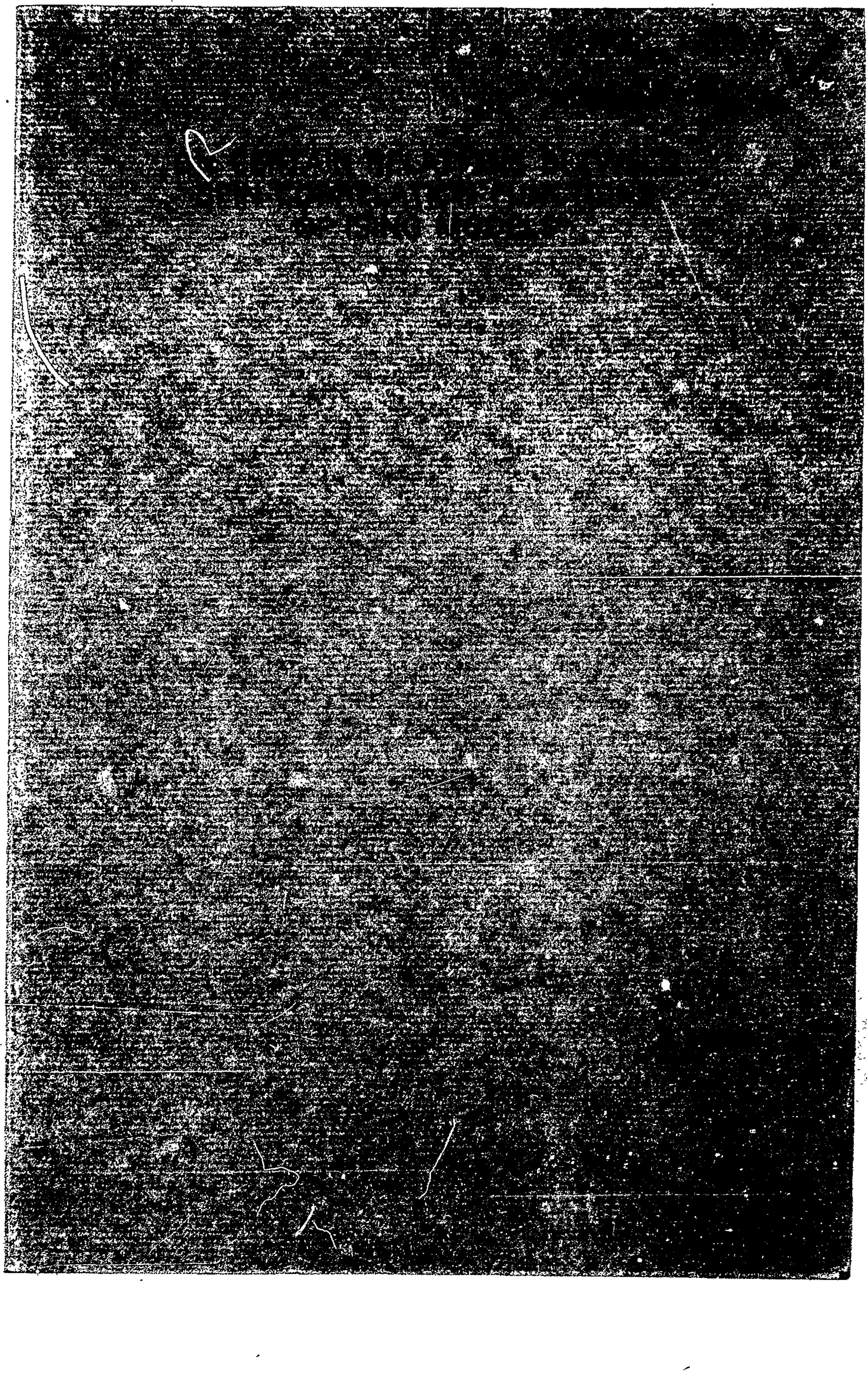

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STELLINGEN

1. Als A een eindige verzameling is bestaande uit n elementen, met n een even getal, en α een familie van n deelverzamelingen van A die ieder uit een even aantal elementen bestaan, dan bevat α (minstens) twee verzamelingen waarvan de doorsnede uit een even aantal elementen bestaat.

2. Voor een Isingmodel met willekeurige n-spininteracties geldt, voor willekeurige verzamelingen spins A, B en C, de volgende correlatiefunctie-ongelijkheid:

$$\sum_{D\subset C} \langle\sigma_D\rangle^2 \geq \sum_{D\subset C} \langle\sigma_A\sigma_D\rangle\langle\sigma_D\sigma_B\rangle ,$$

waarin, voor iedere verzameling spins X, $\sigma_X = \prod_{v\in X} \sigma_v$.

3. Ten onrechte suggereren Hirsch en Smale dat het stelsel differentiaalvergelijkingen:

$$\dot{x} = (A-By-\lambda x)x; \quad \dot{y} = (Cx-D-\mu y)y ,$$

met A, B, C, D, λ en μ groter dan nul, een limietcykel zou kunnen hebben.

M.W. Hirsch en S. Smale
Differential equations, dynamical systems
and linear algebra.

4. Voor een Isingmodel gedefiniëerd op een oneindige planaire graaf in een homogeen magneetveld h geldt voor een randrij $(v_1,\ldots,v_n)$, met n een even getal, de volgende correlatiefunctie-identiteit:

$$\lim_{h\to 0}\left\{ \sum_{k=1}^{n} (-1)^k \langle\sigma_{v_k}\rangle\langle\sigma_{v_k} \prod_{i=1}^{n} \sigma_{v_i}\rangle \right\} = 0.$$

5. Met een eenvoudig voorbeeld kan men laten zien, dat de door Lyons et al. noodzakelijk geachte uitbreiding van het theorema van Faxén voor het geval van niet-lokale viscositeit fout is.

K.B. Lyons, R.C. Mockler and W.J. O'Sullivan
Phys.Rev. A10 (1972) 393.

6. Laat H de hamiltoniaan zijn van een Isingmodel met willekeurige n-spininteracties en C(H) de $2^{|V|}$ bij $2^{|V|}$ matrix met matrixelementen $C_{A,B}(H) = Z_H 2^{-|V|} \langle\sigma_A\sigma_B\rangle_H$, waarin V de verzameling van alle spins is, $A,B \subset V$ en Z_H de kanonieke toestandssom, dan geldt:

$$C(-H) = C(H)^{-1}.$$

7. Zeer recent zijn door Kessel et al. nauwkeurige berekeningen gedaan aan de bandenstructuur van Nb_3Sn in de A-15 fase. Teneinde hun berekeningen te kunnen vergelijken met experimentele gegevens van Nb_3Sn in de supergeleidende toestand, is het zinvol de invloed te bepalen van de verandering van de kristalstructuur, tengevolge van de structurele faseovergang, op de bandenstructuur.

A.T. Kessel, H.W. Myron en F.M. Mueller, preprint.

8. Het is eenvoudiger om de toestandssommen van de tot nu toe opgeloste 8-vertexmodellen op het kwadratisch rooster te berekenen met behulp van de overdrachtsmatrix die het rooster diagonaal opbouwt dan met de overdrachtsmatrix die het rooster rij voor rij opbouwt.

9. Laat, voor een Isingmodel met willekeurige n-spininteracties, H_0 dat deel van de hamiltoniaan zijn, dat de wisselwerking van een gegeven spin v_0 met de overige spins beschrijft. Een eenvoudig argument laat zien dat voor alle verzamelingen spins X die v_0 niet bevatten geldt: $\langle\sigma_X\rangle = \langle\sigma_X e^{-2\beta H_0}\rangle$. De in de hieronder vermelde referenties afgeleide stelsels lineaire relaties tussen de spincorrelatiefuncties zijn equivalent met bovenstaande relaties en dus met elkaar.

R. Dekeyser en J. Rogiers, Physica 59 (1972) 23.
J. Vertogen en A.S. de Vries, Comm.Math.Phys. 29 (1972) 131.
C. Gruber en D. Merlini, Physica 67 (1973) 308.

10. Het verschil dat er bestaat tussen het medische model en het sociologische model voor "afwijkend" gedrag vertoont een sterke overeenkomst met dat tussen ideale modellen en niet-ideale modellen voor fysische systemen. Op grond hiervan kan men vermoeden dat het medische model slechts een nulde-orde benadering is.

11. De basis van het door Bautz geuite vertrouwen in de analogie tussen de regulatie van gen-expressie in procaryoten en die in eucaryoten wordt geheel weggenomen door recente bevindingen over de genetische organisatie van eucaryoten.

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W. Gilbert, Nature 271 (1978) 501.

[illegible].J. Pool Leiden, 1[illegible] juni 19[illegible]

ON CERTAIN RELATIONS BETWEEN SPIN CORRELATION FUNCTIONS OF ISING MODELS

PROEFSCHRIFT

ter verkrijging van de graad van doctor in
de wiskunde en natuurwetenschappen aan de
Rijksuniversiteit te Leiden, op gezag van
de Rector Magnificus Dr. D. J. Kuenen,
hoogleraar in de faculteit der Wiskunde en
Natuurwetenschappen, volgens besluit van
het college van dekanen te verdedigen op
woensdag 14 juni 1978
te klokke 15.15 uur

door

RICHARD JOHN BOEL

geboren te Rotterdam in 1951

Krips Repro – Meppel

Promotor: Prof.dr. P.W. Kasteleyn

Aan hen die dit proefschrift lezen, maar vooral
aan hen die het graag zouden willen kunnen lezen

CONTENTS

Ernst Ising showed a remarkable form of foresight when in 1925 he wrote: "Trotzdem dürfte die vorliegende Untersuchung für das Problem des Ferromagnetismus von einem gewissen Interesse sein."

E. Ising, Z.Phys. 31 (1925) 253.

INTRODUCTION.

One of the main fields of research in experimental and theoretical physics of condensed matter in thermal equilibrium is that of the study of phase transitions. A phase transition is characterized by the fact that the number and nature of thermal equilibrium states of a system are changed by small variations of externally controlled variables. Associated with such a transition is a drastic change in the values of certain macroscopic variables of the system.

As a typical example we consider magnetic crystals (of which certain aspects will form the subject of this thesis), which consist of a large number of particles, occupying the sites of a crystal lattice, each particle having a spin and a corresponding magnetic moment. For certain types of these magnetic crystals (ferromagnets) there is a temperature above which they do not show a magnetization and below which they do (spontaneous magnetization). Associated with this (critical) temperature is a divergence in the values of the susceptibility, and sometimes the specific heat, at zero magnetic field.

The aim of equilibrium statistical mechanics is to give a theoretical description, on the basis of the microscopic structure of the system, of such phenomena. Since a phase transition is due to co-operative behaviour of large groups of interacting particles, the interactions between these particles play a crucial role. This implies that one cannot describe such systems as nearly ideal (i.e. as deviating from ideal systems by the presence of a weak perturbation), which complicates the theoretical treatment considerably.

The first and simplest model proposed to describe the behaviour of a magnetic crystal is the Ising model (for the history of which the reader is referred to a review article by Brush 1)). The model consists of a set of vertices or points (representing the particles of the crystal), and with each vertex v is associated a spin variable σ_v (representing the spin of the particle at v), which can take the values +1 or -1. The spins interact pairwise with each other, and the interaction energy of a pair of spins is $-J_{vv'}\sigma_v\sigma_{v'}$, where $J_{vv'}$ is a real variable representing the strength of the interaction. If $J_{vv'} > 0$, the interaction is called ferromagnetic and the energy is minimal of both spin variables are equal; if $J_{vv'} < 0$, the interaction is called antiferromagnetic and the energy is minimal if the spin variables have opposite values. If the crystal is placed in an external magnetic field, a spin will in addition have an interaction with this field with interaction energy $-mH\sigma_v$, where m is the magnetic moment of the spin and H the magnetic field. The total energy of the system is thus given by

$$H = -\tfrac{1}{2} \sum_{(v,v')} J_{vv'}\sigma_v\sigma_{v'} - mH \sum_{v} \sigma_v$$

where the first sum is over all pairs of vertices, of which the associated spins have an interaction, and the second sum is over all vertices. If one assumes that the system is in thermal equilibrium at temperature T, then, by the standard methods of equilibrium statistical mechanics, the thermodynamic functions such as the free energy, the energy, the entropy and the specific heat can be obtained from the (canonical) partition function.

A major breakthrough in the study of this model is due to Onsager 2), who succeeded in calculating exactly the partition function of the two-dimensional Ising model on a square lattice in zero magnetic field. He also gave an expression for the magnetization. It turned out that the model exhibits a phase transition, but that the nature of the transition is quite different from what existing theories had so far predicted. Until now nobody has succeeded in deriving expressions for the partition function of the two-dimensional Ising model in a field or the three-dimensional model.

At first sight the model seems highly artificial (it was considered as such for a long time) and the results for the two-dimensional case seem to be only of theoretical interest. However, experimental physicists have succeeded in finding compounds in which the interactions between the spins of the particles have a two-dimensional character and can be described by an Ising-like interaction. The experimental results on the specific heat and magnetization of these compounds beautifully confirm the results obtained by Onsager. For details on experimental results, we refer to a review article of De Jongh and Miedema 3) or the Proceedings of the 1976 International Conference on Magnetism 4).

However, it is not only the successful description of a magnetic crystal which makes the Ising model interesting. The model can be considered as describing any system of interacting units, which can be in two different states. These two states can, for instance, be interpreted as corresponding to the presence or absence of a particle at a vertex, and thus one has a model for a lattice gas. Furthermore, it can be used in the case that every vertex can be occupied by two different kinds of atoms, and one has a model for a binary alloy. For some applications of the Ising model to biology (for example the helix-coil transition in DNA) we refer the reader to ref. 5. Recently, the Ising model has also found applications in quantum field theory [6,7].

Although the thermodynamic functions are very important in the description of the model, there is another set of quantities which are of fundamental importance, namely the correlation functions. Correlation functions are thermal expectation values of products of spin variables, and are a measure for the way the spins at different vertices are correlated, and hence for the order and symmetry present in the system. They give a more detailed description of the microscopic properties of the model. Furthermore, the spontaneous magnetization and the magnetic susceptibility in zero magnetic field can be expressed in terms of correlation functions. Since no explicit results are known about thermodynamic functions in case a magnetic field is present, the correlation functions can give important information about the magnetization and susceptibility 8). The susceptibility is also related to cross-sections in neutron scattering on magnetic systems and the correlation functions are important for the interpretation of this kind of experiments.

The results which are known at the moment about correlation functions in Ising models can be divided into two classes. First, there are explicit expressions for the correlation functions. This is the case for the one-dimensional model, which is, however, not very interesting, because it does not exhibit a phase transition. Results for certain correlation functions for some specific lattices (square, honeycomb and triangular) are also known. Very recently, McCoy, Tracy and Wu 9) gave closed expressions for all correlation functions of the two-dimensional model on a square lattice in zero magnetic field. This exhausts the results thus far obtained in this class. Secondly, there are general rigorous results about correlation functions, which again fall into two classes, namely algebraic equations for correlation functions and algebraic inequalities for correlation functions.

Exact algebraic equations for the correlation functions are, for example, the Kirkwood-Salsburg equations 10) and analogous sets of equations due to Gallavotti and Miracle-Sole 11) and to Gruber and Merlini 12). These are linear inhomogeneous equations for the correlation functions in which the coefficients depend explicitly on the interaction parameters of the system. Algebraic equations for correlation functions not explicitly containing the interaction parameters of the system, to be called correlation-function identities, were until now known only for the one-dimensional Ising model and for the Ising model on a tree 13).

Algebraic correlation-function inequalities are statements about the sign of certain algebraic functions of correlation functions. For example, in ferromagnetic systems (in which all interaction parameters are non-negative) the spins tend to align parallel to one another, since this lowers the energy. This tendency is reflected in the statement that the pair-correlation function is positive, which was first proved by Griffiths and is known as the first Griffiths inequality. This inequality is one of a set of inequalities known as the GKS inequalities 14), which are due to Griffiths, Kelly and Sherman. Another inequality for ferromagnetic systems was derived by Griffiths, Hurst and Sherman 14) (and is known as the GHS inequality); it proves the concavity of the magnetization as function of the magnetic field. Further known inequalities are due to, for example, Lebowitz 15), Percus 16), Newman 17) and Sylvester 18).

Both the algebraic equations and the algebraic inequalities have been applied to prove the existence of the thermodynamic limit of correlation functions, the existence or absence of phase transitions, and the analyticity and monotonicity of thermodynamic functions, to derive bounds for correlation functions in terms of lower-order correlation functions (i.e. correlation functions containing fewer spin variables) and to give estimates of critical temperatures. For a general reference, see for instance Ruelle 10) and Lebowitz 19). Furthermore, since some of the inequalities can be generalized to higher-spin systems and continuous-spin systems they have found extensive applications in quantum field theory 6,7). So in the absence of explicit results these algebraic equations and inequalities can be a powerful tool for obtaining information about the system.

In this thesis we shall study certain relations between spin correlation functions in Ising models. Let A be a set of vertices and, if B is a subset of A, let A\B denote the set of vertices in A but not in B. By σ_A we denote the product of all spin variables associated with the vertices of A and by $\langle\sigma_A\rangle$ the thermal average of σ_A. We shall study the family of functions

$$\Lambda_A = \sum_{B \subset A} \lambda_B \langle\sigma_B\rangle\langle\sigma_{A\setminus B}\rangle ,$$

where the sum is over all subsets B of A, and where λ_B is, for all $B \subset A$ an arbitrary complex number which does not depend on the interaction parameters of the system; some generalizations of these functions will also be considered. The main object of this thesis is to find conditions for the λ_B under which Λ_A for a given Ising model (specified by a set of vertices and a set of pairs of spins which have an interaction) is

(a) equal to zero for all possible choices of the values of the interaction parameters

or

(b) nonnegative for all possible choices of nonnegative values of the interaction parameters.

The family of functions Λ_A for which (a) applies then determines a family of correlation-function identities, and the family of functions Λ_A for which (b) applies a family of correlation-function inequalities.

In chapters I, II and III the Ising models which we consider contain only pair interactions. By representing the interaction between two spins by an edge between the corresponding vertices we can, in a natural way, associate with each Ising model a graph (and speak of an Ising model defined on a graph). The coupling parameters associated with the interactions then define a function on the set of edges. The graph-theoretical concepts and lemmas developed and proved, respectively, in chapter I will be used throughout the thesis.

In chapter I we shall give an example of a family of functions Λ_A (specified by the choice of A) for which (a) applies. The Ising models considered will be Ising models on planar graphs.

In chapter II, (a) and (b) will be studied for Ising models on arbitrary graphs. A necessary and sufficient condition on the λ_B will be derived under which (a) or (b) applies. It will turn out that in case (a) these conditions consist in a set of homogeneous linear equations in the λ_B with coefficients 0 or 1, and in case (b) in the corresponding set of homogeneous linear inequalities. The coefficients 0 or 1 are closely related to the way in which the vertices of a given set A are connected by edges of the graph, and can easily be determined. Examples of (a) and (b) will be given.

Chapter III will contain a study of the linear inequalities obtained in the preceding chapter, and results from the theory of linear inequalities. The main conclusion will be that for a given set A, any correlation-function inequality of the form $\Lambda_A \geq 0$ can be written as a positive linear combination of a fixed finite set of "extremal" correlation-function inequalities of this form. Examples of extremal inequalities will be given for the cases that A contains 4 and 6 vertices. Furthermore, some extremal inequalities valid for general A will be derived. For Ising models with pair interactions only, these extremal inequalities will give the best possible upper and lower bounds of correlation functions in terms of lower-order ones within the class of inequalities considered.

In chapter IV, the functions Λ_A for which (a) holds will be studied in the case where general n-spin interactions are present in the system. It is proven that every Λ_A for which (a) holds can be written as a linear combination of simpler functions (namely functions Λ_A for which λ_B takes the values 1, -1 or 0 for all $B \subset A$), each of which satisfies (a). The study of these simpler functions turns out to be rather simple, and a necessary and sufficient condition will be derived under which (a) holds. Again, some examples will be given.

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I. CORRELATION-FUNCTION IDENTITIES FOR GENERAL PLANAR ISING SYSTEMS

Abstract.

A system of exact algebraic relations is derived for the spin correlation functions on any so-called "boundary set" of an Ising model on an arbitrary planar graph. One way of expressing these relations is to say that every higher-order correlation function is equal to a Pfaffian of the pair correlation function on the same set of boundary spins.

1. Introduction.

In this paper we shall derive and study a set of identities which express certain higher-order correlation functions of an arbitrary planar Ising model (in zero magnetic field) in terms of the pair correlation function of this model. These identities do not contain explicitly any interaction parameter of the system and they hold for any values of the parameters.

As is well known, identities of this general nature, i.e. algebraic relations between correlation functions of various orders not involving the interaction parameters, occur in many branches of physics. Most often they are used in order to close an otherwise infinite hierarchy of equations and to obtain a finite set which then can be handled much more simply. For this reason such relations are often called "closure relations".

One distinguishes between exact and approximate closure relations. Perhaps the best known example of the latter is Kirkwood's superposition approximation in the theory of classical fluids and its use, in two different hierarchies, to obtain the (approximate) Kirkwood and Born-Green integral equations, respectively, for the pair correlation function 1).

Exact identities, on the other hand, are far more scarce and they are only known for systems which, in a sense, are exactly solvable. We mention the following examples:

i) The classical one-dimensional system of particles interacting via an arbitrary nearest-neighbour interaction in an arbitrary (inhomogeneous) external potential 2).

ii) It is well known, see e.g. ref. 3, that for an open Ising chain with equal nearest-neighbour interactions in zero field every even-order correlation between Ising spins factorizes into a product of pair correlations each of which, in turn, is a product of nearest-neighbour correlations. *)

iii) Recently, Falk 4) has generalized these relations to the case of the zero-field Ising model on a Cayley tree.

The identities which we shall derive in the present paper can be regarded as a generalization of Falk's result. This connection between the two results will be elucidated in section 5, application 3(a).

The outline of the paper is as follows. In section 2 we define the main

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*) This result can in a straightforward way be generalized so as to cover also the case of an inhomogeneous Ising chain, i.e. an Ising chain with arbitrary nearest-neighbour interactions, in an arbitrary field.

concepts to be used and state the main result (Theorem A). In section 3 some graph-theoretical lemmas on planar graphs are proved. Section 4 contains the proof of a theorem (Theorem B) which is shown to be equivalent to Theorem A. The proof of Theorem B goes by simultaneous induction on two variables, one of which is the number of edges. In section 5 we discuss some corollaries and applications. We end the paper with a few concluding remarks (section 6).

The concepts and methods used in this paper are almost exclusively based on abstract graph theory; only in one place (in the proof of Lemma 1) use is made of topological graphs (i.e. of (plane) embeddings of graphs). Although at first sight one would expect it to be simpler to use embeddings throughout the analysis, it appears that the nonuniqueness of embeddings of graphs of low connectivity (in particular disconnected graphs and graphs with an articulation vertex) would make our proof of Lemmas 2 and 3 more cumbersome; to exclude such graphs is not possible because they may result from the removal of an edge in the proof by induction. Lemma 1 can also be proved without resorting to a plane embedding (viz. by using MacLane's characterization of planar graphs 5)), but in that case it is the abstract proof which is the more lengthy one.

2. Definitions and formulation of the main theorem.

We define a graph G to be a pair $(V(G), E(G))$, where $V(G)$ is a set of elements called vertices and $E(G)$ a set of unordered pairs $\{v,v'\}$ of distinct vertices, called edges. G is finite if $V(G)$, and hence $E(G)$, is finite. For definitions of concepts used but not defined in this paper the reader is referred to ref. 6.

In order to formulate the main result of this paper we introduce the following two concepts.

First, let G be a planar graph, B a subset of $V(G)$, and w a vertex not in $V(G)$. Out of G, B, and w we construct a new graph G_B with vertex set $V(G) \cup \{w\}$ and edge set $E(G) \cup E_B$, where $E_B = \{\{v,w\} \mid v \in B\}$. We call the set B a boundary set of G if G_B is planar.

Second, let $S = (v_1,\dots,v_n)$ be a sequence of (not necessarily distinct) vertices of G, let $w, w_1, \ldots, w_n$ $(n \geq 1)$ be distinct vertices not in $V(G)$, and W the graph (called n-wheel for $n \geq 3$) defined by $V(W) = \{w, w_1, \ldots, w_n\}$, $E(W) = \{\{w, w_1\}\}$ if $n=1$ and $E(W) = \{\{w, w_j\}, \ \{w_j, w_{j+1}\} \mid 1 \leq j \leq n\}$ if $n \geq 2$ (where $w_{n+1} = w_1$). Out of G, S, and W we construct a new graph G_S with vertex set $V(G) \cup V(W)$ and edge set $E(G) \cup E_S \cup E(W)$, where $E_S = \{\{v_j, w_j\} \mid 1 \leq j \leq n\}$. We call the sequence S a *boundary sequence* of G if G_S is planar.

