SEPARABLE INTERACTIONS AND LIQUID ^{3}HE

F.W. NIJHOFF

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FRANK WILLEM NIJHOFF

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Promotiecommissie:

Promotor	:	Prof. dr. H.W. Capel
Referent	:	Dr. J. van der Linden
Overige leden	:	Prof. dr. G. Frossati
		Prof. dr. P.W. Kasteleyn
		Prof. dr. P. Mazur
		Prof. dr. Th.W. Ruijgrok

"Auch in der gelehrten Welt muss man lieben und wählen, um selbst existieren und sich selbst geniessen zu können."

Novalis, Fragmente.

voor Jeroen

CONTENTS

CHAPTER I

INTRODUCTION

1. General Remarks on the Theory of Liquid ^{3}He

The year 1972, in which the superfluid phases of ^{3}He were experimentally discovered by Osheroff et al.[1,2], marks the beginning of an upsurge of interest in the theory of liquid ^{3}He. Before 1972, however, much theoretical work had already been carried out. The success of the BCS (Bardeen, Cooper and Schrieffer) theory of superconductivity [3], initiated many attempts to generalize the underlying ideas [4-12]. It was realized at an early stage that the system of ^{3}He below $\sim$ 100 mK is a Fermi liquid, to which BCS type of theories could be applied [7-12]. It was soon appreciated however, that, in contrast to the case of superconductivity, in which electrons are supposed to form singlet pairs with total spin zero, in ^{3}He due to the strong hardcore repulsive interactomic potential Cooper pairs [13] could only be formed in a state with nonzero spin. Anderson and Morel [10,11] were the first who attempted to determine the possible superfluid phases in ^{3}He. They noted that in many cases such as e.g. in the case of p-wave pairing (i.e. the case that there is a dominant ℓ=1 term in the expansion of the pairing interaction into spherical harmonics), there would be an anisotropic phase, which could give rise to a type of orbital ferromagnetism. The work of Balian and Werthamer [12] was the first treatment, in which all the spin components were properly taken into account. For the case ℓ=1 they showed that the most favourable phase was an isotropic one, which is nowadays referred to as the BW-phase. Still, it was not clear from first principles which contributions could be considered to be dominant in the pairing interaction in ^{3}He. Calculations based on the interatomic potential [9,14], cf. also ref. 15, indicated that there was a strong attractive d-wave interaction and only a very weak p-wave interaction. Emery [16], however, suggested that due to renormalization effects the p-wave pairing was favoured. Nevertheless, up to now there is no conclusive evidence from first-principle calculations that p-wave pairing is dominant in ^{3}He. After 1972, when experimental results became available, many authors [17-23] addressed themselves to the problem of calculating the phases for various types of pairing. These calculations and the comparison with experiment give a strong indication that it is indeed the p-wave pairing which applies to the situation in liquid ^{3}He.

In order to give an idea of what goes on in an anisotropic superfluid we give a brief discussion of the Cooper pair problem [13]. Imagine two identical Fermi particles with mass m, interacting through a potential $\mathcal{V}(\underline{r}_1-\underline{r}_2)$, where $\underline{r}_1$ and $\underline{r}_2$ are the positions of the particles. The Schrödinger equation for this system reads, ($\underline{\nabla}_i=\partial/\partial\underline{r}_i$, i=1,2),

$$[-\frac{\hbar^2}{2m}(\underline{\nabla}_1^2+\underline{\nabla}_2^2)+\mathcal{V}(\underline{r}_1-\underline{r}_2)-E]\,\Psi(\underline{r}_1,\underline{r}_2;\sigma_1,\sigma_2)=0, \qquad (1.1)$$

where $\sigma_1,\sigma_2=\downarrow,\uparrow$ is the spin of the particles. We look at solutions of (1.1) for pair functions of the form

$$\Psi(\underline{r}_1,\underline{r}_2;\sigma_1,\sigma_2)=\phi(\underline{r}_1-\underline{r}_2)\,\chi(\sigma_1,\sigma_2), \qquad (1.2)$$

assuming the total momentum of the particles to be zero. Eq. (1.1) then reduces to

$$(-\frac{\hbar^2}{m}\underline{\nabla}^2+\mathcal{V}(r))\,\phi(\underline{r})=E\phi(\underline{r}),\ \underline{r}=\underline{r}_1-\underline{r}_2,\ r=|\underline{r}|, \qquad (1.3)$$

($\underline{\nabla}=\partial/\partial\underline{r}$). As a consequence of the Pauli principle for the wavefunction $\Psi(r_1, r_2; \sigma_1, \sigma_2)$ we have that $\chi(\sigma_1,\sigma_2)=\chi(\sigma_2,\sigma_1)$ if $\phi(\underline{r})=-\phi(-\underline{r})$ is an odd function of $\underline{r}$ and that $\chi(\sigma_2,\sigma_1)=-\chi(\sigma_1,\sigma_2)$ if $\phi(\underline{r})=\phi(-\underline{r})$ is an even function of $\underline{r}$. Taking Fourier transforms

$$\phi(\underline{r})=\sum_{\underline{k}} e^{i\underline{k}\cdot\underline{r}}\phi_{\underline{k}},\quad V(\underline{k},\underline{k}')=\frac{1}{\Omega}\int d\underline{r}\, e^{-i(\underline{k}-\underline{k}')\cdot\underline{r}}\mathcal{V}(r), \qquad (1.4)$$

we find from eq. (1.3)

$$(2\epsilon_k-\epsilon)\phi_{\underline{k}}=-\sum_{\underline{k}'}V(\underline{k},\underline{k}')\,\phi_{\underline{k}'}, \qquad (1.5)$$

where $\varepsilon_k = \hbar^2 k^2/2m - \varepsilon_F$ is the kinetic energy of the Fermi particle with respect to the Fermi energy ε_F, $k=|\underline{k}|$, $\varepsilon=E-2\varepsilon_F$, and Ω is the volume of the system. From eq. (1.4) we have that $V(\underline{k}, \underline{k}')$ is only a function of $|\underline{k}-\underline{k}'|^2 = k^2+k'^2-kk'\underline{n}\cdot\underline{n}'$, $\underline{n} \equiv \underline{k}/k$, $\underline{n}' \equiv \underline{k}'/k'$, and hence we can expand $V(\underline{k},\underline{k}')$ in terms of Legendre polynomials

$$V(\underline{k},\underline{k}') = \sum_{\ell=0}^{\infty} (2\ell+1)\, V_\ell\,(k,k')\, P_\ell(\underline{n}\cdot\underline{n}'). \tag{1.6}$$

In the limit $\Omega\to\infty$, which implies

$$\frac{1}{\Omega}\sum_{\underline{k}} \to \frac{1}{(2\pi)^3}\int d\underline{k} = \frac{1}{2\pi^2}\int_0^\infty dk\, k^2 \int\frac{d\Omega}{4\pi}\ , \quad d\Omega = \sin\theta d\theta d\psi\ , \tag{1.7}$$

where θ and ψ are the polar angles belonging to $\underline{k}$, i.e. $\underline{k}=k(\sin\theta\sin\psi,\ \sin\theta\cos\psi,\ \cos\theta)$, we can rewrite eq. (1.5) as

$$(2\varepsilon_k-\varepsilon)\ \phi_{\underline{k}} = \frac{\Omega}{2\pi^2}\int_0^\infty dk'\, k'^2 \int\frac{d\Omega'}{4\pi}\ V(\underline{k},\underline{k}')\ \phi_{\underline{k}'}\ . \tag{1.8}$$

If we take the following ansatz for $\phi_{\underline{k}}$

$$\phi_{\underline{k}} = \phi_\ell(k)\ Y_{\ell m}\,(\theta,\psi)\ , \tag{1.9}$$

the $Y_{\ell m}$ being spherical harmonics, and use the property

$$P_\ell(\cos\theta) = (2\ell+1)\int\frac{d\Omega'}{4\pi}\ P_\ell(\underline{n}\cdot\underline{n}')\ Y_{\ell m}\,(\theta',\psi'), \tag{1.10}$$

we finally end up with the equation

$$(2\varepsilon_k-\varepsilon)\ \phi_\ell(k) = -\frac{\Omega}{2\pi^2}\int_0^\infty dk'\ k'^2\ V_\ell(k,k')\ \phi_\ell(k'). \tag{1.11}$$

What we are interested in is solutions of eq. (1.11) with $\varepsilon<0$ corresponding to the two particles forming a bound state in the presence of a filled Fermi sea, i.e. $E<2\varepsilon_F$. Whether or not such solutions exist depends on the detailed behaviour of $V_\ell(k,k')$ as a function of k and k'. In the weak coupling limit one takes

$$V_\ell(k,k') = \begin{cases} -V_\ell/\Omega, & \text{if } |\varepsilon_k|,\ |\varepsilon_{k'}| < \hbar\omega \\ 0\ , & \text{otherwise}\ , \end{cases} \tag{1.12}$$

where $\hbar\omega$ is a cut-off parameter and V_ℓ a constant, i.e. $V_\ell(k,k')$ is zero except for a narrow band around the Fermi surface. In the case (1.12) with $v_\ell > 0$ we always have a solution of eq. (1.11), namely

$$\phi_\ell(k) = C/(2\varepsilon_k - \varepsilon) \ , \ C = -V_\ell \int_{-\hbar\omega}^{\hbar\omega} d\varepsilon' \ \tfrac{1}{2}\mathcal{N}(\varepsilon') \ \phi_\ell(k') \ , \ \varepsilon' = h^2 k'^2/2m - \varepsilon_F . \tag{1.13}$$

where we have introduced the density of states $\mathcal{N}(\varepsilon)$ per unit volume, i.e.

$$\tfrac{1}{2}\mathcal{N}(\varepsilon) = \frac{1}{\Omega}\sum_{\underline{k}} \delta(\varepsilon - \varepsilon_k) = \frac{1}{2\pi^2}\int_0^\infty dk \ k^2 \ \delta(\varepsilon - \varepsilon_k) \tag{1.14}$$

The energy of the Cooper pair follows from inserting the formula for $\phi_\ell(k)$ into the one for C in (1.13), leading to

$$1 = -V_\ell \int_{-\hbar\omega}^{\hbar\omega} d\varepsilon' \ \tfrac{1}{2}\mathcal{N}(\varepsilon') \ (2\varepsilon' - \varepsilon)^{-1} \simeq \tfrac{1}{2}\mathcal{N}(0) \ V_\ell \ \ell n \left(\frac{2\hbar\omega - \varepsilon}{-\varepsilon}\right), \tag{1.15}$$

where we have assumed that $\mathcal{N}(\varepsilon) = \mathcal{N}(0)$, where $\mathcal{N}(0)$ is the density of states at the Fermi surface, in the small interval from $-\hbar\omega$ to $\hbar\omega$. In order to study the nature of the solution for the wave function we use the property.

$$Y_{\ell m}(\theta,\psi) = (-1)^\ell \ Y_{\ell m}(\pi - \theta, \pi + \psi) \ , \tag{1.16}$$

which implies in eq. (1.8), (1.4) that

$$\phi_{-\underline{k}} = (-1)^\ell \ \phi_{\underline{k}} \ , \ \phi(-\underline{r}) = (-1)^\ell \ \phi(\underline{r}) \ . \tag{1.17}$$

Hence we have two possibilities :

i) ℓ is even, $\phi(\underline{r}) = \phi(-\underline{r})$, implying that $\chi(\sigma_1, \sigma_2) = -\chi(\sigma_2, \sigma_1)$, which yields on S = 0 singlet state, i.e.

$$\chi(\sigma_1, \sigma_2) = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \tag{1.18}$$

ii) ℓ is odd, $\phi(\underline{r}) = -\phi(-\underline{r})$, implying that $\chi(\sigma_1, \sigma_2) = \chi(\sigma_2, \sigma_1)$, which yields three possibilities, i.e. a triplet of states with S = 1, i.e.

$$\chi(\sigma_1,\sigma_2) = \begin{cases} |\uparrow\uparrow\rangle \\ \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ |\downarrow\downarrow\rangle \end{cases} \tag{1.19}$$

In order to give a more quantitative basis to these ideas one can (in a second quantized picture) introduce the following hamiltonian

$$\mathcal{H} = \sum_{\underline{k}} \sum_{\alpha} \varepsilon_k a^{\dagger}_{\underline{k}\alpha} a_{\underline{k}\alpha} + \frac{1}{2} \sum_{\underline{k},\underline{k}'} \sum_{\alpha,\beta} V(\underline{k},\underline{k}')\, a^{\dagger}_{\underline{k}\alpha} a^{\dagger}_{-\underline{k}\beta} a_{-\underline{k}'\beta} a_{\underline{k}'\alpha}, (\alpha,\beta=\uparrow,\downarrow) \qquad (1.20)$$

where $a^{\dagger}_{\underline{k}\alpha}$ and $a_{\underline{k}\alpha}$ are creation and annihilation operators of the Fermi-particles obeying canonical anticommutation relations. In a mean-field approximation, the free energy is obtained as the free energy of a hamiltonian

$$\mathcal{H}_{MF} = \sum_{\underline{k}} \sum_{\alpha} \varepsilon_k a^{\dagger}_{\underline{k}\alpha} a_{\underline{k}\alpha} + \sum_{\underline{k}} \sum_{\alpha,\beta} [\Delta_{\alpha\beta}(\underline{k})\, a^{\dagger}_{\underline{k}\alpha} a^{\dagger}_{-\underline{k}\beta} + (\Delta_{\alpha\beta}(\underline{k}))^{*} a_{-\underline{k}\beta} a_{\underline{k}\alpha}]$$

$$- \frac{1}{2} \sum_{\underline{k},\underline{k}'} \sum_{\alpha,\beta} V(\underline{k},\underline{k}') \langle a^{\dagger}_{\underline{k}\alpha} a^{\dagger}_{-\underline{k}\beta} \rangle \langle a_{-\underline{k}'\beta} a_{\underline{k}'\alpha} \rangle \quad , \qquad (1.21)$$

where

$$\Delta_{\alpha\beta}(\underline{k}) = \frac{1}{2} \sum_{\underline{k}'} V(k,k') \langle a_{-\underline{k}'\beta} a_{\underline{k}'\alpha} \rangle \quad , \qquad (1.22)$$

and the brackets denote the (grand)canonical averages with respect to the mean field hamiltonian $\mathcal{H}_{MF}$. As eq. (1.21) is bilinear in the creation- and annihilation operators, one can evaluate the averages in terms of the gapmatrix $\Delta_{\alpha\beta}$, and then, using eq. (1.22), one obtains a set of implicit equations, sometimes referred to as mean-field or "gap"-equations, from which $\langle a_{-\underline{k}'\beta} a_{\underline{k}'\alpha} \rangle$ can be solved. If the solution of the mean-field equations is not unique, one should select that particular solution which leads to the lowest value of the free energy. In this way the free energy corresponding to the hamiltonian (1.20) can be approximated, and nonvanishing values for the anomalous pair correlations $\langle a_{-\underline{k}'\beta} a_{\underline{k}'\alpha} \rangle$ indicate that the system is in an ordered phase. Inserting eq. (1.6) into (1.22) one can distinguish between various types of ordering. For example when only coefficients V_ℓ with even values of ℓ occur, eq. (1.22) implies that

$$\Delta_{\alpha\beta}(\underline{k}) = -\Delta_{\beta\alpha}(\underline{k}) = \Delta_{\alpha\beta}(-\underline{k}) \ , \qquad (1.23)$$

i.e. $\underline{\underline{\Delta}} = (\Delta_{\alpha\beta})$ is an antisymmetric 2x2 matrix and the elements are even functions of $\underline{k}$. In this case we have singlet pairing, as the nonvanishing correlations $2^{-1/2} \langle a_{-\underline{k}\uparrow} a_{-\underline{k}\downarrow} - a_{-\underline{k}\downarrow} a_{\underline{k}\uparrow} \rangle$ correspond to a singlet state. In the other case, when only odd values of ℓ occur in (1.6), we have

$$\Delta_{\alpha\beta}(\underline{k}) = \Delta_{\beta\alpha}(\underline{k}) = -\Delta_{\alpha\beta}(-\underline{k}) \quad , \tag{1.24}$$

i.e. $\underline{\underline{\Delta}}$ is a symmetric 2x2 matrix with as elements odd functions of $\underline{k}$. The only nonvanishing anomalous correlations are then $<a_{-\underline{k}\uparrow}\, a_{\underline{k}\uparrow}>$, $<a_{-\underline{k}\downarrow}\, a_{\underline{k}\downarrow}>$ and $2^{-1/2}<a_{-\underline{k}\uparrow}\, a_{\underline{k}\downarrow} + a_{-\underline{k}\downarrow}\, a_{\underline{k}\uparrow}>$, i.e. the three components of a triplet state. The special case $\ell=0$ corresponds to the BCS theory of superconductivity whereas the case $\ell=1$ is important in the theory of liquid ^{3}He. In the weak coupling limit (1.12) assuming that we can restrict ourselves to a finite number of values, $0\leq\ell\leq L$, in eq. (1.6), the hamiltonian (1.20) can be expressed as

$$\mathcal{H} = \Omega[T-2 \sum_{\ell=0}^{L} (2\ell+1)\, V_{\ell} \sum_{m=-\ell}^{\ell} \sum_{\alpha,\beta} (\omega_{\ell m\alpha\beta})^{\dagger}\, \omega_{\ell m\alpha\beta}] \quad , \tag{1.25}$$

where T is the kinetic energy term divided by Ω, and where

$$\omega_{\ell m\alpha\beta} = \frac{1}{2\Omega} \sqrt{\frac{4\pi}{2\ell+1}} \sum_{\underline{k}>0}{}' \; Y^{*}_{\ell m}(\underline{n})\, [a_{-\underline{k}\beta}\, a_{\underline{k}\alpha} -(-1)^{\ell}\, a_{-\underline{k}\alpha}\, a_{\underline{k}\beta}] \quad . \tag{1.26}$$

In eq. (1.25) use has been made of the expansion of $P_{\ell}(\underline{n}\cdot\underline{n}')$ in terms of spherical harmonics, i.e.

$$P_{\ell}(\underline{n}\cdot\underline{n}') = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\underline{n})\, Y^{*}_{\ell m}(\underline{n}') \tag{1.27}$$

($Y_{\ell m}(\underline{n}) = Y_{\ell m}(\theta,\psi)$). The prime in eq. (1.26) denotes that the summation is restricted to $\underline{k}$ vectors for which $|\varepsilon_{\underline{k}}|<\hbar\omega$, and $\underline{k}>0$ indicates a sum over pairs $(\underline{k},-\underline{k})$ of $\underline{k}$ vectors, from which only one of the two vectors $\underline{k}$ or $-\underline{k}$ is taken into account. Eq. (1.25) has a separable structure in the sense that the interaction term, quartic in the creation- and annihilation operators, can be rewritten as a <u>finite</u> sum of products of one-particle operators, bilinear in the creation- and annihilation operators. This feature allows us to give an exact evaluation of the free energy, via the method of separable interactions, as will be discussed in section 2.

We have seen in the treatment of the Cooper pair phenomenon and the brief summary of the mean-field approach, given above, that the nature of the pairing potential plays an important role for the occurrence of superfluid phases. In contrast to the situation in superconductivity, where the pairing-potential is mainly due to the electron-phonon interaction [24,25], the attempts to calculate the pairing interaction in ^{3}He from first-principles have not been very fruitful [26,15]. However, much information on the nature of the quasi-particles can be gained from Fermi liquid theory. Without going into the details concerning Landau's

Fermi Liquid theory (for a review, see refs. 28-31) we mention that ^{3}He behaves in many respects as a weakly interacting degenerate Fermi gas, notwithstanding the fact that the interactomic distance is comparable to the hard-core radius of the interatomic potential, which leads one to expect that collisions between the ^{3}He atoms would have a very important effect. If one assumes, however, that there is an important indirect exchange due to a cloud of polarized spins on top of the Van der Waals potential it can be shown [32,33] that there is as net effect an attractive triplet pairing, while the singlet-pairing is suppressed. We shall come back to this below, in the discussion of the spin fluctuation model.

We have discussed some theoretical considerations which could give a picture of the superfluidity in ^{3}He. After the experimental discovery in 1972 the theory developed very rapidly (for a review of these theoretical developments we refer to the articles 34-39, cf. also ref. 40). The phase diagram which was found experimentally is sketched qualitatively in fig. 1, see refs. 41-43.

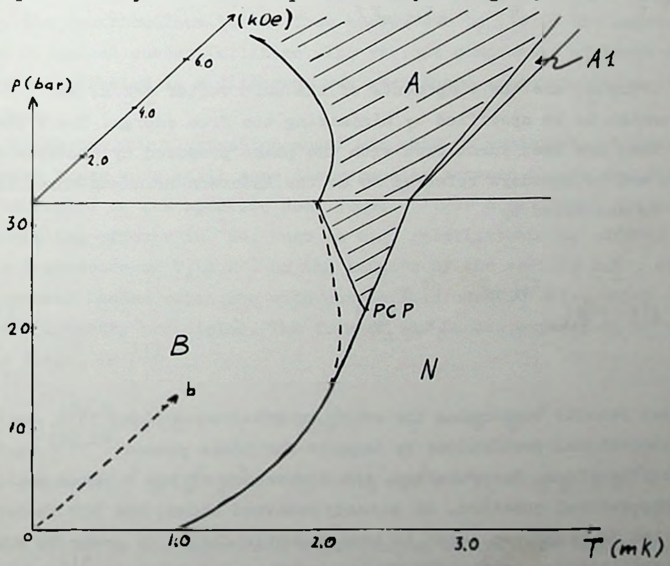

Figure 1. Schematic phase diagram of superfluid ^{3}He. T is the temperature, p is the pressure and b an external magnetic field. N denotes the normal fluid phase, A and B are the superfluid phases, where A occurs in the shaded region, A1 is the phase that occurs in a magnetic field. PCP is the polycritical point (2.4 mk, 21.5 bar). The dashed line indicates the A-B phase transition in a magnetic field of 378 G. Above 33 bar we have solid ^{3}He.

In the absence of a magnetic field, we have at low pressures one superfluid phase, the B phase. Above 22 bar there are two superfluid phases, the B phase and the A phase. Along the melting curve at 33 bar the N-A transition takes place at 2.60 mk and the A-B transition at 2.05 mk cf. refs. 1, 2. When a magnetic field b is applied the polycritical point PCP disappears and the B-A transition extends to much lower pressures.[†)] Furthermore, there occurs another phase, the A1 phase, which,as it were,"splits" the N-A transition line in the phase diagram at fixed pressure p, cf. ref. 47.

From a theoretical point of view the B-phase could be identified for zero field with the BW-phase mentioned above. This phase is described by a gap-matrix of the form [12)]

$$\underline{\underline{\text{BW}}}:\ \underline{\underline{\Delta}}(\underline{k}) = \Delta\begin{pmatrix} -n^x+i\,n^y & n^z \\ n^z & n^x+i\,n^y \end{pmatrix} , \tag{1.28}$$

where n^i, i=x,y,z, are the components of the unit vector $\underline{n}=\underline{k}/k$, and Δ is an order parameter to be specified by minimizing the free energy. The A phase on the other hand has been identified with the phase proposed by Anderson and Morel [11)], and is nowadays referred to as the Anderson-Brinkman-Morel (ABM) phase. It is described by

$$\underline{\underline{\text{ABM}}}:\ \underline{\underline{\Delta}}(\underline{k}) = \Delta\begin{pmatrix} n^x+i\,n^y & 0 \\ 0 & n^x+i\,n^y \end{pmatrix} . \tag{1.29}$$

Experimental results concerning the shift in NMR-frequencies [2,48)], as compared with the theoretical predictions by Leggett for these phases [49,50)] confirmed these identifications. Nevertheless, the occurrence of the A phase posed a puzzling theoretical question. As already remarked above, the BCS theory implies that only the BW-phase can occur in zero magnetic field. In order to account for the stabilitzation of the ABM-phase, Anderson and Brinkman [51)] came up with the idea of a "feedback-mechanism" due to spin fluctuations. The spin fluctuations are considered as arising from a contact interaction of the Hubbard type, i.e.

[†)] This is what is usually referred to as the "profound effect" , see refs.44-46.

$$\mathcal{H}_I = I \int d\underline{x} \;\; \psi^\dagger_\uparrow(\underline{x}) \; \psi_\uparrow(\underline{x}) \; \psi^\dagger_\downarrow(\underline{x}) \; \psi_\downarrow(\underline{x})$$

$$\psi_\alpha(\underline{x}) = \frac{1}{\sqrt{\Omega}} \sum_{\underline{k}} a_{\underline{k}\alpha} \, e^{i\,\underline{k}\cdot\underline{x}}, \; (\alpha = \uparrow,\downarrow). \tag{1.30}$$

In ferromagnetic systems such an interaction tends to suppress singlet-pairing [52], while in normal ^{3}He an interaction of the type (1.30) has been used to interpret the specific heat data [53]. It is natural, therefore, to include (1.30) in the hamiltonian for liquid ^{3}He. In ref. 51 it was shown that the various terms of an effective pairing interaction will have different contributions from (1.30). Moreover, when the system becomes superfluid, the terms of the pairing interaction will change in such a way that an anisotropic state, namely the ABM phase, will be favoured. These ideas were worked out in a more quantitative way in several papers [54-57]. In the spin fluctuation calculations, cf. also ref. 58, the contributions to the free energy due to (1.30) are expressed in terms of dynamical susceptibilities. The various components of these susceptibilities are affected in a different way when there is a transition to a superfluid phase.

Although the spin fluctuation model was able to explain the transition to the A phase there were still some discrepancies between theory and experiment, e.g. with regard to the specific heat jumps [59]. In a different approach to include strong coupling effects in ^{3}He, such as e.g. spinfluctuations effects or a possible dependence of $V_\ell(k,k')$ on the lenghts of the vectors $\underline{k},\underline{k}'$, one starts from a general Landau expansion with unspecified coefficients, which is based on general symmetry principles. The form of the Landau expansion, at zero magnetic field, is [17]

$$\Delta F = \frac{1}{3}\,\alpha(T)\; \mathrm{tr}(\underline{\underline{A}}^\dagger\cdot\underline{\underline{A}}) + \beta_1 \,|\mathrm{tr}(\underline{\underline{A}}\cdot\tilde{\underline{\underline{A}}})|^2 + \beta_2\, (\mathrm{tr}(\underline{\underline{A}}^\dagger\cdot\underline{\underline{A}}))^2$$

$$+ \beta_3\; \mathrm{tr}(\underline{\underline{A}}^\dagger\cdot\underline{\underline{A}}\cdot(\underline{\underline{A}}^\dagger\cdot\underline{\underline{A}})^*) + \beta_4\; \mathrm{tr}((\underline{\underline{A}}^\dagger\cdot\underline{\underline{A}})^2) + \beta_5\; \mathrm{tr}(\tilde{\underline{\underline{A}}}\cdot\underline{\underline{A}}\cdot(\tilde{\underline{\underline{A}}}\cdot\underline{\underline{A}})^*) \tag{1.31}$$

In eq. (1.31) ΔF is the superfluid contribution to the free energy F, $\underline{\underline{A}}$ is a 3x3 order matrix, which is related to the 2x2 matrix $\underline{\underline{\Delta}}(\underline{k})$ by

$$\underline{\underline{\Delta}}(\underline{k}) = \sum_{\alpha,j=x,y,z} n^\alpha A_{j\alpha}\; i\, \underline{\underline{\sigma}}^j\cdot\underline{\underline{\sigma}}^y \;, \tag{1.32}$$

where $\underline{\underline{\sigma}}^j$, j=x,y,z, are the Pauli-matrices, and $\underline{\underline{A}}^\dagger$ and $\tilde{\underline{\underline{A}}}$ denote the hermitean conjugate and transposed matrix respectively. $\beta_1,\ldots,\beta_5$ are unspecified coefficients, and the coefficient $\alpha(T)$ gives the dependence on the temperature. The BCS theory, described by the hamiltonian (1.20), yields the values

$$\beta_1:\beta_2:\beta_3:\beta_4:\beta_5 = 1:-2:-2:-2:2,\ \beta_1 = -\frac{7}{120}\mathcal{N}(0)\ \frac{\zeta(3)}{4\pi^2}\ \beta_c^2, \tag{1.33}$$

whereas the corrections due to spinfluctuations on these values are usually given by [57]

$$\beta_1^{SF}:\beta_2^{SF}:\beta_3^{SF}:\beta_4^{SF}:\beta_5^{SF} = 1:-2.0:0.5:5.5:7.0,\ \beta_1^{SF} = 0.1\delta\ \beta_1, \tag{1.34}$$

in which δ is a parameter to be fitted with experiment, and which one may expect to depend on the pressure p. The value of δ at the polycritical point is $\delta_{PCP} = 20/43$. Calculations of the specific heat jumps at the transition temperature in terms of $\beta_1,\ldots,\beta_5$ enables one to fit these parameters to the experimental values. It turns out that both the BCS-theory values (1.33) as well as the values obtained in spin fluctuation theory (1.34) do not fit the experimental data very well. Some subsequent work [60-63] was able to produce a better fit for the values of these parameters β to experiment, but we shall not go into its details.

In the presence of a magnetic field the situation in ^{3}He is much more complicated and a complete understanding has not yet been obtained. Within the weak coupling theory, the experimentally observed A-A1 splitting of the N-A transition line in a magnetic field [47] could be explained on the basis of an asymmetry in the density of states around the Fermi surface [64]. The treatments of the influence of strong-coupling effects on the phase diagram [65,66], and on thermodynamic quantities [67-71] all were based on certain extrapolations of the ABM-phase or the BW-phase. Although there is a good understanding of the change of the order parameter for these phase due to a magnetic field [72-74], no proper treatment of all the possible breaking mechanisms, caused by the magnetic field, exists. One can imagine that these breakings might have the effect that certain phases, other than the BW-phase or ABM-phase will become favourable. Rather than extrapolating the b=0 results for the phases to finite values of the magnetic field, one should investigate the complete 18 order parameter problem in a finite magnetic field. This is one of the problems that will be discussed in this thesis.

This concludes our short account of the present theoretical understanding of the equilibrium properties of superfluid ^{3}He. We did not dwell upon the nonequilibrium properties and the influence of the dipolar interaction, although there are many very interesting features connected with these problems, such as anomalous NMR-shifts and currents, textures in confined geometries and solitons. For these problems we refer to the review articles 34,37 , and in particular to the papers 75,76 and references therein.

2. The Method of Separable Interactions.

In section 1 we already mentioned some problems concerning the theory of liquid ^{3}He that still have to be clarified. There are two main problems to which we shall address ourselves in this thesis. We already pointed out in section 1 that in the presence of an external magnetic field the situation is still far from clear. The weak coupling theory predicts the occurrence of the BW-phase at b=0, but even a small magnetic field will lead to a breaking of the symmetry that favours an isotropic phase, and the usual arguments for the occurrence of the BW-phase do no longer apply. A systematic determination of the phase diagram on the basis of a rigorous minimization of the Landau expansion in terms of the 18 real order parameters has not been done so far [66]. The same applies to the problem of the strong coupling contributions to the Landau expansion. Barton and Moore [21] have presented a variety of phases, but a systematic derivation of these phases from the gap equations for the minimum of the Landau expansion has not been given. Furthermore, it is very difficult to predict the phase diagram for general coefficients of the Landau expansion. It is therefore necessary to gain some insight in the values of these coefficients, e.g. on the basis of an explicit model calculation, in order to have a grasp on this problem. Inspired by the arguments given in spin fluctuation theory, we choose the model consisting of the BCS hamiltonian (1.20) together with the Hubbard interaction (1.30) for the description of liquid ^{3}He. In order to include the magnetic field we add a Zeeman-term, i.e.

$$\mathcal{H}_Z = -b \sum_{\underline{k}} (a_{\underline{k}\uparrow}^{\dagger} a_{\underline{k}\uparrow} - a_{\underline{k}\downarrow}^{\dagger} a_{\underline{k}\downarrow}) \tag{2.1}$$

to the hamiltonian. For the pairing-interaction we use the weak coupling assumption (1.12). In order to derive the free energy for this model we do not want to follow the lines of reasoning used in the spin fluctuation theory calculations [54-58], but explore the fact that the pairing-interaction is of a separable type. For this type of interactions there exists a rigorous scheme for calculating the free energy, and the results we obtain in this way are exact.

In order to give an outline of the method of separable interactions, we consider eq. (1.25). Eq. (1.25) is of the form

$$\mathcal{H} = \Omega(T - \sum_{S,S'} V_{S,S'} J_S^\dagger J_{S'}) \quad , \tag{2.2}$$

where ΩT and ΩJ_S are sum operators of the form

$$\Omega J_S = \sum_{\underline{k}} J_S(\underline{k}) \quad , \quad \Omega T = \sum_{\underline{k}} T(\underline{k}), \tag{2.3}$$

in which $T(\underline{k})$ and $J_S(\underline{k})$ are one-particle operators, associated with the wave vector $\underline{k}$, and in which the summations over S and S' contain a finite number of terms. Let us define a <u>reference hamiltonian</u> as function of complex parameters h_S associated with the operators J_S, i.e.

$$\mathcal{H}_0(\{h_S\}) = \Omega(T+V) = \Omega[T - \sum_{S,S'} V_{S,S'} (J_S h_{S'}^* + J_{S'}^\dagger h_S)] \ . \tag{2.4}$$

Then, according to the theory of separable interactions [77-83], the free energy per volume corresponding to the hamiltonian , i.e.

$$f \equiv \lim_{\Omega\to\infty} (-\beta\Omega)^{-1} \ln \mathrm{Tr}\, e^{-\beta\mathcal{H}} \quad , \tag{2.5}$$

is given by

$$f = \inf_{\{m_S\}} [g_0(\{m_S\}) - \sum_{S,S'} V_{S,S'} m_S m_S^*] \tag{2.6}$$

in which

$$g_0(\{m_S\}) = \sup_{\{h_S\}} [f_0(\{h_S\}) + \sum_S (h_S^* m_S + m_S^* h_S)] \tag{2.7}$$

is the Legendre transform of the free energy per unit volume corresponding to the reference hamiltonian $\mathcal{H}_0$, i.e.

$$f_0(\{h_S\}) = \lim_{\Omega\to\infty} (-\beta\Omega)^{-1} \ln \mathrm{Tr}\, \exp(-\beta\, \mathcal{H}_0(\{h_S\})) \ . \tag{2.8}$$

In the case of (1.25), for which the reference hamiltonian is a free-particle hamiltonian, the values of m_S at the infimum in eq. (2.6) are given by

$$m_S = \langle J_S \rangle \ , \tag{2.9}$$

where the brackets denote the canonical average with respect to the reference hamiltonian. The derivation of (2.6) is given in refs. 81-83, also for more general cases including more general types of operators T and J_S, such as short range interaction operators [84,85], and also for more general types of interactions such as arbitrary analytic functions $P(J_S, J^{\dagger}_{S'})$ of the set of operators $\{J_S, J^{\dagger}_{S'}\}$, rather than simple quadratic forms.

In the special case that the matrix $V_{S,S'}$ is positive definite, i.e.

$$\sum_{S,S'} e^{*}_S V_{S,S'} e_{S'} > 0, \tag{2.10}$$

for arbitrary complex numbers e_S, one has the fundamental theorem of Bogoliubov Jr. [77], which applies to a broader class of operators. In fact, under the conditions

$$\begin{aligned} &||T|| < M_T \ , \ ||J_S|| < M_S, \\ &||[T\ J_S]|| < \varepsilon_S \ , \ ||[J_S\ J_{S'}]|| < \varepsilon_{S,S'} \end{aligned} \tag{2.11}$$

for the norms of the operators T and J_S and their commutators, in which M_T and M_S are finite numbers, and $\varepsilon_S \to 0$ and $\varepsilon_{S,S'} \to 0$ in the thermodynamic limit $\Omega \to \infty$, it can be shown that

$$f = \min_{\{m_S\}} \ [f_0(\{m_S\}) + \sum_{S,S'} V_{S,S'} m_S m^{*}_{S'}] \ . \tag{2.12}$$

This theorem was formulated in ref. 77, cf. also 78, and is also important in the proof of less general theorems in the case that eq.(2.3) is not satisfied.

Eq. (2.6) is used in chapter II, where it is shown that under the condition $V_\ell < \sup(V_0, V_1)$ there will be no pairing with $\ell \geq 2$, i.e. $\langle \omega_{\ell m \alpha \beta} \rangle = 0$ for $\ell \geq 2$. If, furthermore, $V_1 < V_0$, the hamiltonian (1.20) with (1.6), (1.12) leads to the BCS-theory of superconductivity, whereas in the opposite case $V_0 < V_1$, it leads to the BW-phase given in (1.28). In the remainder of this thesis we will then restrict ourselves to the case $\ell=1$, also in the presence of a magnetic field, i.e. including a term (2.1) in the hamiltonian, thereby ignoring the terms with V_ℓ, $\ell \neq 1$, in the pairing interaction. In chapter II and III we will examine the weak coupling theory, i.e. the theory without taking into account contributions from spin fluctuations. In the absence of the Hubbard term it is straightforward to find with the help of eq. (2.6) an exact expression for the free energy and for arbitrary magnetic fields. In order to derive the features of the phase diagram it is convenient to restrict oneself to a Landau expansion up to fourth order in the 18 real order parameters and with coefficients valid for arbitrary values of the magnetic field. Even in this case a complete minimization of the Landau expansion is a very hard task, and simplifying assumptions, such as the restriction to small magnetic fields and the assumption of a certain rigidity of the directional dependence of the order parameters under changes in the temperature have to be made. Under these assumptions the 18 order-parameter problem can be reduced to the minimization of an appropriate 3 - parameter function in terms of the lengths m_1, m_2, m_3 of three socalled "order vectors", $\underline{m}_1$, $\underline{m}_2$, $\underline{m}_3$, the components of which are related to the elements of the matrix $\underline{\underline{A}}$. cf. eq. (1.32), via the relations

$$m_1^j = -A_{1j} + i\,A_{2j},\; m_2^j = A_{1j} + i\,A_{2j},\; m_3^j = A_{3j}, \quad j=x,y,z. \qquad (2.13)$$

This reduction can only be achieved on the basis of the precise values of the coefficients in the Landau expansion as functions of the magnetic field, even in the fourth order terms. In the phase diagram as obtained from the 3 parameters function there occur three possible phases:

i) an extension of the BW phase to finite values of the magnetic fields, having three types of ordering, ↑↑, ↓↓ and ↑↓,
ii) the ABM phase with two types of ordering, ↑↑ and ↓↓,
iii) the A1 phase with one type of ordering, either ↑↑ or ↓↓, depending on the choice of the asymmetry in the density of states, being positive or negative.

Having, obtained a global understanding of the weak coupling situation, one is able to tackle the problem of including strong-coupling effects. This will be done in chapters IV and V, by including the Hubbard interaction (1.30) in the hamiltonian. In this case we can absorb $\mathcal{H}_T$ into the operator T, and hence include the $\mathcal{H}_I$ in the reference hamiltonian We cannot, however, use now eq.(2.6), because T can no longer be expressed in the form as given in eq. (2.3). Nevertheless, if we restrict ourselves to positive values of the coefficient V_1 of the pairing interaction, we may use the theorem of Bogoliubov Jr. eq. (2.12), where the requirements (2.11) for the operators (1.26) and (1.30) are easily verified. So, even in the case that we include (1.30) in the hamiltonian, we have an exact expression of the free energy in terms of the free energy of a reference system, and in principle a rigorous derivation of the phase diagram from the minimization of a Landau expansion in terms of the proper order parameters. The only difficulty, of course, is now that the reference free-energy f_0 cannot be evaluated exactly, due to the fact that the reference hamiltonian contains the Hubbard interaction. In chapter IV, therefore, we perform a perturbation calculation to obtain the coefficients of the Landau expansion up to second order in the coupling constant I of the Hubbard interaction. The results can be used to study the stability of the phase diagram obtained in chapter III in the weak coupling case, under small perturbations of the Hubbard type. This is done in chapter V, where it will be shown that in the presence of a small Hubbard interaction the phase diagram will contain two new phases, denoted by VI and I, rather than the ABM-phase. It is not necessarily easy to detect these new phases in practise, because phase VI is degenerate with the ABM-phase in the limit that the Hubbard interaction vanishes, and the order parameters will not differ much from the ones in the ABM-phase for small values of the coupling constant I. Furthermore, phase I will only occur in a small region of the phase diagram, if the Hubbard interaction is reasonably small. Nevertheless from a theoretical point of view the difference between the two phase diagrams is appreciable. Furthermore in chaper V we also study the phase diagram for zero magnetic field, but as a function of the Landau coefficients, thereby assuming that the signs of the contributions of the Hubbard interaction are correctly given by the second order results. On the basis of this phase diagram it will be argued that the transition from the B-phase to the A-phase at high pressures may be interpreted as a transition from the BW-phase to the phase VI, rather than to the ABM-phase.

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CHAPTER II

LANDAU EXPANSIONS IN THE WEAK COUPLING-LIMIT

1. Introduction.

The investigation of liquid ^{3}He has received a lot of interest, especially after the experimental observation of the superfluid phases A and B [1]. These phases have been identified with theoretical predictions [2),3)] based on a microscopic theory starting from a hamiltonian which contains a pairing interaction. In fact, phase A has been identified with the (anisotropic) ABM phase first found by Anderson and Morel [2], whereas phase B can be identified with the (isotropic) superfluid BW phase found by Balian and Werthamer [3]. In the presence of a magnetic field there is a phase transition within the A phase [4] leading to a new phase which is called the A1 phase. The phase diagram in the presence of a magnetic field has been explained qualitatively by Ambegaokar and Mermin [5] on the basis of a Landau expansion. A general treatment of the theory can be found in the review paper by Leggett [6] and an excellent review of experimental results was given by Wheatley [7]. For more recent results on ^{3}He, see e.g. ref. 8 and references cited therein.

From a microscopic point of view ^{3}He can be described by a hamiltonian of the type

$$\mathcal{H} = \sum_{\vec{k}} \sum_{\alpha} \varepsilon_k a^\dagger_{\vec{k}\alpha} a_{\vec{k}\alpha} + \tfrac{1}{2} \sum_{\vec{k},\vec{k}'} \sum_{\alpha,\beta} V(\vec{k},\vec{k}') a^\dagger_{\vec{k}\alpha} a^\dagger_{-\vec{k}\beta} a_{-\vec{k}'\beta} a_{\vec{k}'\alpha}$$

$$- b \sum_{\vec{k}} (a^\dagger_{\vec{k}\uparrow} a_{\vec{k}\uparrow} - a^\dagger_{\vec{k}\downarrow} a_{\vec{k}\downarrow}), \quad (\alpha,\beta = \uparrow,\downarrow) . \qquad (1.1)$$

In eq. (1.1) we have a sum of one-particle operators which is diagonal in terms of the fermion operators $a^\dagger_{\vec{k}\alpha}$, $a_{\vec{k}\alpha}$, where $\vec{k}$ denotes the wave vector and α the spin which can be either up ($\uparrow$), or down ($\downarrow$). The ε_k are the one-particle energies measured with respect to the Fermi energy and are assumed to be independent of the direction of $\vec{k}$ and the spin α. The second term in (1.1) is a general pairing interaction between pairs of quasi particles ($\vec{k}$, $-\vec{k}$) and ($\vec{k}'$, $-\vec{k}'$) both having total wave vector 0 and α and β denote the spin components which can be either up or down. The last term in (1.1) describes the effect of a magnetic field in the z direction.

