{1}{\lambda}(\mu - \Delta\psi - Z_2\psi + Z_1\psi),
\end{aligned}
\tag{40}
$$

where the operations Z_1 and Z_2 are defined according to:

$$
\begin{aligned}
Z_1\psi &= \varphi^{\alpha}\frac{\partial\psi}{\partial x_{\alpha}} = g^{\sigma\alpha}\varphi_{\sigma}\frac{\partial\psi}{\partial x_{\alpha}}, \\
Z_2\psi &= f^{\alpha}\frac{\partial\psi}{\partial x_{\alpha}} = g^{\sigma\alpha}f_{\sigma}\frac{\partial\psi}{\partial x_{\alpha}},
\end{aligned}
\tag{41}
$$

and the rest of the definitions were already introduced at the end of §3. From the formulas (28) and (40) we get the folloso scale invariant of the order -1:

$$
\begin{aligned}
\mathscr{L}_1 &= \mathscr{R} + (n-1)\Phi + \frac{(n-1)(n-2)}{4}\varphi, \\
\mathscr{L}_2 &= \varphi + f + 2\mu.
\end{aligned}
\tag{42}
$$

The first of the invariants is the one used by Weyl, $-F$ in his notations (curvature of the tensorial parameters of the space studied by Weyl). The second of the invariants is identically zero for the spaces studied by Weyl.

From the two relative scale invariants of the order -1 and the three relative scale invariants of the order -2 it is useful to get an absolute scale invariant, or a general relative scale invariant of order l according to:

$$
\mathscr{L} = \mathscr{L}_1^{-l}\mathscr{O}\left(\mathscr{L}_1, \mathscr{L}_2, S_1^{1/2}, S_2^{1/2}, S_3^{1/2}\right),
\tag{43}
$$

where $\mathscr{O}$ is a homogeneous function of degree zero. The integral invariant is obviously:

$$
\mathscr{I} = \int \mathscr{M}dV = \int \mathscr{L}_1^{n/2}\mathscr{O}\left(\mathscr{L}_1, \mathscr{L}_2, S_1^{1/2}, S_2^{1/2}, S_3^{1/2}\right)\sqrt{g}\,dV,
$$

where $dV = dx_1 dx_2 \dots dx_n$.

Weyl uses in his notes a particular form of the integral function $\mathscr{M}$, which reads in our notations as:

$$
\mathscr{M} = \mathscr{L}_1^2\sqrt{g}\left(a + b\left(\frac{S_1^{1/2}}{\mathscr{L}_1}\right)^2\right),
$$

where a and b are constants.

3. To conclude let us discuss the question of the physical significance of the geometrical considerations above, although this question is beyond the scope of the present work. By putting the variation of the integral invariant $\mathscr{I}$ to 0 Weyl gets Einstein's equations on the one side and Maxwell's, as they are written with Mie's theory, on the other. In this way Weyl identifies a scale vector with a four-dimensional electromagnetic potential. Using the general considerations about the integral invariant properties, Weyl gets a general form of the equations of electrodynamics. Using a particular form of the world function $\mathscr{M}$ he obtains Maxwell's equations and Mie's theory. As our theory has two scale vectors, it is natural to identify them with a four-dimensional electromagnetic potential and a four-dimensional current vector (Viererstrom). It is easy to obtain, using the infinitesimal scale transformations, the signatures of the Maxwell equations and the additional constraints that replace Mie's theory. It is much more difficult to find a particular form of the world function $\mathscr{M}$ such that it gives the additional constraints, without the controversies that result from Mie's theory.

We leave open the possibility to find the world function $\mathscr{M}$ with the properties just outlined.

A. Friedmann
Professor of Mechanics of Petrograd Polytechnic Institute.
Petrograd, 15th of April, 1922.

References

[1] See Levi-Civita, *Nozione di parallelismo in una varieta qualunque etc.*, Rendic. del Circolo Matem. di Palermo, t. 42 (1917).

[2] A generalization in a certain direction of Weyl's space was made by Eddington in his article published in Proceed. Lond. Roy. Soc. Vol. 1921.

[3] Bianchi, *Lezioni di Geometria Differenziale*, t.1, p.332.