If G is a finite planar graph, we shall understand by an *Ising model on G* a spin system with pair interactions only, defined by the Hamiltonian *)

$$H_G = - \sum_{\{u,v\} \in E(G)} K_{uv} \sigma_u \sigma_v \, , \tag{1}$$

where σ_u is the spin variable associated with the vertex u, assuming the values ±1 only, K_{uv} is the coupling parameter between the spins at u and v (which is allowed to take complex values), and the sum is taken over all edges of G. The unnormalized and normalized correlation functions $(\sigma_A)_G$ and $\langle \sigma_A \rangle_G$ are respectively defined by the equations

$$\begin{aligned} (\sigma_A)_G &= \sum_{\{\sigma_v\}} \sigma_A \, e^{-H_G} \, , \\ \langle \sigma_A \rangle_G &= (\sigma_A)_G \, Z^{-1} \text{ if } Z \neq 0, \end{aligned} \tag{2}$$

where the summation is with respect to all spin variables of the system, $\sigma_A = \Pi_{v \in A} \sigma_v$ for any $A \subset V(G)$, and where Z, the canonical partition function, is defined by $Z = (1)_G$. For brevity we shall often omit reference to the graph G and write (σ_A) and $\langle \sigma_A \rangle$.

Our main result states that each higher-order spin correlation function on any boundary set B of G can be written as a Pfaffian of the pair correlation function on B. More specifically:

Theorem A. If $(v_1, \ldots, v_n)$, with $n \geq 1$, is a boundary sequence of a finite planar graph G, then for any Ising model on G with $Z \neq 0$ the following identity holds

$$\langle \sigma_{v_1} \sigma_{v_2} \ldots \sigma_{v_n} \rangle = \mathrm{Pf}\, C \, , \tag{3}$$

where C is the triangular array of elements given by:

$$C_{ij} = \langle \sigma_{v_i} \sigma_{v_j} \rangle \qquad (1 \leq i < j \leq n).$$

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*) Factors kT have been absorbed into H and K_{uv}.

In the above the Pfaffian is for n even defined by

$$\mathrm{Pf}\, C = \sum_{P}{}' \epsilon_P C_{P_1P_2} C_{P_3P_4} \cdots C_{P_{n-1}P_n} , \qquad (4a)$$

in which the sum is over all permutations P of (1, ..., n) with

$$P_{2i-1} < P_{2i} \qquad (1 \leq i \leq \tfrac{1}{2}n) ,$$

$$P_{2i-1} < P_{2i+1} \qquad (1 \leq i \leq \tfrac{1}{2}n-1) ,$$

and ϵ_P is the signature of P (cf., e.g., ref. 7); for n odd we define

$$\mathrm{Pf}\, C = 0. \qquad (4b)$$

3. Graph-theoretical preliminaries.

In the analysis we shall perform several operations on a graph G: the deletion of an edge, the contraction of an edge {u,v} to a vertex u (or v), the insertion of a vertex u' ($u' \notin V(G)$) into an edge {u,v} (being the addition of the vertex u' to V(G) and the replacement of the edge {u,v} by the two new edges {u,u'} and {u',v}), and the extension of G with an edge {u,v} with $u,v \in V(G)$ (being the replacement of E(G) by $E(G) \cup \{\{u,v\}\}$); note that in the latter case the resulting graph is identical with G if $\{u,v\} \in E(G)$.

If u and v are distinct vertices of G, a chain between u and v is a connected subgraph of G in which u and v have valency 1 and all other vertices have valency 2. (Such a chain exists if and only if u and v belong to the same component of G.) It is convenient to extend this definition by calling the subgraph with vertex set {u} and empty edge set a chain between u and u.

The following elementary properties of boundary sequences of any planar graph G are used in the sequel:

(a) A subsequence of a boundary sequence is a boundary sequence.

(b) Any cyclic permutation of a boundary sequence is a boundary sequence.

(c) A boundary sequence S of G is also a boundary sequence of any subgraph of G containing all vertices of S.

(d) The set of vertices in a boundary sequence is a boundary set of G. (For a converse statement, see Corollary 1b of Lemma 1.)

The following property is used in some applications but not in the proofs of Theorems A and B:

(e) A subset B of V(G) is a boundary set if and only if, for some plane embedding of G, there exists a face R of G such that all vertices of B belong to the (topological) boundary of R.

This property is proven from the fact that two points of the plane lying outside a given open connected region can be joined by a curve lying entirely inside that region if and only if they belong to the boundary of that region.

Lemma 1. If v is a vertex of valency $n \geq 2$ of a finite planar graph G and U is the set of vertices of G adjacent to v, then the vertices of U can be ordered into a sequence $(v_1, \ldots, v_n)$ such that the graph obtained by extending G with the edges $\{v_i, v_{i+1}\}$, $1 \leq i \leq n$, where $v_{n+1} = v_1$, is planar.

Proof. Consider an embedding of G in the plane such that every edge is represented by a straight line segment. Since G is finite we can draw a circle C around v not enclosing any other vertex of G and not intersecting any edge of G except the edges incident with v. Let $w_1, \ldots, w_n$ be the points of intersection of G and the latter edges, in an order in which they are encountered when C is traversed. We consider the figure thus obtained as a new (abstract) graph G'; for $n \geq 3$ the line segments into which C is divided define n distinct edges, for n=2 they define one single edge. By construction G' is planar. For $1 \leq i \leq n$ let v_i be the vertex of V(G) adjacent in G' to w_i. If we now contract, for $1 \leq i \leq n$, the edge $\{v_i, w_i\}$ of G' to the vertex v_i, the graph G" thus obtained is the extension of G with the edges $\{v_i, v_{i+1}\}$, $1 \leq i \leq n$. It can be shown that the contraction of an edge in a planar graph results in a planar graph (cf. ref. 6, p. 61). It follows that G" is planar. ■

Corollary 1a. If v is a vertex of valency ≥ 3 of a finite planar graph G and u is a vertex adjacent to v, there exist two vertices u_1 and u_2 in G, distinct from each other and from u and adjacent to v, such that the graph obtained by extending G with the edges $\{u, u_1\}$ and $\{u, u_2\}$ is planar.

This corollary follows from Lemma 1 by identifying u with some v_i, and u_1 and u_2 with v_{i-1} and v_{i+1}, respectively.

Corollary 1b. The vertices of any boundary set B of a finite planar graph G can be ordered into a boundary sequence of G.

The case $|B| = 1$ [*)] is trivial. For $|B| \geq 2$, the corollary follows from the proof of Lemma 1, applied to the vertex w of G_B, if one puts $S = (v_1, \ldots, v_n)$, $G_S = G'$.

Lemma 2. [**)] If (v_1, v_2, v_3, v_4) is a boundary sequence of a planar graph G, every chain between v_1 and v_3 has a vertex in common with every chain between v_2 and v_4. In particular if $v_1 = v_3$, every chain between v_2 and v_4 must contain v_1; if further $v_2 = v_4$, all four vertices coincide.

Proof. Suppose the lemma is false. Then there exists a chain C_{13} between v_1 and v_3 and a chain C_{24} between v_2 and v_4 which have no vertex in common. If we then form the graph G_S, where $S = (v_1, v_2, v_3, v_4)$, with the aid of a 4-wheel W as introduced in section 2, the subgraph of G_S which is the union of C_{13}, C_{24} and W, extended with the four edges $\{v_i, w_i\}$ $(1 \leq i \leq 4)$, is homeomorphic to the graph K_5. By Kuratowski's theorem this is impossible since G_S is planar (ref. 6, p. 61). This proves the lemma. ■

Lemma 3. If G is a finite connected planar graph and $(v_1, \ldots, v_n)$, with $n \geq 3$, is a boundary sequence of G, then:

(i) $(v_1, \ldots, v_n, v_n)$ is a boundary sequence of G;

(ii) if $v_1 \neq v_n$, there exists a vertex $v' \in V(G)$ such that $\{v_n, v'\} \in E(G)$ and $(v_1, \ldots, v_n, v')$ is a boundary sequence of G.

Proof. (i) It is easily verified that, since G is connected, it contains a vertex u (not necessarily distinct from v_{n-1}, v_n or v_1) such that there exist three chains, one between u and v_{n-1}, one between u en v_n, and one between u and v_1, which have as their intersection the vertex u.

Consider the (planar) graph G_S, with $S = (v_1, \ldots, v_n)$, the (planar) graph G' obtained from G_S by inserting a vertex w_{n+1} into the edge $\{w_n, w_1\}$,

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*) For any set B the number of elements in B is denoted by $|B|$.

**) This lemma has frequently been used in the theory of graphs, e.g. in the theory of vertex colourings of graphs (see, e.g., ref. 6, p. 86).

and the graph G_j obtained by extending G' with an edge $\{w_{n+1},w_j\}$, where $2 \leq j \leq n-1$. There exist in G_j nine chains, one between each of the vertices w_{n-1}, w_n, w_1 on the one hand, and each of the vertices w, w_{n+1}, u on the other hand, such that any two chains have at most an end vertex in common. Therefore G_j contains a subgraph homeomorphic to the graph $K_{3,3}$ and hence is nonplanar by Kuratowski's theorem.

We now consider the four vertices adjacent to w_n, i.e. w_{n-1}, w_{n+1}, w, and v_n. By Corollary 1a of Lemma 1, applied to the vertices w_n and w_{n+1}, we can extend G with two edges, each one between w_{n+1} and another vertex adjacent to w_n, so that the graph thus obtained is planar. These vertices must be w and v_n, since w_{n-1} is excluded by the argument given above that the graph G_{n-1} is nonplanar. The graph obtained from G' by extending it with the edges $\{w_{n+1},w\}$ and $\{w_{n+1},v_n\}$ is just the graph $G_{S'}$, corresponding to the sequence $S' = (v_1,\dots,v_n,v_n)$. Since $G_{S'}$ is planar, S' is a boundary sequence of G.

(ii) Next we consider the set of vertices adjacent to v_n in G', i.e. the set $\{v \mid v \in V(G),\ \{v,v_n\} \in E(G)\} \cup \{w_j \mid 2 \leq j \leq n,\ v_j = v_n\}$. By Corollary 1a of Lemma 1, applied to the vertices v_n and w_{n+1}, we can extend $G_{S'}$ (defined under (a)) with two edges, each one between w_{n+1} and some other vertex adjacent to v_n, so that the graph thus obtained is planar. The edge $\{w_{n+1},w_1\}$ cannot be one of these edges because $v_1 \neq v_n$, so that w_1 is not adjacent to v_n. Nor can either of these edges be of the form $\{w_{n+1},w_j\}$ with $2 \leq j \leq n-1$, again for the reason that G_j is nonplanar. The only remaining candidates are the edges $\{w_{n+1},w_n\}$ and $\{w_{n+1},v'\}$, where v' is some vertex in $V(G)$ adjacent to v_n.

This shows that there exists at least one vertex v' adjacent to v_n in G such that the graph G'' obtained by extending $G_{S'}$ with the edge $\{w_{n+1},v'\}$ is planar. If finally we delete the auxiliary edge $\{w_{n+1},v_n\}$, the graph thus obtained is just $G_{S''}$, where S'' denotes the sequence $(v_1, \dots, v_n, v')$. Since G'', and hence $G_{S''}$, is planar, S'' is a boundary sequence. ■

4. Proof of the main theorem.

We define, quite generally, for any Ising model on a finite planar graph G and any sequence of vertices $(v_1, \dots, v_n, v)$, with $n \geq 0$, of G the following function:

$$f(v_1,\ldots,v_n;\ v|G) = \sum_{j=1}^{n} (-1)^j (\sigma_{v_j}\sigma_v)_G\ (\sigma_v \prod_{\substack{i=1\\ i\neq j}}^{n} \sigma_{v_i})_G\ . \qquad (5)$$

For brevity we shall in most cases omit reference to the graph G in the argument of f and write $f(v_1,\ \ldots,\ v_n;\ v)$.

For later use we list the following simple properties of the correlation functions (σ_A) and the function f:

(A) If G is the disjoint union of the graphs G' and G", and σ_A and σ_B are products of spin variables referring only to the graphs G' and G", respectively, then:

$$(\sigma_A\sigma_B)_G = (\sigma_A)_{G'}(\sigma_B)_{G''}\ .$$

(B) If n is odd and $v_1,\ \ldots,\ v_n$ are vertices of G, then

$$(\prod_{i=1}^{n} \sigma_{v_i})_G = 0.$$

(C) $f(v_1,\ldots,v_n;v) = 0$ if n is zero or odd.

(D) If $n \geq 2$ and n is even, then

$$f(v_1,\ldots,v_n;v) = -f(v_n,v_1,\ldots,v_{n-1};v)\ . \qquad (6)$$

(E) If $n \geq 2$ and $v_1 = v_2$, then

$$f(v_1,\ldots,v_n;v) = f(v_3,v_4,\ldots,v_n;v)\ . \qquad (7)$$

Since f is an entire analytic function in all coupling parameters K_{uv}, where $\{u,v\} \in E(G)$, we can make a convergent Taylor series expansion with respect to all coupling parameters simultaneously of the form:

$$f(v_1,\ldots,v_n;v) = \sum_{m=0}^{\infty} f_m(v_1,\ldots,v_n;v)\ , \qquad (8)$$

where $f_m(v_1,\ldots,v_n;v)$ is a homogeneous polynomial of degree m in the coupling parameters. This defines the functions f_m uniquely.

Lemma 4. For any finite graph G and any sequence of vertices $(v_1,\ldots,v_n,v,w)$ of G with $\{v,w\} \in E(G)$ and n even, the functions f_m defined above satisfy, for $m \geq 1$:

$$\frac{\partial}{\partial K_{vw}} f_m(v_1,\dots,v_n;v) = -f_{m-1}(w,v_1,\dots,v_n,v;v) + f_{m-1}(v_1,\dots,v_n,v,w;w). \quad (9)$$

We note that if G is planar and $(v_1, \dots, v_n, v, w)$ is a boundary sequence of G, the arguments in the f-functions appearing in eq. (9) are, by Lemma 3(i) and property (b), boundary sequences as well.

Proof. By a straightforward calculation, starting from eqs. (2) and (5), using $\sigma_v^2 = \sigma_w^2 = 1$ and $(-1)^n = 1$, we find

$$\frac{\partial}{\partial K_{vw}} f(v_1,\dots,v_n;v) = f(v_1,\dots,v_n,v,w;v) + f(v_1,\dots,v_n,v,w;w). \quad (10)$$

Using eq. (6) we can rewrite this as

$$\frac{\partial}{\partial K_{vw}} f(v_1,\dots,v_n;v) = -f(w,v_1,\dots,v_n,v;v) + f(v_1,\dots,v_n,v,w;w). \quad (11)$$

Comparing terms of the same degree on both sides of this equation we obtain eq. (9). ■

Theorem B. If G is a finite planar graph and $(v_1, \dots, v_n, v)$, with $n \geq 1$, a boundary sequence of G, then for any Ising model on G

$$f(v_1,\dots,v_n;\, v|G) = 0, \quad (12)$$

or equivalently, for all $m \geq 0$,

$$f_m(v_1,\dots,v_n;\, v|G) = 0. \quad (13)$$

Proof. Eqs. (12) and (13) hold trivially if $n=0$ or n is odd (property (C)). Therefore, we may restrict ourselves to the case $n \geq 2$, n even. Furthermore it is sufficient to prove eqs. (12) and (13) only for the case $v_1 \neq v$, since the general case can be reduced to this case or to the case $n=0$ by properties (D) and (E). So let $v_1 \neq v$.

We now prove eq. (13) for all m by induction with respect to the ordered pairs of nonnegative integers (ℓ,m) where $\ell = |E(G)|$. We shall say that a pair (ℓ',m') precedes a pair (ℓ,m) if $\ell' \leq \ell$, $m' \leq m$ and $(\ell',m') \neq (\ell,m)$. We note that in the set of pairs (ℓ,m) ordered in this way every element has (at most) a finite number of predecessors.

We choose a particular fixed pair of nonnegative integers (ℓ,m), a finite planar graph G with $|E(G)| = \ell$, and a boundary sequence $(v_1,\dots,v_n,v)$ of G, with $n \geq 2$, n even, and we take as our induction hypothesis that eq. (13) holds for all cases labelled by a pair (ℓ',m') preceding (ℓ,m) *).

There are two possibilities to be considered: either G is connected or it is disconnected.

First let G be connected. Since by assumption $v_1 \neq v$ we have $\ell = |E(G)| \geq 1$. If m=0, then $f_m(v_1,\dots,v_n;v|G) = f_0(v_1,\dots,v_n;v|G_0)$ where G_0 is the graph defined by $V(G_0) = V(G)$, $E(G_0) = \emptyset$, i.e. the graph obtained from G by deleting all its edges. Since (0,0) precedes $(\ell,0)$ for $\ell \geq 1$, $f_0(v_1,\dots,v_n;v|G_0)$ vanishes by the induction hypothesis and so $f_0(v_1,\dots,v_n;v|G)$ vanishes. Hence we may assume from now on $m \geq 1$.

Since $\ell \geq 1$ and G is connected, v is not an isolated vertex of G. According to Lemma 3(ii) there exists a vertex $w \in V(G)$ such that $\{v,w\} \in E(G)$ and $\{v_1,\dots,v_n,v,w\}$ is a boundary sequence of G (remember that we can assume $n \geq 2$ and $v_1 \neq v$).

We regard $f_m(v_1,\dots,v_n;v|G)$ as a function of K_{vw}. From eq. (9) together with the induction hypothesis we find that this function does not depend on K_{vw} so that we may restrict ourselves to the case that $K_{vw} = 0$. But in that case,

$$f_m(v_1,\dots,v_n;v|G) = f_m(v_1,\dots,v_n;v|G') , \tag{14}$$

where G' is the graph obtained from G by deleting the edge $\{v,w\}$. Since $|E(G')| < |E(G)|$ the right-hand side of eq. (14) vanishes by the induction hypothesis. This proves the validity of eq. (13) for all cases labelled (ℓ,m) where the graph G is connected.

Suppose now that G is not connected, and let G' be the component of G containing v. Let $J = \{j | v_j \in V(G'),\ 1 \leq j \leq n\}$ and $J' = \{1,\dots,n\} \backslash J$. The terms of the sum in the right-hand side of eq. (5) with $j \in J'$ are zero because $(\sigma_{v_j}\sigma_v)_G$ factorizes into vanishing factors (see properties (A) and (B)). The other terms (with $j \in J$) also factorize and we obtain:

$$(1)_{G''} \Big(\prod_{i\in J'} \sigma_{v_i} \Big)_{G''} \sum_{j\in J} (-1)^j \, (\sigma_{v_j}\sigma_v)_{G'} \Big(\sigma_v \prod_{\substack{i\in J \\ i\neq j}} \sigma_{v_i}\Big)_{G'} , \tag{15}$$

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*) This method of induction can easily be reduced to the ordinary method of induction with respect to the number of predecessors of an element. See also ref. 8.

where G'' is defined by $V(G'') = V(G) \setminus V(G')$, $E(G'') = E(G) \setminus E(G')$.

If $|J| = 0$, the sum in (15) is zero, and if $|J|$, and hence $|J'|$, is odd, the second factor of (15) vanishes; in both cases eq. (12) follows trivially. Let now $J = \{j_1, j_2, \ldots, j_p\}$ with $p \geq 2$, p even and $1 \leq j_1 < j_2 < \ldots < j_p \leq n$, and let $K = \{i \mid j_1 < i < j_2\}$ and $L = J' \setminus K$. For all $k \in K$ and $\ell \in L$ it follows from Lemma 2, applied to the boundary sequence $(v_{j_1}, v_k, v_{j_2}, v_\ell)$, that v_k is not in the same component as v_ℓ. Hence $(\Pi_{i\in J'} \sigma_{v_i})_{G''}$ factorizes into two factors, one of which contains only the spin variables σ_{v_k} $(k \in K)$. This factor is zero, and hence eq. (12) holds, if $|K| = j_2 - j_1 - 1$ is odd, so that we may assume that $j_2 - j_1$ is odd. Similarly, we can argue that $j_{r+1} - j_r$ can be assumed to be odd for every r $(1 \leq r \leq p-1)$. In that case $(-1)^{j_r} = (-1)^{j_1 - 1}(-1)^r$. Introducing the notation $u_r = v_{j_r}$ $(1 \leq r \leq p)$ we can write

$$f(v_1, \ldots, v_n; v|G) = (-1)^{j_1 - 1} (1)_{G''} \Big(\prod_{i \in J'} \sigma_{v_i} \Big)_{G''} \sum_{r=1}^{p} (-1)^r (\sigma_{u_r} \sigma_v)_{G'} \Big(\sigma_v \prod_{\substack{s=1 \\ s \neq r}}^{p} \sigma_{u_s}\Big)_{G'}$$

$$= (-1)^{j_1 - 1} (1)_{G''} \Big(\prod_{i \in J'} \sigma_{v_i} \Big)_{G''} f(u_1, \ldots, u_p; v|G'). \tag{16}$$

Since $(u_1, \ldots, u_p, v)$ is a subsequence of $(v_1, \ldots, v_n, v)$ it is a boundary sequence of G (property (a) of section 3) and also of G' (property (c)).

It follows from eq. (16) that $f_m(v_1, \ldots, v_n; v|G)$ can be expressed linearly in terms of $f_{m'}(u_1, \ldots, u_p; v|G')$ with $m' \leq m$, where G' is connected and $|E(G')| \leq |E(G)|$. Hence, from the induction hypothesis together with the above result for connected graphs, we deduce the validity of eq. (13) for all cases labelled (ℓ, m), and hence of eq. (13) and eq. (12) in all cases. ■

In order to prove Theorem A we consider a boundary sequence $(v_1, \ldots, v_n)$ of a planar graph G.

The validity of eq. (3) for n odd is an immediate consequence of eq. (4b) and property (B). Now let n be even and $n \geq 2$. From Lemma 3(i) and property (b) of section 3 we conclude that also $(v_1, \ldots, v_n, v_1)$ is a boundary sequence of G. Applying therefore Theorem B, eq. (12), to this case we obtain, using eq. (5) and dividing out a factor Z (which is non-zero by assumption):

$$\langle \sigma_{v_1} \cdots \sigma_{v_n} \rangle = \sum_{j=2}^{n} (-1)^j \langle \sigma_{v_1} \sigma_{v_j} \rangle \Big\langle \prod_{\substack{i=2 \\ i \neq j}}^{n} \sigma_{v_i} \Big\rangle . \tag{17}$$

This is, for n even, the well-known expansion formula for a Pfaffian of a triangular array with respect to the elements of its first row 7), from which eq. (3) can be obtained by iteration. This completes the proof of the theorem. ■

We remark that, conversely, Theorem B for n even and $Z\neq 0$ follows directly from eq. (17), and hence from Theorem A. To show this we apply eq. (17) to the second factor in the right-hand side of eq. (5). This yields

$$Zf(v_1,\ldots,v_n;v) = \sum_{j=1}^{n}\sum_{k=1}^{n} \varepsilon_{kj}(-1)^{j+k}(\sigma_{v_j}\sigma_v)(\sigma_{v_k}\sigma_v)\Big(\prod_{\substack{i=1\\ i\neq j,k}}^{n}\sigma_{v_i}\Big), \qquad (18)$$

where $\varepsilon_{kj} = \mathrm{sgn}(k-j)$ if $k\neq j$ and $\varepsilon_{kk} = 0$. Since ε_{kj} is antisymmetric in k and j, and the product of the other factors in the terms on the right-hand side of eq. (18) is symmetric, the double sum, and hence $f(v_1,\ldots,v_n;v)$, vanishes. The validity of Theorem B for the case $Z=0$ follows from a continuity argument. ■

Eq. (3) shows a formal analogy with certain results of Green and Hurst 9). These authors showed, on the one hand, that their S-matrix method for calculating partition functions of a class of lattice models (now called vertex models) can be applied to planar lattice graphs only if the coefficients in a certain operator polynomial which generates the high-temperature series expansion of the partition function satisfy certain consistency conditions.

Although a general form of these conditions is not explicitly formulated, the examples given *) show that they are formally identical to the identities (3) (ref. 9, p. 163 ff; ref. 10).

On the other hand, Green and Hurst proved that these consistency conditions are satisfied for "effective" lattice models obtained from an Ising model on a planar graph by summing over the interior states of certain "decorating" subgraphs (ref. 9, p. 214 ff).

The analogy is not accidental. To see this we observe that in the latter case the coefficients in the operator polynomial can be interpreted, apart from a common factor, as correlation functions of the boundary spins of the

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*) There are some annoying misprints in ref. 9, e.g. in eqs. (4.36), (4.40) and (4.41); cf. eqs. (20), (25), (26), and the following three lines of ref.10.

decorating subgraph, considered as a separate Ising model. However, in the present work we adopt a different point of view and we employ methods which are entirely different from those of Green and Hurst.

Firstly, our emphasis is on arbitrary planar graphs and the existence of correlation-function identities as such and not especially with a view to the solution of lattice models via a decoration transformation; for a discussion of the latter problem, see section 5, application 1. Secondly, the proof of Theorem A is essentially graph-theoretical and more or less self-contained, whereas the derivation of the analogous result by Green and Hurst requires the full apparatus of their S-matrix method. Finally, there are a number of differences of a more technical nature.

5. Corollaries and some applications.

The identities (3) and (12) have been derived here under the sole restriction on the coupling parameters K_{uv} that $Z \neq 0$; otherwise all coupling parameters are allowed to assume arbitrary complex values. For the sake of completeness we also consider the case $Z=0$. By a simple continuity argument one deduces from Theorem A the following corollary.

Corollary I. If $Z=0$, n is even, $n \geq 4$, and $(v_1,\ldots,v_n)$ is a boundary sequence of a finite planar graph G, then

$$\mathrm{Pf}\, D = 0, \quad \text{where } D_{ij} = (\sigma_{v_i}\sigma_{v_j}), \quad 1 \leq i \leq j \leq n.$$

Another extension of Theorem A is to the case of infinite graphs. For such graphs neither the partition function nor the unnormalized correlation functions are defined. Nevertheless, as is well known, one still can in these cases attach a meaning to normalized correlation functions; this can be done in various ways. One way is to define a correlation function $\langle \sigma_A \rangle_G$ as the limit, for $n \to \infty$, of the sequence $\langle \sigma_A \rangle_{G_n}$, where the G_n $(n=1,2,\ldots)$ are finite subgraphs of the infinite graph G and $A \subset V(G_n)$ for all n. Theorem A now has the following consequence.