The matrix elements $V(\vec{k},\vec{k}')$ are assumed to arise from a pair potential $\mathcal{V}(r)$ which depends only on the distance r between quasi particles, i.e.

$$V(\vec{k},\vec{k}') = \frac{1}{\Omega}\int d\vec{r}\ \mathcal{V}(r)\ e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}} = V(|\vec{k}-\vec{k}'|)\ . \tag{1.2}$$

The interpretation of the nature of the quasi particles is not trivial and can be given within the context of the Fermi-liquid theory, see e.g. refs. 6 and 9 and references cited therein.

In view of (1.2) the matrix elements $V(\vec{k},\vec{k}')$ can be expanded in terms of Legendre polynomials, i.e.

$$V(\vec{k},\vec{k}') = \sum_{\ell} (2\ell+1)V_\ell(k,k')P_\ell(\vec{n}\cdot\vec{n}'), \qquad \vec{n} = \vec{k}/k, \qquad \vec{n}' = \vec{k}'/k', \tag{1.3}$$

where P_ℓ is the Legrendre polynomial of order ℓ and k and k' are the lengths of the vectors $\vec{k}$ and $\vec{k}'$.

In the case that $V(\vec{k},\vec{k}')$ does not depend on the direction of the vectors $\vec{k}$ and $\vec{k}'$, i.e. $V(\vec{k},\vec{k}') = V(k,k')$, or equivalently $V_\ell(k,k') = V(k,k')$ for $\ell=0$ and $V_\ell(k,k') = 0$ for $\ell\neq 0$, the hamiltonian (1.1) for b=0 reduces to the BCS hamiltonian [10] in the theory of superconductivity. The free energy in ref. 10 was evaluated for general $V(k,k')$ in the framework of the Hartree-Fock approximation. In the so-called weak-coupling limit the coefficients $V(k,k')$ can be taken to be independent of k and k' in a small interval around the Fermi energy, i.e.

$$V(\vec{k},\vec{k}') = \begin{cases} -V/\Omega, & \text{if } |\varepsilon_k|,\ |\varepsilon_{k'}| < \hbar\omega \\ \\ 0\ , & \text{otherwise.} \end{cases} \tag{1.4}$$

Here Ω is the volume of the system and ω is a cut-off parameter of the order of the Debye frequency.

In view of (1.4) the pairing interaction in (1.1) can be considered to be of an extremely long-range type in $\vec{k}$ space, since the coupling constant does not depend on $\vec{k}$ and $\vec{k}'$, provided that ε_k and $\varepsilon_{k'}$ belong to the small interval around the Fermi energy. More specifically, the pairing interaction in (1.1) with (1.4) can be expressed as

$$-\tfrac{1}{2}\Omega V J^\dagger J, \qquad J^\dagger \equiv (1/\Omega) \sum_{\vec{k}}{}' \, a^\dagger_{\vec{k}\uparrow} a^\dagger_{-\vec{k}\downarrow} \quad , \tag{1.5}$$

where the prime indicates that the summation is restricted to $\vec{k}$ vectors satisfying $|\varepsilon_k| < \hbar\omega$. Eq. (1.5) shows that the pairing interaction can be expressed as a product of simpler operators and therefore may be considered as a separable interaction. Using (1.5) it has been shown by various methods, cf. refs. 11-14, that the Hartree-Fock type of approach in which correlations are neglected, leads to a rigorous expression for the free energy per unit volume in the thermodynamic limit, see also refs. 14-16 and references cited therein for more general results. Exactly solvable models which describe the coexistence of superconductivity and ferromagnetism have been treated in refs. 17, 18.

The superfluid phases in liquid ^{3}He almost certainly arise from a pairing hamiltonian with $\ell=1$ pairing [8], i.e. $V_\ell(k,k') = V_1(k,k')$ for $\ell=1$ and $V_\ell(k,k') = 0$ for $\ell\neq 1$. In fact, the ABM phase [2], the BW phase [3] and other superfluid phases [6], in the special case that b=0, have been obtained from (1.1), (1.3) with $V_\ell(k,k') = 0$ for $\ell\neq 1$, in the framework of the Hartree-Fock approximation. Furthermore, it was shown that the BW phase in the weak-coupling limit has the lowest free energy [3]. This is not in agreement with experiments under high pressure and the (observed) stabilization of the A phase has been explained on the basis of spin fluctuations, see e.g. refs. 19-22, arising from a repulsive contact term of the Hubbard type.

In the present chapter we derive an exact expression for the free energy per unit volume in the thermodynamic limit for the general hamiltonian (1.1) with a finite number of ℓ waves in the special case of the weak-coupling limit, i.e.

$$V_\ell(k,k') = \begin{cases} -V_\ell/\Omega, & \text{if } |\varepsilon_k|, |\varepsilon_{k'}| < \hbar\omega \text{ and } \ell \leq L \\ 0, & \text{otherwise} \end{cases} \quad . \tag{1.6}$$

With (1.6) the pairing term in (1.1) can be expressed in terms of separable interactions and in section 2 the free energy per unit volume is evaluated exactly. The free energy is expressed in terms of the absolute minimum over a well defined set of thermodynamic variables of a function containing the Legendre transform of the free energy per unit volume of a reference system. The reference system is described by a hamiltonian which is bilinear in the fermion operators and contains terms like $a^\dagger_{\vec{k}\alpha} a^\dagger_{-\vec{k}\beta}$, $a_{-\vec{k}\beta} a_{\vec{k}\alpha}$ in addition to the diagonal terms $a^\dagger_{\vec{k}\alpha} a_{\vec{k}\alpha}$. The calculation of the reference free energy, as shown in section 3, involves the solution of a 4×4 eigenvalue problem for every $\vec{k}$.

In this chapter we shall in particular consider two cases in which the eigenvalue problem is rather simple. The first case is the case of mixed singlet-triplet spin pairing in the absence of a magnetic field and will be discussed in section 4. The second case is a special case of triplet spin pairing, i.e. the hamiltonian (1.1) with (1.6) and only terms $\ell=1$ ($V_\ell = 0$, $\ell\neq 1$), under the influence of a magnetic field $b\neq 0$. In section 5 we give an expansion of the reference free energy and we perform the Legendre transformation. In section 6 we present the Landau expansion with terms of the 4th degree as a function of 18 real order parameters. In chapter III the different types of ordering will be discussed in detail and special attention will be paid to the generalization of the BW phase for $b\neq 0$.

2. Free energy of the exactly solvable model.

In this section we consider the hamiltonian (1.1) in the weak-coupling limit (1.6). Using (1.3) and (1.6), eq. (1.1) can be rewritten

$$\mathcal{H} = T - (1/2\Omega) \sum_{\ell\leq L} (2\ell+1)V_\ell \sum_{\vec{k},\vec{k}'}{}'' \sum_{\alpha,\beta} P_\ell(\vec{n}\cdot\vec{n}')a^\dagger_{\vec{k}\alpha}\, a^\dagger_{-\vec{k}\beta}\, a_{-\vec{k}'\beta}\, a_{\vec{k}'\alpha} , \qquad (2.1)$$

where the primes indicate that the summation over $\vec{k},\vec{k}'$ is restricted to the small interval $|\varepsilon_k|$, $|\varepsilon_{k'}| < \hbar\omega$ around the Fermi energy and where the operator T is given by

$$T = \sum_{\vec{k}} \left[(\varepsilon_k - b)\, a^\dagger_{\vec{k}\uparrow} a_{\vec{k}\uparrow} + (\varepsilon_k + b)\, a^\dagger_{\vec{k}\downarrow} a_{\vec{k}\downarrow} \right] . \qquad (2.2)$$

In view of the coupling between pairs $(\vec{k},-\vec{k})$ and $(\vec{k}',-\vec{k}')$ with total wave vector 0 in (2.1), it is convenient to decompose the summation over $\vec{k}$ vectors into a summation over pairs $(\vec{k},-\vec{k})$ and a summation over the two $\vec{k}$ vectors $\vec{k}$ and $-\vec{k}$ belonging to the pair $(\vec{k},-\vec{k})$. The summation over pairs will be formally denoted as a summation $\vec{k} > 0$. In view of the property

$$P_\ell(-\vec{n}\cdot\vec{n}') = (-1)^\ell P_\ell(\vec{n}\cdot\vec{n}') \qquad (2.3)$$

the hamiltonian (2.1) can be expressed as

$$\mathcal{H} = T - (1/2\Omega) \sum_{\ell\leq L} (2\ell+1)V_\ell \sum_{\vec{k}>0}{}' \sum_{\vec{k}'>0}{}' \sum_{\alpha,\beta} P_\ell(\vec{n}\cdot\vec{n}') \times$$

$$\times [a^\dagger_{\vec{k}\alpha} a^\dagger_{-\vec{k}\beta} + (-1)^\ell a^\dagger_{-\vec{k}\alpha} a^\dagger_{\vec{k}\beta}][a_{-\vec{k}'\beta} a_{\vec{k}'\alpha} + (-1)^\ell a_{\vec{k}'\beta} a_{-\vec{k}'\alpha}] . \qquad (2.4)$$

Using the addition theorem for spherical harmonics

$$P_\ell(\vec{n}\cdot\vec{n}') = (4\pi/2\ell+1) \sum_{m=-\ell}^{\ell} Y_{\ell m}(\vec{n}) Y^*_{\ell m}(\vec{n}'), \qquad (2.5)$$

where the $Y_{\ell m}(\vec{n})$, for $m=-\ell,\ldots,\ell$, are spherical harmonics, or more generally an orthonormal set of linear combinations of spherical harmonics, and using also the fermion-anticommutation relations, eq. (2.4) can be rewritten

$$\mathcal{H} = T - 2\Omega \sum_{\ell\leq L} (2\ell+1)V_\ell \sum_{m=-\ell}^{\ell} \mathrm{tr}(\underline{\underline{\omega}}^\dagger_{\ell m} \cdot \underline{\underline{\omega}}_{\ell m}) , \qquad (2.6)$$

where $\underline{\underline{\omega}}_{\ell m}$ is a 2×2 operator matrix with elements

$$\omega_{\ell m\alpha\beta} = (1/2\Omega)(4\pi/2\ell+1)^{\frac{1}{2}} \sum_{\vec{k}>0}{}' Y^*_{\ell m}(\vec{n}) [a_{-\vec{k}\beta} a_{\vec{k}\alpha} - (-1)^\ell a_{-\vec{k}\alpha} a_{\vec{k}\beta}] \qquad (2.7)$$

and $\underline{\underline{\omega}}^\dagger_{\ell m}$ is the hermitean adjoint operator-matrix with elements

$$\omega^\dagger_{\ell m\alpha\beta} = (1/2\Omega)(4\pi/2\ell+1)^{\frac{1}{2}} \sum_{\vec{k}>0}{}' Y_{\ell m}(\vec{n}) [a^\dagger_{\vec{k}\beta} a^\dagger_{-\vec{k}\alpha} - (-1)^\ell a^\dagger_{\vec{k}\alpha} a^\dagger_{-\vec{k}\beta}] . \qquad (2.8)$$

Note that the element $\omega^\dagger_{\ell m\alpha\beta}$ of $\underline{\underline{\omega}}^\dagger_{\ell m}$ is the hermitean adjoint operator of the element $\omega_{\ell m\beta\alpha}$ of $\underline{\underline{\omega}}_{\ell m}$. In view of (2.7) we have the property

$$\tilde{\underline{\underline{\omega}}}_{\ell m} = -(-1)^\ell \underline{\underline{\omega}}_{\ell m} \qquad (2.9)$$

for the transposed matrix denoted by the tilde. For odd values of ℓ, the operators $\underline{\underline{\omega}}_{\ell m}$ are symmetric in the spin indices and give rise to triplet-spin pairing, whereas for even ℓ, the $\underline{\underline{\omega}}_{\ell m}$ are antisymmetric in α and β and give rise to singlet-spin pairing.

The interaction term in the hamiltonian (2.6) is a linear combination of products of simpler operators which are bilinear in the $a^\dagger_{\vec{k}\alpha}$, $a_{\vec{k}\alpha}$. In order to establish the relation with the results on systems with separable interactions, it is convenient to introduce (independent) operators $\omega_{\ell m\mu}$, $\omega^\dagger_{\ell m\mu}$, $\mu = 0,1,2,3$, defined by

$$\omega_{\ell m1} = \omega_{\ell m\uparrow\uparrow}, \quad \omega_{\ell m2} = \omega_{\ell m\downarrow\downarrow}, \quad \omega_{\ell m3} = \omega_{\ell m\uparrow\downarrow} = \omega_{\ell m\downarrow\uparrow}, \qquad \ell = \text{odd}, \ \ell\leq L \ ,$$

$$\omega_{\ell m0} = \omega_{\ell m\uparrow\downarrow} = -\ \omega_{\ell m\downarrow\uparrow} \qquad \ell = \text{even}, \quad \ell\leq L \ ,$$

$$\omega_{\ell m\mu} = 0, \quad \text{otherwise.} \tag{2.10}$$

In terms of the operators $\omega_{\ell m\mu}$ the hamiltonian (2.9) reads

$$\mathcal{H} = T - \Omega \sum_{I} \lambda_I \omega_I^\dagger \omega_I \ , \tag{2.11}$$

where $I = \{\ell,m,\mu\}$ characterizes the different possibilities for ℓ, m and μ ($\ell\leq L$, $-\ell \leq m \leq \ell$) and where

$$\lambda_I = \begin{cases} 2(2\ell+1)V_\ell, & \text{if } \ell \text{ is odd and } \mu = 1,2, \\ 4(2\ell+1)V_\ell, & \text{if } \ell \text{ is odd and } \mu = 3, \\ 4(2\ell+1)V_\ell, & \text{if } \ell \text{ is even and } \mu = 0, \\ 0 \qquad\qquad , & \text{otherwise .} \end{cases} \tag{2.12}$$

In terms of the hermitean operators

$$\omega_I^{(1)} = \tfrac{1}{2}(\omega_I + \omega_I^\dagger), \quad \omega_I^{(2)} = -\tfrac{1}{2}i(\omega_I - \omega_I^\dagger) \ , \tag{2.13}$$

eq. (2.11) has the form

$$\mathcal{H} = T + \Omega P\left(\{\omega_I^{(1)}\}, \{\omega_I^{(2)}\}\right) , \tag{2.14}$$

where P is an analytic function of a finite number of hermitean operators which is given by the explicit expression

$$P\left(\{\omega_I^{(1)}\},\{\omega_I^{(2)}\}\right) = - \sum_I \lambda_I\left(\omega_I^{(1)2} + \omega_I^{(2)2} + i[\omega_I^{(1)} , \omega_I^{(2)}]\right) . \tag{2.15}$$

In the general formulation for systems with separable interactions given in refs. 15, 16, the hamiltonian contains a short-range operator T and an analytic function P of a finite number of hermitean short-range operators acting on a N-particle system, see e.g. refs. 23, 24 for the definition of short-range operators. Instead of particles in an N-particle system we have here pairs of wave vectors $(\vec{k},-\vec{k})$ with total wave vector 0, the number of pairs $(\vec{k},-\vec{k})$ being proportional to the volume Ω. Furthermore, the operators T, $\omega_I^{(1)}$, $\omega_I^{(2)}$ in (2.14) are sums of operators acting on the pairs $(\vec{k},-\vec{k})$. Therefore (2.14) corresponds to the special case that all operators T and V in refs. 15, 16 are sums of one-particle operators.

The result for the free energy per unit volume can be described conveniently in terms of the free energy of a reference system which is described by the hamiltonian

$$\mathcal{H}_0 = T - \Omega \sum_I \left(h_I^{(1)} \omega_I^{(1)} + h_I^{(2)} \omega_I^{(2)} \right) , \tag{2.16}$$

which is linear in the operators $\omega_I^{(1)}$, $\omega_I^{(2)}$. The free energy per unit volume of the reference system is defined by

$$\tilde{f}_0\left(\{h_I^{(1)}\},\{h_I^{(2)}\}\right) \equiv \lim_{\Omega\to\infty} (-1/\beta\Omega) \ln \mathrm{Tr} \exp\left\{ -\beta \left[T - \Omega \sum_I \left(h_I^{(1)} \omega_I^{(1)} + h_I^{(2)} \omega_I^{(2)} \right) \right] \right\}. \tag{2.17}$$

The Legendre transform of the free energy of the reference system is defined by

$$\tilde{g}_0\left(\{m_I^{(1)}\},\{m_I^{(2)}\}\right) = \sup_{\{h_I^{(1)}\},\{h_I^{(2)}\}} \left[\sum_I \left(h_I^{(1)} m_I^{(1)} + h_I^{(2)} m_I^{(2)} \right) + \tilde{f}_0\left(\{h_I^{(1)}\},\{h_I^{(2)}\}\right) \right]. \tag{2.18}$$

According to the theorem in ref. 16, see in particular eq. (4.10) of ref. 16, the free energy per unit volume corresponding to the hamiltonian (2.1), cf. (2.11),

$$f \equiv \lim_{\Omega\to\infty} (-1/\beta\Omega) \ln \mathrm{Tr} \exp(-\beta\mathcal{H}) , \tag{2.19}$$

is given by

$$f = \inf_{\{m_I^{(1)}\},\{m_I^{(2)}\}} \left[\tilde{g}_0\left(\{m_I^{(1)}\},\{m_I^{(2)}\}\right) + P\left(\{m_I^{(1)}\},\{m_I^{(2)}\}\right) \right] . \tag{2.20}$$

In order to give a formulation in terms of the matrix operators $\underline{\underline{\omega}}_{\ell m}$ in (2.6), we introduce the (complex) fields

$$h_I = \tfrac{1}{2}(h_I^{(1)} + i h_I^{(2)}), \qquad h_I^{\dagger} = \tfrac{1}{2}(h_I^{(1)} - i h_I^{(2)}) , \tag{2.21}$$

where I characterizes the different possibilities $\{\ell,m,\mu\}$, cf. (2.10), and we introduce also the 2×2 matrices

$$\underline{\underline{h}}_{\ell m} = \begin{cases} h_{\ell m1}\ \tfrac{1}{2}(\underline{\underline{1}} + \underline{\underline{\sigma}}^z) + h_{\ell m2}\ \tfrac{1}{2}(\underline{\underline{1}} - \underline{\underline{\sigma}}^z) + \tfrac{1}{2} h_{\ell m3}\ \underline{\underline{\sigma}}^x , & \ell = \text{odd} , \\ \tfrac{1}{2} h_{\ell m0}\ i\underline{\underline{\sigma}}^y & \ell = \text{even}. \end{cases} \tag{2.22}$$

Then the reference hamiltonian (2.16) can be expressed in matrix notation as

$$\mathcal{H}_0 = T - \Omega \sum_{\ell \leq L} \sum_{m=-\ell}^{\ell} \mathrm{tr}(\underline{\underline{h}}_{\ell m}^{\dagger} \cdot \underline{\underline{\omega}}_{\ell m} + \underline{\underline{h}}_{\ell m} \cdot \underline{\underline{\omega}}_{\ell m}^{\dagger}) \tag{2.23}$$

and for the reference free energy per unit volume, we have

$$\tilde{f}_0\Big(\{h_I^{(1)}\},\{h_I^{(2)}\}\Big) = f_0\Big(\{\underline{\underline{h}}_{\ell m}\},\{\underline{\underline{h}}_{\ell m}\}\Big) \quad , \tag{2.24}$$

where

$$f_0\Big(\{\underline{\underline{h}}_{\ell m}\},\{\underline{\underline{h}}_{\ell m}^{\dagger}\}\Big) \equiv \lim_{\Omega\to\infty} (-1/\beta\Omega) \ln \mathrm{Tr} \exp\Big\{-\beta\big[T - \Omega \sum_{\ell,m} \mathrm{tr}(\underline{\underline{h}}_{\ell m}^{\dagger} \cdot \underline{\underline{\omega}}_{\ell m} + \underline{\underline{h}}_{\ell m} \cdot \underline{\underline{\omega}}_{\ell m}^{\dagger})\big]\Big\} . \tag{2.25}$$

Introducing (complex) thermodynamic variables

$$m_I = m_I^{(1)} + i m_I^{(2)}, \qquad m_I^{\dagger} = m_I^{(1)} - i m_I^{(2)} \tag{2.26}$$

and the matrix notation

$$\underline{\underline{m}}_{\ell m} = \begin{cases} m_{\ell m1}\ \tfrac{1}{2}(\underline{\underline{\mathbb{1}}} + \underline{\underline{\sigma}}^z) + m_{\ell m2}\ \tfrac{1}{2}(\underline{\underline{\mathbb{1}}} + \underline{\underline{\sigma}}^z) + m_{\ell m3}\ \underline{\underline{\sigma}}^x, & \ell = \text{odd} , \\ m_{\ell m0}\ i\underline{\underline{\sigma}}^y & , \ \ell = \text{even}, \end{cases} \tag{2.27}$$

then

$$\sum_I (h_I^{(1)} m_I^{(1)} + h_I^{(2)} m_I^{(2)}) = \sum_{\ell,m} \mathrm{tr}(\underline{\underline{h}}_{\ell m}^{\dagger} \cdot \underline{\underline{m}}_{\ell m} + \underline{\underline{h}}_{\ell m} \cdot \underline{\underline{m}}_{\ell m}^{\dagger}) \tag{2.28}$$

and the Legendre transform (2.18) can be written

$$\tilde{g}_0\Big(\{m_I^{(1)}\},\{m_I^{(2)}\}\Big) = g_0\Big(\{\underline{\underline{m}}_{\ell m}\} , \{\underline{\underline{m}}_{\ell m}^{\dagger}\}\Big) , \tag{2.29}$$

where

$$g_0\Big(\{\underline{\underline{m}}_{\ell m}\},\{\underline{\underline{m}}_{\ell m}^{\dagger}\}\Big) = \sup_{\{\underline{\underline{h}}_{\ell m}\}} \Big[\sum_{\ell,m} \mathrm{tr}(\underline{\underline{h}}_{\ell m}^{\dagger} \cdot \underline{\underline{m}}_{\ell m} + \underline{\underline{h}}_{\ell m} \cdot \underline{\underline{m}}_{\ell m}^{\dagger}) + f_0\Big(\{\underline{\underline{h}}_{\ell m}\},\{\underline{\underline{h}}_{\ell m}^{\dagger}\}\Big) \Big] ,$$

$$\Big(\bar{\underline{\underline{h}}}_{\ell m} = -(-1)^{\ell}\ \underline{\underline{h}}_{\ell m}\Big) . \tag{2.30}$$

In eq. (2.30) it is understood that the supremum is taken over (complex) 2×2 matrices $\underline{\underline{h}}_{\ell m}$ which are symmetric for odd values of ℓ and antisymmetric for even ℓ. Since for the hamiltonian (2.11), we have

$$P\Big(\{m_I^{(1)}\},\{m_I^{(2)}\}\Big) = -\sum_I \lambda_I m_I^{\dagger} m_I = -\sum_{\ell \leq L} \sum_{m=-\ell}^{\ell} (4\ell+2) V_{\ell}\ \mathrm{tr}(\underline{\underline{m}}_{\ell m}^{\dagger} \cdot \underline{\underline{m}}_{\ell m}) , \tag{2.31}$$

cf. (2.15), (2.26), (2.27) and (2.12), the result for the free energy per unit volume can be written

$$f = \inf_{\{\underline{\underline{m}}_{\ell m}\}} \left[g_0\left(\{\underline{\underline{m}}_{\ell m}\},\{\underline{\underline{m}}^{\dagger}_{\ell m}\}\right) - \sum_{\ell} (4\ell+2)V_{\ell}\ \mathrm{tr}(\underline{\underline{m}}^{\dagger}_{\ell m}\cdot\underline{\underline{m}}_{\ell m}) \right] ,$$
$$\left(\tilde{\underline{\underline{m}}}_{\ell m} = -(-1)^{\ell}\ \underline{\underline{m}}_{\ell m}\right) \quad . \tag{2.32}$$

Here it is understood that the infimum should be taken over (complex) 2×2 matrices which are symmetric for odd values of ℓ and antisymmetric for even ℓ and the Legendre transform g_0 is given by (2.30) and (2.25).

Remark: Since the reference hamiltonian (2.24) is a bilinear expression in fermion operators, it does not lead to phase transitions and all first derivatives of f_0 with respect to $\underline{\underline{h}}_{\ell m}$ and $\underline{\underline{h}}^{\dagger}_{\ell m}$ should exist. However, in the derivation of an equation for the first derivatives attention should be paid to the restriction $\tilde{\underline{\underline{h}}}_{\ell m} = -(-1)^{\ell}\ \underline{\underline{h}}_{\ell m}$ on the matrices in the supremum of (2.30). One can e.g. start directly from (2.18). The supremum in the right-hand side leads to

$$m_I^{(1)} = -(\partial/\partial h_I^{(1)}\ \tilde{f}_0), \quad m_I^{(2)} = -(\partial/\partial h_I^{(2)}\ \tilde{f}_0) \ . \tag{2.33}$$

With the explicit formulae (2.21), (2.22), (2.24), (2.26) and (2.27) it can be easily shown that

$$m_{\ell m\alpha\beta} = -\tfrac{1}{2}\left\{ \frac{\partial f_0}{\partial h^*_{\ell m\alpha\beta}} - (-1)^{\ell} \frac{\partial f_0}{\partial h^*_{\ell m\beta\alpha}} \right\} = \langle \omega_{\ell m\alpha\beta} \rangle \quad , \tag{2.34}$$

where the average $\langle\ \ \rangle$ is taken over the reference hamiltonian. In matrix notation, eq. (2.34) can be rewritten

$$\underline{\underline{m}}_{\ell m} = -(\partial/\partial\underline{\underline{h}}^{\dagger}_{\ell m}\ f_0)^S \ , \qquad \ell = \text{odd},$$
$$\underline{\underline{m}}_{\ell m} = (\partial/\partial\underline{\underline{h}}^{\dagger}_{\ell m}\ f_0)^A \ , \qquad \ell = \text{even} \ , \tag{2.35}$$

where $\underline{\underline{A}}^S$ and $\underline{\underline{A}}^A$, for an arbitrary matrix $\underline{\underline{A}}$, denote the symmetric and antisymmetric part, i.e.

$$\underline{\underline{A}}^S = \tfrac{1}{2}(\underline{\underline{A}} + \tilde{\underline{\underline{A}}}), \quad \underline{\underline{A}}^A = \tfrac{1}{2}(\underline{\underline{A}} - \tilde{\underline{\underline{A}}}) \ . \tag{2.36}$$

One can also derive an implicit equation for the $\underline{\underline{h}}_{\ell m}$. From (2.18), (2.20) and (2.15) we have

$$h_I^{(\nu)} = \frac{\partial\tilde{g}_0}{\partial m_I^{(\nu)}} = -\frac{\partial P}{\partial m_I^{(\nu)}} = 2\lambda_I\ m_I^{(\nu)} \ , \quad (\nu = 1,2) \ . \tag{2.37}$$

With eqs. (2.21), (2.22), (2.26), (2.27) and also (2.12), we find

$$\underline{\underline{h}}_{\ell m} = (4\ell+2)V_\ell \, \underline{\underline{m}}_{\ell m} \tag{2.38}$$

and by inserting (2.35) one obtains an implicit equation for the matrices $\underline{\underline{h}}_{\ell m}$.

3. Reference free energy.

In this section we evaluate the free energy per unit volume for the reference system described by the hamiltonian (2.23), cf. also (2.2), (2.6), (2.7) and (2.8). With the property

$$Y_{\ell m}(-\vec{n}) = (-1)^\ell \, Y_{\ell m}(\vec{n}) \tag{3.1}$$

and the anticommutation relations for the operators $a^\dagger_{\vec{k}\alpha}$, $a_{\vec{k}\alpha}$, the hamiltonian (2.23) can be rewritten

$$\mathcal{H}_0 = \sum_{\vec{k}>0} \sum_{\alpha,\beta} (\varepsilon_k \mathbb{1}_{\alpha\beta} - b\sigma^z_{\alpha\beta})(a^\dagger_{\vec{k}\alpha} a_{\vec{k}\beta} - a_{-\vec{k}\alpha} a^\dagger_{-\vec{k}\beta}) +$$
$$+ \sum_{\vec{k}>0}{}' \sum_{\alpha,\beta} \left[\Delta(\vec{n})_{\alpha\beta} \, a^\dagger_{\vec{k}\alpha} a^\dagger_{-\vec{k}\beta} + \Delta^\dagger(\vec{n})_{\alpha\beta} \, a_{-\vec{k}\alpha} a_{\vec{k}\beta} \right] + 2 \sum_{\vec{k}>0} \varepsilon_k \, , \tag{3.2}$$

where $\mathbb{1}$ is the 2×2 unit matrix, $\underline{\underline{\sigma}}^z$ is a Pauli matrix and the matrix $\underline{\underline{\Delta}}(\vec{n})$ is defined by

$$\underline{\underline{\Delta}}(\vec{n}) = - \sum_{\ell,m} (4\pi/2\ell+1)^{\frac{1}{2}} \, Y_{\ell m}(\vec{n}) \underline{\underline{h}}_{\ell m} \, . \tag{3.3}$$

The prime in (3.2) indicates that the summation is restricted to $\vec{k}$ vectors satisfying $|\varepsilon_k| < \hbar\omega$. Since $\tilde{\underline{\underline{h}}}_{\ell m} = -(-1)^\ell \, \underline{\underline{h}}_{\ell m}$, cf. (2.22), we have

$$\underline{\underline{\Delta}}(-\vec{n}) = - \tilde{\underline{\underline{\Delta}}}(\vec{n}) \, . \tag{3.4}$$

In the case of triplet-spin pairing when only odd values of ℓ give a contribution, we have $\underline{\underline{\Delta}}(-\vec{n}) = -\underline{\underline{\Delta}}(\vec{n})$ and $\underline{\underline{\Delta}}(\vec{n}) = \tilde{\underline{\underline{\Delta}}}(\vec{n})$, while in the case of singlet-spin pairing with only a contribution from even ℓ values $\underline{\underline{\Delta}}(-\vec{n}) = \underline{\underline{\Delta}}(\vec{n})$ and $\underline{\underline{\Delta}}(\vec{n}) = -\tilde{\underline{\underline{\Delta}}}(\vec{n})$.

Eq. (3.2) can be easily expressed in normal form with creation operators at the left using the Nambu operators

$$c_{\vec{k}p} = \begin{pmatrix} a_{\vec{k}\uparrow} \\ a_{\vec{k}\downarrow} \\ a^{\dagger}_{-\vec{k}\uparrow} \\ a^{\dagger}_{-\vec{k}\downarrow} \end{pmatrix} , \quad c^{\dagger}_{\vec{k}p} = (a^{\dagger}_{\vec{k}\uparrow} , a^{\dagger}_{\vec{k}\downarrow} , a_{-\vec{k}\uparrow} , a_{-\vec{k}\downarrow}) , \quad (\vec{k} > 0, \; p = 1,2,3,4) . \tag{3.5}$$

The operators $c_{\vec{k}p}$, $c^{\dagger}_{\vec{k}p}$ satisfy the anticommutation relations

$$[c_{\vec{k}p} , c_{\vec{k}'p'}]_{+} = [c^{\dagger}_{\vec{k}p} , c^{\dagger}_{\vec{k}'p'}]_{+} = 0,$$

$$[c_{\vec{k}p} , c^{\dagger}_{\vec{k}'p'}]_{+} = \delta_{\vec{k}\vec{k}'}\delta_{pp'} , \quad (\vec{k} > 0, \; \vec{k}' > 0, \; p,p' = 1,2,3,4) , \tag{3.6}$$

and $\mathcal{H}_0$ is given by

$$\mathcal{H}_0 = \sum_{\vec{k}} \varepsilon_k + \sum_{\vec{k}>0} \sum_{p,p'=1}^{4} c^{\dagger}_{\vec{k}p} D_{\vec{k}pp'} c_{\vec{k}p'} \tag{3.7}$$

with the 4×4 matrix

$$\underline{\underline{D}}_{\vec{k}} = \begin{pmatrix} \varepsilon_k \underline{\underline{\mathbb{1}}} - b\underline{\underline{g}}^z & \underline{\underline{\Delta}}(\vec{n})\eta(\varepsilon_k) \\ \underline{\underline{\Delta}}^{\dagger}(\vec{n})\eta(\varepsilon_k) & -\varepsilon_k \underline{\underline{\mathbb{1}}} + b\underline{\underline{g}}^z \end{pmatrix} , \tag{3.8}$$

where $\eta(\varepsilon_k) = 1$, if $|\varepsilon_k| < \hbar\omega$, and $\eta(\varepsilon_k) = 0$, otherwise.

The free energy of the reference system per unit volume is given by

$$f_0\left(\{\underline{\underline{h}}_{\ell m}\}, \{\underline{\underline{h}}^{\dagger}_{\ell m}\}\right) =$$

$$= \lim_{\Omega\to\infty} (1/\Omega) \left[\sum_{\vec{k}} \varepsilon_k - (1/\beta) \sum_{\vec{k}>0} \sum_{p=1}^{4} \ln(1 + e^{-\beta\lambda_{\vec{k}p}}) \right]$$

$$= \lim_{\Omega\to\infty} (1/\Omega) \left[\sum_{\vec{k}} \varepsilon_k - (1/\beta) \sum_{\vec{k}>0} \mathrm{tr}^{(4)} \left\{ \ln 2 \cosh \tfrac{1}{2}\beta \underline{\underline{D}}_{\vec{k}} \right\} \right] . \tag{3.9}$$

Here $\lambda_{\vec{k}p}$ for $p=1,2,3,4$ are the eigenvalues of the matrix $\underline{\underline{D}}_{\vec{k}}$. $\mathrm{tr}^{(4)}$ denotes the trace of a 4×4 matrix, in contrast with the notation in section 2, where tr without superscript is the trace of a 2×2 matrix.

In (3.9) we still have the restriction $\vec{k} > 0$, i.e. only pairs $(\vec{k},-\vec{k})$ should be taken into account in the summation. However, in the derivation we did not use an explicit convention for selecting the vectors $\vec{k} > 0$. If we would have chosen $-\vec{k}$, rather than $\vec{k}$, as the vector satisfying $-\vec{k} > 0$, we would have obtained (3.9) with $-\vec{k}$ instead of $\vec{k}$ in the right-hand side. As a result eq. (3.9) can be replaced by

$$f_0\left(\{\underline{\underline{h}}_{\ell m}\},\{\underline{\underline{h}}^{\dagger}_{\ell m}\}\right) = \lim_{\Omega\to\infty} (1/\Omega) \left[\sum_{\vec{k}} \epsilon_k - (1/2\beta) \sum_{\vec{k}} \mathrm{tr}^{(4)} \{ \ln 2 \cosh \tfrac{1}{2}\beta \underline{\underline{D}}_{\vec{k}} \} \right]$$

$$= (2\pi)^{-3} \int d\vec{k} \left[\epsilon_k - (1/2\beta)\, \mathrm{tr}^{(4)} \{ \ln 2 \cosh \tfrac{1}{2}\beta \underline{\underline{D}}_{\vec{k}} \} \right] \qquad (3.10)$$

The argument leading to (3.10) can also be presented in more explicit form by considering the matrix $\underline{\underline{D}}^2_{\vec{k}}$ which for $|\epsilon_k| < \hbar\omega$ is given by

$$\begin{pmatrix} (\epsilon_k \underline{\underline{1}} - b\underline{\underline{g}}^z)^2 + \underline{\underline{\Delta}}(\vec{n}) \cdot \underline{\underline{\Delta}}^{\dagger}(n) & -b[\underline{\underline{g}}^z, \underline{\underline{\Delta}}(\vec{n})] \\ -b[\underline{\underline{\Delta}}^{\dagger}(\vec{n}), \underline{\underline{g}}^z] & (\epsilon_k \underline{\underline{1}} - b\underline{\underline{g}}^z)^2 + \underline{\underline{\Delta}}^{\dagger}(\vec{n}) \cdot \underline{\underline{\Delta}}(\vec{n}) \end{pmatrix} . \qquad (3.11)$$

From this it can be easily shown that

$$\left(\underline{\underline{D}}^2_{\vec{k}}\right) = \begin{pmatrix} & \underline{\underline{1}} \\ \underline{\underline{1}} & \end{pmatrix} \left(\underline{\underline{D}}^{*2}_{-\vec{k}}\right) \begin{pmatrix} & \underline{\underline{1}} \\ \underline{\underline{1}} & \end{pmatrix} , \qquad (3.12)$$

implying that the matrices $\underline{\underline{D}}^2_{\vec{k}}$ and $\underline{\underline{D}}^{*2}_{-\vec{k}}$ have the same (real) eigenvalues and equal traces.

In this chapter we shall restrict ourselves to two special cases, i.e.

a) $b = 0$, (3.13a)

b) $b \neq 0$, $\underline{\underline{\Delta}}(\vec{n}) = \tilde{\underline{\underline{\Delta}}}(\vec{n})$. (3.13b)

In these two cases the eigenvalues of the matrix $\underline{\underline{D}}_{\vec{k}}$ occur in pairs $\pm\lambda^{(+)}_{\vec{k}}$, $\pm\lambda^{(-)}_{\vec{k}}$, i.e. the secular problem for the matrix $\underline{\underline{D}}_{\vec{k}}$ contains only even powers of λ. In fact, for $|\epsilon_k| < \hbar\omega$, the condition for only even powers of λ is $b|\Delta_{\uparrow\downarrow}(\vec{n})|^2 = b|\Delta_{\downarrow\uparrow}(\vec{n})|^2$ which includes both cases (3.13a) and (3.13b).

Eq. (3.13a) corresponds to the case of mixed singlet-triplet pairing, for b=0, i.e. the hamiltonian $\mathcal{H}_0$ contains contributions from even as well as from odd values of ℓ. The problem can be easily formulated in terms of 2×2 matrices $\underline{\underline{E}}_{\vec{k}}$ rather than the 4×4 matrices $\underline{\underline{D}}_{\vec{k}}$. For instance, the reference free energy in this case is given by

$$f_0\left(\{\underline{\underline{h}}_{\ell m}\},\{\underline{\underline{h}}^{\dagger}_{\ell m}\}\right) = (2\pi)^{-3} \int d\vec{k} \left[\epsilon_k - (1/\beta)\, \mathrm{tr}\, \{\ln 2 \cosh \tfrac{1}{2}\beta \underline{\underline{E}}_{\vec{k}} \} \right] , \qquad (3.14)$$

where $\underline{\underline{E}}_{\vec{k}}$ is a 2×2 matrix given by

$$\underline{\underline{E}}^2_{\vec{k}} = \varepsilon_k^2 \, \underline{\underline{1}} + \eta(\varepsilon_k)\underline{\underline{\Delta}}(\vec{n}) \cdot \underline{\underline{\Delta}}^\dagger(\vec{n}) \ , \tag{3.15}$$

cf. refs. 3, 6, where a similar matrix notation has been used. More specifically we shall derive in section 4 the Landau expansion including fourth order terms in the matrices $\underline{\underline{m}}_{\ell m}$ for b=0.

Eq. (3.13b) corresponds to the case of triplet pairing in the presence of a magnetic field. The hamiltonian (3.2) contains only contributions from odd values of ℓ leading to matrices $\underline{\underline{\Delta}}(\vec{n})$ which are symmetric.

From (3.11), together with the fact that $[\underline{\underline{\Delta}}(\vec{n}),\underline{\underline{g}}]$ is an antisymmetric 2×2 matrix in this case, we have the relation, for $|\varepsilon_{\vec{k}}| < \hbar\omega$,

$$\det(\underline{\underline{D}}^2_{\vec{k}} - \lambda^2 \underline{\underline{1}}) = \Big[\det \Big\{ (\varepsilon_k \underline{\underline{1}} - b\underline{\underline{g}}^z)^2 + \underline{\underline{\Delta}}(\vec{n})\cdot\underline{\underline{\Delta}}^\dagger(\vec{n}) - \lambda^2 \underline{\underline{1}} \Big\} - \tfrac{1}{2}b^2 \operatorname{tr} \Big\{ [\underline{\underline{g}}^z, \underline{\underline{\Delta}}(\vec{n})]\cdot[\underline{\underline{\Delta}}^\dagger(\vec{n}), \underline{\underline{g}}^z] \Big\} \Big]^2 . \tag{3.16}$$

Therefore, the eigenvalues of $\underline{\underline{D}}_k$ are twofold degenerate, and the eigenvalues of $\underline{\underline{D}}_{\vec{k}}$ occur in pairs $\pm\lambda^{(+)}_{\vec{k}}$, $\pm\lambda^{(-)}_{\vec{k}}$, where $\lambda^{(\pm)}_{\vec{k}}$ are the solutions of the equation

$$\det(\underline{\underline{D}}^2_{\vec{k}} - \lambda^2 \underline{\underline{1}}) = 0 \ . \tag{3.17}$$

These eigenvalues will be used in the derivation of the Landau expansion in sections 5-7.