Corollary II. If B is any finite boundary set of an infinite planar graph G, and $(G_1, G_2,\ldots)$ a sequence of finite subgraphs of G each one containing the set of vertices B, and if for each pair of vertices u, v of B the limit $\langle \sigma_u \sigma_v \rangle_G = \lim_{n\to\infty} \langle \sigma_u \sigma_v \rangle_{G_n}$ exists, then for any $A \subset B$ the corresponding limit $\langle \sigma_A \rangle_G = \lim_{n\to\infty} \langle \sigma_A \rangle_{G_n}$ exists, and these functions satisfy the identities given by eq. (3).

Applications.

1. The first application of the identities (3) concerns the question of the solvability of eight-vertex models by the reduction of such models to equivalent planar Ising models.

It is well known that the partition function of the eight-vertex model on a square-lattice graph is equal to that of an Ising model on a square lattice with properly chosen two-spin and four-spin interactions between spins on each (elementary) square of the lattice (ref. 11, p. 348). Also, the latter partition function is equal (up to a known factor) to that of an Ising model on a (not necessarily planar) graph, obtained from the square-lattice graph by the addition of a set of new vertices and new edges with properly chosen coupling parameters; in fact, one vertex and four edges suffice (see e.g. ref. 12).

If the latter graph is planar, the corresponding partition functions can be evaluated exactly by one of the standard methods. The question arises whether by considering the most general planar graph one would be able to solve some hitherto unsolved cases of the general eight-vertex model. A question equivalent to this, but phrased in the "dual language", can be considered to have been answered in the negative by Green and Hurst in the work discussed in the previous section. The same conclusion is reached using the identities of this paper, as we shall indicate.

Consider a planar graph G decorating a square of the square-lattice graph and having only the four corner vertices in common with this graph. Eq. (3), applied in G, considered as a planar graph by itself, to the four spins on the corner vertices, implies a relation between the two-spin and four-spin coupling parameters in the equivalent generalized Ising model. If in turn we translate this relation in terms of the equivalent eight-vertex model, we obtain in the usual notation (ref. 11, p. 347)

$$\omega_1\omega_2 + \omega_3\omega_4 = \omega_5\omega_6 + \omega_7\omega_8 \ .$$

This is precisely the so-called free-fermion condition. Such a condition applies to the vertex weights at each vertex of the lattice; hence the eight-vertex model is by definition a free-fermion model (ref. 11, p. 452).

This model, which was first considered by Fan and Wu as a special case of the eight-vertex model introduced by them 13), was recognized by these authors to be equivalent (via the high-temperature expansion) to the most general (planar) lattice model which can, at least in principle, be solved by the S-matrix method of Green and Hurst.

Hence, the above-mentioned procedure for solving eight-vertex models by reducing the partition function to that of a planar graph does not lead to the solution of a new class of eight-vertex models. For vertex models on other planar lattices the situation is similar.

Conversely, however, one may use the above method to reduce the calculation of the partition function of certain complicated planar Ising models to that of simple free-fermion models. For example, it is now a simple matter to reduce the calculation of the partition function of the Ising model on the "Union Jack" lattice 14) to that of a homogeneous free-fermion model on a square lattice.

2. It follows from property (C) that the four spins around a square of the square-lattice graph form a boundary set, and this can be ordered into a boundary sequence, (v_1, v_2, v_3, v_4), say. Hence the identity (3), applied to this sequence, expresses the correlation function $\langle \sigma_{v_1}\sigma_{v_2}\sigma_{v_3}\sigma_{v_4} \rangle$ in terms of the pair correlations for pairs of nearest neighbours and pairs of diagonal neighbours. The latter are well known for the translation-invariant square lattice. Similar applications can be made to other planar lattice graphs.

3. A next application concerns outerplanar graphs. A graph is called *outerplanar* if for some embedding all vertices lie in the boundary of one single face. In the terminology of the present paper these are planar graphs for which the set of all vertices constitutes a boundary set (cf. property (e)). It follows from property (a), Corollary 1b and Theorem A, that for any Ising model on an outerplanar graph all correlation functions can be expressed as Pfaffians of the pair correlation function.

Examples of outerplanar graphs are: (a) (Cayley) trees, (b) broken n-wheels (obtained from n-wheels by deleting one edge of the rim), (c) polygons with non-crossing diagonals. We now discuss some of these cases in more detail.

(a) Correlation-function identities for Ising systems on a tree have been obtained by Falk as mentioned in the introduction $^{4)}$. Falk's second decomposition theorem can be derived from the identities (3) or (17), applied to a tree, by using the fact that in general a boundary set on a tree can in more than one way be ordered into a boundary sequence. It is obtained by applying eq. (17) to each boundary sequence corresponding to a given set of vertices, and (linearly) combining the resulting identities. Although this derivation is, of course, not shorter than Falk's, it shows that his identities are a consequence of (i) the identities (3) of this paper applied to a tree, and (ii) the above-mentioned fact that a set of vertices of a tree can be ordered in many different ways into a boundary sequence.

(b) An Ising model on a broken n-wheel is equivalent, via the usual dummy-spin representation (see, e.g., ref. 15, p. 105) to an open Ising chain of n spins in a magnetic field. The odd- (even-)order correlation functions of the latter system are equal to correlation functions of the former system which contain (do not contain) the spin variable associated with the dummy vertex. It follows that for an Ising model with arbitrary nearest-neighbour interactions in a (not necessarily homogeneous) magnetic field the identities (3) remain valid for n even, whereas for n odd they are replaced by

$$\langle \sigma_{v_1} \cdots \sigma_{v_n} \rangle = \mathrm{Pf}\, C^+ , \qquad (19)$$

where C^+ is the triangular array given by

$$C^+_{oj} = \langle \sigma_{v_j} \rangle \qquad (1 \leq j \leq n) ,$$

$$C^+_{ij} = \langle \sigma_{v_i} \sigma_{v_j} \rangle \quad (1 \leq i \leq j \leq n) .$$

(c) Another case where Theorem A can be applied to any set of spins is that of the open Ising chain with arbitrary nearest- and next-nearest-neighbour interactions in zero field; it is easily verified that the graph corresponding to this model is outerplanar.

4. As a last application we consider the conjecture that for ferromagnetic Ising systems in zero field the (even-order) Ursell functions $U_{2\ell}$ (defined as usual) alternate in sign according to the formula

$$(-1)^{\ell-1}\, U_{2\ell}(i_1,\ldots,i_{2\ell}) \geq 0 \qquad (\ell \geq 1) ,$$

where $i_1,\ldots,i_{2\ell}$ label the spins. For $\ell = 1,2,$ and 3 these inequalities are known to be valid (cf., e.g., ref. 16). Also, Setô 17) has shown them to hold for trees for all ℓ.

It can be shown 18), using the identities (3) of this paper, that the above inequalities hold, for all ℓ, for the spins on any boundary set B of any planar (zero-field) Ising model with the property that every pair correlation in this set B is nonnegative (e.g. for a ferromagnetic model, according to Griffiths's first inequality). This generalizes Setô's result.

6. Concluding remarks.

The identities derived in this paper differ from other equations for correlation functions of Ising systems such as the identities derived by Fisher 19) and generalized by Dekeyser and Rogiers 20) and by Gruber and Merlini 21), or a hierarchy such as the Kirkwood-Salsburg equations (see, e.g., ref. 22) in three respects: (1) they are non-linear; (2) they do not contain the interaction parameters explicitly; (3) they are very special in that their validity has been established only for planar Ising systems with pair interactions (counterexamples for nonplanar systems are readily found).

There exists an interesting relationship between our results and some results of Bedeaux et al 3) derived for the one-dimensional Glauber model. One of their results is that certain polynomials of time-dependent correlation functions, called C-functions (which are analogous to the Ursell functions except for the occurrence of minus signs in their definition), vanish more rapidly for time going to infinity than the Ursell functions themselves. Such C-functions can also be defined for the systems considered in this paper. It is interesting to note that the identities (3), translated in terms of these C-functions, are equivalent to the statement that all C-functions of order $n \geq 3$ vanish identically for boundary sets of the equilibrium Ising spin system considered in this paper (see also Kawasaki, ref. 23, p. 467 ff., where this situation is somewhat elucidated).

Throughout this paper we have restricted ourselves to relations between spin correlation functions on boundary sets. It is possible to extend these results and derive certain relations between correlation functions for more general sets of spins in planar Ising models. This enables one in turn to derive relations between odd-spin correlation functions in infinite planar graphs below the critical temperature. For these relations, and for a connection of the identities derived in this paper with inequalities for spin correlation functions we refer to subsequent papers (ref. 24).

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II. CORRELATION-FUNCTION IDENTITIES AND INEQUALITIES FOR ISING MODELS WITH PAIR INTERACTIONS

Abstract.

For Ising models with pair interactions in zero magnetic field a class of linear combinations of products of two correlation functions is studied. We derive sufficient and necessary conditions under which a function in this class is (a) zero for all values of the coupling parameters, or (b) nonnegative for all nonnegative values of the coupling parameters. Examples of correlation-function identities and inequalities of this type are given.

1. Introduction.

In a recent paper [1], to be referred to as I, it was proved that for a (zero-field) Ising model on a planar graph the correlation functions for spins on a so-called boundary set satisfy certain algebraic relations which are valid for all values of the coupling parameters between the spins. These relations can all be derived from a set of identities which can be written in the form

$$\sum_{j=1}^{n} (-1)^{j} \langle \sigma_{v_1}\sigma_{v_j} \rangle \langle \sigma_{v_1}\sigma_{v_j} \prod_{k=1}^{n} \sigma_{v_k} \rangle = 0$$

where $(v_1,\dots,v_n)$ is a sequence of (not necessarily distinct) vertices which is a boundary sequence of the planar graph (cf. I eq. (17)).

The left-hand side of the above equation can be considered as a special case of a function of the type $\Sigma_{B\subset A}\lambda_B \langle \sigma_B \rangle_G \langle \sigma_B\sigma_D \rangle_G$, where G is an arbitrary graph, A is an arbitrary set of vertices of G, D a subset of A, the sum is over all subsets of A, and the coefficients λ_B are independent of the coupling parameters; the case where all vertices in the boundary sequence are different then corresponds to the case D=A, i.e. $\sigma_B\sigma_D = \sigma_{A\setminus B}$.

In this paper we derive sufficient and necessary conditions on the coefficients λ_B under which a function of this general type for an Ising model with pair interactions in zero magnetic field is (a) zero for all values of the coupling parameters, or (b) nonnegative for all nonnegative values of the coupling parameters, respectively.

In section 2 such conditions are derived for the case D=A on the basis of an expansion of the Boltzmann factor with respect to the coupling parameters. They take, respectively, the form of a set of linear equations and a set of linear inequalities for the λ_B with coefficients 0 and 1. In section 3 the results are extended to the general case $D \subset A$. Section 4 is devoted to an analysis of some properties of the sets of correlation-function identities that can be obtained in this way. In section 5 various examples of such identities are given, among which those derived in I. In section 6 we show that some known correlation-function inequalities follow from the general analysis given in this paper and we derive a new inequality. We end this paper with a few concluding remarks.

Since the extension of the partition function and the correlation functions of Ising models to complex values of the coupling parameters is sometimes useful, we allow for these complex values where possible. This implies that the partition function may take the value zero, in which case the normalized correlation functions are not defined. For this reason we shall work almost exclusively with unnormalized correlation functions; the translation of the results to normalized correlation functions is trivial.

2. Conditions for the existence of certain identities and inequalities for correlation functions.

As in I we define a graph to be a pair (V(G), E(G)), where V(G) is a set of elements called vertices and E(G) a set of unordered pairs {v,v'} of distinct vertices, called edges. G is finite if V(G) and E(G) are finite. For definitions used but not defined in this paper the reader is referred to refs. 1 and 2.

An Ising model on a finite graph G is defined as a triple $(G, \mathscr{S}, K)$, where $\mathscr{S}$ is the set of all functions $\sigma\colon\ V(G) \to \{-1,1\}$ (called configurations) and K a complex function on $E(G)$ (called the interaction function). The spin variable σ_v is the value of σ at the vertex v, the coupling parameter K_e is the value of K at the edge e. The set of all interaction functions will be denoted by $\mathscr{K}$, the set of all $K \in \mathscr{K}$ such that $K_e \geq 0$ ($K_e > 0$) for all $e \in E(G)$ by $\mathscr{F}$ ($\mathscr{F}^+$); an Ising model $(G, \mathscr{S}, K)$ with $K \in \mathscr{F}(\mathscr{F}^+)$ is called ferromagnetic (strictly ferromagnetic).

For any set $A \subset V(G)$ we define

$$\sigma_A = \prod_{v \in A} \sigma_v \ ; \tag{1}$$

for $A = \emptyset$ we have $\sigma_\emptyset = 1$. For any edge $e = \{v,v'\}$ and any set $X \subset E(G)$ we define

$$\sigma^e = \sigma_v \sigma_{v'} \tag{2}$$

$$\sigma^X = \prod_{e \in X} \sigma^e \ . \tag{3}$$

The Hamiltonian of an Ising model $(G, \mathscr{S}, K)$ is defined by

$$H_{G,K}(\sigma) = - \sum_{e \in E(G)} K_e \sigma^e \ , \tag{4}$$

the unnormalized and normalized (spin) correlation functions $(\sigma_A)_{G,K}$ and $\langle \sigma_A \rangle_{G,K}$, respectively, for any set $A \subset V(G)$ by

$$\begin{aligned} (\sigma_A)_{G,K} &= \sum_{\sigma \in \mathscr{S}} \sigma_A e^{-H_{G,K}(\sigma)} \ , \\ \langle \sigma_A \rangle_{G,K} &= (\sigma_A)_{G,K}\, Z^{-1} \qquad \text{if } Z \neq 0 \ , \end{aligned} \tag{5}$$

where Z, the canonical partition function, is defined by $Z = (1)_{G,K}$. For brevity, we shall often suppress the index K, and, where no confusion can arise, the index G as well. We have taken $\beta = 1$.

Since the Hamiltonian is quadratic in the σ_v, the correlation function $(\sigma_A)_{G,K}$ vanishes if $|A|$ is odd. Therefore, we shall henceforth consider only correlation functions $(\sigma_A)_{G,K}$ for even sets A, i.e. for sets with $|A|$ even.

We now consider an arbitrary even set $A \subset V(G)$. Let $\lambda = \{\lambda_B\}^{e}_{B \subset A}$ be a set of complex numbers defined for all even sets $B \subset A$, with the restriction $\lambda_B = \lambda_{A \setminus B}$ for all B. We introduce the following quadratic combination of unnormalized correlation functions:

$$\Lambda_A(G,K) = \sum_{B\subset A}^{e} \lambda_B(\sigma_B)_{G,K}(\sigma_{A\setminus B})_{G,K} \quad , \tag{6}$$

where the sum is over all even sets $B\subset A$. For convenience, a function of the type (6) will be referred to as a <u>Λ-function.</u> We shall now derive a condition for λ under which a Λ-function satisfies the equation $\Lambda_A(G,K) = 0$ for all $K\in \mathcal{K}$ (to be referred to as a <u>Λ-identity</u>), and a condition for λ under which it satisfies the inequality $\Lambda_A(G,K) \geq 0$ for all $K\in \mathcal{F}$ (to be referred to as a <u>Λ-inequality</u>).

We first consider an arbitrary product of two unnormalized correlation functions $(\sigma_B)_G(\sigma_C)_G$ with $B, C\subset V(G)$, $B\cap C = \emptyset$. If we expand the Boltzmann factors in this product with the aid of the elementary relation

$$e^{K_e\sigma^e} = c_e + s_e\sigma^e \tag{7}$$

where e is an arbitrary edge, $c_e = \cosh K_e$ and $s_e = \sinh K_e$, we obtain

$$(\sigma_B)_G(\sigma_C)_G = \{c_e(\sigma_B)_{G'} + s_e(\sigma_B\sigma^e)_{G'}\}\{c_e(\sigma_C)_{G'} + s_e(\sigma_C\sigma^e)_{G'}\} \quad , \tag{8}$$

where $(\sigma_B)_{G'}$ etc. are correlation functions of the Ising model $(G', \mathcal{J}, K')$ obtained from $(G, \mathcal{J}, K)$ by deleting e from G and restricting K to $E(G)\setminus e$ (which is equivalent to putting $K_e = 0$ in $H_{G,K}(\sigma)$).

Using $c_e^2 = 1+s_e^2$ we rewrite eq. (8) as

$$\begin{aligned}(\sigma_B)_G(\sigma_C)_G = {} & (\sigma_B)_{G'}(\sigma_C)_{G'} + c_e s_e\{(\sigma_B\sigma^e)_{G'}(\sigma_C)_{G'} + (\sigma_B)_{G'}(\sigma_C\sigma^e)_{G'}\} + \\ & + s_e^2\{(\sigma_B)_{G'}(\sigma_C)_{G'} + (\sigma_B\sigma^e)_{G'}(\sigma_C\sigma^e)_{G'}\} \quad .\end{aligned} \tag{8'}$$

We now repeat this process for all other edges of G. To write the result in a compact form we associate with each term in the resulting expression a function

$$\theta : E(G) \to \{0,1,2\} \quad , \tag{9}$$

where $\theta_e = 0,1,2$ labels the first, second and third term in the right-hand side of eq. (8'), respectively, and the edge sets $L_\theta = \{e\in E(G)|\theta_e = 1\}$ and $M_\theta = \{e\in E(G)|\theta_e = 2\}$. The set of all functions θ is denoted by Θ . We further define

$$g(\theta) = \prod_{e\in E(G)} g_e(\theta_e) \quad , \tag{10}$$

where $g_e(0) = 1$, $g_e(1) = c_e s_e$, $g_e(2) = s_e^2$, and

$$\Gamma_\theta(B,C) = \sum_{X\subset L_\theta}\sum_{Y\subset M_\theta} (\sigma_B\sigma^X\sigma^Y)_{G_\emptyset}(\sigma_C\sigma^{L_\theta\setminus X}\sigma^Y)_{G_\emptyset} \quad , \tag{11}$$

in which $G_\emptyset$ is the graph defined by $V(G_\emptyset) = V(G)$, $E(G_\emptyset) = \emptyset$.

The result of applying eq. (7) to all edges of G is

$$(\sigma_B)_G(\sigma_C)_G = \sum_{\theta\in\Theta} g(\theta)\Gamma_\theta(B,C) \ . \tag{12}$$

A convenient way to characterize the various terms in $\Gamma_\theta(B,C)$ is the following. We define for each θ a graph G_θ by $V(G_\theta) = V(G)$, $E(G_\theta) = L_\theta \cup M_\theta$; i.e. G_θ is obtained from G by deleting all edges e with $\theta_e = 0$. For each pair of sets $X\subset L_\theta$, $Y\subset M_\theta$ we define a function

$$\phi : E(G_\theta) \to Z_2 \times Z_2 \tag{13}$$

where Z_2 is the field of integers modulo 2, by

$$\phi_e = \begin{cases} (1,0) & \text{if } e\in X \\ (0,1) & \text{if } e\in L_\theta\setminus X \\ (1,1) & \text{if } e\in Y \\ (0,0) & \text{if } e\in M_\theta\setminus Y \end{cases} \tag{14}$$

Conversely, every function of the type (13) with the property

$$\phi_e = (\alpha_e,\beta_e) \text{ with } \begin{cases} \alpha_e + \beta_e = 1 & \text{if } e\in L_\theta \\ \alpha_e + \beta_e = 0 & \text{if } e\in M_\theta \end{cases} \tag{15}$$

defines two sets $X\subset L_\theta$, $Y\subset M_\theta$, and hence a term in $\Gamma_\theta(B,C)$, by

$$\begin{aligned} X &= \{e\in L_\theta \mid \phi_e = (1,0)\} \ , \\ Y &= \{e\in M_\theta \mid \phi_e = (1,1)\} \ . \end{aligned} \tag{16}$$

We remark that in the terminology of algebraic graph theory ϕ is a 1-chain of G_θ over $Z_2\times Z_2$ (cf. ref. 3).

Let us now analyse the various terms of $\Gamma_\theta(B,C)$. Since $(\sigma_A)_{G_\emptyset} = \Sigma_\sigma\sigma_A$ and $\Sigma_{\sigma_v=\pm1}\sigma_v = 0$, we have

$$(\sigma_A)_{G_\emptyset} = 2^{|V(G)|}\delta_{A,\emptyset} \text{ for all } A\subset V(G). \tag{17}$$

Hence, the only nonvanishing terms in the right-hand side of eq. (11) are those for which

$$\sigma_B \sigma^X \sigma^Y = \sigma_\emptyset , \qquad \sigma_C \sigma^{L_\theta \setminus X} \sigma^Y = \sigma_\emptyset . \tag{18}$$

The first condition requires that every vertex in B is incident with an odd number of edges of $X \cup Y$ and every vertex in $V(G) \setminus B$ with an even number of edges of $X \cup Y$; the second condition is analogous.

These conditions can easily be translated into a condition on the function ϕ if we introduce the functions $\partial\phi$: $V(G) \to Z_2 \times Z_2$ defined by

$$(\partial\phi)_v = \sum_{e \text{ inc } v} \phi_e , \tag{19}$$

where the sum (taken in $Z_2 \times Z_2$) is over all edges in G_θ incident with the vertex v, and $\chi(B,C) : V(G) \to Z_2 \times Z_2$ defined by

$$\chi_v(B,C) = \begin{cases} (1,0) & \text{if } v \in B \\ (0,1) & \text{if } v \in C \\ (0,0) & \text{if } v \notin B \cup C . \end{cases} \tag{20}$$

$\partial\phi$ and $\chi(B,C)$ are 0-chains of G_θ over $Z_2 \times Z_2$, $\partial\phi$ is called the boundary of ϕ. In terms of these functions, eq. (18) can be written as

$$\partial\phi = \chi(B,C) . \tag{21}$$

The set of all functions ϕ which, for a given choice of θ, B and C ($B \cap C = \emptyset$), satisfy eqs. (15) and (21) will be denoted by $\Phi_\theta(B,C)$. Since every nonvanishing term in the right-hand side of eq. (11) is equal to $2^{2|V(G)|}$ we have

$$\Gamma_\theta(B,C) = 2^{2|V(G)|} |\Phi_\theta(B,C)| . \tag{22}$$

We now proceed to derive a few properties of the sets $\Phi_\theta(B,C)$.

We call a set S of edges of a graph G a (generalized) <u>cycle</u> of G if each vertex of G is incident with an even number of edges of S. The total number of cycles of G_θ, including the empty cycle $\emptyset$, will be denoted by γ_θ.

<u>Lemma 1.</u> If, for $\theta \in \Theta$ and for disjoint sets B, $C \subset V(G)$, $\Phi_\theta(B,C)$ is not empty, then $|\Phi_\theta(B,C)| = \gamma_\theta$.

<u>Proof.</u> Let $\phi \in \Phi_\theta(B,C)$, $S \subset E(G_\theta)$, and define $\phi' = \phi + \psi$, where

$$\psi_e = \begin{cases} (1,1) & \text{if } e \in S \\ (0,0) & \text{if } e \notin S \,. \end{cases} \tag{23}$$

Then it is easily verified that $\phi' \in \Phi_\theta(B,C)$ if and only if S is a cycle of G_θ *). Since this establishes a one-to-one correspondence between the functions ϕ' in $\Phi_\theta(B,C)$ and the cycles S of G_θ, the lemma follows. ■

For any graph G and any set $A \subset V(G)$ let $\pi(A,G)$ denote the partition of A induced by G, i.e. the partition in which two vertices of A are in the same block if and only if they are in the same connected component of G. If H is a spanning subgraph of G, i.e. if $V(H) = V(G)$, $E(H) \subset E(G)$, the partition $\pi(A,H)$ is a refinement of $\pi(A,G)$, i.e., the blocks of $\pi(A,H)$ are subsets of those of $\pi(A,G)$. If H is a subgraph of G such that $\pi(A,H) = \pi(A,G)$ and no proper subgraph of H has this property, we call H a skeleton graph associated with the partition $\pi(A,G)$. Evidently, a skeleton graph is a forest, i.e., it contains no circuits. It is easily seen that for each partition $\pi(A,G)$ with $A \neq \emptyset$ there is at least one skeleton graph.

The set of all partitions of A will be denoted by Π_A, the set of all even partitions of A (i.e. partitions of A into even subsets) by Π_A^e, and the set of all even partitions of A induced by spanning subgraphs of G by $\Pi_A^e(G)$. Obviously, $\Pi_\emptyset = \Pi_\emptyset^e = \Pi_\emptyset^e(G) = \emptyset$.

Lemma 2. If, for $\theta \in \Theta$ and for disjoint sets $B,C \subset V(G)$, $\Phi_\theta(B,C)$ is not empty, then $\pi(B,G_\theta)$ and $\pi(C,G_\theta)$ are even partitions.

Proof: Let H be any connected component of G_θ, $B_H = B \cap V(H)$, and $C_H = C \cap V(H)$. Consider a function $\phi \in \Phi_\theta(B,C)$. By eq. (21) we have

$$\sum_{v \in V(H)} (\partial\phi)_v = \sum_{v \in V(H)} \chi_v(B,C) = \sum_{v \in B_H} (1,0) + \sum_{v \in C_H} (0,1) \,.$$

On the other hand, we have, by the definition of $\partial\phi$,

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*) If S is a cycle of G_θ, ψ is a cycle vector of G_θ 3). If we denote the number of independent cycle vectors (the cycle rank or cyclomatic number) of a graph G by $c(G)$, we have $\gamma_\theta = 2^{c(G_\theta)}$. It is well known that $c(G) = |E(G)| - |V(G)|$ + number of connected components of G.

$$\sum_{v \in V(H)} (\partial\phi)_v = \sum_{v \in V(H)} \sum_{e \text{ inc } v} \phi_e = (0,0) ,$$

since every edge in H is counted twice in the double sum. It follows that $|B_H|$ and $|C_H|$ are even. Since H is arbitrary, the lemma follows. ■

Lemma 3. If, for $\theta \in \Theta$ and for disjoint sets $B, C \subset V(G)$, $\Phi_\theta(\emptyset, B \cup C)$ is not empty and $\pi(B, G_\theta)$ is an even partition, then $\Phi_\theta(B,C)$ is not empty.

Proof. Suppose that $\Phi_\theta(\emptyset, B \cup C)$ is not empty and that $\pi(B, G_\theta)$ is an even partition.

Consider a skeleton graph H associated with $\pi(B, G_\theta)$. The set E(H) is the disjoint union of two uniquely determined sets E_1 and E_2 with the property that if an edge $e \in E_1$ $(e \in E_2)$ is deleted from H the connected component of H containing e breaks up into two components, each one containing an odd (even) number of vertices of B. Each vertex in B is incident with an odd number of edges of E_1, each vertex in $V(H) \setminus B$ is incident with an even number of edges of E_1. We now delete the edges of E_2 from H; in the resulting graph H' the vertices of B are the only vertices of odd valency.

For $v \in B$ we have $\chi_v(\emptyset, B \cup C) = (0,1)$, $\chi_v(B,C) = (1,0)$ and hence

$$\chi_v(B,C) = \chi_v(\emptyset, B \cup C) + (1,1).$$

Consider a function $\phi \in \Phi_\theta(\emptyset, B \cup C)$. We define a function $\phi' \colon E(G_\theta) \to Z_2 \times Z_2$ by

$$\phi'_e = \phi_e + (1,1) \text{ if } e \in E_1 ,$$

$$\phi'_e = \phi_e \qquad \text{if } e \in E(G_\theta) \setminus E_1 .$$

Obviously ϕ' satisfies eq. (15); by the above mentioned property of H' it also satisfies eq. (21). Hence $\phi' \in \Phi_\theta(B,C)$, which proves the lemma. ■

Let, for $A \subset V(G)$ and $\pi \in \Pi^e_A(G)$, $\Theta_\pi(A)$ denote the set of all functions $\theta \in \Theta$ such that $\pi(A,G_\theta) = \pi$ and $\Phi_\theta(\emptyset,A)$ is not empty.

Lemma 4. For any $A \subset V(G)$, and any $\pi \in \Pi^e_A(G)$, $\Theta_\pi(A)$ is not empty.

Proof. If $\Pi^e_A(G)$ is empty, the lemma is trivial, so let $\Pi^e_A(G)$ be non-empty, and $\pi \in \Pi^e_A(G)$. Then, by the definition of $\Pi^e_A(G)$, there is a spanning subgraph G' of G such that $\pi(A,G') = \pi$. Let H be a skeleton graph associated with $\pi(A,G')$, and E_1 and E_2 the corresponding edge sets defined in the proof of Lemma 3.

We now define for each $e \in E(G)$

$$\theta_e = \begin{cases} 0 \text{ if } e \notin E_1 \cup E_2 \\ 1 \text{ if } e \in E_1 \\ 2 \text{ if } e \in E_2 , \end{cases} \tag{24}$$

and for each $e \in E_1 \cup E_2$

$$\phi_e = \begin{cases} (0,1) \text{ if } e \in E_1 \\ (0,0) \text{ if } e \in E_2 . \end{cases} \tag{25}$$

The functions $\theta : E(G) \to \{0,1,2\}$ and $\phi : E(G_\theta) \to Z_2 \times Z_2$ thus defined have the following properties: (a) since $G_\theta = H$, we have $\pi(A,G_\theta) = \pi(A,H) = \pi(A,G') = \pi$; (b) by its definition, ϕ satisfies eq. (15); (c) since the vertices of A are the only vertices of $G_\theta = H$ incident with an odd number of edges of E_1 (cf. the proof of Lemma 3), ϕ satisfies eq. (21) with $B = \emptyset$, $C = A$. It follows from (b) and (c) that $\phi \in \Phi_\theta(\emptyset,A)$, i.e. $\Phi_\theta(\emptyset,A)$ is not empty, and from this fact together with (a) that $\theta \in \Theta_\pi(a)$, which proves the lemma. ■

It follows from Lemmas 1, 2 and 3 that

$$|\Phi_\theta(B,C)| = \begin{cases} |\Phi_\theta(\emptyset, B \cup C)| \text{ if } \pi(B,G_\theta) \text{ is an even partition} \\ 0 \text{ otherwise.} \end{cases} \tag{26}$$

From eqs. (12), (22) and (26) we conclude that

$$(\sigma_B)_G(\sigma_C)_G = \sum_{\pi \in \Pi^e_{B \cup C}(G)} \left\{ \sum_{\theta \in \Theta_\pi(B \cup C)} 2^{2|V(G)|} \gamma_\theta g(\theta) \right\} \eta_\pi(B), \tag{27}$$

where for any set $B \subset A$ and any partition $\pi \in \Pi_A$

$$\eta_\pi(B) = \begin{cases} 1 \text{ if the number of elements of B in every block of } \pi \text{ is even} \\ 0 \text{ otherwise.} \end{cases}$$

We observe that the factor in front of $\eta_\pi(B)$ in eq. (27) depends only on $B \cup C$, not on B and C separately. This implies that the function $\Lambda_A(G,K)$ introduced in eq. (6) can be written in the form

$$\Lambda_A(G,K) = \sum_{\pi\in\Pi^e_A(G)} \left\{ \sum_{\theta\in\Theta_\pi(A)} 2^{2|V(G)|}\gamma_\theta g(\theta) \right\} \left\{ \sum^e_{B\subset A} \lambda_B \eta_\pi(B) \right\} . \tag{28}$$

We now have the following two theorems:

Theorem 1. If A is an even set of vertices of a finite graph G, and $\{\lambda_B\}^e_{B\subset A}$ a set of complex numbers defined for all even sets $B\subset A$, with $\lambda_B = \lambda_{A\setminus B}$ for all B, then

$$\sum^e_{B\subset A} \lambda_B \langle\sigma_B\rangle_G \langle\sigma_{A\setminus B}\rangle_G = 0 \tag{29}$$

for every Ising model on G if and only if

$$\sum^e_{B\subset A} \eta_\pi(B)\lambda_B = 0 \tag{30}$$

for every partition $\pi\in\Pi^e_A(G)$.

Proof. If eq. (30) holds for all $\pi\in\Pi^e_A(G)$, then by eq. (28), $\Lambda_A(G,K)$ vanishes (i.e., eq. (29) holds) for all $K\in\mathcal{K}$. Conversely, suppose that $\Lambda_A(G,K)$ vanishes for all $K\in\mathcal{K}$. Let π be an arbitrary element of $\Pi^e_A(G)$, H a skeleton graph associated with π, E_1 and E_2 the edge sets and θ and ϕ the associated functions defined in the proofs of Lemmas 3 and 4. It was shown in these proofs that θ and ϕ satisfy eqs. (15) and (21). On the other hand, if ϕ' is a function on E(H) satisfying eq. (21), the set of edges e with $\phi'_e \neq \phi_e$ must be a cycle of H. Since the only cycle of H is the empty cycle, we have $\phi' = \phi$, i.e., ϕ is the only function with domain E(H) which satisfies eq. (21). Eq. (15) then shows that the function θ defined by eq. (24) is the only element of $\Theta_\pi(A)$ such that $\theta_e = 0$ for $e\in E(G)\setminus E(H)$, i.e. if $\theta'\in\Theta_\pi(A)$ and $\theta'\neq\theta$ then $\theta_e \neq 0$ for some $e\in E(G)\setminus E(H)$.

Let now K_0 be a positive real constant and K the function on E(G) defined by

$$\begin{aligned} K_e &= K_0 \quad \text{for } e\in E(H) \\ K_e &= 0 \quad \text{for } e\in E(G)\setminus E(H). \end{aligned} \tag{31}$$

By eq. (10) we have in this case

$$g(\theta) = (c_0 s_0)^{|E_1|}(s_0^2)^{|E_2|}, \tag{32}$$

where $c_0 = \cosh K_0$, $s_0 = \sinh K_0$. On the other hand, we have $\gamma_\theta = 1$. For any other function $\theta' \in \Theta_\pi(A)$, $g(\theta')$ contains at least one factor $c_e s_e$ or s_e^2 for $e \in E(G)\setminus E(H)$, and hence vanishes. The same applies to any other $\theta' \in \Theta$ such that $E(G_{\theta'})\setminus E(G_\theta) \neq \emptyset$, in particular to any $\theta' \in \Theta_{\pi'}(A)$ where π' is not a refinement of π.

Consider now a partition $\pi' \in \Pi_A^e(G)$ where $\pi'(\neq\pi)$ is a refinement of π, and a function $\theta' \in \Theta_{\pi'}(A)$ such that $E(G_{\theta'}) \subset E(G_\theta)$. From the definition of the set E_1 and the fact that π' is even it follows that θ' is obtained from θ by replacing the value 2 by the value 0 for one or more properly chosen edges of E_2. Hence,

$$g(\theta') = (c_0 s_0)^{|E_1|} (s_0^2)^{|E_2|-r} \quad \text{with } r \geq 1 . \tag{33}$$

It follows that $2^{-2|V(G)|}(c_0 s_0)^{-|E_1|}\Lambda_A(G,K)$ is a polynomial in s_0^2 of degree $|E_2|$, in which the coefficient of the term of highest degree is $\Sigma^e_{B\subset A}\, \eta_\pi(B)\lambda_B$.

Since $\Lambda_A(G,K) = 0$ for all values of the constant K_0, we must have

$$\sum_{B\subset A}^{e} \eta_\pi(B)\lambda_B = 0,$$

which completes the proof of the theorem. ■

Theorem 2. If A is an even set of vertices of a finite graph G, and $\{\lambda_B\}^e_{B\subset A}$ a set of real numbers defined for all even sets $B\subset A$, with $\lambda_B = \lambda_{A\setminus B}$ for all B, then

$$\sum_{B\subset A}^{e} \lambda_B \langle\sigma_B\rangle_G \langle\sigma_{A\setminus B}\rangle_G \geq 0 \tag{34}$$

for every ferromagnetic Ising model on G if and only if

$$\sum_{B\subset A}^{e} \eta_\pi(B)\lambda_B \geq 0 \tag{35}$$

for every partition $\pi \in \Pi_A^e(G)$.

Proof. For $K \in \mathcal{F}$ we have $g(\theta) \geq 0$ for all $\theta \in \Theta$. Therefore, if eq. (35) holds for all $\pi \in \Pi_A^e(G)$, then, by eq. (28), $\Lambda_A(G,K)$ is nonnegative (i.e., eq. (34) holds) for all $K \in \mathcal{F}$. Conversely, suppose $\Lambda_A(G,K) \geq 0$ for all $K \in \mathcal{F}$. Let π be an arbitrary element of $\Pi_A^e(G)$. Following the lines of the proof of Theorem 1 we construct an interaction function $K \in \mathcal{F}$ for which $2^{-2|V(G)|}(c_0 s_0)^{-|E_1|}\Lambda_A(G,K)$ is a polynomial in s_0^2. Since this polynomial is assumed to be nonnegative for all nonnegative values of K_0, the coefficient of the term of highest degree must be positive, i.e.

$$\sum_{B\subset A}^{e} \eta_\pi(B)\lambda_B \geq 0 \quad ,$$

which completes the proof of the theorem. ■

<u>Corollary.</u> If A is an even set of vertices of a finite graph G, and $\{\lambda_B\}_{B\subset A}^{e}$ a set as defined in Theorem 2, then

$$\sum_{B\subset A}^{e} \lambda_B(\sigma_B)_G(\sigma_{A\setminus B})_G > 0 \qquad (36)$$

for every strictly ferromagnetic Ising model on G if and only if eq. (35) holds for all $\pi \in \Pi_A^e(G)$ and in addition

$$\sum_{B\subset A}^{e} \eta_\pi(B)\lambda_B > 0 \qquad (37)$$

for at least one $\pi \in \Pi_A^e(G)$.

<u>Proof.</u> The corollary follows immediately from Theorems 1 and 2 together with the fact that $\Lambda_A(G,K)$ is an (entire) analytic function of all coupling parameters. ■

3. <u>Generalization of Theorems 1 and 2.</u>

Theorems 1 and 2 can be extended to identities and inequalities for correlation functions on a vertex set A in which the products $(\sigma_B)(\sigma_C)$ refer to subsets B and C of A which satisfy the condition that their <u>symmetric difference</u> (to be denoted by BC) is a given set $D\subset A$; the case discussed thus far, where B and C are disjoint and their union is A, corresponds to the choice D=A.

It is possible to derive these generalizations by a proper extension of the analysis of the preceeding section. However, for reasons of transparency, and in order to show that the general case is, in a certain sense, already included in the special case D=A, we shall present another derivation, starting from the results of section 2.

Let A be a (not necessarily even) subset of a graph G, and D an even subset of A. By $\Pi_A(D)$ we denote the set of all partitions of A in which the number of vertices of D in each block is even, and by $\Pi_A(D,G)$ the set of those partitions in $\Pi_A(D)$ that are induced by spanning subgraphs of G.

<u>Theorem 1</u>*. If A is an arbitrary set of vertices of a finite graph G, D an even subset of A, and $\{\lambda_B\}^e_{B\subset A}$ a set of complex numbers defined for all even sets $B\subset A$, with $\lambda_B = \lambda_{BD}$ for all B, then

$$\sum^e_{B\subset A} \lambda_B(\sigma_B)_G(\sigma_{BD})_G = 0 \tag{38}$$

for every Ising model on G if and only if

$$\sum^e_{B\subset A} \eta_\pi(B)\lambda_B = 0 \tag{39}$$

for all partitions $\pi\in\Pi_A(D,G)$.

<u>Proof.</u> Let $Q = A\backslash D$, and Q' a set of vertices not in V(G) which are in a one-to-one correspondence with the vertices of Q; the vertex in Q' corresponding to $v\in Q$ will be denoted by v'. Let G^* be the graph defined by $V(G^*) = V(G)\cup Q'$, $E(G^*) = E(G)\cup\{\{v,v'\}|v\in Q\}$, and let $A^* = A\cup Q'$. We extend the interaction function K to a function K^* on $E(G^*)$ by defining

$$K^*_e = K_e \quad \text{if } e\in E(G)$$

$$K^*_e = K_0 > 0 \text{ if } e\in E(G^*)\backslash E(G) .$$

It is readily verified that for any set of complex numbers $\{\lambda_B\}^e_{B\subset A}$ we have,

$$\sum^e_{B\subset A} \lambda_B(\sigma_B)_{G^*,K^*}(\sigma_{A^*\backslash B})_{G^*,K^*} = (4 \cosh K_0 \sinh K_0)^{|Q|} \sum^e_{B\subset A} \lambda_B(\sigma_B)_{G,K}(\sigma_{BD})_{G,K}. \tag{40}$$

It follows that the sum in the right-hand side of eq. (40) vanishes if and only if the left-hand side is zero, i.e., by Theorem 1, if and only if $\Sigma^e_{B\subset A}\eta_{\pi^*}(B)\lambda_B = 0$ for every $\pi^*\in\Pi^e_{A^*}(G^*)$; strictly speaking this requires the symmetrization of the set $\{\lambda_B\}^e_{B\subset A}$ with respect to A^*.

Consider a partition $\pi^*\in\Pi^e_{A^*}(G^*)$. Since π^* is an even partition no vertex $v'\in Q'$ can form a block by itself. Therefore every vertex $v'\in Q'$ is in the same block as the corresponding vertex $v\in Q$; hence, the number of elements of D in every block is even. Let now π be the partition of A obtained from π^* by deleting all vertices of Q'. Obviously, $\pi\in\Pi_A(D,G)$. Conversely, every $\pi\in\Pi_A(D,G)$ can be supplemented to a partition $\pi^*\in\Pi^e_{A^*}(G^*)$ by putting each vertex $v'\in Q'$ into the same block as the corresponding vertex $v\in Q$, and hence the theorem follows. ■

The corresponding generalization of Theorem 2 (to be referred to as Theorem 2^*) is obvious and will not be discussed explicitly. Eq. (38) and the analogous generalization of eq. (34) will again be called a Λ-identity and a Λ-inequality, respectively.

Theorems 1^* and 2^* remain valid if the unnormalized correlation functions are replaced by the corresponding normalized correlation functions, provided we restrict ourselves to Ising models with $Z \neq 0$; in Theorem 2^* this condition is always satisfied. The resulting identities and inequalities remain valid if for some edge $e = \{u,v\}$ we take the limit $K_e \to \infty$, i.e. if the edge e is contracted and the spin variables σ_u and σ_v are identified.

Using eqs. (28) and (40) and the relations

$$\frac{\partial}{\partial K_e}(\sigma_A) = (\sigma_A \sigma^e), \quad \frac{\partial g_e(1)}{\partial K_e} = 1 + 2g_e(2), \quad \frac{\partial g_e(2)}{\partial K_e} = 2g_e(1) ,$$

we also find that taking the derivative with respect to any coupling parameter K_e ($e \in E(G)$) in a Λ-identity (Λ-inequality) results in a Λ-identity (Λ-inequality). For Λ-inequalities this implies, in the terminology of Newman 5), that they apply strongly.

4. Some properties of sets of Λ-identities.

It follows from Theorem 1 that for a given graph G and a given set $A \subset V(G)$ the number of linearly independent Λ-identities, with $D=A$, for spin correlation functions on A, to be denoted by $L_A(G)$, is equal to the number of linearly independent solutions of the set of linear equations (30). The latter number depends on G and A only through the set of partitions $\Pi^e_A(G)$. In general, the larger the set $\Pi^e_A(G)$, and hence the number of conditions on λ, the smaller the number of linearly independent solutions of eq. (30), and hence $L_A(G)$. In particular, we have the following theorem.

Theorem 3. If G and G' are finite graphs, and A is an even subset of V(G) and V(G'), then

(i) $L_A(G) = 0$ if $\Pi^e_A(G) = \Pi^e_A$ (41a)

(ii) $L_A(G') \leq L_A(G)$ if $\Pi^e_A(G) \subset \Pi^e_A(G')$ (41b)

iii) $L_A(G) = 2^{|A|-2}$ if $\Pi^e_A(G) = \emptyset$. (41c)

Proof. (i) If $\Pi_A^e(G)=\Pi_A^e$, eq. (30) is required to hold for all even partitions π of A. We shall show that already the set of equations obtained by restricting π to the partitions of A into one or two even subsets has no nontrivial solution.

If P_π and $A\backslash P_\pi$ are the subsets into which A is partitioned by π(where for convenience a one-block partition is considered as a partition of A into $\emptyset$ and A), then for any even set $B \subset A$

$$\eta_\pi(B) = \begin{cases} 1 \text{ if } |B \cap P_\pi| \text{ is even} \\ 0 \text{ if } |B \cap P_\pi| \text{ is odd} , \end{cases}$$

i.e.,

$$\eta_\pi(B) = \tfrac{1}{2}(1 + (-1)^{|B \cap P_\pi|}) .$$

The set of eqs. (30), with the restriction imposed on π, can therefore be written as

$$\sum_{B \subset A}^{e} (1+(-1)^{|B \cap P|})\lambda_B = 0 \tag{42}$$

for all even $P \subset A$; observe that the equations with P and $A\backslash P$ are identical. Multiplying eq. (42) with $(-1)^{|B' \cap P|}$, with $B' \subset A$, $|B'|$ even, summing over P and using the relation

$$\sum_{P \subset A}^{e} (-1)^{|U \cap P|} = 2^{|A|-1}(\delta_{U,\emptyset} + \delta_{U,A})$$

for U=B' and U=BB', we obtain

$$2^{|A|-1} [(\sum_{B \subset A}^{e} \lambda_B)(\delta_{B',\emptyset} + \delta_{B',A}) + (\lambda_{B'} + \lambda_{A\backslash B'})] = 0 \cdot$$

It follows that $\lambda_{B'} + \lambda_{A\backslash B'} = 0$ for all even sets $B' \subset A$. Since we have taken $\lambda_{B'} = \lambda_{A\backslash B'}$, it follows that $\lambda_{B'} = 0$ for all even sets $B' \subset A$. Hence, $L_A(G) = 0$.

(ii) If $\Pi_A^e(G) \subset \Pi_A^e(G')$, the set of linear eqs. (30) for the graph G is a (proper or improper) subset of that of G', and hence the set of identities of the type (29) valid in G' is a subset of the set of identities valid in G. Statement (ii) follows.

(iii) If $\Pi_A^e(G) = \emptyset$ there are no linear equations for the coefficients (except the condition $\lambda_B = \lambda_{A\backslash B}$ for all B), and we have $L_A(G) = \tfrac{1}{2}|\{B|B \subset A, |B| \text{ even }\}| = 2^{|A|-2}$. The simplest set of independent identities then consists of the equations $(\sigma_B)_G(\sigma_{A\backslash B})_G = 0$ for all $K \in \mathcal{K}$

($B\subset A$, $|B|$ even), the validity of which is trivial. ■

Examples of the three cases considered in Theorem 3 are:

(i) G is a complete graph, i.e. $G=(V,E_c)$, with $E_c=\{\{v,v'\}|v, v'\in V, v\neq v'\}$

(ii) G is a spanning subgraph of G'

(iii) G is an empty graph, i.e. $G = (V,\emptyset)$.

If $G = (V,E)$ with $E \neq E_c$ there is at least one set $A\subset V$ for which $L_A(G) > 0$, viz. $A=V$. This is expressed in the following lemma.

Lemma 5. If $G = (V,E)$ with $E = E_c \setminus e$ ($e\in E_c$) we have the following identity

$$\sum^{e}_{B\subset V} (-1)^{|B\cap\{v_1,v_2\}|} (\sigma_B)_G(\sigma_{V\setminus B})_G = 0 \qquad (43)$$

where v_1 and v_2 are the vertices incident with e.

Proof. The equations (30) read in this case

$$\sum^{e}_{B\subset V} \eta_\pi(B)(-1)^{|B\cap\{v_1,v_2\}|} = 0$$

or

$$\sum^{e}_{R\subset V''} \eta_\pi(\{v_1,v_2\}\cup R) - \sum^{o}_{R\subset V''} \eta_\pi(\{v_1\}\cup R) = 0 \qquad (44)$$

where $V'' = V\setminus\{v_1,v_2\}$, and $\Sigma^{o}_{R\subset V''}$ denotes summation over all odd subsets of V''.

Consider first a partition π in which v_1 and v_2 are in the same block. By the structure of G, partitions in which v_1 and v_2 form a block by themselves are not contained in $\Pi^{e}_{V}(G)$. Therefore, if we denote the blocks of π by $U_1,U_2,\ldots,U_r$ (where U_1 is the block containing v_1 and v_2), $U_1\cap V''$ is not empty. The first sum on the left-hand side of (44) is equal to the number of even sets $R\subset V''$ such that $|R\cap U_i|$ is even for $1\leq i\leq r$, the second sum is equal to the number of odd sets $R\subset V''$ such that $|R\cap U_i|$ is even for $2\leq i\leq r$ and odd for $i=1$. Since $U_1\cap V''$ is not empty, the number of even subsets of $U_1\cap V''$ equals the number of odd subsets. Hence, the two sums in the left-hand side of (44) cancel. A similar argument applies to partitions in which v_1 and v_2 are in different blocks. ■

The generalization of Theorem 3 to the case that A is an arbitrary vertex set of G and D an even subset of A is straightforward.