<u>Remark:</u> From eq. (2.35) and (2.38) one can also derive an implicit equation for the matrices $\underline{\underline{\Delta}}(\vec{n})$ occurring in eq. (3.2) for the reference hamiltonian. In fact, from (3.3) and (2.38) one has

$$\underline{\underline{\Delta}}(\vec{n}) = - \sum_{\ell,m} (4\pi/2\ell+1)^{\frac{1}{2}} \, (4\ell+2)V_\ell \, Y_{\ell m}(\vec{n}) \, \underline{\underline{m}}_{\ell m} \ , \tag{3.18}$$

and from (3.3) and (3.10)

$$-\partial/\partial \underline{\underline{h}}^\dagger_{\ell m} f_0 = (-1/2\Omega)(4\pi/2\ell+1)^{\frac{1}{2}} \sum_{\vec{k}} Y^*_{\ell m}(\vec{n}) \times \partial/\partial(\beta\underline{\underline{\Delta}}^\dagger) \operatorname{tr}^{(4)} \{\ln 2 \cosh \tfrac{1}{2}\beta \underline{\underline{D}}_{\vec{k}}\} \ . \tag{3.19}$$

Inserting (2.35) and (3.19) into (3.18) leads to

$$\underline{\underline{\Delta}}(\vec{n}) = (1/2\Omega) \sum_{\ell,m} (4\pi/2\ell+1)V_\ell(4\ell+2) \sum_{\vec{k}'} Y_{\ell m}(\vec{n})Y^*_{\ell m}(\vec{n}') \times$$
$$\times \left\{ \tfrac{1}{2}\left(1-(-1)^\ell\right) \left[\partial/\partial(\beta\underline{\underline{\Delta}}^\dagger)\ \mathrm{tr}^{(4)} \{\ln 2 \cosh \tfrac{1}{2}\beta\underline{\underline{D}}_{\vec{k}'}\}\right]^S \right.$$
$$\left. - \tfrac{1}{2}\left(1+(-1)^\ell\right) \left[\partial/\partial(\beta\underline{\underline{\Delta}}^\dagger)\ \mathrm{tr}^{(4)} \{\ln 2 \cosh \tfrac{1}{2}\beta\underline{\underline{D}}_{\vec{k}'}\}\right]^A \right\} . \tag{3.20}$$

Using the addition theorem (2.5) for spherical harmonics and the decomposition of $V(\vec{n}\cdot\vec{n}')$ into an even and an odd part, cf. (1.3), (1.6) and (2.3),

$$V^e(\vec{n}\cdot\vec{n}') \equiv \tfrac{1}{2}[V(\vec{n}\cdot\vec{n}') + V(-\vec{n}\cdot\vec{n}')] = -\sum_{\ell=\mathrm{even}} (2\ell+1)P_\ell(\vec{n}\cdot\vec{n}')(V_\ell/\Omega),$$
$$V^o(\vec{n}\cdot\vec{n}') \equiv \tfrac{1}{2}[V(\vec{n}\cdot\vec{n}') - V(-\vec{n}\cdot\vec{n}')] = -\sum_{\ell=\mathrm{odd}} (2\ell+1)P_\ell(\vec{n}\cdot\vec{n}')(V_\ell/\Omega) , \tag{3.21}$$

we have

$$\underline{\underline{\Delta}}(\vec{n}) = -\sum_{\vec{k}'} V^o(\vec{n}\cdot\vec{n}')\left[\partial/\partial(\beta\underline{\underline{\Delta}}^\dagger)\ \mathrm{tr}^{(4)} \{\ln 2 \cosh \tfrac{1}{2}\beta\underline{\underline{D}}_{\vec{k}'}\}\right]^S$$
$$+ \sum_{\vec{k}'} V^e(\vec{n}\cdot\vec{n}')\left[\partial/\partial(\beta\underline{\underline{\Delta}}^\dagger)\ \mathrm{tr}^{(4)} \{\ln 2 \cosh \tfrac{1}{2}\beta\underline{\underline{D}}_{\vec{k}'}\}\right]^A . \tag{3.22}$$

Eq. (3.22) is an implicit equation for the "gap parameters" $\underline{\underline{\Delta}}(\vec{n})$ which has been derived exactly in the weak-coupling limit provided that only a finite number of ℓ waves is taken into account. A similar equation may be derived from the hamiltonian (1.1) using an appropriate formulation of the Hartree-Fock approximation, also in the case of more general matrix elements $V(\vec{k},\vec{k}')$. Here it has been shown that the Hartree-Fock approximation is exact in the weak-coupling limit (1.6).

In the special case that b=0 use can be made of (3.15), (3.16) leading to

$$\underline{\underline{\Delta}}(\vec{n}) = -\sum_{\vec{k}'}{}' V^o(\vec{n}\cdot\vec{n}')\left[\tanh \tfrac{1}{2}\beta\underline{\underline{E}}_{\vec{k}'} \cdot \tfrac{1}{2}\underline{\underline{E}}_{\vec{k}'}^{-1} \cdot \underline{\underline{\Delta}}(\vec{n}')\right]^S$$
$$-\sum_{\vec{k}'}{}' V^e(\vec{n}\cdot\vec{n}')\left[\tanh \tfrac{1}{2}\beta\underline{\underline{E}}_{\vec{k}'} \cdot \tfrac{1}{2}\underline{\underline{E}}_{\vec{k}'}^{-1} \cdot \underline{\underline{\Delta}}(\vec{n}')\right]^A , \tag{3.23}$$

where the primes arise as a consequence of the factor $n(\varepsilon_k)$ in the expression (3.8) for $\underline{\underline{D}}_{\vec{k}}$.

4. Singlet-triplet pairing for b=0.

In this section we shall derive in the special case that b=0 a Landau expansion up to the 4th order in the matrices $\underline{\underline{m}}_{\ell m}$, $\underline{\underline{m}}^{\dagger}_{\ell m}$. We start with the free energy per unit volume of the reference system and expand the right-hand side up to 4th order terms in the matrices $\underline{\underline{\Delta}}(\vec{n})$, $\underline{\underline{\Delta}}^{\dagger}(\vec{n})$. From (3.14) and (3.15) it is straightforward to derive

$$f_0\left(\{\underline{\underline{b}}_{\ell m}\},\{\underline{\underline{b}}^{\dagger}_{\ell m}\}\right) - f_n = -\tfrac{1}{2}(2\pi)^{-3}\int d\vec{k}\; n(\varepsilon_k)\Bigg[\frac{\tanh\frac{1}{2}\beta\varepsilon_k}{2\varepsilon_k}\,\mathrm{tr}\{\underline{\underline{\Delta}}(\vec{n})\cdot\underline{\underline{\Delta}}^{\dagger}(\vec{n})\}$$

$$+\tfrac{1}{2}\,\frac{d}{d\varepsilon_k^2}\,\frac{\tanh\frac{1}{2}\beta\varepsilon_k}{2\varepsilon_k}\,\mathrm{tr}\{\underline{\underline{\Delta}}(\vec{n})\cdot\underline{\underline{\Delta}}^{\dagger}(\vec{n})\cdot\underline{\underline{\Delta}}(\vec{n})\cdot\underline{\underline{\Delta}}^{\dagger}(\vec{n})\}\Bigg], \qquad (4.1)$$

where

$$f_n = (2\pi)^{-3}\int d\vec{k}\,[\varepsilon_k - (2/\beta)\ln 2\cosh\tfrac{1}{2}\beta\varepsilon_k] \qquad (4.2)$$

is the free energy per unit volume of the normal liquid, i.e. if all gap parameters $\underline{\underline{\Delta}}(n)$ are zero.

Introducing the density of states per unit volume, i.e.

$$\tfrac{1}{2}\mathcal{N}(\varepsilon) = (1/\Omega)\sum_{\vec{k}}\delta(\varepsilon-\varepsilon_k) = (1/2\pi^2)\int_0^{\infty} dk\, k^2\delta(\varepsilon-\varepsilon_k)\,, \qquad (4.3)$$

we have the relation

$$(2\pi)^{-3}\int d\vec{k}\, g(\varepsilon_k)h(\vec{n}) = \int d\varepsilon\,\tfrac{1}{2}\mathcal{N}(\varepsilon)g(\varepsilon)\int (d\Omega/4\pi)h(\vec{n}),$$
$$d\Omega = \sin\theta\, d\theta d\phi, \qquad (4.4)$$

for arbitrary functions $g(\varepsilon)$ and $h(\vec{n})$. As a consequence f_0 can be written

$$f_0 - f_n = -\tfrac{1}{2}A(T)\int (d\Omega/4\pi)\,\mathrm{tr}\{\underline{\underline{\Delta}}(\vec{n})\cdot\underline{\underline{\Delta}}^{\dagger}(\vec{n})\}$$
$$+\tfrac{1}{4}B(T)\int (d\Omega/4\pi)\,\mathrm{tr}\{\underline{\underline{\Delta}}(\vec{n})\cdot\underline{\underline{\Delta}}^{\dagger}(\vec{n})\cdot\underline{\underline{\Delta}}(\vec{n})\cdot\underline{\underline{\Delta}}^{\dagger}(\vec{n})\}\;. \qquad (4.5)$$

The coefficients A(T) and B(T) in (4.5) are given by

$$A(T) = \int_{-\hbar\omega}^{\hbar\omega} d\varepsilon\,\tfrac{1}{2}\mathcal{N}(\varepsilon)\,\frac{\tanh\frac{1}{2}\beta\varepsilon}{2\varepsilon} \sim \tfrac{1}{2}\mathcal{N}(0)\ln(1.14\,\beta\hbar\omega)\,, \qquad (4.6)$$

where it has been assumed that $\mathcal{N}(\varepsilon)$ = constant = $\mathcal{N}(0)$ in the small interval $|\varepsilon| < \hbar\omega$ and use has been made of $\beta\hbar\omega >> 1$ in the weak-coupling limit,

$$B(T) = - \int_{-\hbar\omega}^{\hbar\omega} d\varepsilon \; \tfrac{1}{2}\mathcal{N}(\varepsilon) \frac{d}{d\varepsilon^2} \left(\frac{\tanh \frac{1}{2}\beta\varepsilon}{2\varepsilon}\right) \sim \frac{7}{16} \frac{\mathcal{N}(0)}{\pi^2} \beta^2 \zeta(3) . \qquad (4.7)$$

Using eq. (3.3) for the "gap" and the orthonormality of spherical harmonics, i.e.

$$\int d\Omega \; Y^*_{\ell m}(\vec{n}) \; Y_{\ell' m'}(\vec{n}) = \delta_{\ell\ell'}\delta_{mm'} , \qquad (4.8)$$

eq. (4.5) can be expressed as

$$f_0 = f_n - \tfrac{1}{2}A(T) \sum_{\ell,m} (2\ell+1)^{-1} \operatorname{tr}(\underline{\underline{h}}_{\ell m} \cdot \underline{\underline{h}}^{\dagger}_{\ell m}) + f_4 , \qquad (4.9)$$

where the fourth-order part f_4 is given by

$$f_4 \equiv f_4\left(\{\underline{\underline{h}}_{\ell m}\},\{\underline{\underline{h}}^{\dagger}_{\ell m}\}\right) =$$

$$= \tfrac{1}{4}B(T) \sum_{\substack{\ell_1,\ell_2,\ell_3,\ell_4 \\ m_1,m_2,m_3,m_4}} S_{\ell_1 m_1\, \ell_2 m_2\, \ell_3 m_3\, \ell_4 m_4} \operatorname{tr}\left(\underline{\underline{h}}_{\ell_1 m_1} \cdot \underline{\underline{h}}^{\dagger}_{\ell_2 m_2} \cdot \underline{\underline{h}}_{\ell_3 m_3} \cdot \underline{\underline{h}}^{\dagger}_{\ell_4 m_4}\right) \qquad (4.10)$$

with

$$S_{\ell_1 m_1\, \ell_2 m_2\, \ell_3 m_3\, \ell_4 m_4} = 4\pi[(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)2\ell_4+1)]^{-\frac{1}{2}} \times$$

$$\times \int d\Omega \; Y_{\ell_1 m_1}(\vec{n}) Y^*_{\ell_2 m_2}(\vec{n}) Y_{\ell_3 m_3}(\vec{n}) Y^*_{\ell_4 m_4}(\vec{n}) . \qquad (4.11)$$

The integral in the right-hand side can be further evaluated in terms of Clebsch-Gordan coefficients, see e.g. ref. 25. In fact, using the relation

$$Y_{\ell_1 m_1}(\vec{n}) Y_{\ell_3 m_3}(\vec{n}) = \sum_{\ell,m} C(\ell_1 \ell_3 \ell;\, m_1 m_3 m) Y_{\ell m}(\vec{n}) , \qquad (4.12)$$

we have

$$S = 4\pi[(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)(2\ell_4+1)]^{-\frac{1}{2}} \times$$

$$\times \sum_{\ell,m} C(\ell_1 \ell_3 \ell; m_1 m_3 m)\, C(\ell_2 \ell_4 \ell;\, m_2 m_4 m) . \qquad (4.13)$$

From (2.35) and (4.9) we find

$$\underline{\underline{m}}_{\ell m} = A(T)(4\ell+2)^{-1} \, \underline{\underline{h}}_{\ell m} + \underline{\underline{H}}_{\ell m}\left(\{\underline{\underline{h}}_{\ell m}\},\{\underline{\underline{h}}^{\dagger}_{\ell m}\}\right) , \qquad (4.14)$$

where the matrix elements of $\underline{\underline{H}}_{\ell m}$ are third-degree polynomials in the elements of all matrices $\underline{\underline{h}}_{\ell m}$, $\underline{\underline{h}}^{\dagger}_{\ell m}$. $\underline{\underline{H}}_{\ell m}$ is symmetric for odd values of ℓ and antisymmetric for even ℓ. The precise form of $\underline{\underline{H}}_{\ell m}$ turns out to be not important for the derivation of a 4th order Landau expansion, as we shall show below.

Inverting (4.14) we have

$$\underline{\underline{h}}_{\ell m} = (4\ell+2)\; A^{-1}(T) \left[\underline{\underline{m}}_{\ell m} - \underline{\underline{H}}_{\ell m}\left(\left\{\frac{4\ell+2}{A(T)}\,\underline{\underline{m}}_{\ell m}\right\},\left\{\frac{4\ell+2}{A(T)}\,\underline{\underline{m}}^{\dagger}_{\ell m}\right\}\right)\right] , \tag{4.15}$$

apart from 5th and higher order terms in $\underline{\underline{m}}_{\ell m}$ which have been neglected.

The Legendre transform (2.30) can be expressed as

$$g_0\left(\{\underline{\underline{m}}_{\ell m}\},\{\underline{\underline{m}}^{\dagger}_{\ell m}\}\right) = \sum_{\ell,m} \mathrm{tr}\left(\underline{\underline{h}}^{\dagger}_{\ell m}\cdot\underline{\underline{m}}_{\ell m} + \underline{\underline{h}}_{\ell m}\cdot\underline{\underline{m}}^{\dagger}_{\ell m}\right)$$

$$- \tfrac{1}{2}A(T)\left[\sum_{\ell,m} (2\ell+1)^{-1}\;\mathrm{tr}\left(\underline{\underline{h}}^{\dagger}_{\ell m}\cdot\underline{\underline{h}}_{\ell m}\right)\right] + f_4\left(\{\underline{\underline{h}}_{\ell m}\},\{\underline{\underline{h}}^{\dagger}_{\ell m}\}\right) , \tag{4.16}$$

where $\underline{\underline{h}}_{\ell m}$ as a function of $\underline{\underline{m}}_{\ell m}$ and $\underline{\underline{m}}^{\dagger}_{\ell m}$ is given by (4.15). Neglecting 6th and higher-order terms we have in the right-hand side of (4.16)

$$\mathrm{tr}\left(\underline{\underline{h}}^{\dagger}_{\ell m}\cdot\underline{\underline{m}}_{\ell m} + \underline{\underline{h}}_{\ell m}\cdot\underline{\underline{m}}^{\dagger}_{\ell m}\right) = (4\ell+2)\; A^{-1}(T)\Big[2\;\mathrm{tr}\left(\underline{\underline{m}}_{\ell m}\cdot\underline{\underline{m}}^{\dagger}_{\ell m}\right)$$

$$- \mathrm{tr}\left(\underline{\underline{H}}_{\ell m}\cdot\underline{\underline{m}}^{\dagger}_{\ell m} + \underline{\underline{H}}^{\dagger}_{\ell m}\cdot\underline{\underline{m}}_{\ell m}\right)\Big] , \tag*{4.17)}$$

$$- \tfrac{1}{2}\left(A(T)/2\ell+1\right)\quad \mathrm{tr}\;\underline{\underline{h}}_{\ell m}\cdot\underline{\underline{h}}^{\dagger}_{\ell m} = -\,(4\ell+2)\; A^{-1}(T)\Big\{\mathrm{tr}\left(\underline{\underline{m}}_{\ell m}\cdot\underline{\underline{m}}^{\dagger}_{\ell m}\right)$$

$$- \mathrm{tr}\left(\underline{\underline{H}}_{\ell m}\cdot\underline{\underline{m}}^{\dagger}_{\ell m} + \underline{\underline{H}}^{\dagger}_{\ell m}\cdot\underline{\underline{m}}_{\ell m}\right)\Big] , \tag{4.18}$$

implying that the contributions from $\underline{\underline{H}}_{\ell m}$ and $\underline{\underline{H}}^{\dagger}_{\ell m}$ to the right-hand side of (4.16) cancel in 4th order. As a result the Legendre transform (4.16) is given by

$$g_0\left(\{\underline{\underline{m}}_{\ell m}\},\{\underline{\underline{m}}^{\dagger}_{\ell m}\}\right) = \left[\sum_{\ell,m} (4\ell+2)\; A^{-1}(T)\;\mathrm{tr}\left(\underline{\underline{m}}_{\ell m}\cdot\underline{\underline{m}}^{\dagger}_{\ell m}\right)\right] + f_4\left(\left\{\frac{4\ell+2}{A(T)}\,\underline{\underline{m}}_{\ell m}\right\},\left\{\frac{4\ell+2}{A(T)}\,\underline{\underline{m}}^{\dagger}_{\ell m}\right\}\right). \tag{4.19}$$

From (2.32) and the explicit expression (4.10) we obtain the final result for the Landau expansion

$$f - f_n = \inf_{\{\underline{\underline{m}}_{\ell m}\}}\Big[\sum_{\ell,m} (4\ell+2)\left(A^{-1}(T) - V_\ell\right)\;\mathrm{tr}\left(\underline{\underline{m}}_{\ell m}\cdot\underline{\underline{m}}^{\dagger}_{\ell m}\right)$$

$$+ 4B(T)A^{-4}(T)\; F\left(\{\underline{\underline{m}}_{\ell m}\},\{\underline{\underline{m}}^{\dagger}_{\ell m}\}\right)\Big] , \tag{4.20}$$

where

$$F\left(\{\underline{\underline{m}}_{\ell m}\},\{\underline{\underline{m}}^{\dagger}_{\ell m}\}\right) = \sum_{\substack{\ell_1,\ell_2,\ell_3,\ell_4\\ m_1,m_2,m_3,m_4}} (2\ell_1+1)(2\ell_2+1)(2\ell_3+1)(2\ell_4+1)\; S_{\ell_1 m_1\,\ell_2 m_2\,\ell_3 m_3\,\ell_4 m_4}\,\times$$

$$\times\;\mathrm{tr}\left(\underline{\underline{m}}_{\ell_1 m_1}\cdot\underline{\underline{m}}^{\dagger}_{\ell_2 m_2}\cdot\underline{\underline{m}}_{\ell_3 m_3}\cdot\underline{\underline{m}}^{\dagger}_{\ell_4 m_4}\right) . \tag{4.21}$$

Note that in (4.20) the function $A^{-1}(T)$, as given by (4.6), is an increasing function of T, so that below a certain temperature one of the coefficients $A^{-1}(T) - V_\ell$ will be negative and the infimum will be reached for an ordered state with at least one of the matrices $\underline{\underline{m}}_{\ell m}$ different from 0.

We shall now investigate in particular the case that

$$V_\ell < V_J \equiv \sup(V_0, V_1) \quad , \quad \text{for } \ell \geq 2, \quad (J=0,1) \quad , \tag{4.22}$$

i.e. none of the ℓ waves with $\ell \geq 2$ is dominant in the expansion (1.3), (1.6) of the potential. For this purpose it is convenient to introduce the 2×2 matrices

$$\underline{\underline{\psi}}(\vec{n}) = \sum_{\ell,m} [4\pi(2\ell+1)]^{\frac{1}{2}} \, Y_{\ell m}(\vec{n}) \, \underline{\underline{m}}_{\ell m} , \tag{4.23}$$

so that the function F given by (4.21) and (4.11) can be expressed as

$$F = \int (d\Omega/4\pi) \, \mathrm{tr} \left\{ \underline{\underline{\psi}}(\vec{n}) \cdot \underline{\underline{\psi}}^\dagger(\vec{n}) \cdot \underline{\underline{\psi}}(\vec{n}) \cdot \underline{\underline{\psi}}^\dagger(\vec{n}) \right\} \quad . \tag{4.24}$$

In order to determine the infimum in (4.20) we first derive a lower bound to the right-hand side. For this purpose we consider an arbitrary 2×2 matrix $\underline{\underline{\Gamma}}(\vec{n})$ with eigenvalues $\lambda_1(\vec{n})$, $\lambda_2(\vec{n})$. From the Schwarz inequality for integrals

$$\left| \int (d\Omega/4\pi) \, f(\vec{n}) \right|^2 \leq \int (d\Omega/4\pi) |f(\vec{n})|^2 \tag{4.25}$$

with equality sign, if and only, if $f(\vec{n})$ is independent of $\vec{n}$, we have

$$\begin{aligned} \int (d\Omega/4\pi) \, \mathrm{tr}\{\underline{\underline{\Gamma}}(\vec{n}) \cdot \underline{\underline{\Gamma}}^\dagger(\vec{n})\} &\geq \int (d\Omega/4\pi)\{|\lambda_1(\vec{n})|^2 + |\lambda_2(\vec{n})|^2\} \geq \\ &\geq \left| \int (d\Omega/4\pi)\lambda_1(\vec{n}) \right|^2 + \left| \int (d\Omega/4\pi)\lambda_2(\vec{n}) \right|^2 \geq \\ &\geq \tfrac{1}{2} \left| \int (d\Omega/4\pi)\{\lambda_1(\vec{n}) + \lambda_2(\vec{n})\} \right|^2 = \tfrac{1}{2} \left| \int (d\Omega/4\pi) \, \mathrm{tr}\{\underline{\underline{\Gamma}}(\vec{n})\} \right|^2 , \end{aligned} \tag{4.26}$$

with equality sign, if and only if

$$\underline{\underline{\Gamma}}(\vec{n}) = \lambda \underline{\underline{\mathbf{1}}}, \tag{4.27}$$

where λ is independent of $\vec{n}$.

Applying (4.25) to the function F in (4.24), cf. also (4.23) and (4.8), we have

$$F \geq \tfrac{1}{2} \left[\int (d\Omega/4\pi) \, \mathrm{tr}\{\underline{\underline{\psi}}(\vec{n}) \cdot \underline{\underline{\psi}}^\dagger(\vec{n})\} \right]^2 = \tfrac{1}{2} \left[\sum_{\ell,m} (2\ell+1) \, \mathrm{tr}(\underline{\underline{m}}_{\ell m} \cdot \underline{\underline{m}}^\dagger_{\ell m}) \right]^2 \tag{4.28}$$

with equality sign, if and only if

$$\underline{\underline{\psi}}(\vec{n}) \cdot \underline{\underline{\psi}}^{\dagger}(\vec{n}) = \lambda \underline{\underline{1}} \ , \tag{4.29}$$

where λ is independent of $\vec{n}$.

We now consider the special case (4.22) with J=0,1. From (4.20) using (4.28), we have

$$f - f_n \geq \inf_x \inf_{\{\underline{\underline{m}}_{\ell m}, \ell \neq J\}} \Big[\tfrac{1}{2} B(T) A^{-4}(T) x^4 + \left(A^{-1}(T) - V_J\right) x^2 + + \sum_{\ell \neq J} \sum_m (4\ell+2)(V_J - V_\ell) \operatorname{tr}\left(\underline{\underline{m}}_{\ell m} \cdot \underline{\underline{m}}_{\ell m}^{\dagger}\right) \Big] \tag{4.30}$$

with

$$x^2 = \sum_{\ell, m} (4\ell+2) \operatorname{tr}\left(\underline{\underline{m}}_{\ell m} \cdot \underline{\underline{m}}_{\ell m}^{\dagger}\right) \ . \tag{4.31}$$

It is obvious that the infimum over $\underline{\underline{m}}_{\ell m}$, $\ell \neq J$, in (4.30) is reached for

$$\underline{\underline{m}}_{\ell m} = 0, \ \ell \neq J, \quad x^2 = \sum_{m=-J}^{J} \operatorname{tr}\left(\underline{\underline{m}}_{Jm} \cdot \underline{\underline{m}}_{Jm}^{\dagger}\right) , \tag{4.32}$$

so that

$$f - f_n \geq \inf_{\{\underline{\underline{m}}_{Jm}\}} \Big\{ \tfrac{1}{2} B(T) A^{-4}(T) \left[\sum_{m=-J}^{J} (4J+2) \operatorname{tr}\left(\underline{\underline{m}}_{Jm} \cdot \underline{\underline{m}}_{Jm}^{\dagger}\right) \right]^2 + + \left(A^{-1}(T) - V_J\right) \sum_{m=-J}^{J} (4J+2) \operatorname{tr}\left(\underline{\underline{m}}_{Jm} \cdot \underline{\underline{m}}_{Jm}^{\dagger}\right) \Big\} \tag{4.33}$$

The right-hand side of eq. (4.33) provides a lower bound to the free energy as given by (4.20). If at the infimum over $\underline{\underline{m}}_{\ell m}$ in (4.33), eq. (4.29) is satisfied, then the right-hand side of (4.33) at the infimum is equal to the right-hand side of (4.20), so that we have also obtained the infimum of (4.20) and hence the correct value for the free energy f.

If $V_0 > V_1$, i.e. J=0, we have

$$\underline{\underline{\psi}}(\vec{n}) \cdot \underline{\underline{\psi}}^{\dagger}(\vec{n}) = \underline{\underline{m}}_{00} \cdot \underline{\underline{m}}_{00}^{\dagger} = \tfrac{1}{2} \operatorname{tr}\left(\underline{\underline{m}}_{00} \cdot \underline{\underline{m}}_{00}^{\dagger}\right) \underline{\underline{1}} \ , \tag{4.34}$$

so that (4.29) is trivially satisfied and the equality sign holds in (4.33). Hence, if $V_0 > V_1$, we have at sufficiently low temperatures $T < T_0$ with T_0 defined by $A^{-1}(T_0) = V_0$, $\ell=0$ singlet pairing with $\underline{\underline{m}}_{00} = -\tilde{\underline{\underline{m}}}_{00} \neq 0$, $\underline{\underline{m}}_{\ell m} = 0$, $\ell \neq 0$.

If $V_1 > V_0$, i.e. J=1, the equality sign in (4.29) is satisfied for the $\ell=1$ state

$$\underline{\underline{\psi}}(\vec{n}) = \tfrac{1}{2} i \sqrt{6} \ \vec{m}\vec{n} \cdot \vec{\underline{\underline{\sigma}}} \sigma^y \ , \tag{4.35}$$

where $\underline{\underline{\vec{\sigma}}}$ is a 3 dimensional vector consisting of the Pauli matrices $\underline{\underline{\sigma}}^x, \underline{\underline{\sigma}}^y, \underline{\underline{\sigma}}^z$. From (4.23) and (4.35) one obtains the (order) matrices

$$\underline{\underline{m}}_{1,1} = -\frac{m}{\sqrt{12}}\,(\underline{\underline{1}} - \underline{\underline{\sigma}}^z), \quad \underline{\underline{m}}_{10} = \frac{m}{\sqrt{6}}\,\underline{\underline{\sigma}}^x, \quad \underline{\underline{m}}_{1,-1} = \frac{m}{\sqrt{12}}\,(\underline{\underline{1}} + \underline{\underline{\sigma}}^z)\,, \tag{4.36}$$

which are characteristic for the BW phase [3]. (A slightly more general representation can be obtained by inserting an orthogonal 3×3 matrix $\underline{\underline{R}}$ between $\vec{n}$ and $\underline{\underline{\vec{\sigma}}}$ in the right-hand side of (4.35)). As a conclusion, if $V_1 > V_0$, we have at sufficiently low temperatures $T < T_1$ with T_1 defined by $A^{-1}(T_1) = V_1$, $\ell=1$ triplet pairing with $\underline{\underline{m}}_{1m}$ of the BW type (4.36) and $\underline{\underline{m}}_{\ell m} = 0$, $\ell \neq 1$.

Finally, in the case that (4.22) is not satisfied, one may also have parameters $\underline{\underline{m}}_{\ell m}$ with $\ell > 1$, see e.g. ref. 26 for a discussion on the possibility of f-wave pairing with $\ell=3$. It is rather unlikely that for absolute minima with non-vanishing order parameters $\underline{\underline{m}}_{\ell m}$, $(\ell>1)$, the equality sign in (4.29) will be satisfied.

5. Reference free energy and Legendre transform for $\ell=1$ pairing with $b\neq 0$.

In this section we consider the reference system for the special case of triplet pairing with $\ell=1$ in the presence of a magnetic field $b\neq 0$. The reference hamiltonian is given by (3.2) with

$$\underline{\underline{\Delta}}(\vec{n}) = -\sum_i n_i\,\underline{\underline{h}}^i\,, \quad i = x,y,z\,, \tag{5.1}$$

as follows from (3.3) with the special choice

$$Y_{1i} = (3/4\pi)^{\frac{1}{2}}\,n_i\,, \quad i = x,y,z, \quad \underline{\underline{h}}_{\ell m} = 0, \quad \ell \neq 1.$$

The reference free energy is given by, cf. (3.10) and (3.13b),

$$f_0\left(\{\underline{\underline{h}}^i\},\{\underline{\underline{h}}^{i\dagger}\}\right) = (2\pi)^{-3}\int d\vec{k}\left[\epsilon_k - (1/\beta)\ln 2\cosh \tfrac{1}{2}\beta\lambda^{(+)}_{\vec{k}} - (1/\beta)\ln 2\cosh \tfrac{1}{2}\beta\lambda^{(-)}_{\vec{k}}\right] \tag{5.2}$$

in which $\pm\lambda^{(\pm)}_{\vec{k}}$ are the eigenvalues of the matrix $\underline{\underline{D}}_{\vec{k}}$. For $|\epsilon_k| < \hbar\omega$ we have, cf. (3.16),

$$\lambda^{(\pm)2}_{\vec{k}} = \tfrac{1}{2}\mathrm{tr}\,\{(\epsilon_k\underline{\underline{1}} - b\underline{\underline{\sigma}})^2 + \underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger\} \pm \tfrac{1}{2}\Big[\left(\mathrm{tr}\,\{(\epsilon_k\underline{\underline{1}} - b\underline{\underline{\sigma}})^2 + \underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger\}\right)^2$$
$$- 4\det\{(\epsilon_k\underline{\underline{1}} - b\underline{\underline{\sigma}})^2 + \underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger\} + 2b^2\,\mathrm{tr}\,\{[\underline{\underline{\sigma}},\underline{\underline{\Delta}}]\cdot[\underline{\underline{\Delta}}^\dagger,\underline{\underline{\sigma}}]\}\Big]^{\frac{1}{2}}, \tag{5.3}$$

with the short-hand notations $\underline{\underline{g}} = \underline{\underline{g}}^z$, $\underline{\underline{\Delta}} \equiv \underline{\underline{\Delta}}(\vec{n})$.

We now expand $\lambda_{\vec{k}}^{(\pm)2}$ up to 4th order terms in $\underline{\underline{\Delta}}$ using the relation

$$\det\left\{(\epsilon_k \underline{\underline{1}} - b\underline{\underline{g}})^2 + \underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger\right\} = (\epsilon_k^2 - b^2)^2 + \det(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger) +$$

$$+ \operatorname{tr}(\epsilon_k \underline{\underline{1}} - b\underline{\underline{g}})^2 \operatorname{tr}(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger) - \operatorname{tr}\left\{(\epsilon_k \underline{\underline{1}} - b\underline{\underline{g}})^2 \cdot \underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger\right\} . \tag{5.4}$$

As a result we have

$$\lambda_{\vec{k}}^{(\pm)2} = (\epsilon_k \pm b)^2 + \tfrac{1}{2}\operatorname{tr}(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger) \pm \Lambda_2 \pm \Lambda_4 \tag{5.5}$$

with

$$\Lambda_2 = \tfrac{1}{4}(4\epsilon_k b)^{-1}\left[-2\operatorname{tr}(\epsilon_k \underline{\underline{1}} - b\underline{\underline{g}})^2 \operatorname{tr}(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger) + 4\operatorname{tr}\left\{(\epsilon_k \underline{\underline{1}} - b\underline{\underline{g}})^2 \cdot \underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger\right\}\right.$$

$$\left. + 2b^2 \operatorname{tr}\left\{[\underline{\underline{g}},\underline{\underline{\Delta}}]\cdot[\underline{\underline{\Delta}}^\dagger,\underline{\underline{g}}]\right\}\right] = -\tfrac{1}{2}\operatorname{tr}(\underline{\underline{g}}\cdot\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger) + \tfrac{1}{2}b^2(4\epsilon_k b)^{-1}\operatorname{tr}\left\{[\underline{\underline{g}},\underline{\underline{\Delta}}]\cdot[\underline{\underline{\Delta}}^\dagger,\underline{\underline{g}}]\right\} , \tag{5.6}$$

$$\Lambda_4 = \tfrac{1}{4}(4\epsilon_k b)^{-1}\left[\left(\operatorname{tr}(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger)\right)^2 - 4\det(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger) - 4\Lambda_2^2\right] . \tag{5.7}$$

From (5.5) it can be shown that

$$\ln 2\cosh \tfrac{1}{2}\beta\lambda_{\vec{k}}^{(\pm)} - \ln 2\cosh\tfrac{1}{2}\beta(\epsilon_k \pm b) =$$

$$= \tfrac{1}{2}\beta\left[\tfrac{1}{2}\operatorname{tr}(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger) \pm \Lambda_2 \pm \Lambda_4\right] p(\epsilon_k \pm b) + \tfrac{1}{2}\beta\left[\tfrac{1}{2}\operatorname{tr}(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger) \pm \Lambda_2\right]^2 \tfrac{1}{2}p_1(\epsilon_k \pm b) , \tag{5.8}$$

$$p(x) = \frac{\tanh\tfrac{1}{2}\beta x}{2x} , \qquad p_1(x) = \frac{dp}{dx^2}(x) , \tag{5.9}$$

and, cf. (5.2) and (4.4),

$$f_0 - f_n = -\tfrac{1}{2}\int_{-\hbar\omega}^{\hbar\omega} d\epsilon\, \tfrac{1}{2}\mathcal{N}(\epsilon)\int \frac{d\Omega}{4\pi}\left\{\tfrac{1}{2}\operatorname{tr}(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger)\,[p(\epsilon+b) + p(\epsilon-b)]\right.$$

$$+ (\Lambda_2 + \Lambda_4)\,[p(\epsilon+b) - p(\epsilon-b)]$$

$$+ \tfrac{1}{2}\operatorname{tr}(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger)\Lambda_2\,[p_1(\epsilon+b) - p_1(\epsilon-b)]$$

$$\left. + \tfrac{1}{2}\left[\tfrac{1}{4}\left(\operatorname{tr}(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger)\right)^2 + \Lambda_2^2\right][p_1(\epsilon+b) + p_1(\epsilon-b)]\right\} , \tag{5.10}$$

where

$$f_n = (2\pi)^{-3}\int d\vec{k}\left[\epsilon_k - (1/\beta)\ln 2\cosh\tfrac{1}{2}\beta(\epsilon_k+b) - (1/\beta)\ln 2\cosh\tfrac{1}{2}\beta(\epsilon_k-b)\right] \tag{5.11}$$

is the free energy per unit volume in the normal liquid with $\underline{\underline{\Delta}}(\vec{n}) = 0$.

Inserting into (5.9) the explicit expressions (5.6), (5.7) and making use of the properties

$$\det(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger) = \tfrac{1}{2}\left(\mathrm{tr}(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger)\right)^2 - \tfrac{1}{2}\,\mathrm{tr}(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger\cdot\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger)\ ,$$

$$\left(\mathrm{tr}(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger)\right)^2 + \left(\mathrm{tr}(\underline{\underline{g}}\cdot\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger)\right)^2 = \mathrm{tr}(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger\cdot\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger) + \mathrm{tr}(\underline{\underline{g}}\cdot\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger\cdot\underline{\underline{g}}\cdot\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger)\ ,$$

$$\mathrm{tr}(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger)\mathrm{tr}(\underline{\underline{g}}\cdot\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger) = \mathrm{tr}(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger\cdot\underline{\underline{g}}\cdot\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger)\ ,$$

we have

$$\begin{aligned}
f_0 - f_n = &-A_1(T,b)\int (d\Omega/4\pi)\ \mathrm{tr}(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger)\\
&-A_2(T,b)\int (d\Omega/4\pi)\ \mathrm{tr}(\underline{\underline{g}}\cdot\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger)\\
&-P(T,b)\int (d\Omega/4\pi)\ \mathrm{tr}\left([\underline{\underline{g}},\underline{\underline{\Delta}}]\cdot[\underline{\underline{\Delta}}^\dagger,\underline{\underline{g}}]\right)\\
&+B_1(T,b)\int (d\Omega/4\pi)\ \mathrm{tr}(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger\cdot\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger)\\
&+B_2(T,b)\int (d\Omega/4\pi)\ \mathrm{tr}(\underline{\underline{g}}\cdot\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger\cdot\underline{\underline{g}}\cdot\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger)\\
&+C(T,b)\int (d\Omega/4\pi)\ \mathrm{tr}(\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger\cdot\underline{\underline{g}}\cdot\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger)\\
&+Q_1(T,b)\int (d\Omega/4\pi)\ \mathrm{tr}\left([\underline{\underline{g}},\underline{\underline{\Delta}}]\cdot[\underline{\underline{\Delta}}^\dagger,\underline{\underline{g}}]\cdot\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger\right)\\
&+Q_2(T,b)\int (d\Omega/4\pi)\ \mathrm{tr}\left([\underline{\underline{g}},\underline{\underline{\Delta}}]\cdot[\underline{\underline{\Delta}}^\dagger,\underline{\underline{g}}]\cdot\underline{\underline{g}}\cdot\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^\dagger\right)\\
&+R(T,b)\int (d\Omega/4\pi)\ \mathrm{tr}\left([\underline{\underline{g}},\underline{\underline{\Delta}}]\cdot[\underline{\underline{\Delta}}^\dagger,\underline{\underline{g}}]\cdot[\underline{\underline{g}},\underline{\underline{\Delta}}]\cdot[\underline{\underline{\Delta}}^\dagger,\underline{\underline{g}}]\right)\ ,
\end{aligned} \tag{5.12}$$

where

$$A_1(T,b) = \tfrac{1}{4}\int d\varepsilon\ \tfrac{1}{2}\mathcal{N}(\varepsilon)\ [\,p(\varepsilon+b) + p(\varepsilon-b)\,]\ ,$$

$$A_2(T,b) = \tfrac{1}{4}\int d\varepsilon\ \tfrac{1}{2}\mathcal{N}(\varepsilon)\ [\,p(\varepsilon+b) - p(\varepsilon-b)\,]\ ,$$

$$P(T,b) = \tfrac{1}{4}b^2\int d\varepsilon\ \tfrac{1}{2}\mathcal{N}(\varepsilon)\ p_2(\varepsilon,b)\ ,$$

$$B_1(T,b) = -\frac{1}{16}\int d\varepsilon\ \tfrac{1}{2}\mathcal{N}(\varepsilon)\ [\,p_1(\varepsilon+b) + p_1(\varepsilon-b) + 2p_2(\varepsilon,b)\,]\ ,$$

$$B_2(T,b) = -\frac{1}{16}\int d\varepsilon\ \tfrac{1}{2}\mathcal{N}(\varepsilon)[\,p_1(\varepsilon+b) + p_1(\varepsilon-b) - 2p_2(\varepsilon,b)\,]\ ,$$

$$C(T,b) = \frac{1}{8}\int d\varepsilon\ \tfrac{1}{2}\mathcal{N}(\varepsilon)\ [\,p_1(\varepsilon+b) - p_1(\varepsilon-b)\,]\ ,$$

$$Q_1(T,b) = -\frac{b^2}{16}\int d\varepsilon\ \tfrac{1}{2}\mathcal{N}(\varepsilon)(\varepsilon b)^{-1}\ [\,p_1(\varepsilon+b) - p_1(\varepsilon-b)\,]\ ,$$

$$Q_2(T,b) = \frac{b^2}{16} \int d\varepsilon \; \tfrac{1}{2}\mathcal{N}(\varepsilon)(\varepsilon b)^{-1} \left[p_1(\varepsilon+b) + p_1(\varepsilon-b) - 2p_2(\varepsilon,b)\right] \;,$$

$$R(T,b) = -\frac{b^4}{128} \int d\varepsilon \; \tfrac{1}{2}\mathcal{N}(\varepsilon)(\varepsilon b)^{-2} \left[p_1(\varepsilon+b) + p_1(\varepsilon-b) - 2p_2(\varepsilon,b)\right] \qquad (5.13)$$

where p and p_1 have been defined by (5.9) and

$$p_2(\varepsilon,b) = [\, p(\varepsilon+b) - p(\varepsilon-b)] \, [\, (\varepsilon+b)^2 - (\varepsilon-b)^2]^{-1} \;. \qquad (5.14)$$

In all expressions (5.13) the integration over ε extends from $-\hbar\omega$ to $\hbar\omega$. Note that the coefficients P, Q_1, Q_2, R in (5.13) arise from the off-diagonal part of the matrix $\underline{\underline{D}}^2_{\vec{k}}$, cf. (3.11).

With regard to the coefficients in (5.13) the following remarks can be made.

i) In view of the properties $p_1(x) = p_1(-x)$, and $\lim_{\varepsilon\to 0} p_2(\varepsilon,x) = p_1(x)$, the factors ε, ε^2 in the denominators of Q_1, Q_2 and R do not give rise to singularities and all coefficients are properly defined.

ii) For a symmetric density of states, i.e. $\mathcal{N}(\varepsilon) = \mathcal{N}(-\varepsilon)$, we would have $A_2(T,b) = C(T,b) = Q_2(T,b) = 0$. However, small deviations from symmetry can lead to qualitative changes in the phase diagram, see also ref. 5, and we shall assume $\mathcal{N}(\varepsilon) \neq \mathcal{N}(-\varepsilon)$ leading to coefficients $A_2(T,b)$, $C(T,b)$ and $Q_2(T,b)$ which are different from 0.

iii) Many of the coefficients vanish in the absence of an external magnetic field. For small values of b we have

$$A_2 \sim b, \quad P \sim b^2, \quad B_2 \sim b^2, \quad C \sim b, \quad Q_1 \sim b^2, \quad Q_2 \sim b^3, \quad R \sim b^4. \qquad (5.15)$$

This can be checked from the explicit expressions (5.14), but it is also obvious from (3.11) for the matrix $\underline{\underline{D}}^2_{\vec{k}}$ in which each factor b is coupled to a matrix $\underline{\underline{\sigma}}$.

iv) For the coefficients we have the inequalities

$$A_1(T,b) > 0 \;, \qquad P(T,b) < 0 \;. \qquad (5.16)$$

From the explicit relation, cf. (5.13),

$$B_1(T,b) = B_2(T,b) - P(T,b)/b^2 \qquad (5.17a)$$

it follows that $B_1(T,b) > 0$, for sufficiently small values of b. This is also obvious, since

$$\lim_{b\downarrow 0} B_1(T,b) = -\tfrac{1}{4} \int d\varepsilon \; \tfrac{1}{2}\mathcal{N}(\varepsilon)(\partial/\partial\varepsilon^2)\, p(\varepsilon) = \tfrac{1}{4}B(T) \;, \qquad (5.17b)$$

cf. (4.6).

Using (5.1) and the integrals

$$\int (d\Omega/4\pi) n_i n_j = \frac{1}{3}\,\delta_{ij} \;,$$

$$\int (d\Omega/4\pi) n_i n_j n_k n_\ell = \frac{1}{15}\,(\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{jk} + \delta_{ik}\delta_{j\ell}) \;, \tag{5.18}$$

it is straightforward to derive the expansion of the reference free energy up to 4th order in $\underline{\underline{h}}^i$, $\underline{\underline{h}}^{i\dagger}$, i.e.

$$f_0 - f_n = -\frac{1}{3} A_1(T,b) \sum_i \mathrm{tr}(\underline{\underline{h}}^i \cdot \underline{\underline{h}}^{i\dagger}) - \frac{1}{3} A_2(T,b) \sum_i \mathrm{tr}(\underline{\underline{g}} \cdot \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{i\dagger})$$
$$- \frac{1}{3} P(T,b) \sum_i \mathrm{tr}\left([\underline{\underline{g}},\underline{\underline{h}}^i] \cdot [\underline{\underline{h}}^{i\dagger},\underline{\underline{g}}]\right) + f_4\left(\{\underline{\underline{h}}^i\},\{\underline{\underline{h}}^{i\dagger}\}\right) \;, \tag{5.19}$$

where

$$f_4\left(\{\underline{\underline{h}}^i\},\{\underline{\underline{h}}^{i\dagger}\}\right) =$$
$$\frac{1}{15} \sum_{i,j} \Big[B_1(T,b)\,\mathrm{tr}\left\{\underline{\underline{h}}^i \cdot \underline{\underline{h}}^{i\dagger} \cdot \underline{\underline{h}}^j \cdot \underline{\underline{h}}^{j\dagger} + \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger} \cdot \underline{\underline{h}}^j \cdot \underline{\underline{h}}^{i\dagger} + \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger} \cdot \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger}\right\}$$
$$+ B_2(T,b)\,\mathrm{tr}\left\{\underline{\underline{g}} \cdot \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{i\dagger} \cdot \underline{\underline{g}} \cdot \underline{\underline{h}}^j \cdot \underline{\underline{h}}^{j\dagger} + \underline{\underline{g}} \cdot \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger} \cdot \underline{\underline{g}} \cdot \underline{\underline{h}}^j \cdot \underline{\underline{h}}^{i\dagger} + \underline{\underline{g}} \cdot \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger} \cdot \underline{\underline{g}} \cdot \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger}\right\}$$
$$+ C(T,b)\,\mathrm{tr}\left\{\underline{\underline{g}} \cdot \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{i\dagger} \cdot \underline{\underline{h}}^j \cdot \underline{\underline{h}}^{j\dagger} + \underline{\underline{g}} \cdot \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger} \cdot \underline{\underline{h}}^j \cdot \underline{\underline{h}}^{i\dagger} + \underline{\underline{g}} \cdot \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger} \cdot \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger}\right\}$$
$$+ Q_1(T,b)\,\mathrm{tr}\left\{\underline{\underline{h}}^i \cdot \underline{\underline{h}}^{i\dagger} \cdot [\underline{\underline{g}},\underline{\underline{h}}^j] \cdot [\underline{\underline{h}}^{j\dagger},\underline{\underline{g}}] + \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger} \cdot [\underline{\underline{g}},\underline{\underline{h}}^j] \cdot [\underline{\underline{h}}^{i\dagger},\underline{\underline{g}}] + \right.$$
$$\left. + \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger} \cdot [\underline{\underline{g}},\underline{\underline{h}}^i] \cdot [\underline{\underline{h}}^{j\dagger},\underline{\underline{g}}]\right\}$$
$$+ Q_2(T,b)\,\mathrm{tr}\left\{\underline{\underline{g}} \cdot \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{i\dagger} \cdot [\underline{\underline{g}},\underline{\underline{h}}^j] \cdot [\underline{\underline{h}}^{j\dagger},\underline{\underline{g}}] + \underline{\underline{g}} \cdot \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger} \cdot [\underline{\underline{g}},\underline{\underline{h}}^j] \cdot [\underline{\underline{h}}^{i\dagger},\underline{\underline{g}}] + \right.$$
$$\left. + \underline{\underline{g}} \cdot \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger} \cdot [\underline{\underline{g}},\underline{\underline{h}}^i] \cdot [\underline{\underline{h}}^{j\dagger},\underline{\underline{g}}]\right\}$$
$$+ R(T,b)\,\mathrm{tr}\left\{[\underline{\underline{g}},\underline{\underline{h}}^i] \cdot [\underline{\underline{h}}^{i\dagger},\underline{\underline{g}}] \cdot [\underline{\underline{g}},\underline{\underline{h}}^j] \cdot [\underline{\underline{h}}^{j\dagger},\underline{\underline{g}}] + [\underline{\underline{g}},\underline{\underline{h}}^i] \cdot [\underline{\underline{h}}^{j\dagger},\underline{\underline{g}}] \cdot [\underline{\underline{g}},\underline{\underline{h}}^j] \cdot [\underline{\underline{h}}^{i\dagger},\underline{\underline{g}}] + \right.$$
$$\left. + [\underline{\underline{g}},\underline{\underline{h}}^i] \cdot [\underline{\underline{h}}^{j\dagger},\underline{\underline{g}}] \cdot [\underline{\underline{g}},\underline{\underline{h}}^i] \cdot [\underline{\underline{h}}^{j\dagger},\underline{\underline{g}}]\right\} \Big]$$
$$(i,j = x,y,z) \;. \tag{5.20}$$

Starting from (5.19) and (5.20) we can also determine the Legendre transform $g_0\left(\{\underline{\underline{m}}^i\},\{\underline{\underline{m}}^{i\dagger}\}\right)$ in the special case of $\ell=1$ pairing for $b\neq 0$. From the definition (2.30) we have

$$g_0\left(\{\underline{\underline{m}}^i\},\{\underline{\underline{m}}^{i\dagger}\}\right) = \sum_i \mathrm{tr}\left[\underline{\underline{h}}^i\left(\{\underline{\underline{m}}^i\},\{\underline{\underline{m}}^{i\dagger}\}\right)\cdot\underline{\underline{m}}^{i\dagger} + \underline{\underline{h}}^{i\dagger}\left(\{\underline{\underline{m}}^i\},\{\underline{\underline{m}}^{i\dagger}\}\right)\cdot\underline{\underline{m}}^i\right] +$$
$$f_0\left(\left\{\underline{\underline{h}}^i\left(\{\underline{\underline{m}}^i\},\{\underline{\underline{m}}^{i\dagger}\}\right)\right\},\ \left\{\underline{\underline{h}}^{i\dagger}\left(\{\underline{\underline{m}}^i\},\{\underline{\underline{m}}^{i\dagger}\}\right)\right\}\right), \qquad (5.21)$$

where $\underline{\underline{h}}^i = \underline{\underline{h}}^i\left(\{\underline{\underline{m}}^i\},\{\underline{\underline{m}}^{i\dagger}\}\right)$ is the solution of the equation

$$\underline{\underline{m}}^i = -[\partial/\partial\underline{\underline{h}}^{i\dagger}\ f_0]^S \quad , \qquad (5.22)$$

cf. (2.35). From (5.19), (5.20) and (5.22) we have

$$\underline{\underline{m}}^i = \frac{1}{3}A_1(T,b)\underline{\underline{h}}^i + \frac{1}{3}A_2(T,b)(\underline{\underline{g}}\cdot\underline{\underline{h}}^i)^S$$
$$+ \frac{1}{3}P(T,b)\ [\underline{\underline{g}},[\underline{\underline{g}},\underline{\underline{h}}^i]]^S + \underline{\underline{H}}^i\left(\{\underline{\underline{h}}^i\},\{\underline{\underline{h}}^{i\dagger}\}\right), \qquad (5.23)$$

where $\underline{\underline{H}}^i$ are symmetric matrices with elements which are third-degree polynomials in the elements of the matrices $\underline{\underline{h}}^i$ and $\underline{\underline{h}}^{i\dagger}$. As we shall show, the precise form of $\underline{\underline{H}}^i$ is not important for the evaluation of g_0 up to 4th order terms in the matrices $\underline{\underline{m}}^i$ and $\underline{\underline{m}}^{i\dagger}$.

Eq. (5.23) can be rewritten in the form

$$\underline{\underline{m}}^i = \frac{1}{3}(A_1+A_2)\frac{1}{4}(\underline{\underline{\mathbb{1}}}+\underline{\underline{g}})\cdot\underline{\underline{h}}^i\cdot(\underline{\underline{\mathbb{1}}}+\underline{\underline{g}}) + \frac{1}{3}(A_1-A_2)\frac{1}{4}(\underline{\underline{\mathbb{1}}}-\underline{\underline{g}})\cdot\underline{\underline{h}}^i\cdot(\underline{\underline{\mathbb{1}}}-\underline{\underline{g}})$$
$$+ \frac{1}{3}(A_1+4P)\frac{1}{2}(\underline{\underline{h}}^i - \underline{\underline{g}}\cdot\underline{\underline{h}}^i\cdot\underline{\underline{g}}) + \underline{\underline{H}}^i\left(\{\underline{\underline{h}}^i\},\{\underline{\underline{h}}^{i\dagger}\}\right), \qquad (5.24)$$

in which the matrices in the first three terms have only nonvanishing contributions from the ↑↑ element, the ↓↓ element and the off-diagonal elements resp. of $\underline{\underline{h}}^i$.

Inverting (5.24) we have

$$\underline{\underline{h}}^i = 3(A_1+A_2)^{-1}\frac{1}{4}(\underline{\underline{\mathbb{1}}}+\underline{\underline{g}})\cdot\left[\underline{\underline{m}}^i - \underline{\underline{H}}^i\left(\{\hat{\underline{\underline{m}}}^i\},\{\hat{\underline{\underline{m}}}^{i\dagger}\}\right)\right]\cdot(\underline{\underline{\mathbb{1}}}+\underline{\underline{g}})$$
$$+ 3(A_1-A_2)^{-1}\frac{1}{4}(\underline{\underline{\mathbb{1}}}-\underline{\underline{g}})\cdot\left[\underline{\underline{m}}^i - \underline{\underline{H}}^i\left(\{\hat{\underline{\underline{m}}}^i\},\{\hat{\underline{\underline{m}}}^{i\dagger}\}\right)\right]\cdot(\underline{\underline{\mathbb{1}}}-\underline{\underline{g}})$$
$$+ 3(A_1+4P)^{-1}\frac{1}{2}\left[\underline{\underline{m}}^i - \underline{\underline{g}}\cdot\underline{\underline{m}}^i\cdot\underline{\underline{g}} - \underline{\underline{H}}^i\left(\{\hat{\underline{\underline{m}}}^i\},\{\hat{\underline{\underline{m}}}^{i\dagger}\}\right) + \underline{\underline{g}}\cdot\underline{\underline{H}}^i\left(\{\hat{\underline{\underline{m}}}^i\},\{\hat{\underline{\underline{m}}}^{i\dagger}\}\right)\cdot\underline{\underline{g}}\right], \qquad (5.25)$$

apart from 5th and higher order terms in the elements of the matrices $\underline{\underline{m}}^i$ and in which the matrix $\hat{\underline{\underline{m}}}^i$, associated with $\underline{\underline{m}}^i$, is defined by

$$\hat{\underline{\underline{m}}}^i = 3(A_1+A_2)^{-1}\frac{1}{4}(\underline{\underline{\mathbb{1}}}+\underline{\underline{g}})\cdot\underline{\underline{m}}^i\cdot(\underline{\underline{\mathbb{1}}}+\underline{\underline{g}}) + 3(A_1-A_2)^{-1}\frac{1}{4}(\underline{\underline{\mathbb{1}}}-\underline{\underline{g}})\cdot\underline{\underline{m}}^i\cdot(\underline{\underline{\mathbb{1}}}-\underline{\underline{g}})$$
$$+ 3(A_1+4p)^{-1}\frac{1}{2}(\underline{\underline{m}}^i - \underline{\underline{g}}\cdot\underline{\underline{m}}^i\cdot\underline{\underline{g}}) \qquad (5.26)$$

More generally with any (symmetric) 2 ×2 matrix $\underline{\underline{B}}$ one can associate a matrix $\hat{\underline{\underline{B}}}$ defined by

$$\hat{\underline{\underline{B}}} = 3(A_1+A_2)^{-1}\,\frac{1}{4}\,(\,\mathbb{1}+\underline{\underline{g}})\cdot\underline{\underline{B}}\cdot(\,\mathbb{1}+\underline{\underline{g}}) + 3(A_1-A_2)^{-1}\,\frac{1}{4}\,(\,\mathbb{1}-\underline{\underline{g}})\cdot\underline{\underline{B}}\cdot(\,\mathbb{1}-\underline{\underline{g}})$$
$$+\ 3(A_1+4P)^{-1}\,\frac{1}{2}\,(\underline{\underline{B}}-\underline{\underline{g}}\cdot\underline{\underline{B}}\cdot\underline{\underline{g}}) \tag{5.27}$$

and eq. (5.25) can be rewritten in matrix notation as

$$\underline{\underline{h}}^i = \hat{\underline{\underline{m}}}^i - \hat{\underline{\underline{H}}}^i\Big(\{\hat{\underline{\underline{m}}}^i\},\{\hat{\underline{\underline{m}}}^{i^\dagger}\}\Big) + \mathcal{O}(m^5)\ , \tag{5.28}$$

where $\mathcal{O}(m^5)$ is a short-hand notation for 5th and higher order terms in the elements of the matrices $\underline{\underline{m}}^i$, $\underline{\underline{m}}^{i^\dagger}$.

For the Legendre transform, we have, cf. (5.21) and (5.19),

$$g_0\Big(\{\underline{\underline{m}}^i\},\{\underline{\underline{m}}^{i^\dagger}\}\Big) = \sum_i \Big[\mathrm{tr}(\underline{\underline{h}}^i\cdot\underline{\underline{m}}^{i^\dagger} + \underline{\underline{h}}^{i^\dagger}\cdot\underline{\underline{m}}^i) + G_i(\underline{\underline{h}}^i,\underline{\underline{h}}^{i^\dagger})\Big]$$
$$+\ f_4\Big(\{\underline{\underline{h}}^i\},\{\underline{\underline{h}}^{i^\dagger}\}\Big) + \mathcal{O}(m^6), \tag{5.29}$$

where $\underline{\underline{h}}^i$ is given by (5.28), f_4 by (5.20) and

$$G_i(\underline{\underline{h}}^i,\underline{\underline{h}}^{i^\dagger}) = -\frac{1}{3}A_1\,\mathrm{tr}(\underline{\underline{h}}^i\cdot\underline{\underline{h}}^{i^\dagger}) - \frac{1}{3}A_2\,\mathrm{tr}(\underline{\underline{g}}\cdot\underline{\underline{h}}^i\cdot\underline{\underline{h}}^{i^\dagger}) - \frac{1}{3}P\,\mathrm{tr}\,\big([\underline{\underline{g}},\underline{\underline{h}}^i]\cdot[\underline{\underline{h}}^{i^\dagger},\underline{\underline{g}}]\big) \tag{5.30}$$

We now first evaluate G_i omitting the explicit dependence of $\hat{\underline{\underline{H}}}^i$, $\hat{\underline{\underline{H}}}^{i^\dagger}$ on $\{\hat{\underline{\underline{m}}}^i\}$ and $\{\hat{\underline{\underline{m}}}^{i^\dagger}\}$,

$$G_i(\underline{\underline{h}}^i,\underline{\underline{h}}^{i^\dagger}) = G_i(\hat{\underline{\underline{m}}}^i,\hat{\underline{\underline{m}}}^{i^\dagger}) + \frac{1}{3}A_1\Big[\mathrm{tr}(\hat{\underline{\underline{m}}}^i\cdot\hat{\underline{\underline{H}}}^{i^\dagger}) + \mathrm{c.c.}\Big] +$$
$$+\ \frac{1}{3}A_2\Big[\mathrm{tr}(\underline{\underline{g}}\cdot\hat{\underline{\underline{m}}}^i\cdot\hat{\underline{\underline{H}}}^{i^\dagger}) + \mathrm{c.c.}\Big] + \frac{1}{3}P\Big[\mathrm{tr}\big([\underline{\underline{g}},\hat{\underline{\underline{m}}}^i]\cdot[\hat{\underline{\underline{H}}}^{i^\dagger},\underline{\underline{g}}]\big) + \mathrm{c.c.}\Big] + \mathcal{O}(m^6) \tag{5.31}$$

In terms containing $\hat{\underline{\underline{H}}}^{i^\dagger}$ we may replace $\hat{\underline{\underline{m}}}^i$ by $\underline{\underline{h}}^i$, cf. (5.28). Using the fact that the matrix $\hat{\underline{\underline{H}}}^i$ is symmetric, we have

$$G_i(\underline{\underline{h}}^i,\underline{\underline{h}}^{i^\dagger}) = G_i(\hat{\underline{\underline{m}}}^i,\hat{\underline{\underline{m}}}^{i^\dagger}) +$$
$$+\ \Big[\mathrm{tr}\Big\{\Big(\frac{1}{3}A_1\underline{\underline{h}}^i + \frac{1}{3}A_2(\underline{\underline{g}}\cdot\underline{\underline{h}}^i)^S + \frac{1}{3}P[\underline{\underline{g}},[\underline{\underline{g}},\underline{\underline{h}}^i]]^S\Big)\cdot\hat{\underline{\underline{H}}}^{i^\dagger}\Big\} + \mathrm{c.c.}\Big] + \mathcal{O}(m^6)\ . \tag{5.32}$$

Using (5.23) and inserting (5.28) again, we have

$$G_i(\underline{\underline{h}}^i,\underline{\underline{h}}^{i^\dagger}) = G_i(\hat{\underline{\underline{m}}}^i,\hat{\underline{\underline{m}}}^{i^\dagger}) + \mathrm{tr}(\underline{\underline{m}}^i\cdot\hat{\underline{\underline{H}}}^{i^\dagger} + \hat{\underline{\underline{H}}}^i\cdot\underline{\underline{m}}^{i^\dagger}) + \mathcal{O}(m^6) =$$
$$= G_i(\hat{\underline{\underline{m}}}^i,\hat{\underline{\underline{m}}}^{i^\dagger}) + \mathrm{tr}(\underline{\underline{m}}^i\cdot\hat{\underline{\underline{m}}}^{i^\dagger} + \hat{\underline{\underline{m}}}^i\cdot\underline{\underline{m}}^{i^\dagger}) - \mathrm{tr}(\underline{\underline{m}}^i\cdot\underline{\underline{h}}^{i^\dagger} + \underline{\underline{h}}^i\cdot\underline{\underline{m}}^{i^\dagger}) + \mathcal{O}(m^6)\ . \tag{5.33}$$

From (5.26) one can derive that

$$\mathrm{tr}(\underline{\underline{m}}^i\cdot\hat{\underline{\underline{m}}}^{i\dagger}) = \mathrm{tr}(\hat{\underline{\underline{m}}}^i\cdot\underline{\underline{m}}^{i\dagger}) = -G_i(\hat{\underline{\underline{m}}}^i,\hat{\underline{\underline{m}}}^{i\dagger}) \quad , \tag{5.34}$$

so that

$$G_i(\underline{\underline{h}}^i,\underline{\underline{h}}^{i\dagger}) + \mathrm{tr}(\underline{\underline{h}}^i\cdot\underline{\underline{m}}^{i\dagger} + \underline{\underline{h}}^{i\dagger}\cdot\underline{\underline{m}}^i) = -G_i(\hat{\underline{\underline{m}}}^i,\hat{\underline{\underline{m}}}^{i\dagger}) \ . \tag{5.35}$$

From (5.35), (5.29) and (5.30) one immediately obtains the final result for the Legendre transform

$$\begin{aligned} g_0\Big(\{\underline{\underline{m}}^i\},\{\underline{\underline{m}}^{i\dagger}\}\Big) &= \tfrac{1}{3} A_1 \ \mathrm{tr}(\hat{\underline{\underline{m}}}^i\cdot\hat{\underline{\underline{m}}}^{i\dagger}) + \tfrac{1}{3} A_2 \ \mathrm{tr}(\underline{\underline{g}}\cdot\hat{\underline{\underline{m}}}^i\cdot\hat{\underline{\underline{m}}}^{i\dagger}) \ + \\ &+ \tfrac{1}{3} P \ \mathrm{tr}\big([\underline{\underline{g}},\hat{\underline{\underline{m}}}^i]\cdot[\hat{\underline{\underline{m}}}^{i\dagger},\underline{\underline{g}}]\big) + f_4\Big(\{\hat{\underline{\underline{m}}}^i\},\{\hat{\underline{\underline{m}}}^{i\dagger}\}\Big) + \mathcal{O}(m^6) \ , \end{aligned} \tag{5.36}$$

where the $\hat{\underline{\underline{m}}}^i$ as function of $\underline{\underline{m}}^i$ are given by (5.26) and the explicit form of f_4 by (5.20).

6. Landau expansion

In this section we consider the hamiltonian (1.1) in the special case of a $\ell=1$ pairing interaction in the weak-coupling limit, cf. (1.3), (1.6),

$$\begin{aligned} \mathcal{H} &= \sum_{\vec{k}} \sum_{\alpha,\beta} (\varepsilon_k \mathbb{1}_{\alpha\beta} - b\sigma^z_{\alpha\beta}) a^\dagger_{\vec{k}\alpha} a_{\vec{k}\beta} \\ &- \tfrac{1}{2}(3V_1/\Omega) \sum''_{\vec{k},\vec{k}'} \sum_{\alpha,\beta} (\vec{n}\cdot\vec{n}') \, a^\dagger_{\vec{k}\alpha} a^\dagger_{-\vec{k}\beta} a_{-\vec{k}'\beta} a_{\vec{k}'\alpha} \ , \end{aligned} \tag{6.1}$$

where the primes indicate that the summations are restricted to $\vec{k}$ vectors satisfying $|\varepsilon_k|$, $|\varepsilon_{k'}| < \hbar\omega$. Eq. (6.1) can be rewritten, cf. (2.6), (2.7), (2.8) and (2.2),

$$\mathcal{H} = T - 6V_1 \, \Omega \sum_i \mathrm{tr}(\underline{\underline{\omega}}^{i\dagger}\cdot\underline{\underline{\omega}}^i) \tag{6.2}$$

with

$$\omega^i_{\alpha\beta} = (1/2\Omega) \sum'_{\vec{k}>0} n^i (a_{-\vec{k}\beta}\, a_{\vec{k}\alpha} + a_{-\vec{k}\alpha}\, a_{\vec{k}\beta}) \ . \tag{6.3}$$

The free energy per unit volume is given by (2.32), i.e.

$$f = \inf_{\{\underline{\underline{m}}^i\}} \Big[g_0\Big(\{\underline{\underline{m}}^i\},\{\underline{\underline{m}}^{i\dagger}\}\Big) - 6V_1 \sum_i \mathrm{tr}(\underline{\underline{m}}^{i\dagger}\cdot\underline{\underline{m}}^i) \Big] \ , \tag{6.4}$$

where the inf should be taken over symmetric 2×2 matrices $\underline{\underline{m}}^i$ and where the Legendre transform is given by (5.36). From (5.26) we have

$$6V_1 \operatorname{tr}(\underline{\underline{m}}^i \cdot \underline{\underline{m}}^{i\dagger}) = \frac{1}{6} V_1[(A_1+A_2)^2 + (A_1-A_2)^2 + 2(A_1+4P)^2] \operatorname{tr}(\hat{\underline{\underline{m}}}^i \cdot \hat{\underline{\underline{m}}}^{i\dagger})$$

$$+ \frac{1}{6} V_1[2(A_1+A_2)^2 - 2(A_1-A_2)^2] \operatorname{tr}(\underline{\underline{g}} \cdot \hat{\underline{\underline{m}}}^i \cdot \hat{\underline{\underline{m}}}^{i\dagger})$$

$$+ \frac{1}{6} V_1[(A_1+A_2)^2 + (A_1-A_2)^2 - 2(A_1+4P)^2] \operatorname{tr}(\underline{\underline{g}} \cdot \hat{\underline{\underline{m}}}^i \cdot \underline{\underline{g}} \cdot \hat{\underline{\underline{m}}}^{i\dagger}) . \tag{6.5}$$

From (6.4), (6.5) and (5.36) we obtain the final result for the Landau expansion

$$f = \inf_{\{\hat{\underline{\underline{m}}}^i\}} \Big\{ \frac{1}{6} \Big[2(A_1+2P) - V_1\{(A_1+A_2)^2 + (A_1-A_2)^2 + 2(A_1+4P)^2\} \Big] \sum_i \operatorname{tr}(\hat{\underline{\underline{m}}}^i \cdot \hat{\underline{\underline{m}}}^{i\dagger})$$

$$+ \frac{1}{6} \Big[2A_2 - 2V_1\{(A_1+A_2)^2 - (A_1-A_2)^2\} \Big] \sum_i \operatorname{tr}(\underline{\underline{g}} \cdot \hat{\underline{\underline{m}}}^i \cdot \hat{\underline{\underline{m}}}^{i\dagger})$$

$$+ \frac{1}{6} \Big[-4P - V_1\{(A_1+A_2)^2 + (A_1-A_2)^2 - 2(A_1+4P)^2\} \Big] \sum_i \operatorname{tr}(\underline{\underline{g}} \cdot \hat{\underline{\underline{m}}}^i \cdot \underline{\underline{g}} \cdot \hat{\underline{\underline{m}}}^{i\dagger})$$

$$+ f_4\Big(\{\hat{\underline{\underline{m}}}^i\}, \{\hat{\underline{\underline{m}}}^{i\dagger}\}\Big) + \mathcal{O}(m^6) \Big\} . \tag{6.6}$$

Note that in (6.6) the infimum over $\underline{\underline{m}}^i$ has been replaced by an infimum over $\hat{\underline{\underline{m}}}^i$, in view of the one-to-one correspondence between the $\underline{\underline{m}}^i$ and $\hat{\underline{\underline{m}}}^i$ in (5.26).

Eq. (6.6) can be expressed equivalently as

$$f = \inf_{\{\hat{\underline{\underline{m}}}^i\}} \Big\{ \frac{1}{3}(A_1+A_2)\big(1 - 2V_1(A_1+A_2)\big) \sum_i |\hat{m}_1^i|^2$$

$$+ \frac{1}{3}(A_1-A_2)\big(1 - 2V_1(A_1-A_2)\big) \sum_i |\hat{m}_2^i|^2$$

$$+ \frac{2}{3}(A_1+4P)\big(1 - 2V_1(A_1+4P)\big) \sum_i |\hat{m}_3^i|^2 + f_4\Big(\{\hat{\underline{\underline{m}}}^i\}, \{\hat{\underline{\underline{m}}}^{i\dagger}\}\Big) + \mathcal{O}(m^6) \Big\} , \tag{6.7}$$

where $\hat{m}_1^i = \hat{m}_{\uparrow\uparrow}^i$, $\hat{m}_2^i = \hat{m}_{\downarrow\downarrow}^i$, $\hat{m}_3^i = \hat{m}_{\uparrow\downarrow}^i = \hat{m}_{\downarrow\uparrow}^i$, cf. (2.27).

Although this Landau expansion has been derived under the assumption that $V_\ell = 0$, for $\ell \neq 1$, it will be valid under more general conditions. In principle one may write down a Landau expansion up to 4th order including all matrices $\underline{\underline{m}}_{\ell m}$ corresponding to a finite number of coefficients

$V_\ell \neq 0$. Starting from the explicit form (3.8) of the matrices $\underline{\underline{D}}_{\vec{k}}$, one can evaluate $\ln 2\cosh \frac{1}{2}\beta \underline{\underline{D}}_{\vec{k}}$ in (3.9) up to fourth order terms in $\underline{\underline{\Delta}}$ using e.g. a Dyson expansion in 4th order perturbation calculation. When the coefficient V_1 is sufficiently large with respect to the other coefficients V_ℓ, one may expect that the infimum over all matrices in such a general Landau expansion will occur under the condition $\underline{\underline{\omega}}_{\ell m} = 0$, for $\ell \neq 1$, so that the free energy per unit volume is correctly described by (6.7). In fact, for b=0, we have shown that it is sufficient to require that $V_\ell < V_1$, for all $\ell \neq 1$. But then we may expect that if $V_\ell < \lambda V_1$ with $\lambda < 1$, for all ℓ, the condition $\underline{\underline{\omega}}_{\ell m} = 0$, for $\ell \neq 1$, will remain valid for sufficiently small values of b.

Remark.

Let us finally consider the special case that all V_ℓ in (1.6) are positive. In that case the free energy per unit volume can be expressed as an absolute minimum of a function containing the reference free energy rather than its Legendre transform. This relation for the free energy is a direct consequence of a fundamental theorem due to Bogolubov Jr. [12)] which has been used in the derivation of (2.20), cf. refs. 15, 16. Here we shall show how this expression for the free energy can be derived from (2.32).

From (2.18) one also has the formula for the inverse Legendre transform, see e.g. refs. 27,28,

$$\bar{f}_0\left(\{h_I^{(1)}\},\{h_I^{(2)}\}\right) = \inf_{\{m_I^{(1)}\},\{m_I^{(2)}\}} \Big[-\sum_I \left(h_I^{(1)} m_I^{(1)} + h_I^{(2)} m_I^{(2)}\right) + \\ + \tilde{g}_0\left(\{m_I^{(1)}\},\{m_I^{(2)}\}\right) \Big] \tag{6.8}$$

From (2.24), (2.28) and (2.29) one then has

$$f_0\left(\{\underline{\underline{h}}_{\ell m}\},\{\underline{\underline{h}}^\dagger_{\ell m}\}\right) = \inf_{\{\underline{\underline{m}}_{\ell m}\}} \Big[-\sum_{\ell,m} \mathrm{tr}\left(\underline{\underline{h}}^\dagger_{\ell m}\cdot\underline{\underline{m}}_{\ell m} + \underline{\underline{h}}_{\ell m}\cdot\underline{\underline{m}}^\dagger_{\ell m}\right) + g_0\left(\{\underline{\underline{m}}_{\ell m}\},\{\underline{\underline{m}}^\dagger_{\ell m}\}\right)\Big] . \tag{6.9}$$

For positive V_ℓ, i.e. $V_\ell > 0$ for $\ell \geq 0$, use can be made of the trivial identity

$$\inf_{\{\underline{\underline{\xi}}_{\ell m}\}} (4\ell+2)V_\ell \;\mathrm{tr}\left\{(\underline{\underline{\xi}}_{\ell m} - \underline{\underline{m}}_{\ell m})\cdot(\underline{\underline{\xi}}^\dagger_{\ell m} - \underline{\underline{m}}^\dagger_{\ell m})\right\} = 0 . \tag{6.10}$$

Adding (6.10) to the right-hand side of (2.32) we have

$$f = \inf_{\{\underline{\underline{m}}_{\ell m}\}} \inf_{\{\underline{\underline{\xi}}_{\ell m}\}} \Big[g_0\Big(\{\underline{\underline{m}}_{\ell m}\},\{\underline{\underline{m}}^\dagger_{\ell m}\}\Big) - \sum_{\ell,m} (4\ell+2)V_\ell \; \mathrm{tr}(\underline{\underline{\xi}}_{\ell m}\cdot\underline{\underline{m}}^\dagger_{\ell m} + \underline{\underline{\xi}}^\dagger_{\ell m}\cdot\underline{\underline{m}}_{\ell m})$$

$$+ \sum_{\ell,m} (4\ell+2)V_\ell \; \mathrm{tr}(\underline{\underline{\xi}}_{\ell m}\cdot\underline{\underline{\xi}}^\dagger_{\ell m})\Big]. \qquad (6.11)$$

Interchanging both infima in (6.11) and using (6.9) for the inverse Legendre transform we have

$$f = \inf_{\{\underline{\underline{\xi}}_{\ell m}\}} \Big[f_0\Big(\{(4\ell+2)V_\ell \; \underline{\underline{\xi}}_{\ell m}\},\{(4\ell+2)V_\ell \; \underline{\underline{\xi}}^\dagger_{\ell m}\}\Big) + \sum_{\ell,m} (4\ell+2)V_\ell \; \mathrm{tr}(\underline{\underline{\xi}}_{\ell m}\cdot\underline{\underline{\xi}}^\dagger_{\ell m})\Big], \qquad (6.12)$$

which one can also prove directly on the basis of the fundamental theorem of Bogolubov.

We now apply (6.12) to the special case that b=0 assuming that now all coefficients V_ℓ are non-negative. From (6.12) and the explicit expression (4.9) for the reference free energy, cf. also (4.10), (4.21), we find

$$f - f_n = \inf_{\{\underline{\underline{m}}_{\ell m}\}} \Big[\sum_{\ell,m} (4\ell+2)[A^{-1}(T) - V_\ell] \; \mathrm{tr}(\underline{\underline{m}}_{\ell m}\cdot\underline{\underline{m}}^\dagger_{\ell m}) \; A(T)V_\ell$$

$$+ 4B(T) \; F\Big(\{V_\ell\underline{\underline{m}}_{\ell m}\},\{V_\ell\underline{\underline{m}}^\dagger_{\ell m}\}\Big) + \mathcal{O}(m^6) \Big]. \qquad (6.13)$$

This expression is equivalent to (4.20), since in the case that $A(T)V_\ell > 1$, we must have

$$A(T)V_\ell = 1 + \mathcal{O}(m^2) \qquad (6.14)$$

in both Landau expansions implying that the difference between (6.13) and (4.20) is of the order $\mathcal{O}(m^6)$, i.e. of the same order as the terms which have been neglected. (If $A(T)V_\ell < 1$ and the 4th order term is a convex function of $\underline{\underline{m}}_{\ell m}$, the absolute minimum will be reached for $\underline{\underline{m}}_{\ell m} = 0$, so that all terms involving $\underline{\underline{m}}_{\ell m}$ can be omitted from the Landau expansion.)

We now consider the case of ℓ=1 pairing in the presence of a magnetic field. Starting from the explicit expression (6.2), we have, cf. (5.19) for the reference free energy,

$$f - f_n = \inf_{\{\underline{\underline{m}}^i\}} \Big[f_0\Big(\{6V_1\underline{\underline{m}}^i\},\{6V_1\underline{\underline{m}}^{i\dagger}\}\Big) + 6V_1 \sum_i \mathrm{tr}(\underline{\underline{m}}^i\cdot\underline{\underline{m}}^{i\dagger})\Big] =$$

$$= \inf_{\{\underline{\underline{m}}^i\}} \Big[6V_1(1 - 2A_1V_1 - 4PV_1) \sum_i \mathrm{tr}(\underline{\underline{m}}^i\cdot\underline{\underline{m}}^{i\dagger}) - 12A_2V_1^2 \sum_i \mathrm{tr}(\underline{\underline{g}}\cdot\underline{\underline{m}}^i\cdot\underline{\underline{m}}^{i\dagger})$$

$$+ 24PV_1^2 \sum_i \mathrm{tr}(\underline{\underline{g}}\cdot\underline{\underline{m}}^i\cdot\underline{\underline{g}}\cdot\underline{\underline{m}}^{i\dagger}) + 1296\, V_1^4 f_4\left(\{\underline{\underline{m}}^i\},\{\underline{\underline{m}}^{i\dagger}\}\right) + \mathcal{O}(m^6)\Big] \cdot \quad 6.15)$$

Eq. (6.15) can be expressed as

$$f - f_n = \inf_{\{\underline{\underline{m}}^i\}} \Big[6V_1\big(1 - 2V\,(A_1+A_2)\big) \sum_i |m_1^i|^2$$

$$+ 6V_1\big(1 - 2V_1(A_1-A_2)\big) \sum_i |m_2^i|^2$$

$$+ 12V_1\big(1 - 2V_1(A_1+4P)\big) \sum_i |m_3^i|^2$$

$$+ 1296\, V_1^4 f_4\left(\{\underline{\underline{m}}^i\},\{\underline{\underline{m}}^{i\dagger}\}\right) + \mathcal{O}(m^6) \Big] , \qquad (6.16)$$

where $m_1^i = m_{\uparrow\uparrow}^i$, $m_2^i = m_{\downarrow\downarrow}^i$, $m_3^i = m_{\uparrow\downarrow}^i$, cf. (2.27).

Eq. (6.16) is equivalent with (6.7), since

$$2(A_1+A_2)V_1 = 1 + \mathcal{O}(m^2) ,$$

$$2(A_1-A_2)V_1 = 1 + \mathcal{O}(m^2) ,$$

$$2(A_1+4P)V_1 = 1 + \mathcal{O}(m^2) , \qquad (6.17)$$

implying that

$$\hat{\underline{\underline{m}}}^i = 6V_1\underline{\underline{m}}^i + \mathcal{O}(m^3) . \qquad (6.18)$$

With (6.17) and (6.18) it is easy to show that the difference between the right-hand sides of (6.7) and (6.16) is of the order $\mathcal{O}(m^6)$, so that both expansions are equivalent as Landau expansions up to the 4th order.

In chapter III eq. (6.7) will be used as a starting point to investigate the different types of ordering which can arise from a pairing hamiltonian with $\ell=1$ terms in the presence of a magnetic field.

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CHAPTER III

EXTREMA OF THE LANDAU EXPANSION AND PHASE DIAGRAM IN THE WEAK COUPLING LIMIT

1. Introduction

The investigation of the phases of liquid ^{3}He is of interest both from a theoretical as well as from an experimental point of view. In many respects the properties of liquid ^{3}He at zero magnetic field are well understood. The two superfluid phases A and B which have been observed experimentally [1] have been conventionally identified with the ABM-phase first found by Anderson and Morel [2] and the (isotropic) BW-phase found by Balian and Werthamer [3]. Theory and experiment until 1975 have been reviewed by Leggett [4] and by Wheatley [5] and a more recent account can be found in refs. 6,7.

The situation in the presence of a magnetic field is much more complicated. Experimentally a phase transition has been fonnd within the A-phase [8] leading to a new phase which is called the A1-phase. The splitting of the A-phase was explained by Ambegaokar and Mermin [9] on the basis of an asymmetry in the density of states around the Fermi level. In addition Ambegaokar and Mermin gave a qualitative description of the phase diagram on the basis of a phenomenological Landau expansion, but no detailed features of the different phases were given. In refs. 10, 11 the effects of spin fluctuations in the presence of a magnetic field were discussed.

A general Landau expansion of liquid ^{3}He is a very complicated expression in terms of 9 complex variables, e.g. the elements of the 3×3 matrices $\underline{\underline{A}}$ used in refs. 12, 13, cf. also refs. 4, 6. In view of that it does not seem to be possible to give a general description of the phase diagram for arbitrary values of the coefficients using e.g. the methods of catastrophe theory [14)-16)].

At zero magnetic field explicit values have been derived for the coefficients starting from a BCS-type hamiltonian with a $\ell=1$ pairing term in the weak-coupling approximation. On the basis of these values it has been shown that the BW-state has always the lowest free energy. (In ref. 17 the same result has been shown to be valid using a Schwarz inequality in the presence of (weaker) pairing terms with $\ell\neq 1$). The general occurrence of the BW-state is not in agreement with experiments under high pressure. The (observed) stabilization of the ABM-phase has been explained on the basis of spin fluctuations arising e.g. from a repulsive contact term of the Hubbard type, see e.g. refs. 18-21. A rather general set of solutions of the gap equations at zero field can be found in ref. 22.

In the presence of a magnetic field the minimization of the Landau expansion is much more difficult. So far no Schwarz inequality has been found and reasonable assumptions such as the restriction to unitary states break down, especially when an asymmetry in the density of states is taken into account. Small deviations from the BW-state due to the presence of a magnetic field and a dipolar interaction term were discussed in refs. 13, 23, 24 to evaluate NMR-shifts. Recently Hasegawa [25] using a treatment based on the generators of the 5-dimensional rotation group, cf. ref. 26, derived implicit equations from which the gap parameters for the ABM- and BW-phases can be solved. The investigation of collective modes in high fields was performed by Tewordt and Schopohl [27], but they did not take into account the asymmetry in the density of states which is essential to explain the A1-phase.

In the present chapter we investigate the behaviour of liquid ^{3}He in a magnetic field on the basis of a Landau expansion which is derived from an exactly solvable model introduced in a previous chapter [17].

The model is described by the hamiltonian

$$\mathcal{H} = \sum_{\underline{k}} \sum_{\alpha\beta} (\varepsilon_k \mathbf{1}_{\alpha\beta} - b\,\sigma^z_{\alpha\beta})\, a^\dagger_{\underline{k}\alpha} a_{\underline{k}\beta} + \tfrac{1}{2} \sum_{\underline{k},\underline{k}'} V(\underline{k},\underline{k}') \sum_{\alpha,\beta} a^\dagger_{\underline{k}\alpha} a^\dagger_{-\underline{k}\beta} a_{-\underline{k}'\beta} a_{\underline{k}'\alpha} \,, \tag{1.1}$$

$(\alpha,\beta = \uparrow,\downarrow)$, with

$$V(\underline{k},\underline{k}') = \begin{cases} -3(V_1/\Omega)\ \underline{k}\cdot\underline{k}'/(kk')\,, & \text{if } |\varepsilon_{\underline{k}}|,|\varepsilon_{\underline{k}'}| < \hbar\omega \\ 0 & , \text{ otherwise} . \end{cases} \tag{1.2}$$

The hamiltonian (1.1) contains a sum of one-particle operators diagonal in the fermion creation and annihilation operators $a^\dagger_{\underline{k}\alpha}$, $a_{\underline{k}\alpha}$, where $\underline{k}$ is the wave vector and α the spin which can be either up (↑) or down (↓). The $\varepsilon_{\underline{k}}$ are one-particle energies relative to the Fermi energy and are assumed to be independent of the direction of $\underline{k}$ and the spin α, thus $\varepsilon_{\underline{k}} = \varepsilon_k$, and b is a magnetic field in the z-direction.

The second term with $V(\underline{k},\underline{k}')$ is a pairing interaction between pairs of quasi particles $(\underline{k},-\underline{k})$ and $(\underline{k}',-\underline{k}')$, both having total wave vector zero and α and β denote the spins of the quasi particles †). In (1.2) we have only taken into account $\ell=1$ pairing and we have assumed the weak-coupling limit, i.e. the matrix elements $V(\underline{k},\underline{k}')$ are non-vanishing and independent of the lengths k,k' of the vectors $\underline{k},\underline{k}'$ in the narrow interval $|\varepsilon_{\underline{k}}|,|\varepsilon_{\underline{k}'}| < \hbar\omega$ around the Fermi energy and Ω is the volume of the system.