5. Examples of Λ-identities.

1) As a first example we discuss in detail the case $|A| = 4$, $D = A$. Let $A = \{v_1, v_2, v_3, v_4\}$. We have, with $\lambda_{ij} = \lambda_{\{i,j\}}$ and $\sigma_i = \sigma_{v_i}$,

$$\Lambda_A(G,K) = 2[\lambda_\emptyset(1)(\sigma_1\sigma_2\sigma_3\sigma_4) + \lambda_{12}(\sigma_1\sigma_2)(\sigma_3\sigma_4) + \\ + \lambda_{13}(\sigma_1\sigma_3)(\sigma_2\sigma_4) + \lambda_{14}(\sigma_1\sigma_4)(\sigma_2\sigma_3)] . \tag{45}$$

The set $\Pi_A^e(G)$ is a subset of the set $\Pi_A^e = \{\pi_i | 0 \leq i \leq 3\}$ where, in an obvious notation,

$\pi_0 = (1234)$, $\pi_1 = (12|34)$, $\pi_2 = (13|24)$, $\pi_3 = (14|23)$. If we define:

$$k_A(\pi) = \tfrac{1}{2} \sum_{B\subset A}^{e} \lambda_B \eta_\pi(B), \tag{46}$$

we find,

$$k_A(\pi_0) = \lambda_\emptyset + \lambda_{12} + \lambda_{13} + \lambda_{14} , \tag{47a}$$

$$k_A(\pi_1) = \lambda_\emptyset + \lambda_{12} , \tag{47b}$$

$$k_A(\pi_2) = \lambda_\emptyset + \lambda_{13} , \tag{47c}$$

$$k_A(\pi_3) = \lambda_\emptyset + \lambda_{14} , \tag{47d}$$

a) If $\Pi_A^e(G) = \Pi_A^e$, then according to Theorem 3 there are no Λ-identities for A, as can easily be checked by putting all $k_A(\pi_i)$, $0 \leq i \leq 3$, equal to zero. As remarked above, this case is realized e.g. if G is a complete graph, with $A \subset V(G)$.

b) We now consider the cases where $|\Pi_A^e(G)| = 3$. One easily verifies that the only possibilities are $\Pi_A^e(G) = \Pi_A^e \backslash \{\pi_i\}$ with $1 \leq i \leq 3$, since π_0 is in $\Pi_A^e(G)$ whenever two of the three partitions π_i $(1 \leq i \leq 3)$ are in $\Pi_A^e(G)$. Without lack of generality we assume $\Pi_A^e(G) = \{\pi_0, \pi_1, \pi_3\}$. The condition for the existence of an identity now consists of the equations $k_A(\pi_i) = 0$ $(i=0,1,3)$, which have as the only solution $\lambda_\emptyset = -\lambda_{12} = \lambda_{13} = -\lambda_{14}$. The resulting identity reads

$$(1)(\sigma_1\sigma_2\sigma_3\sigma_4) - (\sigma_1\sigma_2)(\sigma_3\sigma_4) + (\sigma_1\sigma_3)(\sigma_2\sigma_4) - (\sigma_1\sigma_4)(\sigma_2\sigma_3) = 0 . \tag{48}$$

The graphs G for which this identity holds are characterized by the fact that the partition π_2 of the set A is not induced by any spanning subgraph of G. This implies that every chain between v_1 and v_3 <u>separates</u> v_2 from v_4

(i.e. has a vertex in common with every chain between v_2 and v_4).

The identity (48) is a special case of a general class of identities which formed the subject of I, and to which we shall return later on in this section (example 4).

c) Consider now the cases where $|\Pi^e_A(G)| = 2$. The only possibility is that where $\Pi^e_A(G) = \{\pi_0, \pi_i\}$ for some i $(1 \leq i \leq 3)$. Suppose i=1. From the equations $k_A(\pi_0) = k_A(\pi_1) = 0$ we find $\lambda_\emptyset = -\lambda_{12}$, $\lambda_{13} = -\lambda_{14}$, and hence

$$\lambda_\emptyset[(1)(\sigma_1\sigma_2\sigma_3\sigma_4) - (\sigma_1\sigma_2)(\sigma_3\sigma_4)] + \lambda_{13}[(\sigma_1\sigma_3)(\sigma_2\sigma_4) - (\sigma_1\sigma_4)(\sigma_2\sigma_3)] = 0 \tag{49}$$

for any $\lambda_\emptyset$ and λ_{13}. This implies

$$\begin{aligned} (1)(\sigma_1\sigma_2\sigma_3\sigma_4) &= (\sigma_1\sigma_2)(\sigma_3\sigma_4) \ , \\ (\sigma_1\sigma_3)(\sigma_2\sigma_4) &= (\sigma_1\sigma_4)(\sigma_2\sigma_3). \end{aligned} \tag{50}$$

This case applies when G contains a vertex v (cut vertex or articulation vertex) which separates v_1 and v_2 from v_3 and v_4 (i.e., which is contained in every chain between v_1 and v_3 etc.); v need not be distinct from v_1, v_2, v_3 or v_4. This can be shown by introducing two new vertices u and u', and four new edges $\{u,v_1\}$, $\{u,v_2\}$, $\{u',v_3\}$, $\{u',v_4\}$ and applying Menger's theorem to the vertices u and u' (cf. ref. 2, p. 129).

The relations (50) are trivial in that they follow directly from the factorization of correlation functions in a graph with a cut vertex.

d) If $|\Pi^e_A(G)| = 1$ we have either $\Pi^e_A(G) = \{\pi_0\}$, or $\Pi^e_A(G) = \{\pi_i\}$ with i=1,2 or 3; suppose i=1. In both cases there are three linearly independent Λ-identities. In the former case we find

$$(1)(\sigma_1\sigma_2\sigma_3\sigma_4) = (\sigma_1\sigma_2)(\sigma_3\sigma_4) = (\sigma_1\sigma_3)(\sigma_2\sigma_4) = (\sigma_1\sigma_4)(\sigma_2\sigma_3); \tag{51}$$

this case applies when G contains a cut vertex v (not necessarily distinct from v_1, v_2, v_3 or v_4) which separates every v_i from every v_j $(1 \leq i,j \leq 4, i \neq j)$.

If π_1 is the only partition in $\Pi^e_A(G)$, we find

$$\begin{aligned} (1)(\sigma_1\sigma_2\sigma_3\sigma_4) &= (\sigma_1\sigma_2)(\sigma_3\sigma_4) \\ (\sigma_1\sigma_3)(\sigma_2\sigma_4) &= (\sigma_1\sigma_4)(\sigma_2\sigma_3) = 0 \ . \end{aligned} \tag{52}$$

Here, G is not connected, v_1 and v_2 are in one component and v_3 and v_4 in another one.

e) The case where $\Pi^e_A(G)$ is empty, has been dealt with in Theorem 3.

2) We next consider the case $A = \{v_1, v_2, v_3, v_4\}$, $|D| = 2$, e.g. $D = \{v_1, v_2\}$. The set $\Pi_A(D,G)$ is a subset of the set $\{(1234), (123|4), (124|3), (12|34), (12|3|4)\}$. Again, if $\Pi_A(D,G)$ contains all these partitions, there are no Λ-identities. In contrast with example 1, the deletion of one partition from this maximum set does not necessarily lead to a Λ-identity. If, e.g., the partition $(124|3)$ is missing (which is the case if v_3 separates v_1 and v_2 from v_4), one easily sees that the eqs. (39) have no non-trivial solution.

We discuss only one case in which identities do occur, viz. that where v_3 separates v_1 from v_2 and v_4. In that case, $\Pi_A(D,G) = \{(1234), (123|4)\}$. There are two conditions on the four independent coefficients λ_B, and we find the following identities,

$$\begin{aligned} &(1)(\sigma_1\sigma_2) = (\sigma_1\sigma_3)(\sigma_2\sigma_3) \quad , \\ &(\sigma_1\sigma_4)(\sigma\ \sigma_4) = (\sigma_3\sigma_4)(\sigma_1\sigma_2\sigma_3\sigma_4), \end{aligned} \qquad (53)$$

which also follow, of course, from the factorization property mentioned above.

3) We now turn to the case $|A| = 6$, $D=A$. If $A = \{v_i \mid 1 \le i \le 6\}$, then

$$\Lambda_A(G,K) = 2\{\lambda_\emptyset(1)(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6) + \sum_{\substack{i,j=1\\ i<j}}^{6} \lambda_{ij}(\sigma_i\sigma_j)(\sigma_i\sigma_j\sigma_A)\} . \qquad (54)$$

We give one example of a Λ-identity which can occur in this case, viz. that with

$$\lambda_\emptyset = \lambda_{13} = \lambda_{24} = \lambda_{25} = \lambda_{26} = \lambda_{45} = \lambda_{46} = \lambda_{56} = 1 ,$$

$$\lambda_{12} = \lambda_{14} = \lambda_{15} = \lambda_{16} = \lambda_{23} = \lambda_{34} = \lambda_{35} = \lambda_{36} = -1.$$

The corresponding numbers $k_A(\pi)$ are non-zero only for the partitions $\pi_a = (13|2456)$, $\pi_b = (13|24|56)$, $\pi_c = (13|25|46)$ and $\pi_d = (13|26|45)$. Hence, the corresponding Λ-identity holds if these partitions are not in $\Pi_A^e(G)$. This can occur in various ways, of which we mention the following ones.

a) G is the graph obtained from the complete graph K_6 by deleting the edge $\{v_1, v_3\}$, and $A = V(G)$. This case has been dealt with in Lemma 5.

b) G is a planar graph and the sequence $(v_1, v_2, v_3, v_4, v_5, v_6)$ is a boundary sequence of G. A boundary sequence of a planar graph G is a sequence of (not necessarily distinct) vertices $(u_1, \dots, u_n)$ of G such that the graph G' defined by $V(G') = V(G) \cup \{w_0, w_1, \ldots, w_n\}$ ($w_i \notin V(G)$, $0 \leq i \leq n$) and $E(G') = E(G) \cup \{\{w_0, w_i\}, \{u_i, w_i\}, \{w_i, w_{i+1}\} \mid 1 \leq i \leq n\}$ with $w_{n+1} = w_1$ is planar (see I). It was proven in I that for any boundary sequence (u_1, u_2, u_3, u_4) every chain between u_1 and u_3 separates u_2 from u_4; furthermore, every subsequence of a boundary sequence is a boundary sequence. Applying these properties to the sequences (v_1, v_2, v_3, v_i) ($i=4,5,6$) we see that indeed $\Pi^e_A(G)$ does not contain the partitions π_a, π_b, π_c and π_d.

This example can be generalized, e.g. in the following way. Let $(v_1, v_2, v_3, v_4, w_1', v_5, v_6, w_2')$ be a boundary sequence of a planar graph G_1, G_2 an arbitrary graph such that $V(G_1) \cap V(G_2) = \{v_5, v_6, w_1', w_2'\}$, and $G = G_1 \cup G_2$, the union of G_1 and G_2. A chain between v_1 and v_3 which does not contain the vertices v_2, v_4, v_5, v_6 is either a chain in G_1 or it contains edges of G_2, and hence w_1' and w_2'. In the former case it separates v_2 from v_4, v_5 and v_6; in the latter case it contains a chain between w_1' and v_3, which separates v_4 from v_2, v_5 and v_6. In both cases $\Pi^e_A(G)$ does not contain the partitions π_a, π_b, π_c and π_d.

4) As a final example of Λ-identities we now give an alternative proof of the main result of I mentioned in the introduction. Let G be a finite planar graph and $(v_1, \ldots, v_n)$ a boundary sequence of G. By Theorem 1*, the Λ-identity

$$\sum_{j=1}^{n} (-1)^j (\sigma_{v_1} \sigma_{v_j})(\sigma_{v_1} \sigma_{v_j} \prod_{k=1}^{n} \sigma_{v_k}) = 0 \tag{55}$$

holds if and only if

$$-\eta_\pi(\emptyset) + \sum_{j=2}^{n} (-1)^j \eta_\pi(\{v_1, v_j\}) = 0 \tag{56}$$

for all partitions $\pi \in \Pi_A(D,G)$, with A the set of all vertices occurring in the boundary sequence and D the set of vertices occurring an odd number of times. Let π be such a partition and let $v_{j_1}, v_{j_2}, \ldots, v_{j_m}$ ($1 < j_1 < \ldots < j_m \leq n$, m odd) be the vertices which are in the same block of π as v_1. Obviously, using the fact that $\eta_\pi(\emptyset) = 1$, eq. (56) reduces to

$$\sum_{r=0}^{m} (-1)^{j_r} = 0, \tag{57}$$

where $j_0 = 1$. Since for any r every chain between v_{j_r} and $v_{j_{r+1}}$ separates all v_i with $j_r < i < j_{r+1}$ from all other vertices of the boundary sequence, the set $\{v_i | j_r < i < j_{r+1}\}$ must be a union of blocks of π, and hence even. Therefore, $(-1)^{j_r} = (-1)^{r-1}$, which proves the validity of eq. (56), and hence of eq. (55).

6. Examples of Λ-inequalities.

1) First consider the first GKS inequality $^{4)}$ restricted to ferromagnetic Ising models on a graph, i.e. with pair interactions, which we write in the form

$$(\sigma_\emptyset)(\sigma_A) \geq 0 . \tag{58}$$

Since $\eta_\pi(\emptyset) = 1$ for all $\pi \in \Pi_A^e$, eq. (58) follows from Theorem 2.

2) The second GKS inequality $^{4)}$ for ferromagnetic Ising models reads

$$(\sigma_\emptyset)(\sigma_B\sigma_C) - (\sigma_B)(\sigma_C) \geq 0 \tag{59}$$

for arbitrary sets $B, C \subset V(G)$. The validity of this inequality in the case of pair interactions follows from the fact that $\eta_\pi(\emptyset) = 1$ and $\eta_\pi(B) = 0$ or 1, and hence $\eta_\pi(\emptyset) - \eta_\pi(B) \geq 0$, for all $\pi \in \Pi_{B\cup C}$ (BC,G).

3) We next turn to a set of inequalities recently derived by Newman for ferromagnetic Ising models with pair interactions $^{5)}$. Let A be an even subset of a graph G and X_A a collection of even subsets of A such that every partition of A into pairs is a refinement of some two-block partition $(B|A\setminus B)$ of A with $B \in X_A$. Newman's inequality reads

$$(\sigma_\emptyset)(\sigma_A) \leq \sum_{B \in X_A} (\sigma_B)(\sigma_{A\setminus B}) . \tag{60}$$

By Theorem 2, it is sufficient to prove that

$$\eta_\pi(\emptyset) = 1 \leq \sum_{B \in X_A} \eta_\pi(B) \quad \text{for all } \pi \in \Pi_A^e(G) . \tag{61}$$

Consider any $\pi \in \Pi_A^e(G)$ and let π' be a partition of A into pairs which is a refinement of π. Since, by definition, there is at least one $B \in X_A$ such that π' is a refinement of $(B|A\setminus B)$ every block of π' contains 2 or 0 elements of B. Hence $n_{\pi'}(B) = 1$, and therefore $n_{\pi}(B) = 1$. Since $n_{\pi}(B') \geq 0$ for all other $B' \in X_A$, the inequality (59) holds.

Our general formalism, applied to this particular case, resembles the derivation of eq. (60) given by Sylvester 6).

4) Finally we derive a new Λ-inequality for the case $|A| = 6$, $D = A$, namely:

$$(1)(\sigma_A) + \sum_{j=2}^{6} (\sigma_1\sigma_j)(\sigma_1\sigma_j\sigma_A) \leq \sum_{\substack{i,j=2\\ i<j}}^{6} (\sigma_i\sigma_j)(\sigma_i\sigma_j\sigma_A) . \tag{62}$$

By Theorem 2 it is sufficient to show that for all $\pi \in \Pi_A^e$

$$1 + \sum_{j=2}^{6} n_{\pi}(\{v_1, v_j\}) \leq \sum_{\substack{i,j=2\\ i<j}}^{6} n_{\pi}(\{v_i, v_j\}) . \tag{63}$$

If $\pi = (123456)$, we have $n_{\pi}(\{v_i, v_j\}) = 1$ for all i,j $(i \neq j)$ and hence eq.(63) is valid. If π is of the type $(1j|k\ell mn)$, the left-hand side of eq. (63) is equal to 2, the right-hand side to 6, and again eq. (63) is valid. The remaining two cases, where π is of the type $(1jk\ell|mn)$ or $(1j|k\ell|mn)$, are dealt with in a similar way. The extension of the inequality (62) to the case $|A| = n \geq 6$, n even, is straightforward.

The examples of Λ-inequalities discussed above are valid for any graph G containing the given sets A and D. Evidently for any choice of A and D, Theorem 2^* enables one in principle to derive all Λ-inequalities on A which have the same general validity. The corresponding sets λ, considered as vectors in R^n ($n=2^{|A|-2}$) with components λ_B, form a convex cone. If, for a particular choice of G, $\Pi_A(D,G)$ is a proper subset of $\Pi_A(D)$, the corresponding set of Λ-inequalitites will again form a convex cone, which may contain the convex cone mentioned above as a proper subset.

In a subsequent paper 7) we shall investigate the structure of these convex cones. In particular we shall show that every Λ-inequality can be "decomposed" into a finite number of extremal inequalities. We shall discuss the relation of the inequalities mentioned in the examples with these extremal inequalities, and derive new (extremal) inequalities.

7. Concluding remarks.

1) Formally, the analysis of this paper is restricted to Ising models in zero magnetic field. The case of an arbitrary (not necessarily homogeneous) magnetic field can, however, be easily included by replacing the field by a "dummy" spin interacting with all other vertices 8).

2) Theorems $1(1^*)$, $2(2^*)$ and 3 have been derived for finite graphs. The extension to infinite graphs is straightforward. Consider an infinite graph G, two finite sets of vertices $A \subset V(G)$ and $D \subset A$, and for each partition $\pi \in \Pi_A^e(G)$ a skeleton graph H_π. The edge sets of all H_π are finite. Hence there exists a finite subgraph G_0 of G such that $E(H_\pi) \subset E(G_0)$ for all π. Let $G_0, G_1, G_2, \ldots$ be a sequence of increasing subgraphs of G such that $\lim_{n\to\infty} G_n = G$ and that $\langle \sigma_B \rangle_G = \lim_{n\to\infty} \langle \sigma_B \rangle_{G_n}$ exists for all $B \subset A$. Then $\Pi_A^e(G_n) = \Pi_A^e(G)$ for $n = 0,1,2,\ldots,$ and the validity of the theorems (with the unnormalized correlation functions replaced by normalized correlation functions) for infinite graphs follows.

3) As the reader may have observed, all examples of Λ-identities given in section 5 have the property that all λ_B are equal to 1, -1 or 0. In a forthcoming paper 9), devoted to Λ-identities for Ising models with general (n-spin) interactions, it will be shown that for every choice of G, A and D the set of all Λ-identities can be derived from an independent set of Λ-identities having this property.

4) Many theorems on inequalities include a specification of the condition under which the equality sign holds (cf., e.g.,ref. 10). For correlation-function inequalities such conditions have hitherto not received much attention in the literature (see, however, ref. 11). In the case of Λ-inequalities they are implicit in Theorem 1^*, since the validity of a Λ-equality for all $K \in \mathcal{F}$ implies the validity for all $K \in \mathcal{K}$. The condition takes the form of a condition on $\Pi_A(D,G)$. For the second GKS inequality, e.g., the condition reads (cf. section 6, example 2): $\eta_\pi(B) = 1$ for all $\pi \in \Pi_{B\cup C}$ (BC, G), i.e. all partitions of $B \cup C$ for which $\eta_\pi(B) = 0$ are absent from $\Pi_{B\cup C}(BC, G)$. E.g., if $B = \{v_1, v_2\}$, $C = \{v_3, v_4\}$, the missing partitions are $(13|24)$ and $(14|23)$, in which case G contains a cut vertex separating v_1 and v_2 from v_3 and v_4 (see section 5, example 1c). In case G contains a "dummy" vertex v_0 connected with all other vertices, representing a nonzero magnetic field, v_0 must be the cut vertex. The graph obtained from G by deleting v_0 and the edges incident with it then has v_1 and v_2 in

one component, v_3 and v_4 in another. This specific result was earlier derived by Setô 11).

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III. EXTREMAL A-INEQUALITIES FOR ISING MODELS WITH PAIR INTERACTIONS

1. Introduction.

In this chapter we shall investigate in more detail the class of correlation-function inequalities introduced in the previous chapter. In particular, we shall show that every inequality in this class which refers to a particular set of vertices A can be written as a positive linear combination of so-called extremal inequalities with respect to the same set A. A method will be sketched by which they can, at least in principle, be found. Throughout the chapter we restrict ourselves to the case discussed in Theorem 2 and section 6 of chapter II (i.e. $D=A$). The generalization to the more general case (i.e. $D \subset A$) can in principle be treated in a similar way and will not be discussed here. Some explicit examples of extremal correlation-function identities for general A will be given; for the case $|A|=6$ we shall in addition give three examples not covered by the former ones. These extremal inequalities are the strongest correlation-function inequalities in the subclass determined by A.

2. Definitions and notation.

As in chapter II, a <u>graph</u> G is defined as a pair $(V(G), E(G))$, where $V(G)$ is a set of elements called vertices and $E(G)$ a set of unordered pairs $\{v,v'\}$ of distinct vertices, called edges. G is finite if $V(G)$ and $E(G)$ are finite.

An Ising model on a finite graph is defined as a triple $(G,\mathcal{S},K)$, where $\mathcal{S}$ is the set of all functions $\sigma\colon V(G) \to \{-1,1\}$ and K a complex function on $E(G)$. The spin variable σ_v is the value of σ at the vertex v, the coupling parameter K_e is the value of K at the edge e. An Ising model is called ferromagnetic if $K_e \geq 0$ for all $e \in E(G)$.

For any set $A \subset V(G)$ we define

$$\sigma_A = \prod_{v \in A} \sigma_v , \qquad (1)$$

where for $A=\emptyset$ we have $\sigma_\emptyset=1$.

The Hamiltonian of the Ising model $(G,\mathcal{S},K)$ is defined by

$$H_{G,K}(\sigma) = -\sum_{e\in E(G)} K_e\sigma_e \;, \;, \tag{2}$$

the unnormalized and normalized (spin) correlation functions $(\sigma_A)_{G,K}$ and $\langle\sigma_A\rangle_{G,K}$, respectively, for any set $A\subset V(G)$ by

$$\begin{aligned} (\sigma_A)_{G,K} &= \sum_{\sigma\in\mathcal{S}} \sigma_A\, e^{-H_{G,K}(\sigma)} \\ \langle\sigma_A\rangle_{G,K} &= (\sigma_A)_{G,K}\, Z^{-1} \quad \text{if} \quad Z\neq 0, \end{aligned} \tag{3}$$

where Z, the canonical partition function, is defined by $Z = (1)_{G,K}$. Since the Hamiltonian is quadratic in the σ_v, the correlation function $(\sigma_A)_{G,K}$ vanishes if $|A|$ is odd. Therefore, we shall henceforth consider only correlation functions $(\sigma_A)_{G,K}$ for <u>even</u> sets A, i.e. for sets with $|A|$ even. By $\mathcal{P}_e(A)$ we denote the family of even subsets of A. We shall suppress the index K and, where no confusion arises, the index G as well. We have taken $kT=1$.

Consider a graph G and a set $A\subset V(G)$. By $\pi(A,G)$ we denote the partition of A <u>induced</u> by G, i.e. the partition in which two vertices of A are in the same block if and only if they are in the same connected component of G. If H is a <u>spanning subgraph</u> of G, i.e. if $V(H)=V(G)$, $E(H)\subset E(G)$, the partition $\pi(A,H)$ is a <u>refinement</u> of $\pi(A,G)$, i.e., the blocks of $\pi(A,H)$ are subsets of those of $\pi(A,G)$. The set of all <u>even</u> partitions of A (i.e. partitions of A into even subsets) is denoted by Π_A^e, and the set of all even partitions of A induced by spanning subgraphs of G by $\Pi_A^e(G)$. We furthermore introduce for any set $B\subset A$ and any partition $\pi\in\Pi_A^e$

$$\eta_\pi(B) = \begin{cases} 1 & \text{if the number of elements of } B \text{ in every block of } \pi \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $\eta_\pi(A\backslash B) = \eta_\pi(B)$. In II we have derived the following theorem.

<u>Theorem 1.</u> If A is an even set of vertices of a finite graph G and $\{\lambda_B\}$ a set of real numbers defined for all sets $B\in\mathcal{P}_e(A)$ with $\lambda_B=\lambda_{A\backslash B}$ for all B, then

$$\sum_{B\in\mathcal{P}_e(A)} \lambda_B(\sigma_B)_G(\sigma_{A\backslash B})_G \geq 0 \tag{4}$$

for every ferromagnetic Ising model on G if and only if

$$\sum_{B\in\mathcal{P}_e(A)} \eta_\pi(B)\lambda_B \geq 0 \tag{5}$$

for every partition $\pi\in\Pi_A^e(G)$. The equality sign in eq. (4) holds for every Ising model on G if and only if the equality sign in eq. (5) holds for every partition $\pi\in\Pi_A^e(G)$.

Every inequality of the form eq. (4) will be called a Λ-inequality. The set of eqs. (5) is a finite set of linear inequalities. In the following section we shall present some general properties of such sets of linear inequalities.

3. Polyhedral convex cones.

Consider for any vector $x\in\mathbb{R}^n$ the following set of linear combinations of the components x_k of x, where I is a finite index set,

$$\sum_{k=1}^{n} \alpha_{ik}x_k \; ; \quad i\in I \; , \tag{6}$$

with $\alpha_{ik}\in\mathbb{R}$ for all i and k. We define the set

$$C_\alpha = \left\{x\in\mathbb{R}^n \;\middle|\; \sum_{k=1}^{n} \alpha_{ik}x_k \geq 0 \;\; , \;\; \text{for all } i\in I\right\} . \tag{7}$$

The set C_α is a convex cone in the sense that if $x^{(1)},x^{(2)}\in C_\alpha$ then every <u>positive linear combination</u> of $x^{(1)}$ and $x^{(2)}$ (i.e. every vector $c_1x^{(1)} + c_2x^{(2)}$ with $c_1,c_2 \geq 0$) also belongs to C_α. In view of its definition and the fact that I is finite C_α is called a <u>polyhedral convex cone</u>.

For any $J\subset I$ we define

$$F_J = \left\{x\in\mathbb{R}^n \;\middle|\; \sum_{k=1}^{n} \alpha_{ik}x_k > 0,\; i\in J \; ; \;\; \sum_{k=1}^{n} \alpha_{ik}x_k = 0 \; , \text{ for all } i\in I\backslash J\right\} . \tag{8}$$

If F_J is nonempty it is called a <u>face</u> of C_α; $F_\emptyset$ is called the <u>null-face</u>. If d_J is the number of independent vectors satisfying $\Sigma_{k=1}^{n}\alpha_{ik}x_k = 0$, for all $i\in I\backslash J$, we call F_J a face of dimension d_J. It is clear that there are no faces of dimension smaller then $d=d_\emptyset$, that there is exactly one face of dimension d, and that there are at most $2^{|I|}$ faces.