Inserting (1.2) in (1.1) we obtain an exactly solvable model which can be treated using the general methods for systems with separable interactions [29),30)], cf. also ref. 31. For the free energy per unit volume corresponding to (1.1) we have the rigorous result ($\Omega\to\infty$)

$$f = \inf_{\{\underline{\underline{m}}^i\}} \left[g_0(\{\underline{\underline{m}}^i\},\{\underline{\underline{m}}^{i\dagger}\}) - 6V_1 \sum_{i=x,y,z} \mathrm{tr}(\underline{\underline{m}}^{i\dagger}\cdot\underline{\underline{m}}^i) \right] \qquad (1.3)$$

in which the infimum is taken over all triples $(\underline{\underline{m}}^x,\underline{\underline{m}}^y,\underline{\underline{m}}^z)$ of symmetric 2×2 matrices $\underline{\underline{m}}^i$, $i=x,y,z$, with complex elements; $g_0(\{\underline{\underline{m}}^i\},\{\underline{\underline{m}}^{i\dagger}\})$ is the Legendre transform of the free energy $f_0(\{\underline{\underline{h}}^i\},\{\underline{\underline{h}}^{i\dagger}\})$, i.e.

$$g_0(\{\underline{\underline{m}}^i\},\{\underline{\underline{m}}^{i\dagger}\}) = \sup_{\{\underline{\underline{h}}^i\}} \left[f_0(\{\underline{\underline{h}}^i\},\{\underline{\underline{h}}^{i\dagger}\}) + \sum_{i=x,y,z} \mathrm{tr}(\underline{\underline{m}}^{i\dagger}\cdot\underline{\underline{h}}^i + \underline{\underline{h}}^{i\dagger}\cdot\underline{\underline{m}}^i) \right] \qquad (1.4)$$

of a <u>reference system</u> that is described by the bilinear hamiltonian

$$\mathcal{H}_0 = \sum_{\underline{k}>0} \sum_{\alpha,\beta} (\varepsilon_k \mathbb{1}_{\alpha\beta} - b\,\sigma_{\alpha\beta})(a^\dagger_{\underline{k}\alpha}a_{\underline{k}\beta} - a_{-\underline{k}\alpha}a^\dagger_{-\underline{k}\beta})$$

$$+ \sum_{\underline{k}>0}{}' \sum_{\alpha,\beta} \sum_{i=x,y,z} (-n^i h^i_{\alpha\beta} a^\dagger_{\underline{k}\alpha} a^\dagger_{-\underline{k}\beta} - n^i h^{i*}_{\beta\alpha} a_{-\underline{k}\alpha} a_{\underline{k}\beta}) + 2 \sum_{\underline{k}>0} \varepsilon_{\underline{k}} \, , \qquad (1.5)$$

with $n^i = k^i/k$ the unit vector in the direction of $\underline{k}$ and $\sigma_{\alpha\beta} = \sigma^z_{\alpha\beta}$, cf. equations (II.3.2) and (II.5.1) of chapter II. In eq. (1.5) $k \geq 0$ denotes that the sum is taken over pairs $(-\underline{k},\underline{k})$ and the prime in the second term indicates

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†)Pairs of quasi particles with total wave vector different from zero have been taken into account e.g. in ref. 28.

that the summation is restricted to $\underline{k}$ vectors with $|\varepsilon_k| < \hbar\omega$.

Since the reference system does not have any phase transitions the supremum in (1.4) is reached at:

$$m^i_{\alpha\beta} = -\tfrac{1}{2}\left\{\frac{\partial f_0}{\partial h^{i*}_{\alpha\beta}} + \frac{\partial f_0}{\partial h^{i*}_{\beta\alpha}}\right\} = \frac{1}{2\Omega}\sum_{\underline{k}>0}{}' \frac{k^i}{k}\,\langle a_{-\underline{k}\beta}a_{\underline{k}\alpha} + a_{-\underline{k}\alpha}a_{\underline{k}\beta}\rangle \ . \tag{1.6}$$

By means of an iterative procedure one may solve the matrices $\underline{\underline{h}}^i$, $\underline{\underline{h}}^{i\dagger}$ from (1.6) as functions of the order matrices $\underline{\underline{m}}^i$, $\underline{\underline{m}}^{i\dagger}$ and evaluate the Legendre transform $g_0(\{\underline{\underline{m}}^i\},\{\underline{\underline{m}}^{i\dagger}\})$ in (1.3). In this way we obtained in ref. 17 the complete Landau expansion for the free energy per unit volume up to fourth order terms.

In section 2 of the present chapter the Landau expansion will be expressed in terms of 3 complex vectors $\underline{m}_1$, $\underline{m}_2$, $\underline{m}_3$ associated with the elements $m^i_{\uparrow\uparrow}$, $m^i_{\downarrow\downarrow}$ and $m^i_{\uparrow\downarrow} = m^i_{\downarrow\uparrow}$, $i = x,y,z$, of the 2×2 matrices $\underline{\underline{m}}^i$. Formal expressions which are valid up to any order of the magnetic field b are given. As a consequence of the large number of variables in the Landau expansion it is by no means obvious that the b-dependence of the coefficients of the fourth order terms can be neglected. In fact, the occurrence of certain solutions of the gap equations will depend sensitively on the field-dependent contributions to the fourth-order coefficients, as we shall see later on. In section 2 we also give explicit and simple expressions which are valid up to the order b^2.

The extrema of the Landau expansion can be obtained as the solutions of a set of 3 coupled equations for the complex vectors $\underline{m}_1,\underline{m}_2,\underline{m}_3$ which will be given in section 3. In section 3 we also discuss the "two-dimensional" solutions for which (at least) one of the vectors $\underline{m}_1,\underline{m}_2,\underline{m}_3$ vanishes and in section 4 we deal with the more general class of solutions for which (at least) one of the three vectors $\underline{m}_1,\underline{m}_2,\underline{m}_3$ is real. In section 5 we investigate the possibility of other solutions on the basis of an inertia condition for the three vectors $\underline{m}_1$, $\underline{m}_2$, $\underline{m}_3$. The only possible (new) solution is a 3-dimensional generalization of the ABM-solution which, however, does not lead to a minimum in the weak-coupling limit.

As a result of the treatment in sections 3-5 the absolute minimum of the Landau expansion with the weak-coupling coefficients can be realized by minimizing a three-parameter function $\Phi_{BW}(m_1,m_2,m_3)$ with respect to the lenghts $m_k = (\underline{m}_k\cdot\underline{m}_k^*)^{\frac{1}{2}}$ of the vectors $\underline{m}_k$, $k = 1,2,3$. The minimization of Φ_{BW} leads to three different phases:

i) an $n=1$ phase with one non-vanishing order parameter, i.e. $m_1 \neq 0$, $m_2 \neq m_3 = 0$.
This phase corresponds to the A1-phase.

ii) an $n=2$ phase with two non-vanishing order parameters, i.e. $m_1 \neq 0$, $m_2 \neq 0$, $m_3 = 0$, corresponding to the ABM phase (or to the two-dimensional 2D-phase) in an external field .

iii) an $n=3$ phase with three non-vanishing order parameters $m_1 \neq 0$, $m_2 \neq 0$, $m_3 \neq 0$, which in the limit of zero magnetic field reduces to the BW phase.

In section 6 we discuss the phase diagram as a function of magnetic field b and temperature T using the weak-coupling values of the coefficients in the Landau expansion, and a comparison is made with the phase diagram obtained in ref. 9. In particular a first-order transition is found between the A1-phase and the (generalized) BW-phase at sufficiently small magnetic fields. Furthermore, in the presence of an infinitesimal, but non-vanishing, asymmetry in the density of states the values of m_2/m_1 can be quite different from 1 which is the value of m_2/m_1 for a symmetric density of states. It should be noted that the analysis of the phase diagram is based on the value of the coefficients in the weak-coupling approximation. In practice strong-coupling effects and in particular spin-fluctuation effects will give rise to modifications in the Landau expansion, which would lead to qualitative changes in the phase diagram, cf. refs. 10 and 11. We give a systematic investigation of these problems in chapters IV and V.

2. Landau expansion

In chapter II we have given the complete Landau expansion (II.6.6) up to fourth-order terms as a function of three complex, symmetrical 2×2 matrices $\hat{\underline{\underline{m}}}^i$, i=x,y,z, which are related to the matrices $\underline{\underline{m}}^i$ in (1.3) by:

$$\hat{\underline{\underline{m}}}^i = 3(A_1+A_2)^{-1}\, \tfrac{1}{4}(\underline{\underline{1}}+\underline{\underline{g}})\cdot\underline{\underline{m}}^i\cdot(\underline{\underline{1}}+\underline{\underline{g}}) + 3(A_1-A_2)^{-1}\, \tfrac{1}{4}(\underline{\underline{1}}-\underline{\underline{g}})\cdot\underline{\underline{m}}^i\cdot(\underline{\underline{1}}-\underline{\underline{g}}) +$$
$$+\, 3(A_1+4P)^{-1}\, \tfrac{1}{2}(\underline{\underline{m}}^i - \underline{\underline{g}}\cdot\underline{\underline{m}}^i\cdot\underline{\underline{g}}) \;, \qquad (2.1)$$

cf. (II.5.26). In eq. (2.1), $\underline{\underline{1}}$ is the 2×2 unit matrix, $\underline{\underline{\sigma}}$ is the Pauli $\underline{\underline{\sigma}}^z$-matrix, also given in (1.5), and A_1, A_2 and P are functions of b and T which are explicitly given by (II.5.13), cf. also (2.10).

In order to investigate the possible types of ordering it is convenient to introduce the three complex vectors $\underline{m}_1, \underline{m}_2, \underline{m}_3$ with components:

$$m_1^i = (1/\sqrt{2})\hat{m}^i_{\uparrow\uparrow}\ ,\quad m_2^i = (1/\sqrt{2})\hat{m}^i_{\downarrow\downarrow},\quad m_3^i = (1/\sqrt{2})\hat{m}^i_{\uparrow\downarrow} = (1/\sqrt{2})\hat{m}^i_{\downarrow\uparrow}\ ,\ i=x,y,z\ , \tag{2.2a}$$

with lengths

$$m_k = (\underline{m}_k\cdot\underline{m}_k^*)^{\frac{1}{2}}\ . \tag{2.2b}$$

Inserting (2.2a) into (II.6.6) we find:

$$f = f_n + \inf_{\{\underline{m}_k\}} \Phi(\underline{m}_1,\underline{m}_2,\underline{m}_3)\ , \tag{2.3}$$

where f_n is the free energy per unit volume of the normal Fermi liquid, if $\underline{m}_1 = \underline{m}_2 = \underline{m}_3 = 0$, and $\Phi(\underline{m}_1,\underline{m}_2,\underline{m}_3)$ is given by:

$$\begin{aligned}\Phi(\underline{m}_1,\underline{m}_2,\underline{m}_3) = {} & \frac{1}{3}\left[2A_1+4P-V_1(A_1+A_2)^2 - V_1(A_1-A_2)^2 - 2V_1(A_1+4P)^2\right](m_1^2+m_2^2+2m_3^2) \\ & + \frac{1}{3}\left[2A_2 - 2V_1(A_1+A_2)^2 + 2V_1(A_1-A_2)^2\right](m_1^2-m_2^2) \\ & + \frac{1}{3}\left[-4P - V_1(A_1+A_2)^2 - V_1(A_1-A_2)^2 + 2V_1(A_1+4P)^2\right](m_1^2+m_2^2-2m_3^2) \\ & + 4f_4\left(\left\{\begin{pmatrix} m_1^i & m_3^i \\ m_3^i & m_2^i\end{pmatrix}\right\}, \left\{\begin{pmatrix} m_1^{i*} & m_3^{i*} \\ m_3^{i*} & m_2^{i*}\end{pmatrix}\right\}\right) + \mathcal{O}(m^6)\ ,\end{aligned} \tag{2.4}$$

where f_4 has been given in (II.5.20).

Evaluating f_4, one obtains after a straightforward calculation:

$$\Phi(\underline{m}_1,\underline{m}_2,\underline{m}_3) = \Phi_0(m_1,m_2,m_3) + \Phi_1(\underline{m}_1,\underline{m}_2,\underline{m}_3)\ , \tag{2.5}$$

where Φ_0 depends only on the lengths of the vectors $\underline{m}_k$ in (2.2b) and is explicitly given by

$$\begin{aligned}\Phi_0(m_1,m_2,m_3) = {} & u_1m_1^2 + u_2m_2^2 + 2u_3m_3^2 + 2v_1m_1^4 + 2v_2m_2^4 + 4v_3m_3^4 \\ & + v_4m_1^2m_3^2 + v_5m_2^2m_3^2.\end{aligned} \tag{2.6}$$

For Φ_1 we have:

$$\begin{aligned}\Phi_1(\underline{m}_1,\underline{m}_2,\underline{m}_3) = {} & v_1|\underline{m}_1\cdot\underline{m}_1|^2 + v_2|\underline{m}_2\cdot\underline{m}_2|^2 + 2v_3|\underline{m}_3\cdot\underline{m}_3|^2 \\ & + v_4|\underline{m}_1\cdot\underline{m}_3|^2 + v_4|\underline{m}_1\cdot\underline{m}_3^*|^2 + v_5|\underline{m}_2\cdot\underline{m}_3|^2 + v_5|\underline{m}_2\cdot\underline{m}_3^*|^2 \\ & + 2v_6\,\mathrm{Re}(\underline{m}_1\cdot\underline{m}_2\ \underline{m}_3^*\cdot\underline{m}_3^*) + 4v_6\,\mathrm{Re}(\underline{m}_1\cdot\underline{m}_3^*\ \underline{m}_2\cdot\underline{m}_3^*)\ ,\end{aligned} \tag{2.7}$$

where Re denotes the real part.

The coefficients u_1,u_2,u_3 and $v_1,\ldots,v_6$ in (2.6) and (2.7) are functions of b and T which can be easily expressed in terms of the A_1,A_2, P, B_1,B_2,C, Q_1,Q_2 and R given in (II.5.13). We have:

$$u_1 = \frac{2}{3}(A_1+A_2)\,[1 - 2V_1(A_1+A_2)]\,,$$
$$u_2 = \frac{2}{3}(A_1-A_2)\,[1 - 2V_1(A_1-A_2)]\,, \qquad (2.8)$$
$$u_3 = \frac{2}{3}(A_1+4P)\,[1 - 2V_1(A_1+4P)]\,,$$

and

$$v_1 = \frac{4}{15}(B_1+B_2+C)\,,$$
$$v_2 = \frac{4}{15}(B_1+B_2-C)\,,$$
$$v_3 = \frac{4}{15}(B_1+B_2+4Q_1+16R)\,,$$
$$v_4 = \frac{4}{15}(4B_1+2C+4Q_1+4Q_2)\,,$$
$$v_5 = \frac{4}{15}(4B_1-2C+4Q_1-4Q_2)\,,$$
$$v_6 = \frac{4}{15}(2B_1-2B_2)\,. \qquad (2.9)$$

The functions $A_1,A_2,P,B_1,B_2,C,Q_1,Q_2$ and R can be inferred from:

$$A_1 \pm A_2 = \tfrac{1}{2}\int d\varepsilon\; \tfrac{1}{2}\mathcal{N}(\varepsilon)\,p(\varepsilon\pm b)\,,$$

$$A_1 + 4P = \tfrac{1}{4}\int d\varepsilon\; \tfrac{1}{2}\mathcal{N}(\varepsilon)\,[p(\varepsilon+b) + p(\varepsilon-b) + 4b^2p_2(\varepsilon,b)]\,, \qquad (2.10)$$

and

$$B_1+B_2\pm C = -\tfrac{1}{4}\int d\varepsilon\; \tfrac{1}{2}\mathcal{N}(\varepsilon)\,p_1(\varepsilon\mp b)$$

$$B_1+B_2+4Q_1+16R = -\frac{1}{8}\int d\varepsilon\; \tfrac{1}{2}\mathcal{N}(\varepsilon)\Big[(1+b\varepsilon^{-1})^2\,p_1(\varepsilon+b) +$$
$$+ (1-b\varepsilon^{-1})^2\,p_1(\varepsilon-b) - 2b^2\varepsilon^{-2}\,p_2(\varepsilon,b)\Big]\,,$$

$$4B_1+2C+4Q_1+4Q_2 = -\tfrac{1}{2}\int d\varepsilon\; \tfrac{1}{2}\mathcal{N}(\varepsilon)\Big[(1-b\varepsilon^{-1})p_1(\varepsilon-b) + (1+b\varepsilon^{-1})p_2(\varepsilon,b)\Big]\,,$$

$$4B_1-2C+4Q_1-4Q_2 = -\tfrac{1}{2}\int d\varepsilon\; \tfrac{1}{2}\mathcal{N}(\varepsilon)\Big[(1+b\varepsilon^{-1})p_1(\varepsilon+b) + (1-b\varepsilon^{-1})p_2(\varepsilon,b)\Big]\,,$$

$$2(B_1-B_2) = -\tfrac{1}{2}\int d\varepsilon\ \tfrac{1}{2}\mathcal{N}(\varepsilon)p_2(\varepsilon,b) \quad , \tag{2.11}$$

where the integrations are from $-\hbar\omega$ to $\hbar\omega$, cf. (II.5.13) with the abbreviations:

$$p(\varepsilon) = (2\varepsilon)^{-1}\tanh\tfrac{1}{2}\beta\varepsilon, \quad p_1(\varepsilon) = dp(\varepsilon)/d\varepsilon^2 \ , \quad p_2(\varepsilon,b) = (4\varepsilon b)^{-1}[p(\varepsilon+b) - p(\varepsilon-b)] \tag{2.12}$$

cf. (II.5.9) and (II.5.14).

In eqs. (2.10) and (2.11) $\mathcal{N}(\varepsilon)$ is the density of states per unit volume at the energy $\varepsilon+E_F$ and following ref. 9 we shall assume $\mathcal{N}(\varepsilon)$ to be a linear function of ε in the narrow internal around the Fermi energy E_F, i.e.

$$\begin{aligned} &\mathcal{N}(\varepsilon) = \mathcal{N}(0) + \varepsilon\mathcal{N}'(0) \ , \quad |\varepsilon| < \hbar\omega \ , \\ &\mathcal{N}'(0) \equiv \eta\mathcal{N}(0) \ . \end{aligned} \tag{2.13}$$

The asymmetry parameter η in (2.13) is essential to account for the phase transition within the A-phase.

Inserting (2.13) and the expansions of $p(\varepsilon\pm b)$, $p_1(\varepsilon\pm b)$ and $p_2(\varepsilon,b)$ up to order b^2 in (2.10), (2.11), we obtain:

$$\begin{aligned} &A_1\pm A_2 = \tfrac{1}{2}A(T)(1\mp\eta b) + \tfrac{1}{2}\mathcal{N}(0)\hbar\omega\left(\pm\eta b\, p(\hbar\omega) + b^2p'(\hbar\omega)\right) , \\ &A_1+4P = \tfrac{1}{2}A(T) - b^2B(T) + \tfrac{1}{2}\mathcal{N}(0)\hbar\omega b^2p'(\hbar\omega) , \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} &B_1+B_2\pm C = \tfrac{1}{4}B(T)(1\pm\eta b) + \tfrac{1}{4}\mathcal{N}(0)\hbar\omega\left(\pm\eta bp'(\hbar\omega) - \tfrac{1}{2}b^2p''(\hbar\omega)\right) , \\ &B_1+B_2+4Q_1+16R = \tfrac{1}{4}B(T)(1-4\gamma b^2) - \tfrac{1}{2}\mathcal{N}(0)\hbar\omega\ \tfrac{1}{2}b^2p''(\hbar\omega) , \\ &4B_1\pm 2C+4Q_1\pm 4Q_2 = B(T)(1 - \tfrac{4}{3}\gamma b^2 \pm \tfrac{1}{2}\eta b) + \tfrac{1}{2}\mathcal{N}(0)\hbar\omega\left(\pm\eta bp'(\hbar\omega) - \tfrac{4}{3}b^2p''(\hbar\omega)\right) , \\ &2(B_1-B_2) = \tfrac{1}{2}B(T)(1-\tfrac{2}{3}\gamma b^2) - \tfrac{1}{2}\mathcal{N}(0)\hbar\omega\ \tfrac{1}{3}\ b^2p''(\hbar\omega) \quad , \end{aligned} \tag{2.15}$$

where we have used the abbreviations

$$\begin{aligned} &A(T) = \tfrac{1}{2}\mathcal{N}(0)\int_{-\hbar\omega}^{\hbar\omega} d\varepsilon\ p(\varepsilon) \ , \quad B(T) = -\tfrac{1}{2}\mathcal{N}(0)\int_{-\hbar\omega}^{\hbar\omega} d\varepsilon\ p'(\varepsilon) \ , \\ &\gamma(T) = \tfrac{1}{2}\mathcal{N}(0)B^{-1}(T)\int_{-\hbar\omega}^{\hbar\omega} d\varepsilon\ p''(\varepsilon) \quad , \quad p'(\varepsilon) \equiv \frac{dp(\varepsilon)}{d\varepsilon^2} \ , \ p''(\varepsilon) \equiv \frac{dp_1(\varepsilon)}{d\varepsilon^2} \ . \end{aligned} \tag{2.16}$$

In the extreme limit of weak coupling, i.e. $\beta\hbar\omega\to\infty$, eqs. (2.16) reduce to:

$$A(T) = \tfrac{1}{2}\mathcal{N}(0)\ln(1.14\ \beta\hbar\omega),\quad B(T) = \tfrac{1}{2}\mathcal{N}(0)\ \frac{7}{8}\ (\beta^2/\pi^2)\zeta(3)\ ,$$

$$\gamma = \frac{93}{56}\ (\beta^2/\pi^2)\ \zeta(5)/\zeta(3)\ , \tag{2.17}$$

where $\zeta(s)$ is the Riemann zeta function $\sum n^{-s}$, and

$$\tfrac{1}{2}\mathcal{N}(0)\hbar\omega\, p(\hbar\omega)\, A^{-1}(T)\to 0,\quad \tfrac{1}{2}\mathcal{N}(0)\hbar\omega\ p'(\hbar\omega)\, B^{-1}(T)\to 0\ ,$$

$$\tfrac{1}{2}\mathcal{N}(0)\hbar\,\omega\, p''(\hbar\omega)\,\gamma^{-1}B^{-1}(T)\to 0\ . \tag{2.18}$$

In the neighbourhood of the (poly-)critical point with b=0 and $A(T_c) = A_c = V_1^{-1}$, we may use the aproximations:

$$A(T)\simeq A_c - t\ ,\qquad B(T)\simeq B(T_c) = B_c \equiv 15\ v \tag{2.19}$$

with $t = \tfrac{1}{2}\mathcal{N}(0)(T-T_c)/T_c$.

Inserting (2.15), (2.18) and (2.19) into (2.8), (2.9) and omitting terms which are smaller than t, ηb or b^2, we obtain the simple expressions:

$$\begin{aligned} u_1 &= \frac{1}{3}\ (t+A_c\eta b)\ ,\\ u_2 &= \frac{1}{3}\ (t-A_c\eta b)\ ,\\ u_3 &= \frac{1}{3}\ (t+2b^2B_c)\ , \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} v_1 &= v(1+\eta b)\ , & v_2 &= v(1-\eta b)\ ,\\ v_3 &= v(1-4\gamma b^2)\ , & v_6 &= 2v(1-\frac{2}{3}\ \gamma b^2)\ ,\\ v_4 &= 4v(1+\tfrac{1}{2}\eta b-\frac{4}{3}\ \gamma b^2)\ , & v_5 &= 4v(1-\tfrac{1}{2}\eta b-\frac{4}{3}\ \gamma b^2)\ . \end{aligned} \tag{2.21}$$

The coefficients u_1, u_2, u_3 are in agreement with the ones given in ref. 9, the b-dependent contributions in the coefficients $v_1,\dots,v_6$, which to our knowledge have not been given before, will be important for certain solutions of the gap equations.

Remark: In most treatments of liquid ^{3}He use is made of the 2×2 gap-matrices $\underline{\Delta}(\underline{n})$, $\underline{n} = \underline{k}/k$, which can be identified with the coefficients of the terms $a^\dagger a^\dagger$ and aa in the reference hamiltonian. From (1.5), cf. also (I.5.1) and (I.5.28), and using (2.2) we have, omitting the non-linear terms in (I.5.28):

$$\underline{\underline{\Delta}}(\underline{n}) = - \sum_i n^i h^i \simeq - \sum_i n^i \underline{\underline{m}}^i = - \sqrt{2} \begin{pmatrix} \underline{n}\cdot\underline{m}_1 & \underline{n}\cdot\underline{m}_3 \\ \underline{n}\cdot\underline{m}_3 & \underline{n}\cdot\underline{m}_2 \end{pmatrix} . \tag{2.22}$$

In general a non-vanishing parameter n in (2.20) which is necessary to explain the (observed) phase transition between the A1- and the ABM-phase, leads to a breakdown of unitarity, i.e. the condition

$$\underline{\underline{\Delta}}(\underline{n}) \cdot \underline{\underline{\Delta}}^{\dagger}(\underline{n}) = \Delta(\underline{n}) \underline{\underline{1}} \tag{2.23}$$

is no longer satisfied.

3. The gap equations. One- and two-dimensional solutions

In the remaining part of this chapter we will be concerned with the problem of obtaining the absolute minimum of the function Φ given by (2.5)-(2.7). The extrema of Φ are determined by the conditions $\partial\Phi/\partial\underline{m}_k^* = 0$, $k=1,2,3$, which can be explicitly written as

$$\begin{aligned} &\alpha_1\underline{m}_1 + 2v_1(\underline{m}_1\cdot\underline{m}_1)\underline{m}_1^* + v_6(\underline{m}_3\cdot\underline{m}_3)\underline{m}_2^* \\ &\quad + (2v_6\underline{m}_2^*\cdot\underline{m}_3 + v_4\underline{m}_1\cdot\underline{m}_3^*)\underline{m}_3 + v_4(\underline{m}_1\cdot\underline{m}_3)\underline{m}_3^* = 0 \ , \end{aligned} \tag{3.1a}$$

$$\begin{aligned} &\alpha_2\underline{m}_2 + 2v_2(\underline{m}_2\cdot\underline{m}_2)\underline{m}_2^* + v_6(\underline{m}_3\cdot\underline{m}_3)\underline{m}_1^* \\ &\quad + (2v_6\underline{m}_1^*\cdot\underline{m}_3 + v_5\underline{m}_2\cdot\underline{m}_3^*)\underline{m}_3 + v_5(\underline{m}_2\cdot\underline{m}_3)\underline{m}_3^* = 0 \ , \end{aligned} \tag{3.1b}$$

$$\begin{aligned} &\alpha_3\underline{m}_3 + (2v_3\underline{m}_3\cdot\underline{m}_3 + v_6\underline{m}_1\cdot\underline{m}_2)\underline{m}_3^* + \tfrac{1}{2}v_4(\underline{m}_1\cdot\underline{m}_3)\underline{m}_1^* + \tfrac{1}{2}v_5(\underline{m}_2\cdot\underline{m}_3)\underline{m}_2^* \\ &\quad + \tfrac{1}{2}(2v_6\underline{m}_2\cdot\underline{m}_3^* + v_4\underline{m}_1^*\cdot\underline{m}_3)\underline{m}_1 + \tfrac{1}{2}(2v_6\underline{m}_1\cdot\underline{m}_3^* + v_5\underline{m}_2^*\cdot\underline{m}_3)\underline{m}_2 = 0 \ . \end{aligned} \tag{3.1c}$$

The coefficients $\alpha_1,\alpha_2,\alpha_3$ depend only on the lengths m_1,m_2,m_3:

$$\begin{aligned} \alpha_1 &= u_1 + 4v_1m_1^2 + v_4m_3^2 \\ \alpha_2 &= u_2 + 4v_2m_2^2 + v_5m_3^2 \\ \alpha_3 &= u_3 + 4v_3m_3^2 + \tfrac{1}{2}v_4m_1^2 + \tfrac{1}{2}v_5m_2^2 \ . \end{aligned} \tag{3.2}$$

The values of Φ, cf. (2.5)-(2.7), for an arbitrary solution (s) of eqs. (3.1a)-(3.1c), is given by

$$\Phi_s = \tfrac{1}{2}u_1m_1^2 + \tfrac{1}{2}u_2m_2^2 + u_3m_3^2 \ , \tag{3.3}$$

as is easily seen by multiplying eqs. (3.1a), (3.1b) and (3.1c) by $\underline{m}_1^*, \underline{m}_2^*$ and $\underline{m}_3^*$ respectively. We will, however, in most cases given below, only determine the solutions of eqs. (3.1) for the vectors $\underline{m}_1, \underline{m}_2, \underline{m}_3$ in terms of their lengths; and then use the function Φ in (2.5), considered now as a function only of the lengths m_1, m_2, m_3 (after inserting the obtained expressions), to characterize the various solutions.

In the remaining part of this section we consider simple cases for which at least one of the vectors $\underline{m}_1, \underline{m}_2$ or $\underline{m}_3$ vanishes.

i) If $\underline{m}_2 = \underline{m}_3 = 0$ we have from (3.1a)

$$\alpha_1 \underline{m}_1 + 2v_1(\underline{m}_1 \cdot \underline{m}_1)\underline{m}_1^* = 0 \; . \tag{3.4}$$

From eq. (3.4) we have that either $\underline{m}_1 \| \underline{m}_1^*$ or $\alpha_1 = \underline{m}_1 \cdot \underline{m}_1 = 0$. This leads to two solutions:

$$\underline{m}_1 \| \underline{m}_1^*, \quad |\underline{m}_1 \cdot \underline{m}_1| = m_1^2 = -\tfrac{1}{6} u_1/v_1 \;, \quad \Phi_s = -\tfrac{1}{12} u_1^2/v_1 \;, \quad (u_1 < 0) \tag{3.5a}$$

and

$$\alpha_1 = \underline{m}_1 \cdot \underline{m}_1 = 0 \;, \quad m_1^2 = -\tfrac{1}{4} u_1/v_1 \;, \quad \Phi_s = -\tfrac{1}{8} u_1^2/v_1, \quad (u_1 < 0) \; . \tag{3.5b}$$

It is clear that the solution (3.5a) can never occur, as the corresponding Φ_s is larger than the Φ_s in (3.5b). The solution (3.5b), however, can occur in practice and is characteristic for the so-called A1-phase.

In a similar way one can have a solution with $\underline{m}_1 = \underline{m}_3 = 0$, $\underline{m}_2 \cdot \underline{m}_2 = 0$, $m_2^2 = -\tfrac{1}{4} u_2/v_2$ and $\Phi_s = -\tfrac{1}{8} u_2^2/v_2$. Without loss of generality we assume here and in the following that $nb < 0$, implying that $u_1 < u_2$ so that the solution with $\underline{m}_2 \neq 0$ leads always to a larger Φ_s than the one given in (3.5b).

Finally there is a solution with $\underline{m}_1 = \underline{m}_2 = 0$ (the A3-solution, not to be confused with the one often used in the literature):

$$\alpha_3 = \underline{m}_3 \cdot \underline{m}_3 = 0, \quad m_3^2 = -\tfrac{1}{4} u_3/v_3, \quad \Phi_s = -\tfrac{1}{4} u_3^2/v_3, \quad (u_3 < 0) \; . \tag{3.6}$$

Using the weak-coupling coefficients (2.20) and (2.21) it will be shown in appendix A that the A3-solution does not lead to a minimum of the function Φ in (2.5).

ii) If $\underline{m}_3 = 0$, $\underline{m}_1, \underline{m}_2 \neq 0$, the equations (3.1a) and (3.1b) reduce to two decoupled equations for the vectors $\underline{m}_1$ and $\underline{m}_2$:

$$\begin{aligned} &\alpha_1 \underline{m}_1 + 2v_1(\underline{m}_1 \cdot \underline{m}_1)\underline{m}_1^* = 0 \;, \\ &\alpha_2 \underline{m}_2 + 2v_2(\underline{m}_2 \cdot \underline{m}_2)\underline{m}_2^* = 0 \; . \end{aligned} \tag{3.7}$$

The most favourable solution, i.e. the solution with the lowest value of Φ_s, is the superposition of the solution (3.5b) and a similar solution for $\underline{m}_2$, i.e.:

$$\alpha_1 = \alpha_2 = \underline{m}_1\cdot\underline{m}_1 = \underline{m}_2\cdot\underline{m}_2 = 0, \quad m_1^2 = -\tfrac{1}{4}u_1/v_1, \quad m_2^2 = -\tfrac{1}{4}u_2/v_2, \ (u_1<0,\ u_2<0),$$
$$\Phi_s = -\frac{1}{8}\frac{u_1^2}{v_1} - \frac{1}{8}\frac{u_2^2}{v_2} \ . \tag{3.8}$$

The solution (3.8) has a rather large degeneracy.
One example is given by the representation:

$$\underline{m}_1 = \frac{m_1}{\sqrt{2}}(1,i,0) \ , \quad \underline{m}_2 = -\frac{m_2}{\sqrt{2}}(1,i,0), \quad \underline{m}_3 = 0, \tag{3.9}$$

or in terms of the gap-matrix:

$$\underline{\underline{\Delta}}(\underline{n}) = \begin{pmatrix} -(n^x+in^y)m_1 & 0 \\ 0 & (n^x+in^y)m_2 \end{pmatrix}, \tag{3.10}$$

which can be regarded as the ABM-solution for $b\neq 0$. In fact, when one expresses $\underline{\underline{\Delta}}(\underline{n})$ in the form $\sum i d^j(\underline{n})\underline{\underline{g}}^j\cdot\underline{\underline{g}}^y$ we have $d^x(\underline{n}) = \frac{1}{2}(n^x+in^y)(m_1+m_2)$ and $d^y(\underline{n}) = \frac{1}{2}i(n^x+in^y)(m_1-m_2)$, $d^z(\underline{n}) = 0$, so that in the limit $\eta b\to 0$ with $m_1\to m_2$, $\underline{d}(\underline{n})$ is a complex vector in the x-direction. The solution given in (3.10) is unitary if and only if $m_1=m_2$, or $\eta b=0$.

Another example is:

$$\underline{m}_1 = \frac{m_1}{\sqrt{2}}(1,-i,0) \ , \quad \underline{m}_2 = \frac{m_2}{\sqrt{2}}(-1,-i,0) \ , \quad \underline{m}_3 = 0 \ , \tag{3.11}$$

which leads to

$$\underline{\underline{\Delta}}(\underline{n}) = \begin{pmatrix} -(n^x-in^y)m_1 & 0 \\ 0 & (n^x+in^y)m_2 \end{pmatrix}. \tag{3.12}$$

The components of the $\underline{d}(\underline{n})$ vector are now given by $d^z(\underline{n}) = 0$, $d^x(\underline{n}) = \frac{1}{2}n^x(m_1+m_2) - \frac{1}{2}in^y(m_1-m_2)$, $d^y(n) = \frac{1}{2}n^y(m_1+m_2) + \frac{1}{2}in^x(m_1-m_2)$, and reduce in the limit $\eta b\to 0$, $m_1\to m_2$ to the two-dimensional 2D-state, cf. ref. 4.

iii) If $\underline{m}_2 = 0$, $\underline{m}_1,\underline{m}_3 \neq 0^{+)}$, we have from (3.1b):

$$v_6(\underline{m}_3\cdot\underline{m}_3)\underline{m}_1^* + 2v_6(\underline{m}_1^*\cdot\underline{m}_3)\underline{m}_3 = 0 \ . \tag{3.13}$$

+) The case that $\underline{m}_1 = 0$, $\underline{m}_2,\underline{m}_3 \neq 0$, can be treated in a similar way.

Multiplying (3.13) by $\underline{m}_3$, we have $(\underline{m}_3\cdot\underline{m}_3)(\underline{m}_1^*\cdot\underline{m}_3) = 0$, but since $\underline{m}_1^*$ and $\underline{m}_3$ are assumed to be different from zero, we must have

$$\underline{m}_3\cdot\underline{m}_3 = \underline{m}_1^*\cdot\underline{m}_3 = 0 \; . \tag{3.14}$$

From (3.1a) and (3.1c) we then obtain:

$$\begin{aligned} &\alpha_1\underline{m}_1 + 2v_1(\underline{m}_1\cdot\underline{m}_1)\underline{m}_1^* + v_4(\underline{m}_1\cdot\underline{m}_3)\underline{m}_3^* = 0 \\ &\alpha_3\underline{m}_3 + \tfrac{1}{2}v_4(\underline{m}_1\cdot\underline{m}_3)\underline{m}_1^* = 0 \; . \end{aligned} \tag{3.15}$$

Multiplying the first equation by $\underline{m}_3^*$, or the second one by $\underline{m}_1^*$, we find

$$(\underline{m}_1\cdot\underline{m}_1)(\underline{m}_1^*\cdot\underline{m}_3^*) = 0 \; . \tag{3.16}$$

Noting that $\underline{m}_1\cdot\underline{m}_1$ and $\underline{m}_1\cdot\underline{m}_3$ cannot vanish simultaneously, cf. (3.14), we have the two possibilities:

a) $$\underline{m}_1\cdot\underline{m}_1 = \underline{m}_1\cdot\underline{m}_3^* = \underline{m}_3\cdot\underline{m}_3 = 0 \; . \tag{3.17}$$

Since (3.17) implies that $\underline{m}_1\cdot\underline{m}_3 \neq 0$ we have from the second equation in (3.15) $\underline{m}_3 \parallel \underline{m}_1^*$. This leads to a possible representation:

$$\underline{m}_1 = \frac{m_1}{\sqrt{2}}(1,i,0), \quad \underline{m}_2 = 0 \; , \quad \underline{m}_3 = \frac{m_3}{\sqrt{2}}(1,-i,0) \; , \tag{3.18}$$

characterized by the value

$$\Phi = u_1m_1^2 + 2u_3m_3^2 + 2v_1m_1^4 + 4v_3m_3^4 + 2v_4m_1^2m_3^2 \tag{3.19}$$

inserting (3.18) in (2.5). The solution (3.18) cannot lead to an absolute minimum of Φ in (2.5), since the value (3.19) is also realized for the configuration

$$\underline{m}_1 = \frac{m_1}{\sqrt{2}}(1,i,0), \quad \underline{m}_2 = 0 \; , \quad \underline{m}_3 = \frac{m_3}{\sqrt{2}}(1,i,0) \; , \tag{3.20}$$

which has $\underline{m}_3\cdot\underline{m}_3 = 0$, $\underline{m}_1\cdot\underline{m}_3^* \neq 0$ and therefore does not satisfy (3.14), so that there is a neighbourhood of the point (3.20) in the space of $\underline{m}_1,\underline{m}_2,\underline{m}_3$ where lower values of Φ occur.

b) $$\underline{m}_1\cdot\underline{m}_3 = \underline{m}_1\cdot\underline{m}_3^* = \underline{m}_3\cdot\underline{m}_3 = 0 \tag{3.21}$$

Eq. (3.15) leads to the possible representation

$$\underline{m}_1 = m_1(0,0,1), \quad \underline{m}_2 = 0, \quad \underline{m}_3 = \frac{m_3}{\sqrt{2}}(1,i.0) \tag{3.22}$$

with characteristic value

$$\Phi = u_1m_1^2 + 2u_3m_3^2 + 3v_1m_1^4 + 4v_3m_3^4 + v_4m_1^2m_3^2 \; . \tag{3.23}$$

The solution (3.22) is equivalent to the ε-solution introduced by Barton and Moore [22]. In the weak-coupling limit, however, with the values (2.20) and (2.21) for the coefficients u_1,u_2,u_3 and $v_1,\dots,v_6$, the ε-solution does not lead to a minimum of the function Φ in (2.5) as will be shown in appendix A. When dipolar interactions and strong-coupling effects are taken into account, this situation may be different as suggested by the work of ref. 32.