A face F_J of C_α is called an extremal face of C_α if no vector in F_J can be expressed as a positive linear combination of two vectors in $C_\alpha \setminus F_J$.

We now state without proof the following facts about C_α:

1) The extremal faces of C_α are the d-dimensional null-face $F_\emptyset$ and the (d+1)-dimensional faces (if any). Vectors in (d+1)-dimensional faces we call extremal vectors.

2) If in each extremal face F_J with $d_J = d+1$ we select an arbitrary vector x^J, and if the vectors $x^{(1)},\dots,x^{(d)}$ form a basis of $F_\emptyset$, every vector x in C_α can be written in the form

$$x = \sum_{J \subset I : d_J = d+1} c_J x^J + \sum_{r=1}^{d} c_r x^{(r)} , \tag{9}$$

with $c_J \geq 0$ for all J. For $d > 0$ this decomposition of x in terms of extremal vectors and vectors in the null-face is not unique. For d=0 the null-face consists only of the null-vector, and the decomposition is unique (apart from a positive normalization factor in each of the x^J *); in this case, to which we shall restrict ourselves in this paper, the x^J can be formed by the following procedure. Select in all possible ways a set of n-1 linearly independent equations $\Sigma_{k=1}^{n} \alpha_{ik} x_k = 0$; if a vector x satisfying these equations lies in C_α, it is an extremal vector. For a general reference to linear inequalities see ref. 1.

4. Extremal Λ-inequalities.

In this chapter we shall consider graphs G and sets A for which $\Pi_A^e(G) = \Pi_A^e$, and in this section we shall investigate the set of linear inequalities

$$\sum_{B \in \mathcal{P}_e(A)} \eta_\pi(B) \lambda_B \geq 0 , \quad \pi \in \Pi_A^e \quad . \tag{10}$$

Because $\eta_\pi(B) = \eta_\pi(A \setminus B)$ and $\lambda_B = \lambda_{A \setminus B}$ for all $B \in \mathcal{P}_e(A)$, we select from the even subsets of A a maximal family of subsets $\mathcal{P}'_e(A)$, such that if $B \in \mathcal{P}'_e(A)$ then $A \setminus B \notin \mathcal{P}'_e(A)$; we assume that $\emptyset \in \mathcal{P}'_e(A)$.

Let us denote the two-block partitions of A by $\{B, A \setminus B\}$, $B \in \mathcal{P}'_e(A)$. If for convenience we write the one-block partition $\{A\}$ as $\{\emptyset, A\}$ we can introduce

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*) This proviso will not be repeated explicitly in the sequel.

a square matrix η with elements

$$\eta_{B,B'} = \eta_{\{B,A\setminus B\}}(B') \; ; \quad B,B'\in\mathcal{P}'_e(A) \; . \tag{11}$$

From the definition of $\eta_\pi(B')$ it follows that

$$\eta_{B,B'} = \tfrac{1}{2}(1 + (-1)^{|B\cap B'|}) \; ; \quad B,B'\in\mathcal{P}'_e(A) \; . \tag{12}$$

It is not difficult to see that η has an inverse η^{-1} with elements

$$(\eta^{-1})_{B,B'} = -\delta_{B,\emptyset}\delta_{B',\emptyset} + 2^{-|A|+3}(-1)^{|B\cap B'|} ; \quad B,B'\in\mathcal{P}'_e(A) \; . \tag{13}$$

Let us introduce the set $\lambda = \{\lambda_B | B\in\mathcal{P}'_e(A)\}$, considered as a vector in $\mathbb{R}^n (n = |\mathcal{P}'_e(A)| = 2^{|A|-2})$. The set of vectors λ which satisfy eq. (10) for all $\pi\in\Pi^e_A$ will form a convex cone C_η. It is convenient to carry over the concepts "extremal" and "positive linear combination" from the vectors λ to the corresponding Λ-inequalities.

We can now combine the results of the preceding section with Theorem 1. Since η has an inverse, we are in the case d=0, and the following theorem holds.

Theorem 2. Let G be a finite graph and $A\subset V(G)$. If $\Pi^e_A(G) = \Pi^e_A$, then every Λ-inequality with respect to A is the positive linear combination of a finite number of extremal Λ-inequalities with respect to A.

To illustrate Theorem 2, we consider the case of a set A consisting of four vertices, say $A = \{1,2,3,4\}$. Π^e_A consists of the following partitions (in an obvious notation)

$$(\emptyset|1234), \quad (12|34), \quad (13|24), \quad (14|23). \tag{14}$$

Let $\mathcal{P}'_e(A) = \{\emptyset, \{1,2\}, \{1,3\}, \{1,4\}\}$. The matrix η has the following form

$$\eta = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} , \tag{15}$$

where the order of rows and columns is that of the partitions in (14). The extreme vectors of C_η can be found by selecting any three linearly independent equations from the set of four equations

$$\sum_{B'\in\mathcal{P}'_e(A)} \eta_{B,B'}\lambda_{B'} = 0 \ , \quad B\in\mathcal{P}'_e(A)$$

and requiring that the solution satisfies

$$\sum_{B'\in\mathcal{P}'_e(A)} \eta_{B,B'}\lambda_{B'} \geq 0$$

for the remaining sets B. In this way we easily find the following extremal vectors (normalized so that $|\lambda_\emptyset|=1$):

$$\begin{aligned}
&\lambda_\emptyset = -\lambda_{12} = \lambda_{13} = -\lambda_{14} = 1 \ , \\
&\lambda_\emptyset = -\lambda_{12} = -\lambda_{13} = \lambda_{14} = 1 \ , \\
&\lambda_\emptyset = \lambda_{12} = -\lambda_{13} = -\lambda_{14} = 1 \ , \\
&\lambda_\emptyset = -\lambda_{12} = -\lambda_{13} = -\lambda_{14} = -1 \ .
\end{aligned} \tag{16}$$

The corresponding extremal Λ-inequalities are

$$(1)(\sigma_1\sigma_2\sigma_3\sigma_4) - (\sigma_1\sigma_2)(\sigma_3\sigma_4) + (\sigma_1\sigma_3)(\sigma_2\sigma_4) - (\sigma_1\sigma_4)(\sigma_2\sigma_3) \geq 0, \tag{17a}$$

$$(1)(\sigma_1\sigma_2\sigma_3\sigma_4) - (\sigma_1\sigma_2)(\sigma_3\sigma_4) - (\sigma_1\sigma_3)(\sigma_2\sigma_4) + (\sigma_1\sigma_4)(\sigma_2\sigma_3) \geq 0, \tag{17b}$$

$$(1)(\sigma_1\sigma_2\sigma_3\sigma_4) + (\sigma_1\sigma_2)(\sigma_3\sigma_4) - (\sigma_1\sigma_3)(\sigma_2\sigma_4) - (\sigma_1\sigma_4)(\sigma_2\sigma_3) \geq 0, \tag{17c}$$

$$-(1)(\sigma_1\sigma_2\sigma_3\sigma_4) + (\sigma_1\sigma_2)(\sigma_3\sigma_4) + (\sigma_1\sigma_3)(\sigma_2\sigma_4) + (\sigma_1\sigma_4)(\sigma_2\sigma_3) \geq 0. \tag{17d}$$

The inequalities (17a, b, c) are special cases of inequalities derived by Sherman 2) (see also ref. 3 and ref. 4, Theorem 5). The inequality (17d) is a special case of a set of inequalities due to Newman 4), and is also a consequence of the stronger GHS inequality 5) (see also refs. 6 and 7). By adding all four eqs. (17) we obtain

$$(1)(\sigma_1\sigma_2\sigma_3\sigma_4) \geq 0 \ ; \tag{18}$$

by adding eqs. (17a) and (17b) we obtain

$$(1)(\sigma_1\sigma_2\sigma_3\sigma_4) \geq (\sigma_1\sigma_2)(\sigma_3\sigma_4) \ . \tag{19}$$

Inequalities (18) and (19) are examples of the first and second GKS inequality 5), respectively, which are thus seen not to be extremal. We stress the fact that for the given set A the set of inequalities (17) exhausts the class of extremal (i.e. strongest possible) correlation-function inequalities of the form (4). The example of the GHS inequality shows that there exist stronger inequalities which are not in this class.

To prepare the way for the examples to be discussed in the next section we find it convenient to introduce a change of variables which enables us to write the inequalities in a somewhat simpler form. We define, for $B \in \mathcal{P}'_e(A)$,

$$k_B = \sum_{B' \in \mathcal{P}'_e(A)} \eta_{B,B'} \lambda_{B'} \ . \tag{20}$$

Using the inverse of η it is easily verified that

$$\lambda_B = -\delta_{B,\emptyset} k_\emptyset + 2^{-|A|+3} \sum_{B' \in \mathcal{P}'_e(A)} (-1)^{|B \cap B'|} k_{B'} ; \quad B \in \mathcal{P}'_e(A) \ . \tag{21}$$

Consider a partition $\pi = \{B_1, \dots, B_k\} \in \Pi^e_A$, with $B_1, \dots, B_k$ nonempty. From the definition of η it follows that for $B \in \mathcal{P}'_e(A)$

$$\eta_\pi(B) = \prod_{m=1}^{k} \left\{ \tfrac{1}{2} (1 + (-1)^{|B \cap B_m|}) \right\} . \tag{22}$$

Define $K = \{1, \dots, k\}$, and for $L \subset K$ define

$$B_L = \bigcup_{\ell \in L} B_\ell \ , \tag{23}$$

(in particular, $B_\emptyset = \emptyset$);

$\eta_\pi(B)$ can now be written as

$$\eta_\pi(B) = 2^{-k} \sum_{L \subset K} (-1)^{|B \cap B_L|} \ . \tag{24}$$

Using eqs. (21) and (24) and the fact that

$$\sum_{B \in \mathcal{P}'_e(A)} (-1)^{|B \cap X|} = 2^{|A|-2} \{\delta_{X,\emptyset} + \delta_{X,A}\} \tag{25}$$

for all $X \in \mathcal{P}_e(A)$, we find

$$\begin{aligned}\sum_{B \in \mathcal{P}'_e(A)} \eta_\pi(B) \lambda_B &= -2^{-k} \sum_{B \in \mathcal{P}'_e(A)} \sum_{L \subset K} (-1)^{|B \cap B_L|} \delta_{B,\emptyset} k_\emptyset + \\ &\quad + 2^{-k} 2^{-|A|+3} \sum_{B \in \mathcal{P}'_e(A)} \sum_{B' \in \mathcal{P}'_e(A)} \sum_{L \subset K} (-1)^{|B \cap B'|} (-1)^{|B \cap B_L|} k_{B'} \\ &= -k_\emptyset + 2^{-k+2} \sum_{\substack{L \subset K \\ B_L \in \mathcal{P}'_e(A)}} k_{B_L} \ .\end{aligned} \tag{26}$$

If we introduce

$$\zeta_\pi(B) = \begin{cases} 1 \text{ if } B \text{ is a nonempty union of (nonempty) blocks of } \pi \\ 0 \text{ otherwise} \end{cases}$$

we finally obtain

$$2^{k-2} \sum_{B\in\mathcal{P}'_e(A)} \eta_\pi(B)\lambda_B = -(2^{k-2} - 1)k_\emptyset + \sum_{B\in\mathcal{P}'_e(A)} \zeta_\pi(B)k_B \; . \tag{27}$$

In the new variables the set of inequalities (10), with $\mathcal{P}_e(A)$ replaced by $\mathcal{P}'_e(A)$, reads

$$\sum_{B\in\mathcal{P}'_e(A)} \zeta_\pi(B)k_B \geq (2^{k-2} - 1)k_\emptyset \; , \quad \pi\in\Pi^e_A \; . \tag{28}$$

Note that the set of inequalities (28) includes the set of inequalities $k_B \geq 0$, for all $B\in\mathcal{P}'_e(A)$.

To find the extremal inequalities from (28), we select a set of n-1 independent equations of the form

$$\sum_{B\in\mathcal{P}'_e(A)} \zeta_\pi(B)k_B = (2^{k-2} - 1)k_\emptyset \; , \quad \pi\in\Pi^e_A \; . \tag{29}$$

If the solution of eqs. (29) satisfies the inequalities (28) for all remaining π in Π^e_A , then it will be an extremal inequality.

5. Examples of extremal Λ-inequalities.

In this section we shall give several examples of extremal Λ-inequalities. The examples given do not exhaust the class of all possible extremal Λ-inequalities.

1) We shall first derive three types of extremal Λ-inequalities which are valid for arbitrary sets A with $|A| \geq 4$. To this end we first select a vertex $v\in A$ and choose for $\mathcal{P}'_e(A)$ the set of all even subsets of A not containing v. As in section 4 we define $n = |\mathcal{P}'_e(A)| = 2^{|A|-2}$.

(a) Let C be an arbitrary nonempty element of $\mathcal{P}'_e(A)$, and take

$$k_B = 1 \quad \text{for } B=C \; , \tag{30a}$$

$$k_B = 0 \quad \text{for } B\neq C \; . \tag{30b}$$

Obviously, since $k_\emptyset = 0$, all inequalities (28) are satisfied. Using eq.

(21) we see that the Λ-inequality corresponding to the choice (30), multiplied by a factor $2^{|A|-3}$ in order to avoid fractional coefficients, reads

$$\sum_{B\in\mathcal{P}'_e(A)} (-1)^{|B\cap C|}(\sigma_B)(\sigma_{A\setminus B}) \geq 0 \ . \tag{31}$$

Since the n-1 eqs. (30b) are manifestly linearly independent, the inequality (31) is an extremal Λ-inequality. Any Λ-inequality with $k_\emptyset=0$ and $k_B\neq 0$ for more than one $B\in\mathcal{P}'_e(A)$ is a positive linear combination of Λ-inequalities of the type (31) with strictly positive coefficients, and hence not extremal.

It is easy to verify that the inequality (31) is of the type derived by Sherman [2] (see also ch. IV).

(b) Next we select a vertex $v'\in A$, $v'\neq v$, and we take

$$k_B = 1 \text{ if } v'\notin B \text{ and } B \neq A\setminus\{v,v'\} \ , \tag{32a}$$

$$k_B = 0 \text{ otherwise.} \tag{32b}$$

Obviously, this set of k_B-values satisfies the inequalities (28) for all π such that $k = |\pi| \leq 2$. If $|A| \geq 6$ we further consider a partition $\pi = \{B_1,\ldots,B_k\}$ with $k > 2$, where the order of the blocks is chosen so that $v,v'\in B_1\cup B_2$. There are two cases to be distinguished: (α) v and v' are in the same block, say B_1, and (β) v and v' are in different blocks, say v in B_1, and v' in B_2. In case (α), the left-hand side of eq. (28) is equal to $2^{k-1}-1$ (if $B_1 \neq \{v,v'\}$) or equal to $2^{k-1}-2$ (if $B_1 = \{v,v'\}$). Since $k_\emptyset = 1$ and $k > 2$, the inequality (28) is satisfied. In case (β), the left-hand side is equal to $(2^{k-1}-1) - 2^{k-2} = 2^{k-2}-1$, and hence the inequality (28) is again satisfied.

To show that the choice (32) of k_B-values satisfies a set of n-1 linearly independent equations of the type (29) we consider, for $|A| \geq 6$, the partitions $\pi = \{B_1,B_2,B_3\}$, with $v\in B_1$, $v'\in B_2$. For such a partition the equation (29) reads

$$k_{B_2\cup B_3} + k_{B_2} + k_{B_3} = k_\emptyset \tag{33}$$

which reduces, by eq. (32b), to the equation $k_{B_3} = k_\emptyset$. Since for any $B \neq A\setminus\{v,v'\}$ not containing v' such a partition with $B_3 = B$ can be found, and since the full set of equations for the k_B thus obtained is linearly independent, and has (32) as a solution, the corresponding Λ-inequality is extremal. By using eq. (21) one easily verifies that the coefficients λ_B

(again multiplied by a factor $2^{|A|-3}$ in order to avoid fractional values) are

$$\lambda_B = \begin{cases} 2^{|A|-3} - 1 & \text{if } B = A\setminus\{v,v'\} , \\ 1 & \text{if } B \text{ contains } v' , \\ -1 & \text{otherwise.} \end{cases} \tag{34}$$

(c) Finally we take

$$k_B = 0 \text{ if } |B| > \tfrac{1}{2}|A| - 1 , \tag{35a}$$

$$k_B = 1 \text{ if } |B| = \tfrac{1}{2}|A| - 1 , \tag{35b}$$

$$k_B = 2 \text{ if } |B| < \tfrac{1}{2}|A| - 1 . \tag{35c}$$

Again, this set of k_B-values satisfies the inequalities (28) for $k = |\pi| \leq 2$. If $|A| \geq 6$ we further consider a partition $\pi = \{B_1,\ldots,B_k\}$ with $k > 2$ and $v \in B_1$, and we define $M = \{2,\ldots,k\}$. Using the definition (23) we write eq. (28) for this partition as

$$k_{B_M} + \sum_{\substack{L \subset M \\ L \neq \emptyset, M}} \tfrac{1}{2}(k_{B_L} + k_{B_{M\setminus L}}) \geq (2^{k-2} - 1)k_{\emptyset} . \tag{36}$$

Since for any $L \neq \emptyset, M$ we have

$$|B_L| + |B_{M\setminus L}| = |A| - |B_1| \leq |A| - 2,$$

$|B_L|$ and $|B_{M\setminus L}|$ cannot be both larger than $\frac{1}{2}|A|-1$. Hence we have, by (35), $k_{B_L} + k_{B_{M\setminus L}} \geq 2$. Since the number of sets $L \neq \emptyset, M$ is $2^{|M|} - 2 = 2^{k-1} - 2$, and since $k_{\emptyset} = 2$, the inequality (36) is satisfied.

We shall now show that the set of k_B defined by (35) satisfies a set of equations of the form (29) with $k = |\pi| \leq 3$, among which n-1 are linearly independent. First, the eqs. (35a) are of this form, and they are independent. If $|A| \geq 6$, we further consider the set of simultaneous equations of the form (29) where $\pi = \{B_1, B_2, B_3\}$, with $B_1 = \{v,v'\}$, where v' is an arbitrary vertex not equal to v. They have the form

$$k_{B_2 \cup B_3} + k_{B_2} + k_{B_3} = k_{\emptyset} ,$$

which reduces to

$$k_{B_2} + k_{B_3} = k_{\emptyset} , \tag{37}$$

since $|B_2 \cup B_3| = |A| - 2 > \frac{1}{2}|A| - 1$. If $|B_2| > |B_3|$, then $k_{B_2} = 0$, and eq. (37) further reduces to the equation $k_{B_3} = k_{\emptyset}$. If $|B_2| = |B_3|$ we consider a vertex $v'' \in B_2$ and the sets B_1' and B_2' obtained from B_1 and B_2 by interchanging v' and v''. For the partition $\{B_1', B_2', B_3\}$ eq. ([illegible]) reads

$$k_{B_2'} + k_{B_3} = k_\emptyset \quad . \tag{38}$$

From eqs. (37) and (38) it follows that $k_{B_2'} = k_{B_2}$. Repeating this argument we find that $k_{B'} = k_B$ for any two sets B, B' with $|B| = |B'|$. In the case $|B_2| = |B_3|$ considered we conclude that $k_{B_2} = k_{B_3} = \frac{1}{2} k_\emptyset$. This shows that if $k_\emptyset$ is fixed, the solution of the set of equations considered is unique. Putting $k_\emptyset = 2$ we obtain eqs. (35a,b,c). Consequently, the corresponding A-inequality is extremal. The λ_B can be found from eqs. (21) and (35), but the general expression is not very illuminating.

2) After having discussed these general types of extremal inequalities we now turn to a special case, viz. $|A| = 6$; let $A = \{1,2,3,4,5,6\}$. In this case, it is convenient to choose for $\mathcal{P}_e'(A)$ the set of all even subsets B of A such that $|B| \leq 2$. Using the notation $k_{ij} = k_{\{i,j\}}$ we can write the eqs. (28) as

$$k_\emptyset \geq 0 \tag{39a}$$

$$k_{ij} \geq 0 \quad \text{for all pairs } i,j \text{ such that } 1 \leq i < j \leq 6 \tag{39b}$$

$$k_{ij} + k_{k\ell} + k_{mn} \geq k_\emptyset \quad \begin{array}{l} \text{for all permutations } ijk\ell mn \text{ of } 123456 \\ \text{such that } i < j, \;\; k < \ell, \;\; m < n; \;\; i < k < m. \end{array} \tag{39c}$$

In this case, there is exactly one extremal A-inequality (up to a permutation of the vertices) of each one of the three general types derived under 1(a), (b), (c). Searching semi-systematically we have, in addition, found three special extremal inequalities (again counting cases which differ only by a permutation of the vertices as a single case). The way in which they have been arrived at suggests that there are no more extremal inequalities in this case. We hope to present a more complete analysis in the future.

The six extremal inequalities are represented in Tables 1a and 1b. Table 1a gives the values of the k_B, $B \in \mathcal{P}_e'(A)$, Table 1b those of the λ_B (multiplied by a common factor in order to avoid fractional values). The first columns label the type of inequality, the last column of Table 1b gives the number N of distinct extremal inequalities obtained from the one represented by a permutation of the numbers 1, 2, 3, 4, 5, 6.

Table 1a. Values of the k_B, $B \in \mathcal{P}''_e(A)$, for extremal Λ-inequalities in the case $A = \{1,2,3,4,5,6\}$.

	$k_\emptyset$	k_{12}	k_{13}	k_{14}	k_{15}	k_{16}	k_{23}	k_{24}	k_{25}	k_{26}	k_{34}	k_{35}	k_{36}	k_{45}	k_{46}	k_{56}
a	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
b	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
c	2	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1
d	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
e	1	1	0	1	1	1	0	1	1	1	0	0	0	0	0	0
f	1	0	0	1	1	1	0	1	1	1	1	1	1	0	0	0

Table 1b. Values of the λ_B, $B \in \mathcal{P}'_e(A)$, for extremal Λ-inequalities in the case $A = \{1,2,3,4,5,6\}$. The last column gives the number N of distinct extremal inequalities obtained from the one represented by a permutation of the *numbers* 1,2,3,4,5,6.

	$\lambda_\emptyset$	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{16}	λ_{23}	λ_{24}	λ_{25}	λ_{26}	λ_{34}	λ_{35}	λ_{36}	λ_{45}	λ_{46}	λ_{56}	N
a	1	1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	15
b	-1	7	1	1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	15
c	-1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	6
d	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	1	1	1	1	6
e	0	-1	0	0	0	0	0	0	0	0	1	1	1	0	0	0	60
f	1	-1	-1	1	1	1	-1	1	1	1	1	1	1	-1	-1	-1	60

The extremal Λ-inequality labeled a is, of course, one of the type derived by Sherman, the one labeled c is a special case of Newman's inequalities, referred to in section 4; the other ones are new, as far as we know.

Remark: One can prove that the inequalities (c) and (17d) are the only Newman inequalities which are extremal Λ-inequalities.

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IV. CORRELATION-FUNCTION IDENTITIES FOR GENERAL ISING MODELS

Abstract.

For Ising models with general interactions correlation-function identities of the form $\Sigma_B \lambda_B \langle \sigma_B \rangle \langle \sigma_B \sigma_A \rangle = 0$ are studied. It is shown that each identity of this form is equivalent to a set of special identities, in which the coefficients λ_B take only the values -1, 0, 1. A necessary and sufficient condition for the validity of such a special identity is derived.

1. Introduction.

In two previous papers, to be denoted by I$^{1)}$ and II$^{2)}$, a class of correlation-function identities for Ising models with pair interactions was studied. These identities have the form

$$\sum_{B\subset A} \lambda_B \langle \sigma_B \rangle \langle \sigma_B \sigma_A \rangle = 0 \qquad (1)$$

where λ_B, for all B, is independent of the interaction parameters of the system, and they hold for all values of these parameters. In II a necessary and sufficient condition was derived under which the identity (1) holds. It turns out that this condition consists in a simple set of linear equations in the λ_B's, with coefficients 0 and 1.

In this paper we shall show that if, for a given Ising model with general n-spin interactions, one has an identity of the above-mentioned form, the left-hand side of eq. (1) can be written as a linear combination of special functions, each of which is zero for all values of the interaction parameters. These functions have the same form as the left-hand side of eq. (1), but the coefficients λ_B take only the values -1, 0, 1. We then proceed to give a necessary and sufficient condition under which these special functions satisfy the identity (1).

As an application, it will be shown that in the case of Ising models with only pair interactions this condition is equivalent to that of II, Theorem 1* (applied to the special functions mentioned). We also study in some detail the equation $\langle \sigma_X \sigma_Y \rangle - \langle \sigma_X \rangle \langle \sigma_Y \rangle = 0$. Finally, it will be shown that identities of the special type mentioned are readily constructed for arbitrary Ising models.

2. Definitions and notation.

Let V be a finite set of elements called vertices and $\mathcal{P}(V)$ the family of subsets of V. Elements of $\mathcal{P}(V)$ will be denoted by A, B, C,..., and subsets of $\mathcal{P}(V)$ by α, β, γ, ..., except a specially chosen subset (and some derived sets), called the bond set, which will be denoted by $\mathcal{B}$. We give $\mathcal{P}(V)$ the structure of a group by defining the product AB of two elements A and B of $\mathcal{P}(V)$ as the symmetric difference $(A\setminus B)\cup(B\setminus A)$. The unit element is the empty set and every element is of order two.

For $\alpha \subset \mathcal{P}(V)$ we define

$$V_\alpha = \bigcup_{A\in\alpha} A \quad , \quad P_\alpha = \prod_{A\in\alpha} A , \tag{2}$$

where $V_\emptyset = P_\emptyset = \emptyset$; further $g(\alpha)$ denotes the smallest subgroup of $\mathcal{P}(V)$ having α as a subset.

A set $\alpha \subset \mathcal{P}(V)$ will be called a *cycle* if $P_\alpha = \emptyset$, and *cycle-free* if there is no $\beta \subset \alpha$, $\beta \neq \emptyset$, such that $P_\beta = \emptyset$. Observe that by these definitions the set $\emptyset$ is both a cycle and cycle-free, whereas the set $\{\emptyset\}$ is a cycle and not cycle-free.

If $A \in \mathcal{P}(V)$ and $\beta \subset \mathcal{P}(V)$, a set $\alpha \subset \beta$ will be called *a factorization of A with respect to β* if α is cycle-free and $P_\alpha = A$; the elements of α will be called *β-factors* of A. A_β denotes the set of all factorizations of A with respect to β, and β_A the set of all β-factors of A. Obviously, A_β is not empty if and only if $A \in g(\beta)$.

For a given finite set V and a given set $\mathcal{B} \subset \mathcal{P}(V)$ we define *an Ising model on $\mathcal{B}$* as a pair $(\mathcal{B}, K)$ where K is a mapping from $\mathcal{B}$ into the complex numbers. Let $\mathcal{S}$ be the set of all mappings $\sigma : V \to \{1,-1\}$. The Hamiltonian of $(\mathcal{B},K)$ is defined by

$$H_{\mathcal{B},K}(\sigma) = -\sum_{X\in\mathcal{B}} K_X \sigma_X \quad , \qquad \sigma \in \mathcal{S} \quad , \tag{3}$$

where σ_X is defined by

$$\sigma_X = \prod_{v\in X} \sigma_v \quad . \tag{4}$$

$\mathcal{B}$ is called the bond set of $(\mathcal{B},K)$ and K_X the interaction parameter associated with the set X. The normalized and unnormalized correlation functions $(\sigma_A)_{\mathcal{B},K}$ and $\langle \sigma_A \rangle_{\mathcal{B},K}$, respectively, are, for a given set $A \in \mathcal{P}(V)$, given by

$$\begin{aligned} (\sigma_A)_{\mathcal{B},K} &= \sum_{\sigma\in\mathcal{S}} \sigma_A \, e^{-H_{\mathcal{B},K}(\sigma)} \quad , \\ \langle \sigma_A \rangle_{\mathcal{B},K} &= (\sigma_A)_{\mathcal{B},K} \, Z^{-1}_{\mathcal{B},K} \quad \text{if } Z_{\mathcal{B},K} \neq 0 \quad , \end{aligned} \tag{5}$$

where $Z_{\mathcal{B},K}$, the canonical partition function, is defined by $Z_{\mathcal{B},K} = (1)_{\mathcal{B},K}$. We have taken $kT = 1$.

If for a given interaction function K on $\mathcal{B}$, $K_X = 0$ for a given $X \in \mathcal{B}$, and $\mathcal{B}' = \mathcal{B} \setminus \{X\}$ and K' is the restriction of K to $\mathcal{B}'$, then

$$(\sigma_A)_{\mathcal{B},K} = (\sigma_A)_{\mathcal{B}',K'} \; . \tag{6}$$

Henceforth, we shall omit reference to K. As in II we shall work exclusively with unnormalized correlation functions, the translation to identities for normalized correlation functions being trivial.

3. General formalism.

We first prove a simple lemma about $(\sigma_A)_{\mathcal{B}}$.

Lemma 1. If V is a finite set, $\mathcal{B} \subset \mathcal{P}(V)$ and $A \in \mathcal{P}(V)$, then $(\sigma_A)_{\mathcal{B}} = 0$ for every Ising model on $\mathcal{B}$ if and only if $A \notin g(\mathcal{B})$.

Proof. Since for any $X \in \mathcal{B}$

$$e^{K_X \sigma_X} = c_X + s_X \sigma_X \; , \tag{7}$$

where $c_X = \cosh K_X$ and $s_X = \sinh K_X$, we can write

$$(\sigma_A)_{\mathcal{B}} = \sum_{\beta \subset \mathcal{B}} \left\{ \prod_{X \in \mathcal{B} \setminus \beta} c_X \prod_{X' \in \beta} s_{X'} \right\} (\sigma_A \sigma_{P_\beta})_{\emptyset} \; . \tag{8}$$

From eq. (8) one easily derives that $(\sigma_A)_{\mathcal{B}} = 0$ for every Ising model on $\mathcal{B}$ if and only if $(\sigma_A \sigma_{P_\beta})_{\emptyset} = 0$ for all $\beta \subset \mathcal{B}$, i.e., because of the relation

$$(\sigma_D)_{\emptyset} = 2^{|V|} \delta_{D,\emptyset} \quad \text{for all } D \in \mathcal{P}(V) \; , \tag{9}$$

if and only if $A \neq P_\beta$ for all $\beta \subset \mathcal{B}$, which proves the lemma. ■

Let us now, for a given Ising model $(\mathcal{B},K)$, consider the function

$$\Lambda^{\alpha}(\mathcal{B}\,|\,A) = \sum_{B \in g(\alpha)} \lambda_B (\sigma_B)_{\mathcal{B}} (\sigma_B \sigma_A)_{\mathcal{B}} \; , \tag{10}$$

where $A \in \mathcal{P}(V)$, $\alpha \subset \mathcal{P}(V)$, and where, for all B, λ_B is an arbitrary complex number not depending on the interaction parameters of the system. By Lemma 1 we may, without loss of generality, assume $\alpha \subset g(\mathcal{B})$ and $A \in g(\mathcal{B})$.

We shall study the following identity

$$\Lambda^{\alpha}(\mathcal{B}|A) = 0 \text{ for all Ising models on } \mathcal{B}. \tag{11}$$

For this purpose we first apply to the right-hand side of eq. (10) a Fourier transform with respect to the group $g(\alpha)$. Since $g(\alpha)$ is abelian and of order two, its characters satisfy $\chi(B) \in \{1,-1\}$ for all $B \in g(\alpha)$. We can construct the characters, labelled by the elements of $g(\alpha)$, in the following way. Let α' be an arbitrary but fixed maximal cycle-free subset of α (i.e. a minimal generating set of $g(\alpha)$) and $C \in g(\alpha)$. Further, let $\gamma \subset \alpha'$ be the unique factorization of C with respect to α'. We define for $B \in \alpha'$

$$\chi_C(B) = \begin{cases} -1 & \text{if } B \in \gamma \\ 1 & \text{if } B \in \alpha' \backslash \gamma \end{cases} \tag{12}$$

and for $B \in g(\alpha) \backslash \alpha'$

$$\chi_C(B) = \prod_{D \in \beta} \chi_C(D) \quad , \tag{13}$$

where β is the unique factorization of B with respect to α'. It is easy to see that

$$\chi_C(B)\chi_C(B') = \chi_C(BB') \tag{14}$$

and hence that χ_C is a character of $g(\alpha)$. Observe that the labelling of the characters depends on the choice of α'. The orthogonality relation for the characters reads

$$\frac{1}{|g(\alpha)|} \sum_{B \in g(\alpha)} \chi_C(B)\chi_{C'}(B) = \delta_{C,C'} \quad . \tag{15}$$

Finally we have $\chi_B(C) = \chi_C(B)$.

If we define

$$\Lambda^{\alpha}_C(\mathcal{B}|A) = \sum_{B \in g(\alpha)} \chi_C(B)(\sigma_B)_{\mathcal{B}}(\sigma_B\sigma_A)_{\mathcal{B}} \tag{16}$$

and

$$\hat{\lambda_C} = \sum_{B \in g(\alpha)} \chi_C(B)\lambda_B \quad , \tag{17}$$

we can write $\Lambda^{\alpha}(\mathcal{B}|A)$ as follows:

$$\Lambda^{\alpha}(\mathcal{B}|A) = \frac{1}{|g(\alpha)|} \sum_{C \in g(\alpha)} \hat{\lambda_C}\Lambda^{\alpha}_C(\mathcal{B}|A) \quad . \tag{18}$$

We now have the following theorem.

Theorem 1. Let V be a finite set, $\mathcal{B}\subset\mathcal{P}(V)$, $A\in g(\mathcal{B})$ and $\alpha\subset g(\mathcal{B})$. If $\Lambda^{\alpha}(\mathcal{B}|A)=0$ for every Ising model on $\mathcal{B}$, then there exists a $\gamma\subset g(\alpha)$ such that $\Lambda^{\alpha}_{C}(\mathcal{B}|A)=0$ for every Ising model on $\mathcal{B}$ for all $C\in\gamma$, and $\hat{\lambda}_{C}=0$ for all $C\in g(\alpha)\setminus\gamma$. Furthermore, $\emptyset\notin\gamma$.

Proof. Since $\Lambda^{\alpha}(\mathcal{B}|A)=0$ for all K, we have

$$\frac{\partial^2}{\partial K_X^2}\,\Lambda^{\alpha}(\mathcal{B}|A) = 0 \text{ for all } K \text{ and all } X\in\mathcal{B},$$

from which it can easily be seen that

$$\sum_{B\in g(\alpha)} \lambda_B(\sigma_B\sigma_X)_{\mathcal{B}}(\sigma_B\sigma_A\sigma_X)_{\mathcal{B}} = 0 \quad \text{for all } K \text{ and all } X\in\mathcal{B}. \tag{19}$$

Repeating this argument, we find that eq. (19) holds for all $X\in g(\mathcal{B})$, and in particular for all $X\in g(\alpha)$.

Multiplying eq. (19) by $\chi_C(X)$, with $C\in g(\alpha)$, using eq. (14) and summing over all $X\in g(\alpha)$ we find

$$\sum_{X\in g(\alpha)}\ \sum_{B\in g(\alpha)} \chi_C(BX)\chi_C(B)\lambda_B(\sigma_{BX})_{\mathcal{B}}(\sigma_{BX}\sigma_A)_{\mathcal{B}} = 0\ . \tag{20}$$

Using the fact that $g(\alpha)$ is closed under multiplication we find from eq. (20) that

$$\hat{\lambda}_C\Lambda^{\alpha}_C(\mathcal{B}|A) = 0 \quad \text{for all } C\in g(\alpha). \tag{21}$$

The first part of the theorem follows directly from eq. (21).

From eq. (14) it follows that $\chi_B(\emptyset)=1$ for all $B\in g(\alpha)$, and that hence

$$\Lambda^{\alpha}_{\emptyset}(\mathcal{B}|A) = \sum_{B\in g(\alpha)} (\sigma_B)_{\mathcal{B}}(\sigma_B\sigma_A)_{\mathcal{B}}\ .$$

Now suppose $\emptyset\in\gamma$, i.e. $\Lambda^{\alpha}_{\emptyset}(\mathcal{B}|A)=0$ for all K. Let β be a factorization of A with respect to $\mathcal{B}$. It follows that for $X\in\beta$

$$\begin{aligned}\frac{\partial\Lambda^{\alpha}_{\emptyset}(\mathcal{B}|A)}{\partial K_X} &= \sum_{B\in g(\alpha)}\Big\{(\sigma_B\sigma_X)_{\mathcal{B}}(\sigma_B\sigma_A)_{\mathcal{B}} + (\sigma_B)_{\mathcal{B}}(\sigma_B\sigma_A\sigma_X)_{\mathcal{B}}\Big\} = \\ &= 2^{n(\alpha,\{X\})}\sum_{B\in g(\alpha\cup\{X\})}(\sigma_B)_{\mathcal{B}}(\sigma_B\sigma_A\sigma_X)_{\mathcal{B}} = 0\end{aligned} \tag{22}$$

where, for $\delta\subset\mathcal{B}$, $n(\alpha,\delta) = |\delta| - r(\alpha\cup\delta) + r(\alpha)$, with $r(\alpha) = |\alpha'|$ for any maximal cycle-free set $\alpha'\subset\alpha$ (which is easily seen to be independent of the choice of α') The factor $2^{n(\alpha,\{X\})}$ arises from the fact that if $X \in g(\alpha)$, we have to take into account a factor 2. From eq. (22) it follows, that if we repeat the argument for all $X \in \beta$ we find

$$\sum_{B \in g(\alpha \cup \beta)} (\sigma_B)_{\mathcal{B}} (\sigma_B \sigma_A \sigma_{P_\beta})_{\mathcal{B}} = \sum_{B \in g(\alpha \cup \beta)} (\sigma_B)^2_{\mathcal{B}} = 0 \ .$$

This would, for any real interaction function, imply that $(\sigma_B)_{\mathcal{B}}=0$ for all $B \in g(\alpha \cup \beta)$, which is impossible by Lemma 1, since $\alpha, \beta \subset \mathcal{B}$. Hence the second part of the theorem follows. ■

From Theorem 1 and eq. (18) it follows that if $\Lambda^{\alpha}(\mathcal{B}|A)$ is zero for all Ising models on $\mathcal{B}$, it is a linear combination of functions $\Lambda^{\alpha}_C(\mathcal{B}|A)$ which are themselves zero for all Ising models on $\mathcal{B}$.

Now every $\Lambda^{\alpha}_C(\mathcal{B}|A)$ with $C \neq \emptyset$ will consist of terms with $\chi_C(B)=1$ and terms with $\chi_C(B)=-1$, $B,C \in g(\alpha)$. If we introduce, for a given $C \in g(\alpha)$, the set $g(\alpha,C) = \{B \in g(\alpha) | \chi_C(B) = 1\}$, which is easily seen to be a group, and if D is any element of $g(\alpha)$ with $\chi_C(D) = -1$, we can write $\Lambda^{\alpha}_C(\mathcal{B}|A)$ as

$$\Lambda^{\alpha}_C(\mathcal{B}|A) = \sum_{B \in g(\alpha,C)} \left\{ (\sigma_B)_{\mathcal{B}} (\sigma_B \sigma_A)_{\mathcal{B}} - (\sigma_B \sigma_D)_{\mathcal{B}} (\sigma_B \sigma_A \sigma_D)_{\mathcal{B}} \right\} , \qquad (23)$$

which is a special case of the following function

$$\Lambda_{\gamma}(\mathcal{B}|A,D) := \sum_{B \in g(\gamma)} \left\{ (\sigma_B)_{\mathcal{B}} (\sigma_B \sigma_A)_{\mathcal{B}} - (\sigma_B \sigma_D)_{\mathcal{B}} (\sigma_B \sigma_A \sigma_D)_{\mathcal{B}} \right\} , \qquad (24)$$

where γ is any subset of $\mathcal{P}(V)$ and A and D are arbitrary subsets of V. This kind of functions has already been studied in connection with correlation-function inequalities. Sherman [3)] and Ginibre [4)] have shown that for any ferromagnetic Ising model on $\mathcal{B}$

$\Lambda_{\gamma}(\mathcal{B}|A,D) \geq 0$ for all $\gamma \in g(\mathcal{B})$ and $A,D \in \mathcal{P}(V)$.

We now proceed to give a necessary and sufficient condition under which $\Lambda_{\gamma}(\mathcal{B}|A,D)=0$ for all Ising models on $\mathcal{B}$.

Theorem 2.

Let V be a finite set, $\mathcal{B}, \gamma \subset \mathcal{P}(V)$, and $A,D \in \mathcal{P}(V)$. Then $\Lambda_{\gamma}(\mathcal{B}|A,D)=0$ for all Ising models on $\mathcal{B}$ if and only if

$$D \in g(\alpha \cup \gamma) \text{ for all } \alpha \in A_{\mathcal{B}} \ . \qquad (25)$$

In words, condition (25) expresses that every factorization of A with respect to $\mathcal{B}$ contains a subset which can be supplemented by a subset of γ to yield a factorization of D with respect to $\mathcal{B} \cup \gamma$.

This theorem is a consequence of two lemmas which we shall prove first. Observe that if $A \notin g(\mathcal{B})$, $A_{\mathcal{B}}$ is empty and the identity holds for any D.

<u>Lemma 2.</u> Let V be a finite set, $\mathcal{B}$, $\gamma \subset \mathcal{P}(V)$ and A, $D \in \mathcal{P}(V)$. Then $\Lambda_\gamma(\mathcal{B}|A,D) = 0$ for all Ising models on $\mathcal{B}$ if and only if $\Lambda_{\gamma\cup\alpha}(\emptyset|AP_{\alpha'},D) = 0$ for all $\alpha \subset \mathcal{B}$ and all $\alpha' \subset \alpha$.

<u>Proof.</u> Using eq. (7) and the fact that $c_X^2 = 1+s_X^2$, we find by a straightforward calculation that for any $X \in \mathcal{B}$

$$\Lambda_\gamma(\mathcal{B}|A,D) = \Lambda_\gamma(\mathcal{B}'|A,D) + c_X s_X\, 2^{n(\gamma,\{X\})}\, \Lambda_{\gamma\cup\{X\}}(\mathcal{B}'|AX,D) +$$

$$+\, s_X^2\, 2^{n(\gamma,\{X\})}\, \Lambda_{\gamma\cup\{X\}}(\mathcal{B}'|A,D)\,, \qquad (26)$$

where $\mathcal{B}' = \mathcal{B}\backslash\{X\}$ and $n(\gamma,\{X\})$ is defined as in the proof of Theorem 1. If we apply eq. (7) to all $X \in \mathcal{B}$ and iterate eq. (26) we obtain

$$\Lambda_\gamma(\mathcal{B}|A,D) = \sum_{\alpha\subset\mathcal{B}} \sum_{\alpha'\subset\alpha} \left\{ \prod_{X\in\alpha'} c_X s_X \prod_{X'\in\alpha\backslash\alpha'} s_{X'}^2 \right\} 2^{n(\gamma,\alpha)}\, \Lambda_{\gamma\cup\alpha}(\emptyset|AP_{\alpha'},D). \qquad (27)$$

If $\Lambda_{\gamma\cup\alpha}(\emptyset|AP_{\alpha'},D) = 0$ for all $\alpha \subset \mathcal{B}$ and all $\alpha' \subset \alpha$ it follows that $\Lambda_\gamma(\mathcal{B}|A,D) = 0$ for all Ising models on $\mathcal{B}$. Conversely, suppose $\Lambda_\gamma(\mathcal{B}|A,D) = 0$ for all Ising models on $\mathcal{B}$. We select an $\alpha \in \mathcal{B}$ and consider an interaction function K such that $K_X = 0$ if $X \in \mathcal{B}\backslash\alpha$, $K_X \neq 0$ if $X \in \alpha$. From eq. (27) it follows that in that case

$$\Lambda_\gamma(\mathcal{B}|A,D) = 0 = \sum_{\alpha'\subset\alpha} \left\{ \prod_{X\in\alpha'} c_X s_X \prod_{X'\in\alpha\backslash\alpha'} s_{X'}^2 \right\} 2^{n(\gamma,\alpha)}\, \Lambda_{\gamma\cup\alpha}(\emptyset|AP_{\alpha'},D). \qquad (28)$$

If we divide out a factor $\Pi_{X\in\alpha} c_X s_X$, we obtain a polynomial in the variables $t_X = c_X^{-1} s_X$, $X \in \alpha$. This polynomial is (by continuity) zero for all values of the variables K_X, $X \in \alpha$, even if $\Pi_{X\in\alpha} c_X s_X = 0$. We select an $\alpha' \subset \alpha$ and consider the restriction K' of K to α. We now take $K'_X = 0$ if $X \in \alpha'$ and $K'_X = K_0 \neq 0$ if $X \in \alpha\backslash\alpha'$ and obtain a polynomial in tanh K_0 of which the coefficient of the term of highest degree is $\Lambda_{\gamma\cup\alpha}(\emptyset|AP_{\alpha'},D)$. Since this polynomial is zero for all values of K_0, it follows that $\Lambda_{\gamma\cup\alpha}(\emptyset|AP_{\alpha'},D) = 0$. Since α and α' were arbitrary, the lemma follows. ■

Lemma 3. $\Lambda_\gamma(\emptyset|A,D) = 0$ for a given set $\gamma\subset\mathcal{P}(V)$ and $A,D\in\mathcal{P}(V)$ if and only if $A=\emptyset$ implies $D\in g(\gamma)$.

Proof. Using eq. (9) we find

$$\begin{aligned}\Lambda_\gamma(\emptyset|A,D) &= 2^{2|V|}\sum_{B\in g(\gamma)}(\delta_{B,\emptyset}\delta_{A,B} - \delta_{D,B}\delta_{AD,B}) = \\ &= 2^{2|V|}\,\delta_{A,\emptyset}\,[1 - \sum_{B\in g(\gamma)}\delta_{D,B}]\ .\end{aligned} \tag{29}$$

The lemma directly follows from eq. (29). ■

Proof of Theorem 2. From Lemmas 2 and 3 it follows that $\Lambda_\gamma(\mathcal{B}|A,D) = 0$ for given $A,D\in\mathcal{P}(V)$, γ, $\mathcal{B}\subset\mathcal{P}(V)$ and all Ising models on $\mathcal{B}$ if and only if the relation $AP_{\alpha'}=\emptyset$ implies $D\in g(\gamma\cup\alpha)$ where $\alpha'\subset\alpha\subset g(\mathcal{B})$. From this it follows that $\Lambda_\gamma(\mathcal{B}|A,D) = 0$ for all Ising models on $\mathcal{B}$ if and only if for all $\alpha\subset\mathcal{B}$ such that $A\in g(\alpha)$, D is an element of $g(\alpha\cup\gamma)$. The theorem follows from the last statement. ■

Theorem 2 can be used in several ways. First, it states that for fixed $\mathcal{B}$, γ and A there is an identity of the form $\Lambda_\gamma(\mathcal{B}|A,D) = 0$ for every set D in the intersection (to be denoted by $g_A(\gamma)$) of all sets $g(\alpha\cup\gamma)$ with $\alpha\in A_\mathcal{B}$. It should, however, be observed that not all these identities are distinct, and that some of them are of the form $0=0$. Suppose $D,D'\in g_A(\gamma)$ and $DD'\in g(\gamma\cup\{A\})$, then it is easy to see that $\Lambda_\gamma(\mathcal{B}|A,D) = \Lambda_\gamma(\mathcal{B}|A,D')$; in this case D and D' will be called <u>equivalent</u>. Furthermore, if $D=\emptyset$, terms of opposite sign in $\Lambda_\gamma(\mathcal{B}|A,D)$ cancel pairwise so that the identity holds trivially.

Alternatively, it follows from Theorem 2 that for fixed $\mathcal{B}$, A and D there is an identity of the form $\Lambda_\gamma(\mathcal{B}|A,D) = 0$ for every set $\gamma\subset\mathcal{P}(V)$ having a non-empty intersection with all sets $\{B\in\mathcal{P}(V)|B=DX$ for some $X\in g(\alpha)\}$. Obviously, if $\Lambda_\gamma(\mathcal{B}|A,D) = 0$, then $\Lambda_{\gamma'}(\mathcal{B}|A,D) = 0$ for every $\gamma'\supset\gamma$.

4. Applications.

1) As a first application we consider the case $g(\gamma) = \{\emptyset\}$, i.e. we study the equation

$$(\sigma_\emptyset)_\mathcal{B}(\sigma_X\sigma_Y)_\mathcal{B} = (\sigma_X)_\mathcal{B}(\sigma_Y)_\mathcal{B}\ . \tag{30}$$

The equation has been studied by Setô for some special cases of X and Y in ferromagnetic systems $^{5)}$. Setô found, that for a ferromagnetic system with pair interactions and in a sufficiently general magnetic field $\langle \sigma_1\sigma_2 \rangle = \langle \sigma_1 \rangle \langle \sigma_2 \rangle$ and $\langle \sigma_1\sigma_2\sigma_3\sigma_4 \rangle = \langle \sigma_1\sigma_2 \rangle \langle \sigma_3\sigma_4 \rangle$ if and only if spin 1 "decouples" from spin 2 in the former case and spins 1 and 2 "decouple" from spins 3 and 4 in the latter case. We shall now generalize these results to Ising models with arbitrary (not necessarily ferromagnetic) n-spin interactions. If X or $Y=\emptyset$ eq. (30) is trivially true; furthermore, if $XY=\emptyset$ and $X\neq\emptyset$, Theorem 2 tells us that eq. (30) can not be valid for all Ising models on $\mathcal{B}$. So we assume X, Y, XY $\neq \emptyset$ and furthermore X, $Y \in g(\mathcal{B})$. We have the following lemmas.

Lemma 4. Let V be a finite set, $\mathcal{B} \subset \mathcal{P}(V)$ and X, $Y \in g(\mathcal{B})$. The following two statements are equivalent.

(a) For all $\gamma \subset \mathcal{B}$, $XY \in g(\gamma)$ implies $X \in g(\gamma)$.

(b) $\mathcal{B}_X \cap \mathcal{B}_Y = \emptyset$.