4. Three-dimensional solutions of the gap equations with real vector

In this section we discuss solutions of (3.1a)-(3.1c) with $\underline{m}_1,\underline{m}_2,\underline{m}_3 \neq 0$, for which one of the vectors $\underline{m}_k$ is real. We have two cases either $\underline{m}_3 \| \underline{m}_3^*$, $\underline{m}_1,\underline{m}_2 \neq 0$, or $\underline{m}_1 \| \underline{m}_1^*$, $\underline{m}_2,\underline{m}_3 \neq 0$.

i) If $\underline{m}_3 \| \underline{m}_3^*$ then we immediately have

$$|\underline{m}_3\cdot\underline{m}_3| = m_3^2, \quad |\underline{m}_1\cdot\underline{m}_3| = |\underline{m}_1\cdot\underline{m}_3^*| \;, \; |\underline{m}_2\cdot\underline{m}_3| = |\underline{m}_2\cdot\underline{m}_3^*| \tag{4.1}$$

For sufficiently small b we have the inequality

$$2v_4|\underline{m}_1\cdot\underline{m}_3^*|^2 + 2v_5|\underline{m}_2\cdot\underline{m}_3^*|^2 + 4v_6\,\mathrm{Re}(\underline{m}_1\cdot\underline{m}_3^*\;\underline{m}_2\cdot\underline{m}_3^*) =$$
$$= \left(v_4 + v_6\sqrt{\frac{v_4}{v_5}}\right)|\underline{m}_1\cdot\underline{m}_3^* + \sqrt{\frac{v_5}{v_4}}\,\underline{m}_2^*\cdot\underline{m}_3|^2 + \left(v_4 - v_6\sqrt{\frac{v_4}{v_5}}\right)|\underline{m}_1\cdot\underline{m}_3^* - \sqrt{\frac{v_5}{v_4}}\,\underline{m}_2^*\cdot\underline{m}_3|^2 \geq 0 \tag{4.2}$$

the coefficients being positive in view of (2.21).
From (4.1) and (4.2) we have the lower bound

$$\Phi_1 = v_1|\underline{m}_1\cdot\underline{m}_1|^2 + v_2|\underline{m}_2\cdot\underline{m}_2|^2 + 2v_3m_3^4 + 2v_4|\underline{m}_1\cdot\underline{m}_3^*|^2 + 2v_5|\underline{m}_2\cdot\underline{m}_3^*|^2$$
$$+ 2v_6m_3^2\,\mathrm{Re}(\underline{m}_1\cdot\underline{m}_2) + 4v_6\,\mathrm{Re}(\underline{m}_1\cdot\underline{m}_3^*\;\underline{m}_2\cdot\underline{m}_3^*)$$
$$\geq \Phi_2 \equiv v_1|\underline{m}_1\cdot\underline{m}_1|^2 + v_2|\underline{m}_2\cdot\underline{m}_2|^2 + 2v_3m_3^4 + 2v_6m_3^2\,\mathrm{Re}(\underline{m}_1\cdot\underline{m}_2) \;, \tag{4.3}$$

where we have chosen $\underline{m}_3\cdot\underline{m}_3$ to be real. The equality sign in (4.3) occurs if:

$$\underline{m}_1\cdot\underline{m}_3^* = \underline{m}_1\cdot\underline{m}_3 = \underline{m}_2\cdot\underline{m}_3^* = \underline{m}_2\cdot\underline{m}_3 = 0 \;. \tag{4.4}$$

For the function Φ_2 we have the lower bound:

$$\Phi_2 \geq 2v_3m_3^4 - 2v_6m_3^2m_1m_2 \;, \tag{4.5}$$

with the equality sign if

$$\underline{m}_1\cdot\underline{m}_1 = \underline{m}_2\cdot\underline{m}_2 = 0 , \quad \underline{m}_1\cdot\underline{m}_2 = -m_1m_2 \tag{4.6}$$

Combining (4.3) and (4.6) we have

$$\begin{aligned}\Phi \geq \Phi_S &= \Phi_0 + 2v_3m_3^4 - 2v_6m_3^2m_1m_2 = \\ &= u_1m_1^2 + u_2m_2^2 + 2u_3m_3^2 + 2v_1m_1^4 + 2v_2m_2^4 + 6v_3m_3^4 \\ &\qquad + v_4m_1^2m_3^2 + v_5m_2^2m_3^2 - 2v_6m_3^2m_1m_2\end{aligned} \tag{4.7}$$

in which, for given m_1,m_2,m_3, the conditions (4.4) and (4.6) for the equality sign can be realized choosing

$$\underline{m}_1 = \frac{m_1}{\sqrt{2}}(1,-i,0), \quad \underline{m}_2 = -\frac{m_2}{\sqrt{2}}(1,i,0), \quad \underline{m}_3 = -m_3(0,0,1) . \tag{4.8}$$

This leads to the following $\underline{\underline{\Delta}}$-matrix:

$$\underline{\underline{\Delta}}(\underline{n}) = \begin{pmatrix} -m_1(n^x-in^y) & \sqrt{2}\, m_3n^z \\ \sqrt{2}\, m_3n^z & m_2(n^x+in^y) \end{pmatrix} \tag{4.9}$$

which corresponds to a unitary state only in the limit $\eta b \to 0$, $m_1 \to m_2$. For $m_3=0$ (4.8) reduces to the 2D-state (3.12) but not to (3.10). Using $\underline{\underline{\Delta}}(\underline{n}) = \sum_{(j)} id^j(\underline{n})\underline{\underline{\sigma}}^j\cdot\sigma^y$, the $\underline{d}$ vector is given by

$$\begin{aligned} d^x(\underline{n}) &= \tfrac{1}{2}n^x(m_1+m_2) - \tfrac{1}{2}in^y(m_1-m_2), \\ d^y(\underline{n}) &= \tfrac{1}{2}n^y(m_1+m_2) + \tfrac{1}{2}in^x(m_1-m_2), \\ d^z(\underline{n}) &= \sqrt{2}\, n^zm_3 . \end{aligned} \tag{4.10}$$

From (4.10) we see that (4.8) reduces in the limit $b\to 0$, $m_1 \to m_2$ and $m_1 \to m_3\sqrt{2}$ to the BW-phase with $\underline{d}(\underline{n}) = m\underline{n}$. Therefore,(4.8) provides the appropriate generalization of the BW-solution in a magnetic field.

ii) If $\underline{m}_1 \parallel \underline{m}_1^*$,we shall show that also $\underline{m}_2 \parallel \underline{m}_2^*$. In fact,use can be made of the following inequality,(note that $|\underline{m}_1\cdot\underline{m}_1| = m_1^2$; $|\underline{m}_1\cdot\underline{m}_3| = |\underline{m}_1\cdot\underline{m}_3^*|$,

$$\begin{aligned}\Phi_1 &= v_1m_1^4 + v_2|\underline{m}_2\cdot\underline{m}_2|^2 + 2v_3|\underline{m}_3\cdot\underline{m}_3|^2 + 2v_4|\underline{m}_1\cdot\underline{m}_3^* + \frac{v_6}{v_4}\underline{m}_2^*\cdot\underline{m}_3|^2 \\ &+ \left(v_5 - 2\frac{v_6^2}{v_4}\right)|\underline{m}_2^*\cdot\underline{m}_3|^2 + v_5|\underline{m}_2\cdot\underline{m}_3|^2 + 2v_6\,\mathrm{Re}(\underline{m}_1\cdot\underline{m}_2\,\underline{m}_3^*\cdot\underline{m}_3^*)\end{aligned}$$

$$\geq \Phi_3 \equiv v_1 m_1^4 + v_2|\underline{m}_2\cdot\underline{m}_2|^2 + 2v_3|\underline{m}_3\cdot\underline{m}_3|^2 +$$

$$+ \left(v_5 - 2\frac{v_6^2}{v_4}\right)|\underline{m}_2\cdot\underline{m}_3^*|^2 + v_5|\underline{m}_2\cdot\underline{m}_3|^2 + 2v_6\ \mathrm{Re}(\underline{m}_1\cdot\underline{m}_2\ \underline{m}_3^*\cdot\underline{m}_3^*) \ . \qquad (4.11)$$

The equality sign in (4.11) holds for

$$\underline{m}_1\cdot\underline{m}_3^* + \frac{v_6}{v_4}\,\underline{m}_2^*\cdot\underline{m}_3 = 0 \ . \qquad (4.12)$$

We shall now show that in the absolute minimum of $\Phi_0+\Phi_3$ the condition (4.12) is satisfied, so that we obtain the lowest value of $\Phi_0+\Phi_1$ for the case of real $\underline{m}_1$. In fact, in the absolute minimum of $\Phi_0+\Phi_3$ we have the equations

$$(\alpha_1+2v_1m_1^2)\underline{m}_1 + v_6(\underline{m}_3\cdot\underline{m}_3)\underline{m}_2^* = 0 \qquad (4.13a)$$

$$\alpha_2\underline{m}_2 + 2v_2(\underline{m}_2\cdot\underline{m}_2)\underline{m}_2^* + \left(v_5 - 2\frac{v_6^2}{v_4}\right)(\underline{m}_2\cdot\underline{m}_3^*)\underline{m}_3$$

$$+ v_5(\underline{m}_2\cdot\underline{m}_3)\underline{m}_3^* + v_6(\underline{m}_3\cdot\underline{m}_3)\underline{m}_1^* = 0 \qquad (4.13b)$$

$$\alpha_3\underline{m}_3 + (2v_3\underline{m}_3\cdot\underline{m}_3 + v_6\underline{m}_1\cdot\underline{m}_2)\underline{m}_3^* + \tfrac{1}{2}\left(v_5 - 2\frac{v_6^2}{v_4}\right)(\underline{m}_2^*\cdot\underline{m}_3)\underline{m}_2$$

$$+ \tfrac{1}{2}v_5(\underline{m}_2\cdot\underline{m}_3)\underline{m}_2^* = 0 \ . \qquad (4.13c)$$

For $\underline{m}_2\neq 0$ eq. (4.13a) has two possible solutions, either $\alpha_1+2v_1m_1^2 = \underline{m}_3\cdot\underline{m}_3 = 0$, or $\underline{m}_1 \parallel \underline{m}_2^*$. The first possibility with $\underline{m}_3\cdot\underline{m}_3 = 0$, however, cannot lead to an absolute minimum of $\Phi_0+\Phi_3$. In fact, with $\underline{m}_3\cdot\underline{m}_3 = 0$, the function $\Phi_0+\Phi_3$ does not depend at all on the orientation of the vector $\underline{m}_1$. On the other hand eq. (4.13c) does. Starting from a solution with $\underline{m}_3\cdot\underline{m}_3 = 0$, $\underline{m}_2 \neq 0$, of the eqs. (4.13a)-(4.13c) one can easily find another configuration with a different $\underline{m}_1$ that does not satisfy (4.13c), but does lead to the same value of $\Phi_0+\Phi_3$. It is clear then that this value cannot correspond to the absolute minimum .

We can therefore restrict ourselves to the case that $\underline{m}_2^* \parallel \underline{m}_1$ implying

$$\underline{m}_2 \parallel \underline{m}_2^* \parallel \underline{m}_1 \parallel \underline{m}_1^* \qquad (4.14)$$

and hence

$$\Phi_3 = v_1m_1^4 + v_2m_2^4 + 2v_3|\underline{m}_3\cdot\underline{m}_3|^2 + \left(2v_5 - 2\frac{v_6^2}{v_4}\right)|\underline{m}_2\cdot\underline{m}_3|^2 + 2v_6\ \mathrm{Re}(\underline{m}_1\cdot\underline{m}_2\ \underline{m}_3^*\cdot\underline{m}_3^*), \qquad (4.15)$$

which can be minimized for fixed m_1, m_2, m_3 by choosing

$$\underline{m}_3\cdot\underline{m}_3 + \frac{v_6}{2v_3}\,\underline{m}_1\cdot\underline{m}_2 = 0\ , \tag{4.16}$$

and $\underline{m}_2\cdot\underline{m}_3 = 0$. From this together with (4.14) we find

$$\underline{m}_2\cdot\underline{m}_3 = \underline{m}_2^*\cdot\underline{m}_3 = \underline{m}_1\cdot\underline{m}_3 = \underline{m}_1^*\cdot\underline{m}_3 = 0\ , \tag{4.17}$$

so that in particular (4.12) is satisfied. Then the absolute minimum of $\Phi_0+\Phi_3$ is equal to the minimum of Φ in (2.5) for the case $\underline{m}_1$ real.

Eqs. (4.16) and (4.17) are satisfied if we choose

$$\begin{aligned}&\underline{m}_1 = m_1(0,0,1),\quad \underline{m}_2 = -m_2(0,0,1),\quad \underline{m}_3 = m_3(\cos\phi\,,\ i\sin\phi\,,0)\ ,\\ &\cos 2\phi = v_6 m_1 m_2(2v_3 m_3^2)^{-1}\ ,\end{aligned} \tag{4.18}$$

under the condition $m_3^2 \geq \frac{v_6}{2v_3}\,m_1 m_2$ for the lengths. Eq. (4.18) can be considered to be a 3-dimensional generalization with $m_2 \neq 0$ of the ε-phase, cf. (3.22). It leads to the value

$$\begin{aligned}\Phi_s = {}& u_1 m_1^2 + u_2 m_2^2 + 2u_3 m_3^2 + 3v_1 m_1^4 + 3v_2 m_2^4 + 4v_3 m_3^4\\ &+ v_4 m_1^2 m_3^2 + v_5 m_2^2 m_3^2 - \frac{v_6^2}{2v_3}\,m_1^2 m_2^2\ .\end{aligned} \tag{4.19}$$

The extrema of (4.19) are given by the equations

$$\begin{aligned}&m_1\left(u_1 + 6v_1 m_1^2 + v_4 m_3^2 - \frac{v_6^2}{2v_3}\,m_2^2\right) = 0\ ,\\ &m_2\left(u_2 + 6v_2 m_2^2 + v_5 m_3^2 - \frac{v_6^2}{2v_3}\,m_1^2\right) = 0\ ,\\ &m_3\left(u_3 + 4v_3 m_3^2 + \tfrac{1}{2}v_4 m_1^2 + \tfrac{1}{2}v_5 m_2^2\right) = 0\ .\end{aligned} \tag{4.20}$$

The interpretation of the solution with $m_1,m_2,m_3 \neq 0$ is not without difficulties, since, if we replace the coefficients $v_1,\dots,v_6$ in (4.20) by the values given in (2.21) with b=0, the determinant of the coefficients of m_1^2,m_2^2,m_3^2 will vanish, and eqs. (4.20) will not have a solution. However, if we consider the full expressions (2.21) for these coefficients $v_1,\dots,v_6$ eqs. (4.20) can have a solution under appropriate conditions for u_1,u_2,u_3. The corresponding matrix $\underline{\underline{\Delta}}(\underline{n})$, cf. (2.22), is in general not unitary, also in the limit $\eta b \to 0$.

Furthermore, in the weak-coupling limit with the full expressions (2.21) for the coefficients $v_1,\dots,v_6$ the solutions with $m_3 \neq 0$ of eqs. (4.20) will not lead to a minimum of the function Φ_s in (4.19), as will be shown in appendix A. This means that the 3-dimensional ε-solution (4.18) with $m_1, m_2, m_3 \neq 0$, the ε-solution (3.22) with $m_2=0$, $m_1, m_3 \neq 0$, and the A3-solution (3.6) with $m_1=m_2=0$, $m_3\neq 0$, will not occur in the weak-coupling limit. In this limit the absolute minimum of (4.20) can only occur for $m_3=0$, $m_1 \parallel m_1^*$, $m_2 \parallel m_2^*$, cf. (4.18) with $m_3=0$, but this solution leads to a larger value of Φ_s than the solution (3.8) corresponding to the ABM- or 2D-solution.

5. Other solutions of the gap equations

Here we investigate the possibility of other solutions of the gap equations (3.1a)-(3.1c). We shall take into account that the geometrical configuration of $\underline{m}_1, \underline{m}_2, \underline{m}_3$ is fixed in the sense that

$$
\begin{aligned}
\underline{m}_1 \cdot \underline{m}_3 &= \lambda_1(m_1^2, m_2^2, m_3^2)m_1 m_3 \ , \\
\underline{m}_2 \cdot \underline{m}_3 &= \lambda_2(m_1^2, m_2^2, m_3^2)m_2 m_3 \ , \\
\underline{m}_1 \cdot \underline{m}_3^* &= \mu_1(m_1^2, m_2^2, m_3^2)m_1 m_3 \ , \\
\underline{m}_2 \cdot \underline{m}_3^* &= \mu_2(m_1^2, m_2^2, m_3^2)m_2 m_3 \ ,
\end{aligned}
\tag{5.1}
$$

where $\lambda_1, \lambda_2, \mu_1$ and μ_2 are functions depending only on the lengths of the vectors $\underline{m}_1, \underline{m}_2$ and $\underline{m}_3$, but not on u_1, u_2, u_3. The λ_1, λ_2, μ_1 and μ_2 in (5.1) are to be determined self-consistently by the requirement that $(\underline{m}_1, \underline{m}_2, \underline{m}_3)$ be a solution of (3.1a)-(3.1c).

Eq. (5.1) may be regarded as a weak version of an <u>inertia</u> condition for the geometrical configuration with respect to changes of the magnetic field b and the temperature T. A much stronger inertia condition would be $\underline{m}_i = \psi(T,b)\underline{\mathcal{M}}_i$, where the $\underline{\mathcal{M}}_i$, i=1,2,3, are constant vectors and $\psi(T,b)$ is a function of T and b, independent of i. This condition which is equivalent to the one discussed in ref. 22, cf. also ref. 33, is characteristic for a description with essentially one order parameter. Such a description is not sufficient to explain the phase diagram in the presence of a magnetic field. On the other hand we may conjecture that eq. (5.1) leads to the physically interesting cases.

Inserting (5.1) in the function Φ of (2.5) we have

$$\Phi = \hat{\Phi} = \hat{\Phi}_0 + v_1|\underline{m}_1\cdot\underline{m}_1|^2 + v_2|\underline{m}_2\cdot\underline{m}_2|^2 + 2v_3|\underline{m}_3\cdot\underline{m}_3|^2 + 2v_6\ \mathrm{Re}(\underline{m}_1\cdot\underline{m}_2\ \underline{m}_3^*\cdot\underline{m}_3^*)\ , \tag{5.2}$$

where

$$\hat{\Phi}_0 = u_1m_1^2 + u_2m_2^2 + 2u_3m_3^2 + 2v_1m_1^4 + 2v_2m_2^4 + 4v_3m_3^4$$
$$+ v_4m_1^2m_3^2 + v_5m_2^2m_3^2 + \phi(m_1^2,m_2^2,m_3^2) \tag{5.3}$$

with

$$\phi(m_1^2,m_2^2,m_3^2) = v_4(|\lambda_1|^2 + |\mu_1|^2)m_1^2m_3^2 + v_5(|\lambda_2|^2 + |\mu_2|^2)m_2^2m_3^2$$
$$+ 4v_6m_3^2m_1m_2\ \mathrm{Re}(\mu_1\mu_2)\ . \tag{5.4}$$

In the absolute minimum of $\hat{\Phi}$ we have $\partial\hat{\Phi}/\partial\underline{m}_k^* = 0$, $k=1,2,3$. Thus

$$\hat{\alpha}_1\underline{m}_1 + 2v_1(\underline{m}_1\cdot\underline{m}_1)\underline{m}_1^* + v_6(\underline{m}_3\cdot\underline{m}_3)\underline{m}_2^* = 0\ , \tag{5.5a}$$

$$\hat{\alpha}_2\underline{m}_2 + 2v_2(\underline{m}_2\cdot\underline{m}_2)\underline{m}_2^* + v_6(\underline{m}_3\cdot\underline{m}_3)\underline{m}_1^* = 0\ , \tag{5.5b}$$

$$\hat{\alpha}_3\underline{m}_3 + (2v_3\underline{m}_3\cdot\underline{m}_3 + v_6\underline{m}_1\cdot\underline{m}_2)\underline{m}_3^* = 0\ , \tag{5.5c}$$

where $\hat{\alpha}_k = \alpha_k + \partial\phi/\partial m_k^2$, $k=1,2,3$, depending only on the lengths m_k. Eq. (5.5c) has solutions

$$2v_3\underline{m}_3\cdot\underline{m}_3 + v_6\underline{m}_1\cdot\underline{m}_2 = 0\ , \qquad \underline{m}_1\cdot\underline{m}_2 \neq 0\ , \tag{5.6a}$$

or

$$\underline{m}_3\cdot\underline{m}_3 = \underline{m}_1\cdot\underline{m}_2 = 0\ , \tag{5.6b}$$

apart from the solution $\underline{m}_3 \parallel \underline{m}_3^*$ which has been treated in section 4. Furthermore in appendix B it will be shown that eq. (5.6a) does not lead to new solutions. We are left with the case (5.6b). In that case (5.5a) and (5.5b) have the solution

$$\hat{\alpha}_1 = \hat{\alpha}_2 = 0, \qquad \underline{m}_1\cdot\underline{m}_1 = \underline{m}_2\cdot\underline{m}_2 = 0\ , \tag{5.7}$$

apart from solutions with $\underline{m}_1 \parallel \underline{m}_1^*$ or $\underline{m}_2 \parallel \underline{m}_2^*$, which have also been treated in section 4. Proceeding with the case (5.7) we consider the equation

$$\alpha_1\underline{m}_1 + (2v_6\underline{m}_2^*\cdot\underline{m}_3 + v_4\underline{m}_1\cdot\underline{m}_3^*)\underline{m}_3 + v_4(\underline{m}_1\cdot\underline{m}_3)\underline{m}_3^* = 0\ , \tag{5.8}$$

which can be derived from (3.1a) and (5.5a). From (5.8), taking into account that $\underline{m}_3\cdot\underline{m}_3 = 0$, we find $\underline{m}_1\cdot\underline{m}_3 = 0$ or $\alpha_1+v_4m_3^2 = 0$. $\underline{m}_1\cdot\underline{m}_3 = 0$ together with (5.6) and (5.7) leads to a new solution which can be represented by

$$\underline{m}_1 = \frac{m_1}{\sqrt{2}}(1,i,0)\ , \qquad \underline{m}_2 = -\frac{m_2}{\sqrt{2}}(1,i,0)\ , \qquad \underline{m}_3 = \frac{m_3}{\sqrt{2}}(1,i,0)\ . \tag{5.9}$$

The corresponding $\underline{\underline{\Delta}}(\underline{n})$ matrix is given by

$$\underline{\underline{\Delta}}(\underline{n}) = \begin{pmatrix} -m_1(n^x+in^y) & -m_3(n^x+in^y) \\ -m_3(n^x+in^y) & m_2(n^x+in^y) \end{pmatrix}, \tag{5.10}$$

and reduces for $m_3=0$ to the ABM result (3.10). In view of this, solution (5.9) will be called the 3-dimensional ABM-solution. The corresponding Φ_s is given by

$$\Phi_s = u_1m_1^2 + u_2m_2^2 + 2u_3m_3^2 + 2v_1m_1^4 + 2v_2m_2^4 + 4v_3m_3^4$$
$$+ 2v_4m_1^2m_3^2 + 2v_5m_2^2m_3^2 - 4v_6m_3^2m_1m_2 \; . \tag{5.11}$$

The other possibility $\alpha_1+v_4m_3^2 = 0$ leads with (5.8) after multiplication by $\underline{m}_3^*$ to a new solution which can be represented by $\underline{m}_1 = (m_1/\sqrt{2})\,(1,i,o)$, $\underline{m}_2 = (-m_2/\sqrt{2})\,(1,i,0)$, $\underline{m}_3 = (\pm m_3/\sqrt{2})\,(1,-i,0)$. This solution yields a Φ_s which is given by (5.11) without the term $-4v_6m_3^2m_1m_2$ and therefore it does not correspond to an absolute minimum of the Φ in (2.5).

We now consider the expression (5.11) in some detail. Substituting $\lambda \equiv m_2/m_1$ and introducing the scaled variables $x = [2(v_1+\lambda^4v_2)]^{\frac{1}{4}}m_1$, $y = (4v_3)^{\frac{1}{4}}m_3$, eq. (5.11) can be expressed in the form

$$\Phi = u_x x^2 + u_y y^2 + x^4 + y^4 + px^2y^2 \; , \tag{5.12}$$

where

$$p = 2(v_4+v_5\lambda^2-2v_6\lambda)[2(v_1+\lambda^4v_2)4v_3]^{-\frac{1}{2}} \; , \tag{5.13}$$

and where the coefficients u_x and u_y can be easily expressed in terms of u_1,u_2,u_3 and λ. For fixed λ , eq. (5.12) corresponds to a special section containing even terms in x and y of the so-called double cusp catastrophe[14),15)]. A Landau expansion of the type (5.12) has been used by Liu and Fisher[34)] in a mean field description of the phase diagram for an antiferromagnet. The minimization of (5.12) with respect to x and y can lead to four different phases, a paramagnetic phase (PM) with x=y=0, an "antiferromagnetic" phase (AFM) with $x\neq0$, $y=0$, a "spin-flop" phase (SF) with $x=0$, $y\neq0$, and an intermediate phase (IM) with $x\neq0$, $y\neq0$. For $p>2$ the intermediate phase does not occur and at $u_x=u_y=0$, we have a bicritical point, where two second-order lines (PM$\rightarrow$AFM) and (SF$\rightarrow$PM) meet together with the first-order line (AFM$\rightarrow$SF). For $p<2$ we have at $u_x=u_y=0$ a tetracritical point, where four second-order lines (PM$\rightarrow$AFM), (AFM$\rightarrow$IM), (IM$\rightarrow$SF) and (SF$\rightarrow$PM) meet together.

In our case a phase with both m_1 and m_3 different from zero is equivalent to the intermediate phase, so that the 3-dimensional ABM-solution can only occur under the condition $p<2$. In the weak-coupling limit, however, it can be shown that

$$p > 2 \,, \quad \lambda < 1 \,. \tag{5.14}$$

In fact using the values (2.21) for the coefficients $v_1,\ldots,v_6$ and omitting terms of order ηb^3 and b^4, we have the inequality

$$\begin{aligned}
&(v_1+v_2\lambda^4)v_3 - \frac{1}{8}(v_4+v_5\lambda^2-2v_6\lambda)^2 = \\
&= v^2\Big[\big((1+\lambda^4) - \eta b(\lambda^4-1)\big)(1-4\gamma b^2) + \\
&\qquad - 2\big((1+\lambda^2)(1-\tfrac{4}{3}\gamma b^2) + \tfrac{1}{2}\eta b(1-\lambda^2) - \lambda(1-\tfrac{2}{3}\gamma b^2)\big)^2\Big] = \\
&= v^2(1-\lambda)^2(1-4\gamma b^2)\Big[-(1-\lambda)\big(1+\eta b - \lambda(1-\eta b)\big) - \frac{8}{3}\gamma b^2(1+\lambda^2-\lambda)\Big] < 0 \,,
\end{aligned} \tag{5.15}$$

where we have taken into account the inequalities $0<\lambda<1$ and the estimate

$$\frac{m_1^2-m_2^2}{m_1^2+m_2^2} = \frac{1-\lambda^2}{1+\lambda^2} \sim \mathcal{O}\left(\frac{u_1-u_2}{u_1+u_2}\right) = \mathcal{O}(A_c\eta b t^{-1}) \,, \tag{5.16}$$

implying that $|\eta b| \ll (1-\lambda)(1+\lambda)^{-1}$, for sufficiently small t.

Eq. (5.15), which is equivalent to (5.14) implies that for $\lambda<1$ the minimum of (5.11) is reached for $m_3=0$ or $m_1=m_2=0$. The solution with $m_3=0$ corresponds to the ABM-solution (3.8); the solution with $m_1=m_2=0$, $m_3\neq 0$ leads to the A3-solution (3.6).

We finally consider the special case that $\lambda=1$ which corresponds to a proper solution of the gap equations if $\eta b=0$. For $\lambda=1$ we have, inserting (2.20) and (2.21) in (5.11)

$$\Phi_s = \frac{2}{3}t(m_1^2+m_3^2) + \frac{4}{3}b^2B_c m_3^2 + \left(\frac{4}{15}\right)B_c\left[m_1^2+m_3^2(1-2\gamma b^2)\right]^2 \tag{5.17}$$

neglecting terms of order b^4. Eq. (5.17) leads to a double cusp [+)] with the form (5.12) with p=2.

In catastrophe theory the case p=2 is usually excluded, as it is not possible to find a phase diagram in terms of a finite number of variables which is structurally stable. In the case (5.17) the lowest value of Φ_s

+) Another example of a double cusp with p=2 occurs in a special case of the 3-dimensional ε-solution, when one uses the b=0 values of the coefficients in (2.21) and the values with $\eta b=0$ for the coefficients in (2.20) and $m_1=m_2$ in (4.19).

is reached for $m_3=0$ and (5.9) reduces to the ABM-solution, provided that $B_c + \gamma t > 0$.

Remark: When we consider the gap equations corresponding to an extremum with $m_1,m_2,m_3 \neq 0$ of (5.11) in the special case that $m_1=m_2$ we find

$$(u_1+u_2) + 4(v_1+v_2)m_1^2 + 2(v_4+v_5-2v_6)m_3^2 = 0 \,,$$
$$2u_3 + 2(v_4+v_5-2v_6)m_1^2 + 8v_3m_3^2 = 0 \,, \tag{5.18}$$

and eq. (5.18) does not have a solution, if we use (2.20) and neglect the terms of order b^4 in the determinant of the coefficients of m_1^2 and m_3^2.

6. The BW solutions

In the preceeding sections we have shown that in the weak-coupling limit (2.20), (2.21) the only possible candidate for the absolute minimum of Φ in (2.5) can be found by minimizing the 3-parameter function

$$\Phi_{BW} = u_1m_1^2 + u_2m_2^2 + 2u_3m_3^2 + 2v_1m_1^4 + 2v_2m_2^4 + 6v_3m_3^4$$
$$+ v_4m_1^2m_3^2 + v_5m_2^2m_3^2 - 2v_6m_3^2m_1m_2 \,, \tag{6.1}$$

with respect to the lengths m_1,m_2,m_3.

The gap equations

$$\partial\Phi_{BW}/\partial m_1 = 2m_1(u_1 + 4v_1m_1^2 + v_4m_3^2) - 2v_6m_3^2m_2 = 0 \,,$$
$$\partial\Phi_{BW}/\partial m_2 = 2m_2(u_2 + 4v_2m_2^2 + v_5m_3^2) - 2v_6m_3^2m_1 = 0 \,,$$
$$\partial\Phi_{BW}/\partial m_3 = 4m_3(u_3 + 6v_3m_3^2 + \tfrac{1}{2}v_4m_1^2 + \tfrac{1}{2}v_5m_2^2 - v_6m_1m_2) = 0 \,, \tag{6.2}$$

lead to the following ordered phases:

a) the A1-phase (3.5b) with $m_1\neq 0$, $m_2,m_3=0$,
b) the ABM-phase (3.9) or 2D-phase (3.11), with $m_1,m_2\neq 0$, $m_3=0$,
c) the BW-phase (4.8) with $m_1\neq 0$, $m_2\neq 0$, $m_3\neq 0$.

We now investigate the condition that the solutions of the gap equations correspond to a minimum of Φ_{BW} (stability condition). This condition says that the matrix of second derivatives, i.e.

$$\left(\frac{\partial^2\Phi_{BW}}{\partial m_i \partial m_j}\right) = \begin{vmatrix} 2\alpha_1+16v_1m_1^2 & -2v_6m_3^2 & 4m_3(v_4m_1-v_6m_2) \\ -2v_6m_3^2 & 2\alpha_2+16v_2m_2^2 & 4m_3(v_5m_2-v_6m_1) \\ 4m_3(v_4m_1-v_6m_2) & 4m_3(v_5m_2-v_6m_1) & 4\alpha_3+56m_3^2-4v_6m_1m_2 \end{vmatrix} \tag{6.3}$$

is positive definite.

For the A1-phase, with $m_1^2 = -\frac{1}{4}u_1/v_1$, $m_2=m_3=0$, eq. (6.3) requires that

$$u_1 \leq 0, \quad u_2 \geq 0 \tag{6.4a}$$

$$4u_3 - \frac{1}{2}\frac{v_4}{v_1}u_1 = 2(2u_3-u_1-u_2) + 2\left(1-\frac{1}{4}\frac{v_4}{v_1}\right)u_1 + 2u_2 \geq 0 . \tag{6.4b}$$

Using the values (2.20) for u_1,u_2,u_3 and (2.21) with $b=0$ for $v_1,\dots,v_6$ we see that (6.4b) is implied by (6.4a). If (6.4) is not satisfied,the A1-phase becomes unstable.

For the ABM- or 2D-phase with $m_1^2 = -\frac{1}{4}u_1/v_1$, $m_2^2 = -\frac{1}{4}u_2/v_2$, $m_3=0$,eq. (6.3) requires:

$$4u_3 - \frac{1}{2}\frac{v_4}{v_1}u_1 - \frac{1}{2}\frac{v_5}{v_2}u_2 - v_6\left(\frac{u_1u_2}{v_1v_2}\right)^{\frac{1}{2}} \geq 0 , \tag{6.5}$$

in addition to the conditions $u_1 \leq 0$, $u_2 \leq 0$ (existence conditions).

Finally, using (6.2) for the BW-solution with $m_1,m_2,m_3 \neq 0$, we have

$$\det\left(\frac{\partial^2\Phi_{BW}}{\partial m_i \partial m_j}\right) = 1024\, m_3^2m_1^2m_2^2v_1v_2\left[12v_3 - \frac{(v_4-v_6m_2/m_1)^2}{4v_1} - \frac{(v_5-v_6m_1/m_2)^2}{4v_2}\right]$$

$$+ 32\, m_3^4m_1m_2v_6\left[48\, v_3\left(v_1\frac{m_1^2}{m_2^2} + v_2\frac{m_2^2}{m_1^2}\right) - \left(v_4\frac{m_1}{m_2} + v_5\frac{m_2}{m_1} - 2v_6\right)^2\right] \geq 0 . \tag{6.6}$$

Let us now discuss the phase transitions which follow from the minimalization of (6.1).

a) *Phase transitions between the BW-phase and the 2D-phase*

There is a second-order transition between the 2D-phase and the BW-phase, if

$$2u_3 - \frac{1}{4}\frac{v_4}{v_1}u_1 - \frac{1}{4}\frac{v_5}{v_2}u_2 - \frac{1}{2}v_6\left(\frac{u_1u_2}{v_1v_2}\right)^{\frac{1}{2}} = 0 \tag{6.7}$$

and

$$12v_3 - \frac{(v_4-v_6\lambda)^2}{4v_1} - \frac{(v_5-v_6/\lambda)^2}{4v_2} \geq 0 , \tag{6.8}$$

$\lambda = m_2/m_1$. Eq. (6.7) corresponds to the equality sign in the stability condition (6.5) for the 2D-phase, but can also be derived from the third equation (6.2) inserting $m_3 \downarrow 0$, $m_1^2 = -\frac{1}{4}u_1/v_1$, $m_2^2 = -\frac{1}{4}u_2/v_2$. Eq. (6.8) is the stability condition (6.6) for the BW-solution in the limit $m_3 \downarrow 0$. Restricting ourselves to λ values satisfying $0 \leq \lambda \leq 1$, eq. (6.8) implies

$$1 \geq \lambda \geq \lambda_0, \tag{6.9}$$

where λ_0 is the solution of

$$12v_3 - \frac{(v_4 - v_6\lambda)^2}{4v_1} - \frac{(v_5 - v_6/\lambda)^2}{4v_2} = 0 \ . \tag{6.10}$$

In the special case of the weak-coupling values (2.21) for b=0, i.e. $v_1 = v_2 = v_3 = \frac{1}{4}v_4 = \frac{1}{4}v_5 = \frac{1}{2}v_6 = v$, eq. (6.10) has the solution

$$\lambda_0 = 1 + \tfrac{1}{2}\sqrt{10} - \tfrac{1}{2}\sqrt{10 + 4\sqrt{10}} \simeq 0.202. \tag{6.11}$$

We now consider the first-order transitions between the BW-phase (with $m_1, m_2, m_3 \neq 0$) and the 2D-phase. Rewriting the gap equations (6.2) in terms of $\lambda = m_2/m_1$ we have:

$$\begin{aligned}
&u_1 + 4v_1m_1^2 + (v_4 - v_6\lambda)m_3^2 = 0 \\
&u_2\lambda + 4v_2\lambda^3m_1^2 + (v_5\lambda - v_6)m_3^2 = 0 \\
&2u_3 + (v_4 + v_5\lambda^2 - 2v_6\lambda)m_1^2 + 12v_3m_3^2 = 0 \ .
\end{aligned} \tag{6.12}$$

The function Φ corresponding to the BW-solution should be equal to the Φ of the 2D-solution at the first-order line, hence

$$(u_1 + \lambda^2u_2)m_1^2 + 2u_3m_3^2 = -\tfrac{1}{4}\frac{u_1^2}{v_1} - \tfrac{1}{4}\frac{u_2^2}{v_2} \ . \tag{6.13}$$

Solving u_1, u_2 and u_3 from (6.12) and inserting the result in (6.13) we are left with the condition for the first-order transition, which turns out to be identical to eq. (6.10), where now λ is a parameter. Ignoring the dependence of the coefficients $v_1, \ldots, v_6$ on b, we find that the first-order transitions between BW and 2D take place at a constant value of λ, i.e. independent of u_1, u_2, u_3, so that m_2/m_1 in the BW-phase is constant at the first-order line. This constant value λ_0 of (6.11) is much smaller than the value $m_2/m_1 = 1$ which would occur for a symmetric density of states, i.e. $\eta = 0$ in eq. (2.13). The first-order and second-order line meet together at a tricritical point (TC), which is given by

$$\lambda = \lambda_0, \quad \left(\frac{u_2}{u_1}\right)_{TC} = \lambda_0^2 \; , \quad \left(\frac{u_3}{u_1}\right)_{TC} = \frac{1}{8}\,\frac{v_4}{v_1} + \frac{1}{8}\,\frac{v_5}{v_2}\,\lambda_0^2 - \frac{1}{4}\,v_6(v_1v_2)^{-\frac{1}{2}}\lambda_0 \simeq 0.42 \; , \tag{6.14}$$

where we have used (6.11) and the b=0 values for the coefficients $v_1,\ldots,v_6$, in eq. (6.7). The value of $(u_2/u_1)_{TC}$ has been given before by Ambegaokar and Mermin [9], together with the line of second-order transitions (6.7), which in view of (2.20) using b=0 values for $v_1,\ldots,v_6$ may be rewritten as

$$t^2 = (A_c nb)^2 + (4b^2B_c)^2 \; . \tag{6.15}$$

And for the tricritical point we have in particular

$$A_c nbt^{-1} \simeq 0.922 \qquad 4b^2B_c t^{-1} \simeq -0.387 \; . \tag{6.16}$$

To show that we have actually a tricritical point one may set up a Landau expansion in one variable m_3. Inserting into the expression for Φ_{BW} $m_1=m_1^0+\delta m_1$, $m_2=m_2^0+\delta m_2$ with $(m_1^0)^2=-\frac{1}{4}u_1/v_1$, $(m_2^0)^2=-\frac{1}{4}u_2/v_2$, which are the m_1 and m_2 values in the 2D-phase, and minimizing with respect to δm_1 and δm_2 one obtains an expansion of the form $\Phi_{BW} = p_2m_3^2 + p_4m_3^4 + p_6m_3^6$, where $p_2=0$, $p_4=0$ determine the multicritical point and where $p_6>0$ should hold in order to check that this is really a tricritical point (or a subcase of a butterfly catastrophe). After a laborious but straightforward calculation we have indeed checked that $p_6>0$, but details of the calculation will not be given here. An example with a negative 6th order coefficient and a positive 8th order coefficient may be inferred from the theory of metamagnetism, see Kincaid and Cohen, ref. 35.

The condition for first-order transitions between BW and 2D can also be expressed in terms of u_1,u_2 and u_3. In fact,for an arbitrary solution of eq. (6.12) we have

$$\frac{2u_3}{u_1}\Big[v_2(v_4-v_6\lambda)\lambda^3 - v_1(v_5\lambda-v_6)\Big] + \frac{u_2}{u_1}\Big[\;12v_1v_3\lambda - \tfrac{1}{4}\lambda(v_4-v_6\lambda)(v_4+v_5\lambda^2-2v_6\lambda)\Big]$$
$$+\Big[-12v_2v_3\lambda^3 + \tfrac{1}{4}(v_5\lambda-v_6)(v_4+v_5\lambda^2-2v_6\lambda)\Big] = 0 \; , \tag{6.17}$$

and the first-order line is determined by $\lambda=\lambda_0$. The first-order transitions between BW and 2D can only exist for $u_2\geq 0$, and at $u_2=0$ we have a <u>critical endpoint</u>. Using the b=0 values for $v_1,\ldots,v_6$ the coordinates at the critical endpoint are given by, cf. (2.20)

$$A_c n b t^{-1} = 1 \qquad b^2 B_c t^{-1} \simeq -0.103 \; . \tag{6.18}$$

At this point two equal phases, namely the A1-phase and the 2D-phase, are in equilibrium with a third phase, i.e. the BW-phase.

b) *Phase transitions between the BW-phase and the ABM-phase*

So far we have discussed the phase transitions between the 2D-phase and the BW-phase. Let us now consider the ABM-phase which is described by (3.9). From (3.9) it is clear that the ABM-phase cannot have a second-order transition to the BW-phase, since the geometrical configuration of $\underline{m}_2$ is changed; in the ABM-phase we have $\underline{m}_2 \| \underline{m}_1$, whereas in the BW-phase $\underline{m}_2 \| \underline{m}_1^*$. With the second-order line from 2D to BW, there corresponds a first-order line from ABM to BW, but it is a first-order line without any jumps in m_1^2, m_2^2 or m_3^2 and without latent heat. With the first-order line from 2D to BW there corresponds a first-order line from ABM to BW with discontinuities in m_1^2, m_2^2 and m_3^2. Obviously (6.14) is also a special critical point for the transition from ABM to BW, but it is certainly not a tricritical point. (In the presence of strong-coupling effects the ABM-phase will be stabilized with respect to the 2D-phase. One can expect a first-order transition with latent heat between the ABM-phase and the BW-phase and there will be no tricritical point, see ref. 36.)

c) *Phase transitions between the BW-phase and the A1-phase*

For a second-order transition between BW and A1 one should have the condition $2u_3 - \frac{1}{4}v_4 u_1/v_1 = 0$, $u_2 \geq 0$, but this is in contradiction with (6.4) for $b \neq 0$. First-order transitions occur under the condition (6.12), i.e. we have a BW-solution with $m_1, m_2, m_3 \neq 0$, together with the condition that the Φ of the BW-solution is equal to

$$(u_1 + \lambda^2 u_2) m_1^2 + 2u_3 m_3^2 = -\frac{1}{4} \frac{u_1^2}{v_1} \; . \tag{6.19}$$

Inserting the solutions for u_1, u_2, u_3 from (6.12) we find

$$-4v_2 \lambda^4 m_1^4 - 2(v_5\lambda - v_6)\lambda m_1^2 m_3^2 + \left[-12v_3 + \frac{(v_4 - v_6\lambda)^2}{4v_1}\right] m_3^4 = 0 \; . \tag{6.20}$$

From the first two equations of (6.12) one finds

$$m_3^2 = \frac{-v_2\lambda^3 u_1 + v_1 \lambda u_2}{v_2(v_4 - v_6\lambda)\lambda^3 - v_1(v_5\lambda - v_6)} \; , \tag{6.21}$$

$$m_1^2 = \frac{1}{4}\,\frac{(v_5\lambda - v_6)u_1 - (v_4 - v_6\lambda)\lambda u_2}{v_2(v_4-v_6\lambda)\lambda^3 - v_1(v_5\lambda - v_6)} ,$$

where $\lambda^2 \geq v_1u_2/(v_2u_1)$. Inserting (6.21) in (6.20) we have

$$p(\lambda)u_2^2 + q(\lambda)u_2u_1 - \frac{1}{2}\frac{v_2}{v_1}\,q(\lambda)\lambda^2u_1^2 = 0 \qquad (6.22)$$

with

$$p(\lambda) = -\tfrac{1}{4}v_2\lambda^4(v_4-v_6\lambda)^2 + \tfrac{1}{2}v_1\lambda(v_4-v_6\lambda)(v_5\lambda-v_6) + v_1^2\left[-12v_3 + \frac{(v_4-v_6\lambda)^2}{4v_1}\right],$$

$$q(\lambda) = -\tfrac{1}{2}v_1(v_5\lambda-v_6)^2 - 2v_1v_2\lambda^2\left[-12v_3 + \frac{(v_4-v_6\lambda)^2}{4v_1}\right]. \qquad (6.23)$$

The following relation is easily verified:

$$4\lambda^2p(\lambda)v_2 + 2q(\lambda)v_1 = -\left[v_2\lambda^3(v_4-v_6\lambda) - v_1(v_5\lambda-v_6)\right]^2 . \qquad (6.24)$$

Using (6.24) we find from (6.22)

$$\lambda^2u_1 = \left\{1 \pm \left(-2q(\lambda)v_1\right)^{-\frac{1}{2}}\left[v_1(v_5\lambda-v_6) - v_2\lambda^3(v_4-v_6\lambda)\right]\right\}\frac{v_1}{v_2}\,u_2 , \qquad (6.25)$$

where the -sign can be disregarded for values of $v_1,\dots,v_6$ close to the ones in (2.21), as u_1/u_2 has to be negative. The corresponding u_3/u_1 can be inferred from (6.17).

In order to investigate the limiting value of λ in the limit $b\to 0$ we consider the inequality $2u_3-u_1-u_2 = \frac{4}{3}b^2B_c > 0$. From (6.17) one can then derive an inequality for $(u_1-u_2)/(u_1+u_2)$ which can be expressed in terms of λ using (6.25). In the case that one uses the b=0 values for $v_1,\dots,v_6$ with $q(\lambda) = -2v^3[(\lambda-1)^4 - 10\lambda^2]$ we have †)

$$[(\lambda-1)^4 - 10\lambda^2]^{\frac{1}{2}} \leq 1 - 3\lambda - 2\lambda^2 - \lambda^3 . \qquad (6.26)$$

Obviously this inequality is satisfied for $\lambda=\lambda_0$. but not for $\lambda=0$. There is a value λ_p with $0 < \lambda_p < \lambda_0$ such that the equality sign in (6.26) holds. At the first-order transition between BW and A1 the λ value decreases from λ_0 to $\lambda_p \simeq 0.14$ at the polycritical point.

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†) A more complicated expression for general coefficients $v_1,\dots,v_6$ has also been derived, but will not be given here.

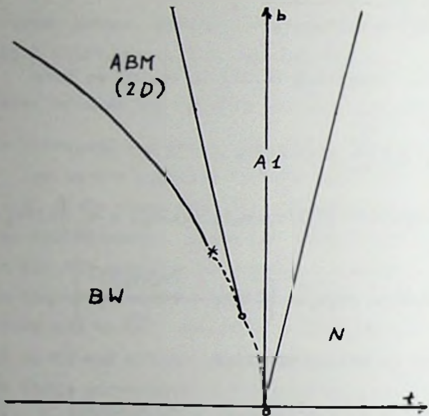

*Figure 1. Schematic phase diagram for liquid ^3He in the weak-coupling limit, $t = \frac{1}{2}\mathcal{N}(0)(T-T_c)/T_c$ is the reduced temperature and b the external field. N, A1, ABM (2D) and BW denote various phases mentioned in the text. The solid lines correspond to second-order phase transitions, the dashed lines to first-order transitions. * is a tricritical point (TC) and o a critical end-point (CE).*

From the treatment given above we arrive at the following phase diagram near the polycritical point (fig. 1). There are second-order transitions between the normal phase (N) and the A1-phase, between the A1-phase and the ABM (or 2D)-phase for b larger than the value at the critical endpoint (CE). Furthermore for b values larger than the one at the tricritical point (TC) there is a second-order transition between the 2D-phase and the BW-phase, and a first-order transition without latent heat between ABM and BW. The ABM-solution and the 2D-solution lead to the same value of Φ, so that these solutions can coexist within the context of the weak-coupling approximation (2.20), (2.21). (The presence of spin-fluctuation effects will lead to qualitative changes in the phase diagram, cf. refs. 10, 11, 36). For b-values between those at the critical endpoint and the tricritical point there is a first-order transition between ABM (2D) and BW, in contrast to previous suggestions in the literature. For b-values smaller than the values at the critical endpoint there is a first-order transition between A1 and BW.