Proof. (a)→(b). Suppose $\mathcal{B}_X \cap \mathcal{B}_Y \neq \emptyset$. Then there are $\alpha \in X_{\mathcal{B}}$, $\beta \in Y_{\mathcal{B}}$ with common $\mathcal{B}$- factors , i.e. there are sets $\alpha' \subset \alpha$, $\alpha' \neq \alpha$ and $\beta' \subset \beta$, $\beta \neq \beta$, such that $P_{\alpha\backslash\alpha'} = P_{\beta\backslash\beta'}$. Let α' and β' be such sets with the property that $|\alpha\backslash\alpha'|$ is maximal. We have

$$XY = P_{\alpha'} P_{\beta'} \tag{31}$$

and hence $XY \in g(\alpha' \cup \beta')$. It follows from (a) that $X \in g(\alpha' \cup \beta')$, i.e. there are sets $\alpha'' \subset \alpha'$, $\beta'' \subset \beta'$ such that $X = P_{\alpha''} P_{\beta''}$, and hence $Y = P_{\alpha'\backslash\alpha''} P_{\beta'\backslash\beta''}$. It follows that

$$P_{\alpha\backslash x'} P_{\alpha'\backslash\alpha''} = P_{\beta''} \; . \tag{32}$$

Since α is cycle-free, $P_{\beta''} \neq \emptyset$; since β is cycle-free, $P_{\alpha'\backslash\alpha''} \neq \emptyset$. This leads to a contradiction because $|\alpha'\backslash\alpha''|$ has been assumed to be maximal. Therefore, $B_X \cap B_Y = \emptyset$.

(b)→(a). Take any cycle-free γ such that $XY = P_\gamma$ and let $\beta \in Y_{\mathcal{B}}$; then $X = P_{\beta\backslash\gamma} P_{\gamma\backslash\beta}$. Since $\mathcal{B}_X \cap \mathcal{B}_Y = \emptyset$, there is a $\gamma' \subset \gamma\backslash\beta$ such that $P_{\gamma\backslash\beta} = P_{\gamma'}$, and hence $X = P_{(\gamma\backslash\beta)\backslash\gamma'}$. Since $(\gamma\backslash\beta)\backslash\gamma' \subset \gamma$, this completes the proof of the lemma. ■

The following lemma tells us something about the structure of $\mathcal{B}_X$.

Lemma 5. Let V be a finite set, $X \in \mathcal{P}(V)$, $X \neq \emptyset$, and $\mathcal{B} \in \mathcal{P}(V)$. Then every $Z \in g(\mathcal{B}_X)$ belongs to a factorization of X with respect to $\mathcal{B}_X \cup \{Z\}$, i.e., there is a $\gamma \subset \mathcal{B}_X$ such that $ZP_\gamma = X$ and there is no $\gamma' \subset \gamma$ such that $Z = P_{\gamma'}$.

Proof. Let $\beta \subset \mathcal{B}_X$ such that $Z = P_\beta$. It follows that $(\mathcal{B}_X)_Z = (\mathcal{B}_X)_Z \cap (\mathcal{B}_X)_X \neq \emptyset$, and hence, by Lemma 4, applied to $\mathcal{B}_X$ instead of $\mathcal{B}$, there is a $\gamma \subset \mathcal{B}_X$ such that $ZX \in g(\gamma)$ and $Z \notin g(\gamma)$, from which the lemma follows. $\blacksquare$

Lemma 6. Let V be a finite set, $X, Y \in \mathcal{P}(V)$ and $\mathcal{B} \subset \mathcal{P}(V)$; then $\mathcal{B}_X \cap \mathcal{B}_Y = \emptyset$ if and only if $g(\mathcal{B}_X) \cap g(\mathcal{B}_Y) = \{\emptyset\}$.

Proof. If $g(\mathcal{B}_X) \cap g(\mathcal{B}_Y) = \{\emptyset\}$, then trivially $\mathcal{B}_X \cap \mathcal{B}_Y = \emptyset$. Suppose now $\mathcal{B}_X \cap \mathcal{B}_Y = \emptyset$ but $g(\mathcal{B}_X) \cap g(\mathcal{B}_Y) \neq \{\emptyset\}$, i.e., there are sets $\alpha \subset \mathcal{B}_X$, $\beta \subset \mathcal{B}_Y$, $\alpha, \beta \neq \emptyset$, such that $\emptyset \neq P_\alpha = P_\beta \in g(\mathcal{B}_X)$. From Lemma 5 it then follows that there is a $\gamma \subset \mathcal{B}_X$ such that $P_\beta P_\gamma = X$, and there is no $\gamma' \subset \gamma$ such that $P_\beta = P_{\gamma'}$; hence $\beta \cap \mathcal{B}_X \neq \emptyset$, which leads to a contradiction and ends the proof of the lemma. $\blacksquare$

Lemma 7. Let V be a finite set, $X, Y \in \mathcal{P}(V)$ and $\mathcal{B} \subset \mathcal{P}(V)$. If $\mathcal{B}_X \cap \mathcal{B}_Y = \emptyset$ and $\mathcal{B}' = \mathcal{B} \setminus (\mathcal{B}_X \cup \mathcal{B}_Y)$, then $g(\mathcal{B}') \cap g(\mathcal{B}_X \cup \mathcal{B}_Y) = \{\emptyset\}$.

Proof. Suppose $g(\mathcal{B}') \cap g(\mathcal{B}_X \cup \mathcal{B}_Y) \neq \{\emptyset\}$, i.e., there is a $Z \in g(\mathcal{B}')$, $\alpha \subset \mathcal{B}_X$, $\beta \subset \mathcal{B}_Y$, $\alpha, \beta \neq \emptyset$, such that $Z = P_\alpha P_\beta$. It follows that $ZP_\beta = P_\alpha \in g(\mathcal{B}_X)$. By Lemma 5, there is then a $\gamma \subset \mathcal{B}_X$ such that $ZP_\beta P_\gamma = X$, and there is no $\gamma' \subset \gamma$ such that $ZP_\beta = P_{\gamma'}$. Since $Z \notin g(\mathcal{B}_X) \cup g(\mathcal{B}_Y)$ it follows that $\beta \cap \mathcal{B}_X \neq \emptyset$, which leads to a contradiction. $\blacksquare$

From Theorem 1 and Lemmas 4 and 6 we immediately obtain the following theorem.

Theorem 3. Let V be a finite set, $\mathcal{B} \subset \mathcal{P}(V)$ and $X, Y \in g(\mathcal{B})$, $X, Y \neq \emptyset$. Then $\langle\sigma_\emptyset\rangle_{\mathcal{B}} \langle\sigma_X \sigma_Y\rangle_{\mathcal{B}} = \langle\sigma_X\rangle_{\mathcal{B}} \langle\sigma_Y\rangle_{\mathcal{B}}$ for every Ising model on $\mathcal{B}$ if and only if $g(\mathcal{B}_X) \cap g(\mathcal{B}_Y) = \{\emptyset\}$.

From Theorem 3 together with Lemma 7 it follows that if

$(\sigma_\emptyset)_{\mathcal{B}}(\sigma_X\sigma_Y)_{\mathcal{B}} = (\sigma_X)_{\mathcal{B}}(\sigma_Y)_{\mathcal{B}}$, the bond set is the union of three pairwise disjoint subsets $\mathcal{B}_X$, $\mathcal{B}_Y$ and $\mathcal{B}' = \mathcal{B}\setminus(\mathcal{B}_X\cup\mathcal{B}_Y)$, such that $g(\mathcal{B}_X)\cap g(\mathcal{B}_Y) = \{\emptyset\}$ and $g(\mathcal{B}')\cap g(\mathcal{B}_X\cup\mathcal{B}_Y) = \{\emptyset\}$. It follows that one can extend $\mathcal{B}$ with all elements from $g(\mathcal{B}_X)$, $g(\mathcal{B}_Y)$ and $g(\mathcal{B}')$ to a new bond set $\mathcal{B}''$ with the property that $(\sigma_\emptyset)_{\mathcal{B}''}(\sigma_X\sigma_Y)_{\mathcal{B}''} = (\sigma_X)_{\mathcal{B}''}(\sigma_Y)_{\mathcal{B}''}$.

2) In our second application we consider Ising models with pair interactions only, i.e., with $\mathcal{B}\subset\mathcal{P}_2(V) := \{B\in\mathcal{P}(V)\,\|\,|B| = 2\}$. This case was treated in I and II. We shall show that Theorem 1^* of II, restricted to functions of the type (24), is equivalent to Theorem 2 of the present paper, restricted to $\mathcal{B}\subset\mathcal{P}_2(V)$; by Lemma 1 and Theorem 1, the corresponding identities are the only ones that need to be considered.

With each Ising model with $\mathcal{B}\subset\mathcal{P}_2(V)$ we can associate a graph $G = (V,\mathcal{B})$ with vertex set V and edge set $\mathcal{B}$. Let $A\subset V$, $|A|$ even. G defines a set of partitions of A, denoted by $\Pi_A(G)$, in the following way. If π is a partition of A, i.e. a family of pairwise disjoint non-empty subsets of A whose union is A, then $\pi\in\Pi_A(G)$ if there is a set $\beta\subset\mathcal{B}$ such that two elements are in the same block of π if they are in the same connected component of the spanning subgraph $G_\beta = (V,\beta)$ of G. Conversely, every spanning subgraph G_β of G defines a partition of A, which will be denoted by $\pi(A,G_\beta)$.

The following lemma was used in the proof of Lemma 3 of II:

<u>Lemma 8.</u> Let $A\in\mathcal{P}(V)$ and $\beta\subset\mathcal{P}_2(V)$. If $\pi(A,G_\beta)$ is even, there is a cycle-free $\alpha\subset\beta$ such that $A=P_\alpha$.

<u>Proof.</u> Take $\alpha\subset\beta$ such that $\pi(A,G_\alpha)$ is even and $|\alpha|$ minimal. Then α is cycle-free and each $B\in\alpha$ separates two odd subsets of A in G_α. It follows that each vertex in A (in $V\setminus A$) is incident with an odd (even) number of elements of α, i.e., $A=P_\alpha$. ■

Let us define, for any set $B\in\mathcal{P}(V)$ and any partition π of a set $X\in\mathcal{P}(V)$,

$$\eta_\pi(B) = \begin{cases} 1 & \text{if the number of elements of B in every block of } \pi \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we define $\Pi_X(A,G) = \{\pi\in\Pi_X(G)\,|\,\eta_\pi(A) = 1\}$. We now have the following proposition.

<u>Proposition.</u> Let $\mathcal{B}\subset\mathcal{P}_2(V)$, $\gamma\subset\mathcal{P}(V)$, $A\in g(\mathcal{B})$, $D\in\mathcal{P}(V)$. The following two statements are equivalent:

(a) $D\in g(\alpha\cup\gamma)$ for all $\alpha\in A_{\mathcal{B}}$;

(b) $\Sigma_{B\in g(\gamma)}\{\eta_\pi(B)-\eta_\pi(BD)\} = 0$ for all $\pi\in\Pi_V(A,G)$. (33)

<u>Proof.</u> (a) → (b). Let $\pi\in\Pi_V(A,G)$ and $\beta\subset\mathcal{B}$ such that $\pi(V,G_\beta)=\pi$. Since $\pi(A,G_\beta)$ is even there is, by Lemma 8, a cycle-free $\alpha\subset\beta$ such that $A=P_\alpha$. Hence, $\alpha\in A_{\mathcal{B}}$, and hence, by (a), $D\in g(\alpha\cup\gamma)$, i.e., there are sets $\alpha'\subset\alpha$, $\gamma'\subset\gamma$ such that $D=P_{\alpha'}P_{\gamma'}$. Let $\pi'=\pi(V,G_\alpha)$; then $\eta_{\pi'}(DP_{\gamma'}) = \eta_{\pi'}(P_{\alpha'}) = 1$ because each component of G_α contains an even number of vertices of $P_{\alpha'}$. Since π' is a refinement of π, we also have $\eta_\pi(DP_{\gamma'})=1$, and hence $\eta_\pi(BDP_{\gamma'}) = \eta_\pi(B)$ for every $B\in g(\gamma)$. Since $P_{\gamma'}\in g(\gamma)$, we conclude that

$$\sum_{B\in g(\gamma)}\eta_\pi(BD) = \sum_{B\in g(\gamma)}\eta_\pi(BDP_{\gamma'}) = \sum_{B\in g(\gamma)}\eta_\pi(B),$$

which proves (b).

(b) →(a). Let $\alpha\in A_{\mathcal{B}}$ and $\pi=\pi(V,G_\alpha)$; $\pi\in\Pi_V(A,G)$ because $A=P_\alpha$. Since $\eta_\pi(\emptyset)=1$, we have $\Sigma_{B\in g(\gamma)}\eta_\pi(B) \geq 1$, and hence, by (b), $\Sigma_{B\in g(\gamma)}\eta_\pi(BD) \geq 1$. So there is a $B\in g(\gamma)$ such that $\eta_\pi(BD) = 1$, i.e., BD is partitioned by π into even subsets. In other words, $\pi(BD, G_\alpha)$ is even. By Lemma 8, $BD=P_{\alpha'}$ for some $\alpha'\subset\alpha$, and hence $D=P_{\alpha'}B\in g(\alpha\cup\gamma)$. ■

It is easily seen that the set of equations (33) is equivalent to any set obtained from (33) by replacing V by X, where $V_\gamma\cup A\cup D\subset X\subset V$. This suffices to show that the proposition establishes the relation between Theorem 2 and Theorem 1^* of II referred to above.

A convenient way to find, for a given $\mathcal{B}\subset\mathcal{P}(V)$ and $A,D\in\mathcal{P}(V)$, a family of sets γ such that $\Lambda_\gamma(\mathcal{B}|A,D) = 0$ for all Ising model on $\mathcal{B}$ is the following.

Consider the set $\Pi_A(A,G)$, ordered by refinement. Let $\Pi_A^m(A,G)$ be the set of minimal elements of this ordered set, $\alpha\subset A_{\mathcal{B}}$ and π a refinement of $\pi(A,G_\alpha)$ such that $\pi\in\Pi_A^m(A,G)$. Since every two points v_1,v_2 which are in the same block of π are connected in G_α by a chain, $\{v_1,v_2\}\in g(\alpha)$. Using this fact it is easy to see that $g(\pi)\subset g(\alpha)$ and that, hence, if $\gamma\subset\mathcal{B}$ then $g(\pi\cup\gamma)\subset g(\alpha\cup\gamma)$. It follows that

$$\psi := \bigcap_{\pi\in\Pi_A^m(A,G)} g(\pi\cup\gamma) \subset \bigcap_{\alpha\in A_{\mathcal{B}}} g(\alpha\cup\gamma) ,$$

so that if γ is such that $D\in\psi$, then $\Lambda_\gamma(\mathcal{B}|A,D)=0$ for all Ising models on $\mathcal{B}$.

Since ψ can be found relatively simply (because it does not depend on the detailed structure of the graph) examples of identities of the form $\Lambda_\gamma(\mathcal{B}|A,D) = 0$ can easily be constructed from the condition $D \in \psi$. For example, let G be a planar graph and $\{v_1,v_2,v_3,v_4,v_5,v_6\}$ a boundary sequence of G (for the definition of a boundary sequence, see I). Let $A = \{v_1,v_2,v_3,v_4,v_5,v_6\}$ and $D = \{v_1,v_2\}$. The set of minimal partitions consists in this case of $\pi_1 = (12|34|56)$, $\pi_2 = (23|45|16)$, $\pi_3 = (14|23|56)$, $\pi_4 = (25|34|16)$ and $\pi_5 = (12|45|36)$. If we now construct the sets $\phi_i := \{B \subset A | B = DX \text{ for some } X \in g(\pi_i)\}$ (each of which consists of eight elements), then the sets γ such that $\gamma \cap \phi_i \neq \emptyset$ for all i are readily constructed.

3) As a last example we consider a set of special identities which can readily be constructed for Ising models with n-spin interactions.

Let V be a finite set, $\mathcal{B} \subset \mathcal{P}(V)$. Suppose that V is the union of three pairwise disjoint sets V_1, V_2 and V_s and that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_s \cup \mathcal{B}_2$, with $\mathcal{B}_1 \subset \mathcal{P}(V_1 \cup V_s) \setminus \mathcal{P}(V_s)$, $B_2 \subset \mathcal{P}(V_2 \cup V_s) \setminus \mathcal{P}(V_s)$ and $\mathcal{B}_s \subset \mathcal{P}(V_s)$, i.e.,there is no $B \in \mathcal{B}$ containing both a vertex of V_1 and a vertex of V_2. We further assume that $V_1, V_2 \neq \emptyset$. V_s is then called a <u>separating set.</u>

Consider now non-empty sets $X_1, X_2 \in \mathcal{P}(V)$ such that $X_1 \cap V_2 = \emptyset = X_2 \cap V_1$ and $X_1X_2 \in g(\mathcal{B})$. Take $\alpha \subset \mathcal{B}$ such that $X_1X_2 = P_\alpha$, and let $\alpha_i = \alpha \cap \mathcal{B}_i$ $(i=1,2,s)$. Then we have $X_1P_{\alpha_1} = X_2P_{\alpha_2}P_{\alpha_s}$. Since $X_1P_{\alpha_1} \cap V_2 = \emptyset$ and $X_2P_{\alpha_2}P_{\alpha_s} \cap V_1 = \emptyset$ we have $X_1P_{\alpha_1} \subset V_s$, or $X_1 \in g(\alpha \cup \mathcal{P}(V_s))$. By Theorem 2,we therefore have the identity

$$\sum_{S \in \mathcal{P}(V_s)} \left\{ \langle \sigma_S \rangle_{\mathcal{B}} \langle \sigma_S \sigma_{X_1} \sigma_{X_2} \rangle_{\mathcal{B}} - \langle \sigma_S \sigma_{X_1} \rangle_{\mathcal{B}} \langle \sigma_S \sigma_{X_2} \rangle_{\mathcal{B}} \right\} = 0 \;. \qquad (34)$$

That separating sets are readily constructed can be seen as follows. Consider any set $A_1 \subset V$. Let A_s be the set of all vertices u, $u \notin A_1$, such that there is a set $B \in \mathcal{B}$ containing u and a vertex v in A_1. Let $A_2 = V \setminus (A_1 \cup A_s)$. Then any $V_s \supset A_s$ such that $A_1 \setminus V_s$ and $A_2 \setminus V_s$ are not empty is a separating set.

As an illustration, consider the following example. Let $v_1, v_2 \in V$ and define $\mathcal{B} = \mathcal{P}_2(V) \setminus \{v_1,v_2\}$. The set $V_s = V \setminus \{v_1,v_2\}$ is a separating set and we have the identity

$$\sum_{S \in \mathcal{P}_e(V_s)} \langle \sigma_S \rangle_{\mathcal{B}} \langle \sigma_S \sigma_{V_1} \sigma_{V_2} \sigma_{V_s} \rangle_{\mathcal{B}} - \sum_{S \in \mathcal{P}_o(V_s)} \langle \sigma_S \sigma_{V_1} \rangle_{\mathcal{B}} \langle \sigma_S \sigma_{V_2} \sigma_{V_s} \rangle_{\mathcal{B}} = 0 \qquad (35)$$

where $\mathcal{P}_e(V_S)$ and $\mathcal{P}_o(V_S)$ are the families of even and odd subsets of V_S, respectively.

It can easily be seen that eq. (35) can be written as

$$\sum_{S\in\mathcal{P}_e(V)} (-1)^{|S\cap\{v_1,v_2\}|} (\sigma_S)_{\mathcal{B}}(\sigma_{V\setminus S})_{\mathcal{B}} = 0 \;, \tag{35'}$$

a relation which was already derived in II. It is clear that eq. (35') remains valid, if we extend the bond set with those $B\in\mathcal{P}(V)$ which do not contain both v_1 and v_2.

5. Concluding remarks.

1) At first sight, an obvious generalization of the identity $\Lambda^{\alpha}(\mathcal{B}|A) = 0$ would be an identity of the form

$$\sum_{B\in g(\alpha)} \lambda_B(\sigma_B\sigma_A)_{\mathcal{B}}(\sigma_B\sigma_{A'})_{\mathcal{B}} = 0 \;, \tag{36}$$

with A and A' arbitrary subsets of V.

If $A,A'\in g(\mathcal{B})$, it is clear from Lemma 1 that we can restrict ourselves to the case $g(\alpha)\subset g(\mathcal{B})$. In that case, however, one finds by the use of multiple differentiation (see the proof of Theorem 1) that eq. (36) is equivalent to the identity $\Lambda^{\alpha}(\mathcal{B}|AA') = 0$. If A and A' are not both elements of $g(\mathcal{B})$, there can only be non-vanishing terms $(\sigma_B\sigma_A)_{\mathcal{B}}(\sigma_B\sigma_{A'})_{\mathcal{B}}$ if $AA'\in g(\mathcal{B})$; furthermore, a term is nonzero only if B is of the form B=AC with $C\in g(\mathcal{B})$, and it is easy to see that eq. (36) again reduces to an identity of the form $\Lambda^{\alpha}(\mathcal{B}|AA') = 0$.

2) Lemma 1 and Theorem 1 can be combined to the statement that every function of the form

$$\sum_{B\in\gamma} \lambda_B(\sigma_B)_{\mathcal{B}}(\sigma_B\sigma_A)_{\mathcal{B}} \;, \tag{37}$$

with $\mathcal{B}$, $\gamma\subset\mathcal{P}(V)$ and $A\in g(\mathcal{B})$, which is zero for every Ising model on $\mathcal{B}$, can be written as

$$\sum_{B\in\gamma\setminus g(\mathcal{B})} \lambda_B(\sigma_B)_{\mathcal{B}}(\sigma_B\sigma_A)_{\mathcal{B}} + \sum_{C\in g(\gamma\cap g(\mathcal{B}))} \lambda_{\hat{C}}\Lambda_C^{\gamma\cap g(\mathcal{B})}(\mathcal{B}|A) \tag{38}$$

where $\lambda_{\hat{C}}$ and $\Lambda_C^{\gamma\cap g(\mathfrak{s})}(\mathfrak{s}|A)$ are defined by eqs. (16) and (17) with $\lambda_B=0$ for $B\in g(\gamma\cup g(\mathfrak{s}))\setminus\gamma$, and where each individual term is zero. This shows that Theorem 2 together with Lemma 1 covers the most general case of an identity of the form (11).

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SAMENVATTING.

In dit proefschrift worden bepaalde relaties tussen spincorrelatiefuncties in Ising modellen onderzocht. Deze relaties worden bepaald door een klasse van functies (gemakshalve Λ-functies genoemd) die gedefiniëerd zijn als lineaire combinaties van produkten van twee spincorrelatiefuncties. De coëfficiënten (aangegeven met λ's) van deze produkten zijn willekeurige complexe getallen die niet van de koppelingsparameters van het systeem afhangen.

Onderzocht wordt wanneer voor een gegeven Ising model (gespecificeerd door een verzameling punten die de spins voorstellen en een verzameling lijnen tussen punten, die aangeven tussen welke spins er wisselwerking is) een gegeven Λ-functie

(a) nul is voor alle waarden van de koppelingsparameters (de relatie die hierdoor bepaald wordt noemen wij een Λ-identiteit),

of

(b) niet-negatief is voor alle niet-negatieve waarden van de koppelingsparameters (de relatie die hierdoor bepaald wordt noemen wij een Λ-ongelijkheid).

Het blijkt dat een nodige en voldoende voorwaarde waaronder (a) geldt, bestaat in een stel homogene lineaire vergelijkingen in de bovengenoemde coëfficiënten λ, met coëfficiënten 0 en 1. Een nodige en voldoende voorwaarde waaronder (b) geldt wordt gegeven door het corresponderende stel homogene lineaire ongelijkheden. De coëfficiënten 0 en 1 hangen nauw samen met bepaalde connectiviteitseigenschappen van een groep punten in de graaf die men op een natuurlijke manier aan ieder Ising model met paarwisselwerkingen kan toevoegen, en zijn eenvoudig te bepalen.

In hoofdstuk I wordt een voorbeeld gegeven van een klasse van Λ-identiteiten, die gelden voor Ising modellen die gedefiniëerd zijn op planaire grafen.

In hoofdstuk II worden bovengenoemde nodige en voldoende voorwaarden afgeleid en worden enkele voorbeelden van Λ-identiteiten en Λ-ongelijkheden gegeven.

Hoofdstuk III bevat een meer gedetailleerde studie van de Λ-ongelijkheden. Een van de resultaten uit dit hoofdstuk is dat iedere Λ-ongelijkheid (uit een bepaalde klasse) te schrijven is als een positief lineaire combinatie van een vast eindig stel "extremale" Λ-ongelijkheden (uit dezelfde klasse). Deze extremale Λ-ongelijkheden geven, binnen de beschouwde klasse, de scherpste boven- en ondergrenzen van spincorrelatiefuncties in termen van correlatiefuncties van lagere orde. Voorbeelden van extremale Λ-ongelijkheden worden afgeleid.

In hoofdstuk IV worden Λ-identiteiten onderzocht die gelden wanneer er n-spin interacties ($n > 2$) in het Ising model aanwezig zijn. Het blijkt dat iedere Λ-functie, die een Λ-identiteit bepaalt, geschreven kan worden in termen van eenvoudiger Λ-functies (waarin de bovengenoemde coëfficiënten , -1, 1 of 0 zijn), die ieder voor zich een Λ-identiteit bepalen. Een nodige en voldoende voorwaarde wordt afgeleid waaronder voor deze eenvoudiger functies (a) geldt. Enkele toepassingen worden daarna behandeld.

STUDIEOVERZICHT.

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Once, when T.A. Edison showed one of his new inventions, somebody said to him: "It's marvellous, Mr. Edison, but what is the use of it?" Edison answered: "And what is the use of the newborn child?"