From the results above it follows that the m_2/m_1-values in the BW-phase can be much smaller than the value 1 which would follow from a symmetric density of states, (2.13) with $\eta=0$. In fact, we have $m_2/m_1 = \lambda_0$ and (6.11) along the first-order line between ABM (2D) and BW, and $\lambda_0 \geq m_2/m_1 \geq \lambda_p$ along the first-order line between A1 and BW, independently of the values of η, provided that $\eta<0$. The values of m_2/m_1 should also show up in thermodynamic quantities and correlation functions.

Appendix A

In this appendix we show that the solutions with $m_3 \neq 0$ of (4.20) do not lead to a minimum of the function Φ_s in (4.19) when we use the weak-coupling values (2.21) for the coefficients $v_1,\dots,v_6$.

We first consider the 3-dimensional ε-solution with $m_1,m_2,m_3 \neq 0$. At the extrema the matrix of second derivatives of the function Φ_s of (4.19) is given by, (omitting the subscript s),

$$\left(\frac{\partial^2\Phi}{\partial m_i \partial m_j}\right) = \begin{pmatrix} 2\alpha_1 + 28v_1m_1^2 - v_6^2m_2^2/v_3 & -2v_6^2m_1m_2/v_3 & 4v_4m_1m_3 \\ -2v_6^2m_1m_2/v_3 & 2\alpha_2 + 28v_2m_2^2 - v_6^2m_1^2/v_3 & 4v_5m_2m_3 \\ 4v_4m_1m_3 & 4v_5m_2m_3 & 4\alpha_3 + 32v_3m_3^2 \end{pmatrix}. \tag{A1}$$

Using (4.20) for $m_1,m_2,m_3 \neq 0$ to eliminate u_1,u_2,u_3 we find

$$\det\left(\frac{\partial^2\Phi}{\partial m_i \partial m_j}\right) = 32m_1^2m_2^2m_3^2(576\, v_1v_2v_3 - 12v_5^2v_1 - 12v_4^2v_2 - 4v_6^4v_3^{-1} - 2v_4v_5v_6^2v_3^{-1}). \tag{A.2}$$

Inserting the expression (2.21) we have, omitting terms of order b^3 and higher,

$$\det\left(\frac{\partial^2\Phi}{\partial m_i \partial m_j}\right) = -4(8v)^3m_1^2m_2^2m_3^2\left(4\eta^2b^2 + \frac{64}{3}\gamma b^2\right) < 0 , \tag{A.3}$$

implying that the matrix of second derivatives is not positive definite. Hence, the 3-dimensional ε-solution is unstable.

We next consider the ε-solution (3.22) with $m_2=0$, $m_1,m_3 \neq 0$. From (A.1) and (4.20) with $m_2=0$, $m_1,m_3 \neq 0$ we have

$$\frac{\partial^2\Phi}{\partial m_2^2} = 2u_2 + 2v_5m_3^2 - \frac{v_6^2}{v_3}m_1^2 =$$

$$= 2(u_1+u_2-2u_3) + 2(v_4+v_5-8v_3)m_3^2 + \left(12v_1 - \frac{v_6^2}{v_3} - 2v_4\right)m_1^2 . \tag{A.4}$$

Inserting (2.20) and (2.21) for the coefficients u_1, u_2, u_3 and $v_1, \ldots, v_6$ we have

$$\frac{\partial^2 \phi}{\partial m_2^2} = \frac{8}{15} B_c \eta b m_1^2 + \frac{8}{3} b^2 B_c \left(-1 + \frac{16}{15} \gamma m_3^2\right) < 0 \tag{A.5}$$

for sufficiently small m_3^2, in view of $\eta b < 0$. Eq. (A.5) implies that the ε-solution is not a minimum of (4.19).

Finally, for the A3-solution (3.6) with $m_1 = m_2 = 0$, $m_3 \neq 0$, we have from (A.1) and (4.20)

$$\frac{\partial^2 \phi}{\partial m_1^2} + \frac{\partial^2 \phi}{\partial m_2^2} = 2(u_1 + u_2 - 2u_3) + 2(v_4 + v_5 - 8v_3) m_3^2 \tag{A.6}$$

which, with (2.20) and (2.21), can be rewritten as

$$\frac{\partial^2 \phi}{\partial m_1^2} + \frac{\partial^2 \phi}{\partial m_2^2} = \frac{8}{3} b^2 B_c \left(-1 + \frac{16}{15} \gamma m_3^2\right) < 0 \,. \tag{A.7}$$

Eq. (A.7) implies that the A3-solution is not a minimum of (4.19).

Appendix B

In this appendix we show that the case (5.6a) does not lead to new solutions. Without loss of generality $(\underline{m}_1 \cdot \underline{m}_2)$ may be chosen to be real and from (5.5a) and (5.5b) we have:

$$\hat{\alpha}_1 \hat{\alpha}_2 + 4 v_1 v_2 (\underline{m}_1^* \cdot \underline{m}_1^*)(\underline{m}_2^* \cdot \underline{m}_2^*) - (\tfrac{1}{2} v_6^2 v_3^{-1}\, \underline{m}_1 \cdot \underline{m}_2)^2 = 0 \tag{B.1}$$

and

$$\hat{\alpha}_2 2 v_1 \underline{m}_1 \cdot \underline{m}_1 + \hat{\alpha}_1 2 v_2 \underline{m}_2^* \cdot \underline{m}_2^* = 0 \,, \tag{B.2}$$

apart from a solution with $\underline{m}_1 \| \underline{m}_1^*$ which has been treated in section 4. Multiplying eq. (5.5a) by $\hat{\alpha}_2 \underline{m}_2^*$ and the complex conjugate of eq. (5.5b) by $\hat{\alpha}_1 \underline{m}_1$, one can show that

$$\left(2 v_1 \hat{\alpha}_2 + \frac{v_6^2}{2 v_3} \hat{\alpha}_1\right) \underline{m}_1 \cdot \underline{m}_1 = \left(2 v_2 \hat{\alpha}_1 + \frac{v_6^2}{2 v_3} \hat{\alpha}_2\right) \underline{m}_2^* \cdot \underline{m}_2^* \,. \tag{B.3}$$

From (B.2) and (B.3) we have the possibilities

$$\underline{m}_1 \cdot \underline{m}_1 = \underline{m}_2 \cdot \underline{m}_2 = 0 \,, \tag{B.4a}$$

or

$$\frac{v_6^2}{2v_3}(2v_2\hat{\alpha}_1^2 + 2v_1\hat{\alpha}_2^2) + 8v_1v_2\hat{\alpha}_1\hat{\alpha}_2 = 0 \ . \tag{B.4b}$$

When we use the weak-coupling values (2.21) for the coefficients $v_1,\dots,v_6$ we have the inequality

$$\frac{1}{4v_1v_2}\left(\frac{v_6^2}{2v_3}\right)^2 = \left(1 - \frac{2}{3}\gamma b^2\right)^4\cdot(1-4\gamma b^2)^{-2}\,(1-\eta^2b^2)^{-1} > 1 \ , \tag{B.5}$$

implying with (B.4b) that $\hat{\alpha}_1=\hat{\alpha}_2=0$. From (B.1) on the one hand and multiplying (5.5a), (5.5b) by $\underline{m}_2^*,\underline{m}_1^*$ resp. on the other hand one can derive two different expressions for $|\underline{m}_1\cdot\underline{m}_1|^2, |\underline{m}_2\cdot\underline{m}_2|^2$ which contradict each other in view of (B.5) and $\underline{m}_1\cdot\underline{m}_2 \neq 0$. Hence, on account of (B.5), eq. (5.6a) leads to (B.4a). This conclusion is also valid in the opposite case when

$$\kappa^2 = 1 - \frac{1}{4v_1v_2}\left(\frac{v_6^2}{2v_3}\right)^2 \tag{B.6}$$

is non-negative and $\underline{m}_1\cdot\underline{m}_2$ is non-vanishing. In fact, when $\underline{m}_1,\underline{m}_2 \neq 0$, $\underline{m}_1\cdot\underline{m}_1 \neq 0$, one can derive after a lengthy calculation the explicit expressions

$$4v_1v_2(\pm2\kappa)\left[(\underline{m}_1\cdot\underline{m}_2)^2 - \tfrac{1}{2}m_1^2m_2^2\right] = \frac{v_6^2}{2v_3}(v_1m_1^4 - v_2m_2^4) \ ,$$

$$2v_1(\pm2\kappa)\left[|\underline{m}_1\cdot\underline{m}_1|^2 - \tfrac{1}{2}m_1^4\right] = 2v_1m_1^4 - \frac{1}{4v_1v_2}\left(\frac{v_6^2}{2v_3}\right)^2(v_1m_1^4 + v_2m_2^4) \ ,$$

$$2v_2(\pm2\kappa)\left[|\underline{m}_2\cdot\underline{m}_2|^2 - \tfrac{1}{2}m_2^4\right] = -2v_2m_2^4 + \frac{1}{4v_1v_2}\left(\frac{v_6^2}{2v_3}\right)^2(v_1m_1^4 + v_2m_2^4) \ . \tag{B.7}$$

In combination with the condition that $\underline{m}_2$ is a linear combination of $\underline{m}_1$ and $\underline{m}_1^*$, cf. (5.5a) with $\underline{m}_3\cdot\underline{m}_3 \neq 0$, or

$$\begin{vmatrix} \underline{m}_1\cdot\underline{m}_1 & m_1^2 & \underline{m}_1\cdot\underline{m}_2 \\ m_1^2 & \underline{m}_1^*\cdot\underline{m}_1^* & \underline{m}_1^*\cdot\underline{m}_2 \\ \underline{m}_2\cdot\underline{m}_1 & \underline{m}_2\cdot\underline{m}_1^* & \underline{m}_2\cdot\underline{m}_2 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} \underline{m}_1\cdot\underline{m}_1 & m_1^2 & \underline{m}_1\cdot\underline{m}_2 \\ m_1^2 & \underline{m}_1^*\cdot\underline{m}_1^* & \underline{m}_1^*\cdot\underline{m}_2 \\ \underline{m}_2^*\cdot\underline{m}_1 & \underline{m}_2^*\cdot\underline{m}_1^* & m_2^2 \end{vmatrix} = 0 \ , \tag{B.8}$$

eqs. (B.7) would yield an extra relation between m_1^2 and m_2^2 that is independent of u_1,u_2,u_3 , thereby ensuring that (B.4b) does not lead to appropriate solutions of eqs. (3.1a)-(3.1c) in the case that κ^2 in (B.6) is positive

We now proceed with the case (B.4a). If $\underline{m}_1 \cdot \underline{m}_2 \neq 0$ we have from (5.5a) $\underline{m}_2^* \| \underline{m}_1$, which implies also $\underline{m}_1 \cdot \underline{m}_2^* = 0$. Substracting eqs. (5.5a), (5.5b) from (3.1a), (3.1b), we have

$$(\alpha_1 - \hat{\alpha}_1)\underline{m}_1 + (2v_6 \underline{m}_2^* \cdot \underline{m}_3 + v_4 \underline{m}_1 \cdot \underline{m}_3^*)\underline{m}_3 + v_4(\underline{m}_1 \cdot \underline{m}_3)\underline{m}_3^* = 0,$$

$$(\alpha_2 - \hat{\alpha}_2)\underline{m}_2 + (2v_6 \underline{m}_1^* \cdot \underline{m}_3 + v_5 \underline{m}_2 \cdot \underline{m}_3^*)\underline{m}_3 + v_5(\underline{m}_2 \cdot \underline{m}_3)\underline{m}_3^* = 0 \ . \qquad \text{(B.9)}$$

Using $\underline{m}_1 \cdot \underline{m}_1 = \underline{m}_2 \cdot \underline{m}_2 = \underline{m}_1 \cdot \underline{m}_2^* = 0$, and taking inner products one can derive from (B.9) a set of four equations for the inner products $\underline{m}_1 \cdot \underline{m}_3$, $\underline{m}_2 \cdot \underline{m}_3$, $\underline{m}_1 \cdot \underline{m}_3^*$ and $\underline{m}_2 \cdot \underline{m}_3^*$. For $\eta b \neq 0$, i.e. $v_4 \neq v_5$, this set has only the trivial solution

$$\underline{m}_1 \cdot \underline{m}_3 = \underline{m}_2 \cdot \underline{m}_3 = \underline{m}_1 \cdot \underline{m}_3^* = \underline{m}_2 \cdot \underline{m}_3^* = 0 \qquad \text{(B.10)}$$

which reduces to the BW-solution (4.8). Hence, no new solutions have been obtained from (5.6b).

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CHAPTER IV

LANDAU EXPANSION IN THE PRESENCE OF A HUBBARD INTERACTION

1. Introduction

In two previous chapters [1,2] we have investigated the phases of superfluid ^{3}He in the presence of a magnetic field, from the point of view of an exactly solvable model. This model consists of a pairing-interaction of the BCS-type, apart from the usual kinetic energy and Zeeman term. The hamiltonian reads

$$\mathcal{H}_0 = T + \tfrac{1}{2} \sum_{\underline{k},\underline{k}'} \sum_{\alpha,\beta} V(\underline{k}-\underline{k}') a^{\dagger}_{\underline{k}\alpha} a^{\dagger}_{-\underline{k}\beta} a_{-\underline{k}'\beta} a_{\underline{k}'\alpha} , \tag{1.1}$$

where

$$T = \sum_{\underline{k}} \sum_{\alpha,\beta} (\varepsilon_k \mathbb{1}_{\alpha\beta} - b\,\sigma^z_{\alpha\beta})\, a^{\dagger}_{\underline{k}\alpha} a_{\underline{k}\beta} , \tag{1.2}$$

and where $\underline{k}$ denotes the wave vector, ε_k is the kinetic energy of the quasiparticles, b is an external magnetic field in the z direction, and $a_{\underline{k}\alpha}$ and $a^{\dagger}_{\underline{k}\alpha}$ are fermion creation- and annihilation operators. The spin α can be either up ($\uparrow$) or down ($\downarrow$), $\underline{\underline{\sigma}}^i$ (i=x,y,z) are the Pauli matrices and $\underline{\underline{\mathbb{1}}}$ is the 2×2 unit matrix. The pairing potential $V(\underline{k}-\underline{k}')$ can be expanded in spherical harmonics. Although the case that a finite number of different ℓ values occurs can be treated as well, we restrict ourselves to the $\ell=1$ term which gives the important contribution in the case of ^{3}He. (For a survey of the theory for superfluid ^{3}He we refer the reader to the reviews by Leggett [3] and Anderson and Brinkman [4]). As a further restriction we impose the weak-coupling limit on the pairing potential, that is the assumption that the coefficients in the expansion in spherical harmonics can be taken to be independent of the lengths k and k' of the vectors $\underline{k}$ and $\underline{k}'$ in a small interval around the Fermi energy, and otherwise zero. The pairing potential can then be written as

$$V(\underline{k}-\underline{k}') = \begin{cases} -3(V_1/\Omega)\underline{n}\cdot\underline{n}' , & \text{if } |\varepsilon_k|,|\varepsilon_{k'}| < \hbar\omega \\ 0 , & \text{otherwise} , \end{cases} \tag{1.3}$$

where $\underline{n} = \underline{k}/k$ and $\underline{n}' = \underline{k}'/k'$ are the unit vectors in the directions of $\underline{k}$ and $\underline{k}'$, and where Ω is the volume of the system. The hamiltonian (1.1), (1.2) together with (1.3) for the pairing potential describes an exactly solvable model. In fact, the interaction term in (1.1) is separable, which means that it can be written as a product of sums of one-particle operators. This property makes it possible to give an exact expression for the free energy of the system [5-8], cf. also refs. 9-11. On the basis of this expression we have derived in ref. 1 the Landau expansion up to fourth order terms in the 18 real order parameters of the system, and for arbitrary magnetic fields. As the coefficients of the Landau expansion turn out to be complicated functions of the temperature T and the magnetic field b, it is difficult to establish the features of the phase diagram for arbitrary b. In ref. 2, therefore, we have used expansions for these coefficients up to second order in b, and we have given a detailed investigation of the phase diagram as well as a comparison with the earlier results of ref. 12 in that approximation. As a result it turned out that the occurrence of certain solutions of the "gap equations" (by which we mean the implicit equations for the order parameter describing the extrema of the free energy functional) depends very sensitively on the behaviour of the coefficients in the Landau expansion as functions of the magnetic field, even in the fourth order terms.

One of the shortcomings of the BCS theory, which we treated so far, is that it does not give a correct explanation of the occurrence of the A phase in zero magnetic field under high pressure, as observed experimentally in liquid ^{3}He. (For a review of experimental results, see refs. 13 and 14). Already in ref. 15 it was shown that for $\ell=1$ in zero magnetic field the isotropic Balian-Werthamer (BW) state has always the lowest free energy, and this result remains true for mixtures of different ℓ-values provided that the $\ell=1$ contribution is dominant [1]. It has become common practice to adopt the spinfluctuation model [16-18] as a starting point for the understanding of the occurrence of the A phase. From a microscopic point of view this model can be obtained starting from a contact term of the Hubbard type, i.e.

$$\mathcal{H}_I = I \int d\underline{x}\ \psi_\uparrow^\dagger(\underline{x})\psi_\uparrow(\underline{x})\psi_\downarrow^\dagger(\underline{x})\psi_\downarrow(\underline{x}) , \tag{1.4a}$$

where

$$\psi_\alpha(\underline{x}) = \frac{1}{\sqrt{\Omega}} \sum_{\underline{k}} e^{i\underline{k}\cdot\underline{x}} a_{\underline{k}\alpha}, \quad \alpha=\uparrow,\downarrow , \tag{1.4b}$$

and using some statistical approximations in order to express the free energy in terms of dynamical susceptibilities. In ^3He, above the superfluid transition temperature T_c, (1.4) leads to an effective exchange interaction [19,20], whereas below T_c the various susceptibilities are changed in such a way that an anisotropic state is favoured [21]. Various self-consistent approximation schemes [22-25] have been used in order to give a more quantitative basis to these considerations, yielding explicit values for the coefficients in the generalized Landau expansion [26,27] which differ appreciably from those of the weak-coupling theory. These approaches led to an explanation of the experimentally observed A phase in terms of the Anderson-Brinkman-Morel (ABM) phase [28]. Applications of the spin fluctuation model to the calculation of the specific heat jumps, NMR shifts and superfluid densities can be found in refs. 29-32. The estimates which these theories provide for the specific heat discontinuities at the transition point as compared to experiment [33] could be improved by combining the results of the theories with paramagnon-induced pairing together with the information on the scattering amplitudes from Fermi liquid theory, cf. e.g. refs. 34-36.

In the presence of a magnetic field the situation is far more complicated. A first estimate of the effect of a magnetic field in the context of the spin fluctuation model was given in refs. 29, 30, and a more detailed investigation can be found in refs. 37 and 38. In these papers only the possibility of BW or ABM types of ordering was considered. The influence of magnetic field effects in the order parameter on NMR shifts and collective modes was studied in refs. 39-42, but some important aspects of the problem (such as the asymmetry in the density of states, necessary to explain the A-A_1 splitting in a magnetic field) were not considered there. In all the treatments mentioned above only specific phases such as the BW phase and the ABM (or planar) phase were taken into consideration, although it is by no means clear to which extent these phases correspond to the most favourable solutions of the 18 order parameter problem in the presence of a Hubbard interaction.

In the present chapter we give a derivation of the Landau expansion for the free energy with an explicit evaluation of the coefficients based on the model for ^3He described by the hamiltonian (1.1)-(1.3) together with (1.4). The Landau expansion can then be used to investigate to which extent the phase diagram in the weak-coupling limit (without spin fluctuations) undergoes some qualitative changes in the presence of (small) perturbations of the Hubbard type. The approach we follow is quite different from the one based on the spin fluctuation model. Our approach is based on a fundamental theorem of Bogoliubov Jr. [9-11], cf. also refs. 5-8, by which the free energy associated with (1.1), (1.4) can be expressed rigorously in terms of the free energy of an associated reference system. The associated reference hamiltonian contains a bilinear part arising from T in (1.2) as well as from a bilinearization of the pairing interaction in $\mathcal{H}_0$, but also the complete Hubbard interaction itself. This implies that in order to evaluate the reference free energy one has to rely on certain approximations. In the present chapter we shall evaluate the reference free energy using a perturbation calculation with respect to the Hubbard interaction, so that the result is at least correct up to a certain power of the coupling constant I. One of the advantages of the Bogoliubov Jr. theorem is that one has a self-consistent way of obtaining the free energy by minimizing over a well-defined free energy functional in terms of the 18 real order parameters of the problem. As a consequence one can then investigate, at least in principle, which one of the ordered phases will lead to the absolute minimum of the free energy functional without having to rely on an ad-hoc choice of certain phases.

More explicitly the outline of the calculation is as follows. The hamiltonian (1.1)-(1.3) together with (1.4) can be rewritten in the following form

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I = (T + \mathcal{H}_I) - 6V_1\Omega \sum_{i=1}^{3} \mathrm{tr}(\underline{\underline{\omega}}^{i\dagger} \cdot \underline{\underline{\omega}}^{i}) \ , \tag{1.5}$$

where the $\underline{\underline{\omega}}^i$, i=1,2,3, are 2×2 matrix operators with elements

$$\omega^i_{\alpha\beta} = \frac{1}{2\Omega} \sum_{\underline{k}>0} \eta(\varepsilon_k) n^i (a_{-\underline{k}\beta} a_{\underline{k}\alpha} + a_{-\underline{k}\alpha} a_{\underline{k}\beta}) \ . \tag{1.6}$$

The summation $\underline{k}>0$ means that in the summation over the wave vector $\underline{k}$ only one vector belonging to a pair $(\underline{k},-\underline{k})$ is taken into account, and $\eta(\varepsilon_k) = 1$ for $|\varepsilon_k| < \hbar\omega$, $\eta(\varepsilon_k) = 0$ otherwise. The associated reference hamiltonian is given by

$$\mathcal{H}_r(\{\underline{\underline{h}}^i\}) = T - \Omega \sum_{i=1}^{3} \mathrm{tr}(\underline{\underline{h}}^{i\dagger} \cdot \underline{\underline{\omega}}^{i} + \underline{\underline{\omega}}^{i\dagger} \cdot \underline{\underline{h}}^{i}) \ , \tag{1.7}$$

where the $\underline{\underline{h}}^i$ are symmetric 2×2 matrices with c-number entries, and the corresponding reference free energy per unit volume is defined by

$$f_r(\{\underline{\underline{h}}^i\}) = \lim_{\Omega\to\infty} -(\beta\Omega)^{-1} \ln \mathrm{Tr} \exp\left(-\beta\mathcal{H}_r(\{\underline{\underline{h}}^i\})\right) \quad . \tag{1.8}$$

For an attractive pairing interaction, i.e. $V_1>0$, taking into account some bounds on the operators $\underline{\underline{\omega}}^i$ and their commutators, one can then apply the theorem of Bogoliubov Jr. [9]. This theorem provides a rigorous expression for the free energy of the system in the thermodynamic limit

$$\begin{aligned} f &\equiv \lim_{\Omega\to\infty} -(\beta\Omega)^{-1} \ln \mathrm{Tr} \exp\left(-\beta(\mathcal{H}_0 + \mathcal{H}_I)\right) \\ &= \min_{\{\underline{\underline{m}}^i\}} \left[f_r(\{\underline{\underline{m}}^i\}) + \frac{1}{6V_1} \sum_{i=x,y,z} \mathrm{tr}(\underline{\underline{m}}^i \cdot \underline{\underline{m}}^{i\dagger}) \right] \quad . \end{aligned} \tag{1.9}$$

The three 2×2 complex symmetric matrices $\underline{\underline{m}}^i$ form the 18 real order parameters of the system. (In the absence of the Hubbard interaction eq. (1.9) can be inferred from eq. (6.12) of ref. 1 for $\ell=1$ with the replacements $6V_1\underline{\underline{\xi}}^i \to \underline{\underline{m}}^i$).

In section 2 we evaluate the contributions to the reference free energy up to second order in the coupling constant I, in terms of the unperturbed Green's functions, corresponding to the bilinear part of the reference hamiltonian. In section 3 the Green's functions are expanded up to fourth order terms in the matrices $\underline{\underline{h}}^i$, in order to find the reference free energy also up to fourth order. Finally in section 4 we apply eq. (1.9) to find the Landau expansion up to fourth order in terms of the order parameters $\underline{\underline{m}}^i$. The form of the Landau expansion in higher order perturbation is discussed, and some typical differences with the results obtained in spin fluctuation theory are mentioned in section 5.

In a following chapter [43] the Landau expansion will be used in order to investigate the stability of the phase diagram obtained in ref. 2 under influence of small perturbations of the Hubbard type. From the full Landau expansion in terms of 18 real order parameters one may at least in principle determine which phases under appropriate values of the coefficients will lead to the absolute minimum of the free energy function. Although in practice we have not been able to give a complete solution to the 18 real order parameter problem in the presence of a magnetic field, it will be shown that,apart from the BW phase and the ABM phase, also some other phases which have not been considered before should be taken into account. This problem, however, will be discussed in the next chapter.

2. Perturbation calculation

In this section we evaluate the reference free energy associated with the hamiltonian $\mathcal{H}_r(\{\underline{\underline{h}}^i\})$. The reference hamiltonian can be expressed as

$$\mathcal{H}_r(\{\underline{\underline{h}}^i\}) = \mathcal{H}_{0,r}(\{\underline{\underline{h}}^i\}) + \mathcal{H}_I \ , \tag{2.1}$$

where, cf. ref. 1,

$$\mathcal{H}_{0,r}(\{\underline{\underline{h}}^i\}) = \sum_{\underline{k}} \varepsilon_k + \sum_{\underline{k}>0} \sum_{p,p'} c^\dagger_{\underline{k}p} D_{\underline{k}\,pp'} c_{\underline{k}p'} \tag{2.2}$$

In eq. (2.2) the summation over $\underline{k}>0$ again means that in the summation over the vectors $\underline{k}$ only one term belonging to a pair $(\underline{k},-\underline{k})$ is taken into account, and $c_{\underline{k}p}$ and $c^\dagger_{\underline{k}p}$, $p=1,2,3,4$, are the components of the Nambu operator

$$\underline{c}_{\underline{k}} = \begin{pmatrix} a_{\underline{k}\uparrow} \\ a_{\underline{k}\downarrow} \\ a^\dagger_{-\underline{k}\uparrow} \\ a^\dagger_{-\underline{k}\downarrow} \end{pmatrix} , \quad \underline{c}^\dagger_{\underline{k}} = \left(a^\dagger_{\underline{k}\uparrow},\ a^\dagger_{\underline{k}\downarrow},\ a_{-\underline{k}\uparrow},\ a_{-\underline{k}\downarrow}\right) . \tag{2.3}$$

The Nambu operators satisfy the anticommutation relations

$$[c_{\underline{k}p},\ c^\dagger_{\underline{k}'p'}]_+ = \delta_{\underline{k},\underline{k}'}\delta_{p,p'} \ , \quad [c_{\underline{k}p},\ c_{\underline{k}'p'}]_+ = [c^\dagger_{\underline{k}p},\ c^\dagger_{\underline{k}'p'}]_+ = 0 \ , \quad \underline{k},\underline{k}'>0 \ . \tag{2.4}$$

The hermitean 4×4 matrix $\underline{\underline{D}}_{\underline{k}}$ in (2.2) is given by

$$\underline{\underline{D}}_{\underline{k}} = \begin{pmatrix} \varepsilon_{\underline{k}} \underline{\underline{1}} - b\underline{\underline{\sigma}}^z & \eta(\varepsilon_k)\underline{\underline{\Delta}}(\underline{n}) \\ \eta(\varepsilon_k)\underline{\underline{\Delta}}^\dagger(\underline{n}) & -\varepsilon_k \underline{\underline{1}} + b\underline{\underline{\sigma}}^z \end{pmatrix} , \tag{2.5}$$

where, as before, $\eta(\varepsilon_k)=1$ if $|\varepsilon_k|<\hbar\omega$, $\eta(\varepsilon_k)=0$ otherwise, $\underline{\underline{1}}$ is the 2×2 unit matrix, $\underline{n}$ is the unit vector $n=\underline{k}/k$, and the "gap matrix" $\underline{\underline{\Delta}}(\underline{n})$ is given by

$$\underline{\underline{\Delta}}(\underline{n}) = -\sum_{i=1}^{3} n^i \underline{\underline{h}}^i \ , \tag{2.6}$$

in which the $\underline{\underline{h}}^i$ are symmetric 2×2 matrices with c-number entries.

In contrast to the situation investigated in ref. 1, $f_r(\{\underline{\underline{h}}^i\})$ defined in eq. (1.8) cannot be evaluated exactly as a function of the $\underline{\underline{h}}^i$, due to the presence of the Hubbard term in eq. (2.1). It is, however, possible to derive an approximate expression for the reference free energy, e.g. by a second order perturbation expansion in the coupling constant I, taking into account, up to this order, the full dependence on the matrices $\underline{\underline{h}}^i$. One may anticipate this to be sufficient for studying the stability of the phase diagram under small perturbations of the Hubbard type, which will be done in a following chapter[43]. In order to evaluate the reference free energy up to second order in I, it is convenient to rewrite eq. (1.4) in terms of the Nambu operators

$$\underline{\Psi}(\underline{x}) = \begin{pmatrix} \psi_\uparrow(\underline{x}) \\ \psi_\downarrow(\underline{x}) \\ \psi^\dagger_\uparrow(\underline{x}) \\ \psi^\dagger_\downarrow(\underline{x}) \end{pmatrix} , \quad \underline{\Psi}^\dagger(\underline{x}) = \left(\psi^\dagger_\uparrow(\underline{x}),\ \psi^\dagger_\downarrow(\underline{x}),\ \psi_\uparrow(\underline{x}),\ \psi_\downarrow(\underline{x})\right) , \tag{2.7}$$

namely

$$\mathcal{H}_I = \tfrac{1}{4} I \int d\underline{x}\ \left(\underline{\Psi}^\dagger(\underline{x})\cdot\underline{\underline{O}}\cdot\underline{\Psi}(\underline{x})\right)\left(\underline{\Psi}^\dagger(\underline{x})\cdot\tilde{\underline{\underline{O}}}\cdot\underline{\Psi}(\underline{x})\right) , \tag{2.8}$$

in which the 4×4 matrix $\underline{\underline{O}}$ is given by

$$\underline{\underline{O}} = \begin{pmatrix} 0 & i\underline{\underline{\sigma}}^y \\ 0 & 0 \end{pmatrix} , \tag{2.9}$$

and $\tilde{\underline{\underline{O}}}$ is the transposed matrix. The Nambu operators given in (2.7) are the Fourier transforms of the operators given in (2.3)

$$\underline{\Psi}(\underline{x}) = \frac{1}{\sqrt{\Omega}} \sum_{\underline{k}} e^{i\underline{k}\cdot\underline{x}}\ \underline{c}_{\underline{k}} , \tag{2.10}$$

cf. eq. (1.4b), and their anticommutation relations are given by

$$[\underline{\Psi}(\underline{x}),\underline{\Psi}^\dagger(\underline{x})]_+ = \underline{\underline{1}}^{(4)}\delta(\underline{x}-\underline{x}') , \tag{2.11}$$

where $\underline{\underline{1}}^{(4)}$ is the 4×4 unit matrix. Furthermore we have the symmetry property

$$\tilde{\underline{\Psi}}^{\dagger}(\underline{x}) = \underline{\underline{J}}\cdot\underline{\Psi}(\underline{x}) \ , \quad \underline{\underline{J}} \equiv \begin{pmatrix} 0 & \underline{\underline{\mathbb{1}}} \\ \underline{\underline{\mathbb{1}}} & 0 \end{pmatrix} , \tag{2.12}$$

where again the tilde denotes matrix transposition, i.e. $\tilde{\underline{\Psi}}^{\dagger}$ is a column vector.

In order to calculate the contributions from the Hubbard interaction to the reference free energy $f_r(\{\underline{\underline{h}}^i\})$ we use the standard linked cluster expansion

$$f_r(\{\underline{\underline{h}}^i\}) - f_{0,r}(\{\underline{\underline{h}}^i\})$$

$$= \lim_{\Omega\to\infty} \frac{1}{\Omega} \left\{ \langle \mathcal{H}_I \rangle - \frac{1}{2} \int_0^{\beta} d\tau \left[\langle \mathcal{H}_I(\tau)\mathcal{H}_I \rangle - \langle \mathcal{H}_I(\tau)\rangle\langle \mathcal{H}_I \rangle \right] + O(I^3) \right\} , \tag{2.13}$$

in which $f_{0,r}$ denotes the free energy associated with the hamiltonian (2.2) and the brackets are the expectation values with respect to the same hamiltonian $\mathcal{H}_{0,r}(\{\underline{\underline{h}}^i\})$, i.e.

$$\langle A \rangle = \mathrm{Tr}(e^{-\beta\mathcal{H}_{0,r}} A) \,/\, \mathrm{Tr}(e^{-\beta\mathcal{H}_{0,r}}) \tag{2.14}$$

for an arbitrary operator A, and the τ-dependence is defined as

$$A(\tau) = e^{\tau\mathcal{H}_{0,r}} A \, e^{-\tau\mathcal{H}_{0,r}} \ . \tag{2.15}$$

After inserting eq. (2.8) in (2.13) the expectation values can be expressed using the thermodynamic Wick theorem [44-46] in terms of contractions of the type $\langle \underline{\Psi}(\underline{x},\tau)\underline{\Psi}^{\dagger}(\underline{x}')\rangle$. These contractions are evaluated in appendix A, and the result is

$$\langle \underline{\Psi}(\underline{x},\tau)\underline{\Psi}^{\dagger}(\underline{x}')\rangle = \frac{1}{\Omega} \sum_{\underline{k}} e^{i\underline{k}\cdot(\underline{x}-\underline{x}')} \, \underline{\underline{G}}(\underline{k},\tau) \ , \tag{2.16}$$

where the 4×4 matrix $\underline{\underline{G}}(\underline{k},\tau)$ is given by the following expression in terms of the matrix $\underline{\underline{D}}_{\underline{k}}$, cf. eq. (2.5) ,

$$\underline{\underline{G}}(k,\tau) = \frac{e^{-\tau\underline{\underline{D}}_{\underline{k}}}}{\underline{\underline{\mathbb{1}}}^{(4)} + e^{-\beta\underline{\underline{D}}_{\underline{k}}}} \ . \tag{2.17}$$

The Green's function $\underline{\underline{G}}(\underline{k},\tau)$ obeys the following symmetry property

$$\underline{\underline{J}}\cdot\tilde{\underline{\underline{G}}}(\underline{k},\tau)\cdot\underline{\underline{J}} = \underline{\underline{G}}(-\underline{k},\beta-\tau) \ , \tag{2.18}$$

which follows from the relations

$$\underline{\underline{\Delta}}(\underline{n}) = -\underline{\underline{\Delta}}(-\underline{n}) \ , \quad \tilde{\underline{\underline{\Delta}}}(\underline{n}) = \underline{\underline{\Delta}}(\underline{n}) \ , \tag{2.19}$$

leading to

$$\underline{\underline{J}}\cdot\tilde{\underline{\underline{D}}}_{\underline{k}}\cdot\underline{\underline{J}} = -\underline{\underline{D}}_{-\underline{k}} \ , \tag{2.20}$$

cf. eqs. (2.5) and (2.6). Using eq. (2.12) it follows that all possible contractions occurring in the Wick decomposition of eq. (2.13) can be expressed in terms of the one given in eq. (2.17), namely

$$\langle\underline{\Psi}(\underline{x},\tau)\tilde{\underline{\Psi}}(\underline{x}')\rangle = \langle\underline{\Psi}(\underline{x},\tau)\underline{\Psi}^{\dagger}(\underline{x}')\rangle\cdot\underline{\underline{J}} \ , \tag{2.21a}$$

$$\langle\tilde{\underline{\Psi}}^{\dagger}(\underline{x},\tau)\underline{\Psi}^{\dagger}(\underline{x}')\rangle = \underline{\underline{J}}\cdot\langle\underline{\Psi}(\underline{x},\tau)\underline{\Psi}^{\dagger}(\underline{x}')\rangle \ , \tag{2.21b}$$

$$\langle\tilde{\underline{\Psi}}^{\dagger}(\underline{x},\tau)\tilde{\underline{\Psi}}(\underline{x}')\rangle = \underline{\underline{J}}\cdot\langle\underline{\Psi}(\underline{x},\tau)\underline{\Psi}^{\dagger}(\underline{x}')\rangle\cdot\underline{\underline{J}} \ . \tag{2.21c}$$

It is now straightforward to evaluate (2.13) in terms of the matrices $\underline{\underline{G}}(k,\tau)$, using eqs. (2.16) and (2.21) and also the standard representation of the δ-function, leading to

$$\int d\underline{x}\int d\underline{x}'\ e^{i\underline{k}\cdot(\underline{x}-\underline{x}')} = (2\pi)^3\Omega\delta(\underline{k}) \ . \tag{2.22}$$

Some of the details of the calculation are worked out in appendix B. The result up to second order in I is

$$\begin{aligned}
&f_r(\{\underline{\underline{h}}^i\}) - f_{0,r}(\{\underline{\underline{h}}^i\}) = \\
&= \lim_{\Omega\to\infty}\Big[\tfrac{1}{2}I\Omega^{-2}\sum_{\underline{k}_1,\underline{k}_2}\mathrm{tr}^{(4)}\{\underline{\underline{G}}(\underline{k}_1,0)\cdot\tilde{\underline{\underline{Q}}}\cdot\underline{\underline{G}}(-\underline{k}_2,\beta)\cdot\underline{\underline{Q}}\} - \tfrac{1}{2}I^2\Omega^{-4}(2\pi)^3\int_0^{\beta}d\tau\sum_{\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{k}_4}\times \\
&\times\Big[-\mathrm{tr}^{(4)}\{\underline{\underline{G}}(\underline{k}_1,0)\cdot\tilde{\underline{\underline{Q}}}\cdot\underline{\underline{G}}(\underline{k}_2,\tau)\cdot\tilde{\underline{\underline{Q}}}\cdot\underline{\underline{G}}(-\underline{k}_3,\beta)\cdot\underline{\underline{Q}}\cdot\underline{\underline{G}}(-\underline{k}_4,\beta-\tau)\cdot\underline{\underline{Q}}\}\ \delta(\underline{k}_2+\underline{k}_4) \\
&\quad+\mathrm{tr}^{(4)}\{\underline{\underline{G}}(\underline{k}_1,0)\cdot\tilde{\underline{\underline{Q}}}\cdot\underline{\underline{G}}(\underline{k}_2,\tau)\cdot\underline{\underline{Q}}\cdot\underline{\underline{G}}(\underline{k}_3,0)\cdot\tilde{\underline{\underline{Q}}}\cdot\underline{\underline{G}}(-\underline{k}_4,\beta-\tau)\cdot\underline{\underline{Q}}\}\ \delta(\underline{k}_2+\underline{k}_4) \\
&\quad-\mathrm{tr}^{(4)}\{\underline{\underline{G}}(\underline{k}_1,\tau)\cdot\underline{\underline{Q}}\cdot\underline{\underline{G}}(-\underline{k}_2,\beta-\tau)\cdot\tilde{\underline{\underline{Q}}}\cdot\underline{\underline{G}}(\underline{k}_3,\tau)\cdot\tilde{\underline{\underline{Q}}}\cdot\underline{\underline{G}}(-\underline{k}_4,\beta-\tau)\cdot\underline{\underline{Q}}\}\ \delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4) \\
&\quad+\tfrac{1}{4}\mathrm{tr}^{(4)}\{\underline{\underline{G}}(\underline{k}_1,\tau)\cdot\underline{\underline{Q}}\cdot\underline{\underline{G}}(-\underline{k}_2,\beta-\tau)\cdot\underline{\underline{Q}}\}\ \mathrm{tr}^{(4)}\{\underline{\underline{G}}(\underline{k}_3,\tau)\cdot\tilde{\underline{\underline{Q}}}\cdot\underline{\underline{G}}(-\underline{k}_4,\beta-\tau)\cdot\tilde{\underline{\underline{Q}}}\}\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4) \\
&\quad+\tfrac{1}{4}\mathrm{tr}^{(4)}\{\underline{\underline{G}}(\underline{k}_1,\tau)\cdot\tilde{\underline{\underline{Q}}}\cdot\underline{\underline{G}}(-\underline{k}_2,\beta-\tau)\cdot\underline{\underline{Q}}\}\ \mathrm{tr}^{(4)}\{\underline{\underline{G}}(\underline{k}_3,\tau)\cdot\underline{\underline{Q}}\cdot\underline{\underline{G}}(-\underline{k}_4,\beta-\tau)\cdot\tilde{\underline{\underline{Q}}}\}\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4)\ \Big]\Big] \ .
\end{aligned} \tag{2.23}$$

Inserting the decomposition of the matrix $\underline{\underline{G}}(\underline{k},\tau)$

$$\underline{\underline{G}}(\underline{k},\tau) = \begin{pmatrix} \underline{\underline{G}}_1(\underline{k},\tau) & \underline{\underline{G}}_3(\underline{k},\tau) \\ \underline{\underline{G}}_3^\dagger(\underline{k},\tau) & \underline{\underline{G}}_2(\underline{k},\tau) \end{pmatrix} \tag{2.24}$$

in terms of the 2×2 matrices $\underline{\underline{G}}_1$, $\underline{\underline{G}}_2$, $\underline{\underline{G}}_3$, eq. (2.23) can be rewritten as

$$f_r(\{\underline{\underline{h}}^i\}) - f_{0,r}(\{\underline{\underline{h}}^i\}) =$$

$$= \lim_{\Omega\to\infty} \Big[\tfrac{1}{2} I\Omega^{-2} \sum_{\underline{k}_1,\underline{k}_2} \mathrm{tr}\{\underline{\underline{G}}_2(\underline{k}_1,0)\cdot\underline{\underline{g}}^y\cdot\underline{\underline{G}}_1(-\underline{k}_2,\beta)\cdot\underline{\underline{g}}^y\} + \tfrac{1}{2} I^2\Omega^{-4}(2\pi)^3 \int_0^\beta d\tau \sum_{\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{k}_4} \times$$

$$\times \Big[\mathrm{tr}\{\underline{\underline{G}}_2(\underline{k}_1,0)\cdot\underline{\underline{g}}^y\cdot\underline{\underline{G}}_3(\underline{k}_2,\tau)\cdot\underline{\underline{g}}^y\cdot\underline{\underline{G}}_1(-\underline{k}_3,\beta)\cdot\underline{\underline{g}}^y\cdot\underline{\underline{G}}_3^\dagger(-\underline{k}_4,\beta-\tau)\cdot\underline{\underline{g}}^y\}\delta(\underline{k}_2+\underline{k}_4)$$

$$- \mathrm{tr}\{\underline{\underline{G}}_2(\underline{k}_1,0)\cdot\underline{\underline{g}}^y\cdot\underline{\underline{G}}_1(\underline{k}_2,\tau)\cdot\underline{\underline{g}}^y\cdot\underline{\underline{G}}_2(\underline{k}_3,0)\cdot\underline{\underline{g}}^y\cdot\underline{\underline{G}}_1(-\underline{k}_4,\beta-\tau)\cdot\underline{\underline{g}}^y\}\delta(\underline{k}_2+\underline{k}_4)$$

$$+ \mathrm{tr}\{\underline{\underline{G}}_3^\dagger(\underline{k}_1,\tau)\cdot\underline{\underline{g}}^y\cdot\underline{\underline{G}}_2(-\underline{k}_2,\beta-\tau)\cdot\underline{\underline{g}}^y\cdot\underline{\underline{G}}_3(\underline{k}_3,\tau)\cdot\underline{\underline{g}}^y\cdot\underline{\underline{G}}_1(-\underline{k}_4,\beta-\tau)\cdot\underline{\underline{g}}^y\}\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4)$$

$$- \tfrac{1}{4}\mathrm{tr}\{\underline{\underline{G}}_3^\dagger(\underline{k}_1,\tau)\cdot\underline{\underline{g}}^y\cdot\underline{\underline{G}}_3^\dagger(-\underline{k}_2,\beta-\tau)\cdot\underline{\underline{g}}^y\}\ \mathrm{tr}\{\underline{\underline{G}}_3(\underline{k}_3,\tau)\cdot\underline{\underline{g}}^y\cdot\underline{\underline{G}}_3(-\underline{k}_4,\beta-\tau)\cdot\underline{\underline{g}}^y\}\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4)$$

$$- \tfrac{1}{4}\mathrm{tr}\{\underline{\underline{G}}_2(\underline{k}_1,\tau)\cdot\underline{\underline{g}}^y\cdot\underline{\underline{G}}_1(-\underline{k}_2,\beta-\tau)\cdot\underline{\underline{g}}^y\}\ \mathrm{tr}\{\underline{\underline{G}}_1(\underline{k}_3,\tau)\cdot\underline{\underline{g}}^y\cdot\underline{\underline{G}}_2(-\underline{k}_4,\beta-\tau)\cdot\underline{\underline{g}}^y\}\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4)\Big]\Big]. \tag{2.25}$$

In the following section the Green's functions $\underline{\underline{G}}_1$, $\underline{\underline{G}}_2$, $\underline{\underline{G}}_3$ will be expanded up to fourth order terms in the matrices $\underline{\underline{h}}^i$, $\underline{\underline{h}}^{i\dagger}$, in order to obtain in section 4 the Landau expansion in fourth order with explicit coefficients up to second order in the coupling constant I of the Hubbard interaction (1.4).

3. Reference Free Energy

In this section we evaluate the reference free energy (2.25) up to fourth order in terms of the matrices $\underline{\underline{h}}^i$, $\underline{\underline{h}}^{i\dagger}$, occurring in eq. (2.6) for the gap matrix $\underline{\underline{\Delta}}(\underline{n})$, in order to find explicit coefficients in the Landau expansion later on.

As a first step we evaluate the matrices $\underline{\underline{G}}_1$, $\underline{\underline{G}}_2$, $\underline{\underline{G}}_3$, defined in eq. (2.24), up to fourth order terms in $\underline{\underline{\Delta}}(\underline{n})$, $\underline{\underline{\Delta}}^{+}(\underline{n})$. For convenience we shall ignore the b-dependence in the contributions to the reference free energy from the Hubbard interaction, and hence we can take b=0 in the expressions for the matrices $\underline{\underline{G}}_1$, $\underline{\underline{G}}_2$, $\underline{\underline{G}}_3$. (In principle one can of course take into account the b-dependence of the matrices $\underline{\underline{G}}_1$, $\underline{\underline{G}}_2$, $\underline{\underline{G}}_3$ but this leads to very complicated formulae. From the investigation of the phase diagram, as will be carried out in a following chapter [43], it is not likely that the contributions from the magnetic field in the presence of a contrast term of the Hubbard type are really important for small b, in contrast to the case treated in refs. 1 and 2 in which the Hubbard term is absent). In the case b=0 the matrix $\underline{\underline{D}}_{\underline{k}}^2$ has the diagonal form

$$\underline{\underline{D}}_{\underline{k}}^2 = \begin{pmatrix} \varepsilon_k^2 \underline{\underline{1}} + \underline{\underline{\Delta}}(\underline{n})\cdot\underline{\underline{\Delta}}^{+}(\underline{n}) & 0 \\ 0 & \varepsilon_k^2 \underline{\underline{1}} + \underline{\underline{\Delta}}^{+}(\underline{n})\cdot\underline{\underline{\Delta}}(\underline{n}) \end{pmatrix} , \tag{3.1}$$

cf. eq. (2.5). Using the expansion

$$f(x,\tau) \equiv \frac{e^{-\tau x}}{1+e^{-\beta x}} = \sum_n a_n x^n \tag{3.2}$$

the Green's functions can be expanded as

$$\underline{\underline{G}}(k,\tau) = f(\underline{\underline{D}}_{\underline{k}},\tau) = \sum_n \begin{pmatrix} (a_{2n} + \varepsilon_k a_{2n+1})\, \underline{\underline{E}}_{\underline{k}}^{2n} & a_{2n+1}\, \underline{\underline{E}}_{\underline{k}}^{2n}\cdot\underline{\underline{\Delta}}(\underline{n}) \\ a_{2n+1}\, \bar{\underline{\underline{E}}}_{\underline{k}}^{2n}\cdot\underline{\underline{\Delta}}^{+}(\underline{n}) & (a_{2n}-\varepsilon_k a_{2n+1})\, \bar{\underline{\underline{E}}}_{\underline{k}}^{2n} \end{pmatrix} \tag{3.3}$$

where

$$\underline{\underline{E}}_{\underline{k}}^2 \equiv \varepsilon_k^2 \underline{\underline{1}} + \underline{\underline{\Delta}}(\underline{n})\cdot\underline{\underline{\Delta}}^{+}(\underline{n}), \tag{3.4}$$

and $\bar{\underline{\underline{E}}}_{\underline{k}}^2$ is the transposed matrix.
From eq. (3.3) we immediately have the expansions for the 2x2 matrices $\underline{\underline{G}}_1$, $\underline{\underline{G}}_2$, $\underline{\underline{G}}_3$, viz.

$$\underline{\underline{G}}_1(\underline{k},\tau) = f(\varepsilon_k,\tau)\underline{\underline{1}} + f_1(\varepsilon_k,\tau)\, \underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^{+} + \tfrac{1}{2} f_2(\varepsilon_k,\tau)\, \underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^{+}\cdot\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^{+} + O(\Delta^6) \tag{3.5}$$

$$\underline{\underline{G}}_2(\underline{k},\tau) = f(-\varepsilon_k,\tau)\underline{\underline{1}} + f_1(-\varepsilon_k,\tau)\, \underline{\underline{\Delta}}^{+}\cdot\underline{\underline{\Delta}} + \tfrac{1}{2} f_2(-\varepsilon_k,\tau)\, \underline{\underline{\Delta}}^{+}\cdot\underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^{+}\cdot\underline{\underline{\Delta}} + O(\Delta^6) \tag{3.6}$$

$$\underline{\underline{G}}_3(\underline{k},\tau) = p(\varepsilon_k,\tau)\, \underline{\underline{\Delta}} + p_1(\varepsilon_k,\tau)\, \underline{\underline{\Delta}}\cdot\underline{\underline{\Delta}}^{+}\cdot\underline{\underline{\Delta}} + O(\Delta^5) , \tag{3.7}$$

where

$$f_1(\varepsilon_k,\tau) = \sum_n n(a_{2n} + \varepsilon_k a_{2n+1})\varepsilon_k^{2(n-1)}$$

$$= \frac{\partial}{\partial\varepsilon_k^2}\left(\frac{f(\varepsilon_k,\tau) + f(-\varepsilon_k,\tau)}{2}\right) + \varepsilon_k \frac{\partial}{\partial\varepsilon_k^2}\left(\frac{f(\varepsilon_k,\tau) - f(-\varepsilon_k,\tau)}{2\varepsilon_k}\right) \quad (3.8)$$

$$f_2(\varepsilon_k,\tau) = \sum_n \tfrac{1}{2}n(n-1)\ (a_{2n}+\varepsilon_k a_{2n+1})\varepsilon_k^{2(n-2)}$$

$$= \frac{\partial^2}{(\partial\ \varepsilon_k^2)^2}\left(\frac{f(\varepsilon_k,\tau) + f(-\varepsilon_k,\tau)}{2}\right) + \varepsilon_k \frac{\partial^2}{(\partial\varepsilon_k^2)^2}\left(\frac{f(\varepsilon_k,\tau) - f(-\varepsilon_k,\tau)}{2\varepsilon_k}\right)$$

(3.9)

and

$$p(\varepsilon_k,\tau) = \sum_n a_{2n+1}\varepsilon_k^{2n} = \frac{f(\varepsilon_k,\tau) - f(-\varepsilon_k,\tau)}{2\varepsilon_k} \quad (3.10)$$

$$p_1(\varepsilon_k,\tau) = \sum_n n a_{2n+1}\varepsilon_k^{2(n-1)} = \frac{\partial}{\partial\varepsilon_k^2}\ p(\varepsilon_k,\tau)\ . \quad (3.11)$$

After inserting eqs. (3.5)-(3.7) into eq. (2.25) we take the thermodynamic limit $\Omega\to\infty$ by replacing the summations over the $\underline{k}$-vectors by integrations, using the relations

$$\frac{1}{\Omega}\sum_{\underline{k}} g(k)\ h(\underline{n}) \to \frac{1}{(2\pi)^3}\int d\underline{k}\ g(k)\ h(\underline{n}) = \int d\varepsilon\ \tfrac{1}{2}\mathcal{N}(\varepsilon)\ g(\varepsilon)\int\frac{d\Omega}{4\pi}\ h(\underline{n}) \quad (3.12)$$

where $d\Omega=\sin\theta d\theta d\phi$ is the solid angle associated with the polar coordinates of the unit vector $\underline{n}=(\sin\theta\ \cos\phi,\ \sin\theta\ \sin\phi,\ \cos\theta)$ and $\mathcal{N}(\varepsilon)$, given by

$$\tfrac{1}{2}\mathcal{N}(\varepsilon) = \frac{1}{\Omega}\sum_{\underline{k}}\delta(\varepsilon-\varepsilon_k) \to \frac{1}{2\pi^2}\int_0^\infty dk\ k^2\ \delta(\varepsilon-\varepsilon_k) \quad (3.13)$$

is the density of states per unit volume. The dependence on the unit vectors $\underline{n}_r=\underline{k}_r/k_r$ stems from the gap matrices $\underline{\underline{\Delta}}(\underline{n})$ and the delta functions in eq. (2.25). In order to find the reference free energy as an explicit function of the matrices $\underline{\underline{h}}^i$, $\underline{\underline{h}}^{i\dagger}$, we have to deal with the delta functions in the integrations over the solid angels $d\Omega_r$. For that purpose we use the following integrals

$$\left[\prod_{r=1}^{4} \int \frac{d\Omega_r}{4\pi}\right] n_1^i n_1^j \, (2\pi)^3 \, \delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4) = \frac{1}{3} H_{11}\delta_{ij} \tag{3.14a}$$

$$\left[\prod_{r=1}^{4} \int \frac{d\Omega_r}{4\pi}\right] n_1^i n_2^j \, (2\pi)^3 \, \delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4) = \frac{1}{3} H_{12}\delta_{ij} \tag{3.14b}$$

and

$$\left[\prod_{r=1}^{4} \int \frac{d\Omega_r}{4\pi}\right] n_1^i n_2^j n_3^k n_4^l \, (2\pi)^3 \, \delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4) = \frac{1}{15} H_{1234}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \tag{3.15a}$$

$$\left[\prod_{r=1}^{4} \int \frac{d\Omega_r}{4\pi}\right] n_1^i n_1^j n_2^k n_3^l \, (2\pi)^3 \, \delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4) = \frac{1}{9} H_{1123}\,\delta_{ij}\delta_{kl} + \frac{1}{15} H'_{1123}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \tag{3.15b}$$

$$\left[\prod_{r=1}^{4} \int \frac{d\Omega_r}{4\pi}\right] n_1^i n_1^j n_2^k n_2^l \, (2\pi)^3 \, \delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4) = \frac{1}{9} H_{1122}\,\delta_{ij}\delta_{kl} + \frac{1}{15} H'_{1122}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \tag{3.15c}$$

$$\left[\prod_{r=1}^{4} \int \frac{d\Omega_r}{4\pi}\right] n_1^i n_1^j n_1^k n_2^l \, (2\pi)^3 \, \delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4) = \frac{1}{15} H_{1112}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \tag{3.15d}$$

$$\left[\prod_{r=1}^{4} \int \frac{d\Omega_r}{4\pi}\right] n_1^i n_1^j n_1^k n_1^l \, (2\pi)^3 \, \delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4) = \frac{1}{15} H_{1111}(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \, , \tag{3.15e}$$

where the functions $H_{ab}(k_1, k_2, k_3, k_4)$ and $H_{abcd}(k_1, k_2, k_3, k_4)$ are only functions of the lengths k_r of the vectors $\underline{k}_r$. The explicit expressions of these functions are ($\partial_r = \partial/\partial k_r$, $r=1,\ldots,4$)

$$H_{11}(k_1,k_2,k_3,k_4) = -(\partial_1^2+2k_1^{-1}\partial_1)\, h_2(k_1,k_2,k_3,k_4) \tag{3.16a}$$

$$H_{12}(k_1,k_2,k_3,k_4) = -\partial_1\partial_2 h_2\ (k_1,k_2,k_3,k_4) \tag{3.16b}$$

and

$$H_{1234}(k_1,k_1,k_3,k_4) = \partial_1\partial_2\partial_3\partial_4\ h_4(k_1,k_2,k_3,k_4) \tag{3.17a}$$

$$H_{1123}(k_1,k_2,k_3,k_4) = (3k_1^{-1}\partial_1)\partial_2\partial_3\ h_4(k_1,k_2,k_3,k_4) \tag{3.17b}$$

$$H'_{1123}(k_1,k_2,k_3,k_4) = (\partial_1^2 - k_1^{-1}\partial_1)\partial_2\partial_3\ h_4(k_1,k_2,k_3,k_4) \tag{3.17c}$$

$$H_{1122}(k_1,k_2,k_3,k_4) = 3(k_1^{-1}\partial_1\partial_2^2 + k_2^{-1}\partial_2\partial_1^2 + k_1^{-1}k_2^{-1}\partial_1\partial_2)h_4(k_1,k_2,k_3,k_4) \tag{3.17d}$$

$$H'_{1122}(k_1,k_2,k_3,k_4) = (\partial_1^2 - k_1^{-1}\partial_1)(\partial_2^2 - k_2^{-1}\partial_2)\ h_4(k_1,k_2,k_3,k_4) \tag{3.17e}$$

$$H_{1112}(k_1,k_2,k_3,k_4) = (\partial_1^3 + 2k_1^{-1}\partial_1^2 - 2k_1^{-2}\partial_1)\partial_2\ h_4(k_1,k_2,k_3,k_4) \tag{3.17f}$$

$$H_{1111}(k_1,k_2,k_3,k_4) = (\partial_1^4 + 4k_1^{-1}\partial_1^3)\ h_4(k_1,k_2,k_3,k_4) \tag{3.17g}$$

with

$$h_{2n}(k_1,k_2,k_3,k_4) = -\frac{(-1)^n\pi^2}{(2n+1)!}\sum_{\nu_1,\nu_2,\nu_3,\nu_4=\pm 1}\frac{\theta(\nu_1k_1+\nu_2k_2+\nu_3k_3+\nu_4k_4)}{k_1k_2k_3k_4}\times$$

$$\times\ (\nu_1k_1+\nu_2k_2+\nu_3k_3+\nu_4k_4)^{2n+1}\ . \tag{3.18}$$

In eq. (3.18) θ is the Heaviside step-function. Eqs. (3.14) and (3.15), together with (3.16), (3.17), are derived in appendix C.

We can now insert the expressions for the matrices $\underline{\underline{G}}_1$, $\underline{\underline{G}}_2$, $\underline{\underline{G}}_3$, eqs. (3.5)-(3.7) into eq. (2.25) and perform the integrations over the solid angles, associated with the vectors $\underline{k}_r$, using (3.14) and (3.15). Collecting all the terms in different orders of $\underline{\underline{h}}^i$, $\underline{\underline{h}}^{i\dagger}$ we find after a straightforward but tedious calculation the following result for the reference free energy per unit volume

$$f_r(\{\underline{\underline{h}}^i\}) - f_{0,r}(\{\underline{\underline{h}}^i\}) = \Delta f_n + \frac{1}{3} S \sum_{i=1}^{3} \mathrm{tr}\,(\underline{\underline{h}}^i \cdot \underline{\underline{h}}^{i\dagger}) +$$

$$+ \sum_{i,j=1}^{3} \Big[\frac{1}{15} P_1 \, \mathrm{tr}(\underline{\underline{h}}^i \cdot \underline{\underline{h}}^{i\dagger} \cdot \underline{\underline{h}}^j \cdot \underline{\underline{h}}^{j\dagger} + \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger} \cdot \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger} + \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger} \cdot \underline{\underline{h}}^j \cdot \underline{\underline{h}}^{i\dagger})$$

$$+ \frac{1}{9} P_2 \, \mathrm{tr}(\underline{\underline{h}}^{i\dagger} \cdot \underline{\underline{h}}^i \cdot \underline{\underline{\sigma}}^y \cdot \underline{\underline{h}}^j \cdot \underline{\underline{h}}^{j\dagger} \cdot \underline{\underline{\sigma}}^y)$$

$$+ \frac{1}{15} P_3 \, \mathrm{tr}(\underline{\underline{h}}^{i\dagger} \cdot \underline{\underline{h}}^i \cdot \underline{\underline{\sigma}}^y \cdot \underline{\underline{h}}^j \cdot \underline{\underline{h}}^{j\dagger} \cdot \underline{\underline{\sigma}}^y + \underline{\underline{h}}^{i\dagger} \cdot \underline{\underline{h}}^j \cdot \underline{\underline{\sigma}}^y \cdot \underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger} \cdot \underline{\underline{\sigma}}^y + \underline{\underline{h}}^{i\dagger} \cdot \underline{\underline{h}}^j \cdot \underline{\underline{\sigma}}^y \cdot \underline{\underline{h}}^j \cdot \underline{\underline{h}}^{i\dagger} \cdot \underline{\underline{\sigma}}^y$$

$$+ \frac{1}{15} P_4 \{ \mathrm{tr}(\underline{\underline{h}}^{i\dagger} \cdot \underline{\underline{\sigma}}^y \cdot \underline{\underline{h}}^{i\dagger} \cdot \underline{\underline{\sigma}}^y) \, \mathrm{tr}(\underline{\underline{h}}^j \cdot \underline{\underline{\sigma}}^y \cdot \underline{\underline{h}}^j \cdot \underline{\underline{\sigma}}^y) + \mathrm{tr}(\underline{\underline{h}}^{i\dagger} \cdot \underline{\underline{\sigma}}^y \cdot \underline{\underline{h}}^{j\dagger} \cdot \underline{\underline{\sigma}}^y) \times$$

$$\times \, \mathrm{tr}(\underline{\underline{h}}^i \cdot \underline{\underline{\sigma}}^y \cdot \underline{\underline{h}}^j \cdot \underline{\underline{\sigma}}^y) + \mathrm{tr}(\underline{\underline{h}}^{i\dagger} \cdot \underline{\underline{\sigma}}^y \cdot \underline{\underline{h}}^{j\dagger} \cdot \underline{\underline{\sigma}}^y) \, \mathrm{tr}(\underline{\underline{h}}^j \cdot \underline{\underline{\sigma}}^y \cdot \underline{\underline{h}}^i \cdot \underline{\underline{\sigma}}^y) \}$$

$$+ \frac{1}{9} P_5 \, \mathrm{tr}(\underline{\underline{h}}^i \cdot \underline{\underline{h}}^{i\dagger}) \mathrm{tr}(\underline{\underline{h}}^j \cdot \underline{\underline{h}}^{j\dagger})$$

$$+ \frac{1}{15} P_6 \{ \mathrm{tr}(\underline{\underline{h}}^i \cdot \underline{\underline{h}}^{i\dagger}) \, \mathrm{tr}(\underline{\underline{h}}^j \cdot \underline{\underline{h}}^{j\dagger}) + \mathrm{tr}(\underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger}) \mathrm{tr}(\underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger}) + \mathrm{tr}(\underline{\underline{h}}^i \cdot \underline{\underline{h}}^{j\dagger}) \mathrm{tr}(\underline{\underline{h}}^j \cdot \underline{\underline{h}}^{i\dagger}) \}$$

$$+ \frac{1}{9} P_7 \, \mathrm{tr}(\underline{\underline{h}}^i \cdot \underline{\underline{h}}^{i\dagger} \cdot \underline{\underline{h}}^j \cdot \underline{\underline{h}}^{j\dagger}) \Big] , \qquad (3.19)$$

up to fourth order terms in the matrices $\underline{\underline{h}}^i$, $\underline{\underline{h}}^{i\dagger}$. In eq. (3.19) Δf_n is the correction to the normal free energy per unit volume due to the Hubbard interaction. In contrast to the situation for the superfluid contributions to the reference free energy, involving matrices $\underline{\underline{\Delta}}$, $\underline{\underline{\Delta}}^\dagger$, it is easy to specify the normal fluid contributions for arbitrary values of b. This can be done by inserting in (2.25) the normal Green's functions, which are given by

$$\underline{\underline{G}}_1^{(n)}(\underline{k},\tau) = f(\underline{\underline{\varepsilon}}_k,\tau) \; , \; \underline{\underline{G}}_2^{(n)}(\underline{k},\tau) = f(-\underline{\underline{\varepsilon}}_k,\tau) \; , \; \underline{\underline{G}}_3^{(n)}(\underline{k},\tau) = 0,$$

$$\underline{\underline{\varepsilon}}_k = \varepsilon_k \underline{\underline{1}} - b\underline{\underline{\sigma}}^z . \qquad (3.20)$$

The result is

$$\Delta f_n = \tfrac{1}{2}I \int d\varepsilon_1\, \tfrac{1}{2}\mathcal{N}(\varepsilon) \int d\varepsilon_2\, \tfrac{1}{2}\mathcal{N}(\varepsilon_2)\, \mathrm{tr}\{f(-\underline{\underline{\varepsilon}}_1,0)\cdot\underline{\underline{\sigma}}^y\cdot f(\underline{\underline{\varepsilon}}_2,\beta)\cdot\underline{\underline{\sigma}}^y.\}$$

$$-\tfrac{1}{2}I^2 \int d\varepsilon_1\, \tfrac{1}{2}\mathcal{N}(\varepsilon_1) \int d\varepsilon_2\, \tfrac{1}{2}\mathcal{N}(\varepsilon_2) \int d\varepsilon_3\, \tfrac{1}{2}\mathcal{N}(\varepsilon_3) \int_0^\beta d\tau\, \mathrm{tr}\,\{f(-\underline{\underline{\varepsilon}}_1,0)\cdot\underline{\underline{\sigma}}^y\cdot f(\underline{\underline{\varepsilon}}_2,\tau)\cdot\underline{\underline{\sigma}}^y\cdot$$

$$f(-\underline{\underline{\varepsilon}}_3,0)\cdot\underline{\underline{\sigma}}^y\cdot f(\underline{\underline{\varepsilon}}_2,\beta-\tau)\cdot\underline{\underline{\sigma}}^y\}$$

$$-\tfrac{1}{8}I^2\left[\prod_{r=1}^{4} \int d\varepsilon_r\, \tfrac{1}{2}\mathcal{N}(\varepsilon_r)\right]\int_0^\beta d\tau\, h_0\, \mathrm{tr}\,\{f(-\underline{\underline{\varepsilon}}_1,\tau)\cdot\underline{\underline{\sigma}}^y\cdot f(-\underline{\underline{\varepsilon}}_2,\beta-\tau)\cdot\underline{\underline{\sigma}}^y\}\times$$

$$\times\, \mathrm{tr}\{f(\underline{\underline{\varepsilon}}_3,\tau)\cdot\underline{\underline{\sigma}}^y\cdot f(-\underline{\underline{\varepsilon}}_4,\beta-\tau)\cdot\underline{\underline{\sigma}}^y\}\ , \qquad (3.21)$$

where $h_0\ (k_1,k_2,k_3,k_4)$ is given in eq. (3.18) and $\underline{\underline{\varepsilon}}_r \equiv \varepsilon_r \underline{\underline{1}} - b\ \underline{\underline{\sigma}}^z$. Note that there occurs a definite first order contribution in eq. (3.21), which was not considered in refs. 16-18. The coefficients S, P_1, .., P_7 in eq. (3.19) can be expressed in terms of integrals over the functions f and p, introduced in eqs. (3.2), (3.8) -(3.11), using the abbreviations

$$K(\tau) = \int d\varepsilon\, \tfrac{1}{2}\mathcal{N}(\varepsilon)\ \ f(\varepsilon,\tau)\ ,\ K_{1,2}\,(\tau) = \int d\varepsilon\, \tfrac{1}{2}\mathcal{N}(\varepsilon)\ f_{1,2}(\varepsilon,\tau)\ . \qquad (3.22)$$

They are given by

$$\begin{aligned} S = {} & I\, K(\beta)\, K_1(\beta) + \tfrac{1}{2}I^2\, K(\beta)^2 \int_0^\beta d\tau \int d\varepsilon\, \tfrac{1}{2}\mathcal{N}(\varepsilon)\, p(\varepsilon,\tau)\, p(\varepsilon,\beta-\tau) \\ & - I^2 K(\beta)\, K_1(\beta) \int_0^\beta d\tau \int d\varepsilon\, \tfrac{1}{2}\mathcal{N}(\varepsilon)\, f(\varepsilon,\tau)\, f(\varepsilon,\beta-\tau) \\ & - I^2 K(\beta)^2 \int_0^\beta d\tau \int d\varepsilon\, \tfrac{1}{2}\mathcal{N}(\varepsilon)\ f_1(\varepsilon,\tau)\, f(\varepsilon,\beta-\tau) \\ & + \tfrac{1}{2}I^2 \int_0^\beta d\tau \left[\prod_{r=1}^{4} \int d\varepsilon_r\, \tfrac{1}{2}\mathcal{N}(\varepsilon_r)\right]\{H_{13}\, p(\varepsilon_1,\tau)\, f(\varepsilon_2,\beta-\tau)\, p(\varepsilon_3,\tau)\, f(\varepsilon_4,\beta-\tau) \\ & \qquad -2\, H_{11}\, f_1(\varepsilon_1,\beta-\tau)\, f(\varepsilon_2,\beta-\tau)\, f(\varepsilon_3,\tau)\, f(\varepsilon_4,\tau)\} \end{aligned} \qquad (3.23)$$

$$P_1 = \tfrac{1}{2} I\, K(\beta)\, K_2(\beta) + I^2\, K(\beta)^2 \int_0^\beta d\tau \int d\varepsilon\ \tfrac{1}{2}\mathcal{N}(\varepsilon)\, p_1(\varepsilon,\tau)\, p(\varepsilon,\beta-\tau)$$

$$- \tfrac{1}{2} I^2 K(\beta)\, K_2(\beta) \int_0^\beta d\tau \int d\varepsilon\ \tfrac{1}{2}\mathcal{N}(\varepsilon)\, f(\varepsilon,\tau)\, f(\varepsilon,\beta-\tau)$$

$$- \tfrac{1}{2} I^2 K(\beta)^2 \int_0^\beta d\tau \int d\varepsilon\ \tfrac{1}{2}\mathcal{N}(\varepsilon) \left[f_2(\varepsilon,\tau)\, f(\varepsilon,\beta-\tau) + f_1(\varepsilon,\tau)\, f_1(\varepsilon,\beta-\tau)\right]$$

$$+ \tfrac{1}{2} I^2 \int_0^\beta d\tau \left[\prod_{r=1}^{4} \int d\varepsilon_r\ \tfrac{1}{2}\mathcal{N}(\varepsilon_r)\right] \{2H_{1113}\, p_1(\varepsilon_1,\tau)\, f(\varepsilon_2,\tau)\, p(\varepsilon_3,\tau)\, f(\varepsilon_4,\beta-\tau)$$

$$- H_{1111}\, f_2(\varepsilon_1,\tau)\, f(\varepsilon_2,\beta-\tau)\, f(\varepsilon_3,\tau)\, f(\varepsilon_4,\beta-\tau)\} \tag{3.24}$$

$$P_2 = \tfrac{1}{2} I\, K_1(\beta)^2 + I^2\, K(\beta)\, K_1(\beta) \int_0^\beta d\tau \int d\varepsilon\ \tfrac{1}{2}\mathcal{N}(\varepsilon)\, p(\varepsilon,\tau)\, p(\varepsilon,\beta-\tau)$$

$$- I^2\, K(\beta)\, K_1(\beta) \int_0^\beta d\tau \int d\varepsilon\ \tfrac{1}{2}\mathcal{N}(\varepsilon)\, 2\, f_1(\varepsilon,\tau)\, f(\varepsilon,\beta-\tau) +$$

$$+\tfrac{1}{2} I^2 \int_0^\beta d\tau \left[\prod_{r=1}^{4} \int d\varepsilon_r\ \tfrac{1}{2}\mathcal{N}(\varepsilon_r)\right] \{2H_{2213}\, p(\varepsilon_1,\tau)\, f_1(\varepsilon_2,\tau)\, p(\varepsilon_3,\tau)\, f(\varepsilon_4,\beta-\tau)$$

$$- H_{1122}\, f_1(\varepsilon_1,\beta-\tau)\, f_1(\varepsilon_2,\beta-\tau)\, f(\varepsilon_3,\tau)\, f(\varepsilon_4,\tau)\} \tag{3.25}$$

$$P_3 = \tfrac{1}{2} I^2 \int_0^\beta d\tau \left[\prod_{r=1}^{4} \int d\varepsilon_r\ \tfrac{1}{2}\mathcal{N}(\varepsilon_r)\right] \{2H'_{2213}\, p(\varepsilon_1,\tau)\, f_1(\varepsilon_2,\tau)\, p(\varepsilon_3,\tau)\, f(\varepsilon_4,\beta-\tau)$$

$$- H'_{1122}\, f_1(\varepsilon_1,\beta-\tau)\, f_1(\varepsilon_2,\beta-\tau)\, f(\varepsilon_3,\tau)\, f(\varepsilon_4,\tau)\} \tag{3.26}$$

$$P_4 = -\tfrac{1}{2} I^2 \int_0^\beta d\tau \left[\prod_{r=1}^{4} \int d\varepsilon_r\ \tfrac{1}{2}\mathcal{N}(\varepsilon_r)\right] \tfrac{1}{4}\, H_{1234}\, p(\varepsilon_1,\tau)\, p(\varepsilon_2,\beta-\tau)\, p(\varepsilon_3,\tau)\, p(\varepsilon_4,\beta-\tau) \tag{3.27}$$

$$P_5 = -\tfrac{1}{2}\, I^2 \int_0^\beta d\tau \left[\prod_{r=1}^{4} \int d\varepsilon_r \, \tfrac{1}{2}\mathcal{N}(\varepsilon_r) \right] H_{1133}\, f_1(\varepsilon_1,\beta-\tau) f(\varepsilon_2,\beta-\tau) f_1(\varepsilon_3,\tau) f(\varepsilon_4,\tau) \tag{3.28}$$

$$P_6 = -\tfrac{1}{2}\, I^2 \int_0^\beta d\tau \left[\prod_{r=1}^{4} \int d\varepsilon_r \, \tfrac{1}{2}\mathcal{N}(\varepsilon_r) \right] H'_{1133}\, f_1(\varepsilon_1,\beta-\tau) f(\varepsilon_2,\beta-\tau) f_1(\varepsilon_3,\tau) f(\varepsilon_4,\tau) \tag{3.29}$$

$$P_7 = -\tfrac{1}{2}\, I^2\, K_1(\beta)^2 \int_0^\beta d\tau \int d\varepsilon \, \tfrac{1}{2}\mathcal{N}(\varepsilon_r)\, f(\varepsilon,\tau)\, f(\varepsilon,\beta-\tau)\ . \tag{3.30}$$

The various terms in eq. (3.19) are not all independent. In fact we have the relations

$$\sum_{i,j,k,l=1}^{3} (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \Big[\mathrm{tr}\,(\underline{\underline{h}}^{i}\cdot\underline{\underline{\sigma}}^{y}\cdot\underline{\underline{h}}^{j}\cdot\underline{\underline{\sigma}}^{y})\ \mathrm{tr}\,(\underline{\underline{h}}^{k\dagger}\cdot\underline{\underline{\sigma}}^{y}\cdot\underline{\underline{h}}^{l\dagger}\cdot\underline{\underline{\sigma}}^{y})$$

$$-2\ \mathrm{tr}\,(\underline{\underline{h}}^{i\dagger}\cdot\underline{\underline{h}}^{j}\cdot\underline{\underline{\sigma}}^{y}\cdot\underline{\underline{h}}^{k}\cdot\underline{\underline{k}}^{l\dagger}\cdot\underline{\underline{\sigma}}^{y}) \Big] = 0 \tag{3.31a}$$

$$\sum_{i,j,k,l=1}^{3} (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \Big[\mathrm{tr}\,(\underline{\underline{h}}^{i}\cdot\underline{\underline{h}}^{j\dagger})\ \mathrm{tr}\,(\underline{\underline{h}}^{k}\cdot\underline{\underline{h}}^{l\dagger})$$

$$-\ \mathrm{tr}\,(\underline{\underline{h}}^{i}\cdot\underline{\underline{h}}^{j\dagger}\cdot\underline{\underline{h}}^{k}\cdot\underline{\underline{h}}^{l\dagger}) - \mathrm{tr}\,(\underline{\underline{h}}^{i\dagger}\cdot\underline{\underline{h}}^{j}\cdot\underline{\underline{\sigma}}^{y}\cdot\underline{\underline{h}}^{k}\cdot\underline{\underline{h}}^{l\dagger}\cdot\underline{\underline{\sigma}}^{y}) \Big] = 0 \tag{3.31b}$$

$$\sum_{i,j=1}^{3} \Big[\mathrm{tr}\,(\underline{\underline{h}}^{i}\cdot\underline{\underline{h}}^{i\dagger}\cdot\underline{\underline{h}}^{j}\cdot\underline{\underline{h}}^{j\dagger}) - \mathrm{tr}\,(\underline{\underline{h}}^{i}\cdot\underline{\underline{h}}^{i\dagger})\ \mathrm{tr}\,(\underline{\underline{h}}^{j}\cdot\underline{\underline{h}}^{j\dagger})$$

$$+\ \mathrm{tr}\,(\underline{\underline{h}}^{i\dagger}\cdot\underline{\underline{h}}^{i}\cdot\underline{\underline{\sigma}}^{y}\cdot\underline{\underline{h}}^{j}\cdot\underline{\underline{h}}^{j\dagger}\cdot\underline{\underline{\sigma}}^{y}) \Big] = 0 \tag{3.31c}$$

for symmetric 2x2 matrices $\underline{\underline{h}}^{i}$, $\underline{\underline{h}}^{i\dagger}$. Using the relations (3.31), eq. (3.19) can be reduced to the following form

$$f_r(\{\underline{\underline{h}}^{i}\}) - f_{0,r}(\{\underline{\underline{h}}^{i}\}) = \Delta f_n + \frac{1}{3} S \sum_{i=1}^{3} \mathrm{tr}\,(\underline{\underline{h}}^{i}\ \underline{\underline{h}}^{i\dagger})$$

$$+ \frac{1}{15}(P_1+P_6) \sum_{i,j=1}^{3} \mathrm{tr}\,(\underline{\underline{h}}^{i}\cdot\underline{\underline{h}}^{i\dagger}\cdot\underline{\underline{h}}^{j}\cdot\underline{\underline{h}}^{j\dagger} + \underline{\underline{h}}^{i}\cdot\underline{\underline{h}}^{j\dagger}\cdot\underline{\underline{h}}^{i}\cdot\underline{\underline{h}}^{j\dagger} + \underline{\underline{h}}^{i}\cdot\underline{\underline{h}}^{j\dagger}\cdot\underline{\underline{h}}^{j}\cdot\underline{\underline{h}}^{i\dagger})$$

$$+ \frac{1}{9}(P_2+P_5) \sum_{i,j=1} \mathrm{tr}\,(\underline{\underline{h}}^{i\dagger}\cdot\underline{\underline{h}}^{i}\cdot\underline{\underline{\sigma}}^{y}\cdot\underline{\underline{h}}^{j}\cdot\underline{\underline{h}}^{j\dagger}\cdot\underline{\underline{\sigma}}^{y})$$

$$+ \frac{1}{15}(P_3+2P_4+P_6) \sum_{i,j=1}^{3} \mathrm{tr}(\underline{\underline{h}}^{i\dagger}\cdot\underline{\underline{h}}^{i}\cdot\underline{\underline{g}}^{y}\cdot\underline{\underline{h}}^{j}\cdot\underline{\underline{h}}^{j\dagger}\cdot\underline{\underline{g}}^{y} + \underline{\underline{h}}^{i\dagger}\cdot\underline{\underline{h}}^{j}\cdot\underline{\underline{g}}^{y}\cdot\underline{\underline{h}}^{j}\cdot\underline{\underline{h}}^{i\dagger}\cdot\underline{\underline{g}}^{y} + \underline{\underline{h}}^{i\dagger}\cdot\underline{\underline{h}}^{j}\cdot\underline{\underline{g}}^{y}\cdot h^{i}\cdot\underline{\underline{h}}^{j\dagger}\cdot\underline{\underline{g}}^{y})$$

$$+ \frac{1}{9}(P_5+P_7) \sum_{i,j=1}^{3} \mathrm{tr}\,(\underline{\underline{h}}^{i}\cdot\underline{\underline{h}}^{i\dagger}\cdot\underline{\underline{h}}^{j}\cdot\underline{\underline{h}}^{j\dagger}) \qquad (3.32)$$

Eq. (3.32) together with (3.23)-(3.30) gives the contribution to the reference free energy from the Hubbard term for b=0 and up to the order I^2. If we use the expression for $f_{0,r}$, given in formulae (5.19), (5.20) of ref. 1, we obtain the full reference free energy up to the fourth order in terms of the matrices $\underline{\underline{h}}^i, \underline{\underline{h}}^i$. In the following section the theorem of Bogoliubov Jr. will be used in order to calculate the free energy of the system itself in a Landau expansion up to fourth order in the order parameters.

4. Landau Expansion

According to the theorem of Bogoliubov Jr. , the free energy per unit volume f corresponding to the hamiltonian (1.1)-(1.3), (1.4) is given by

$$f = \min_{\{\underline{\underline{m}}^i\}} \left[f_{0,r}(\{\underline{\underline{m}}^i\}) + \frac{1}{6V_1}\sum_{i=1}^{3} \mathrm{tr}\,(\underline{\underline{m}}^i\cdot\underline{\underline{m}}^{i\dagger}) + \left(f_r(\{\underline{\underline{m}}^i\}) - f_{0,r}(\{\underline{\underline{m}}^i\})\right)\right] \qquad (4.1)$$

Here $f_r(\{\underline{\underline{m}}^i\})-f_{0,r}(\{\underline{\underline{m}}^i\})$ can be inferred for b=0 up to fourth order terms in $\underline{\underline{m}}^i,\underline{\underline{m}}^{i\dagger}$ from eq. (3.32), and the first two terms in the right hand side constitute the Landau expansion of the free energy due to the pairing interaction in eq. (1.1) but in the absence of the Hubbard term, as studied in refs. 1 2. Rather than working with the 2x2 symmetric matrices $\underline{\underline{m}}^i$, it is convenient to introduce the three complex vectors $\underline{m}_1$, $\underline{m}_2$, $\underline{m}_3$ with components m_1^i, m_2^i, m_3^i defined by

$$\sqrt{2}\,m_1^i = m_{\uparrow\uparrow}^i\,, \quad \sqrt{2}\,m_2^i = m_{\uparrow\uparrow}^i\,, \quad \sqrt{2}\,m_3^i = m_{\uparrow\downarrow}^i = m_{\downarrow\uparrow}^i\,, \quad i=1,2,3, \qquad (4.2)$$

with lengths

$$m_a = (\underline{m}_a\cdot\underline{m}_a^*)^{\frac{1}{2}}\,, \quad a=1,2,3. \qquad (4.3)$$

As shown in ref. 2

$$f_{0,r}(\{\underline{\underline{m}}^i\}) + \frac{1}{6V_1}\sum_{i=1}^{3} \mathrm{tr}(\underline{\underline{m}}^i\cdot\underline{\underline{m}}^{i+}) = \Phi_0(m_1, m_2, m_3) + \Phi_1(\underline{m}_1, \underline{m}_2, \underline{m}_3) \,, \tag{4.4}$$

in which $\Phi_0(m_1, m_2, m_3)$, depending only on the lengths of the vectors $\underline{m}_a$, and $\Phi_1(\underline{m}_1, \underline{m}_2, \underline{m}_3)$, have been given in eqs. (2.6) and (2.7) of ref. 2 for arbitrary values of the magnetic field b. Furthermore we define

$$f_r(\{\underline{\underline{m}}^i\}) - f_{0,r}(\{\underline{\underline{m}}^i\}) = \Delta f_n + \Phi_2(\underline{m}_1, \underline{m}_2, \underline{m}_3) \,, \tag{4.5}$$

where Δf_n has been given in eq. (3.21) and where $\Phi_2(\underline{m}_1, \underline{m}_2, \underline{m}_3)$ is the contribution of the Hubbard hamiltonian which in the case b=0 can be derived immediately from eqs.(3.32), (4.2) and (4.3).
The result is

$$\begin{aligned}
\Phi_2(\underline{m}_1, \underline{m}_2, \underline{m}_3) = {} & \frac{2}{3} S\,(m_1^2 + m_2^2 + 2m_3^2) \\
& + w_0\,[2m_1^4 + |\underline{m}_1\cdot\underline{m}_1|^2 + 2m_2^4 + |\underline{m}_2\cdot\underline{m}_2|^2 + 4m_3^4 + \\
& + 2|\underline{m}_3\cdot\underline{m}_3|^2 + 4m_1^2m_3^2 + 4|\underline{m}_1\cdot\underline{m}_3|^2 + 4|\underline{m}_1\cdot\underline{m}_3^*|^2 \\
& + 4m_2^2\,m_3^2 + 4|\underline{m}_2\cdot\underline{m}_3|^2 + 4|\underline{m}_3\cdot\underline{m}_3^*|^2 \\
& + 4\mathrm{Re}\,(\underline{m}_1\cdot\underline{m}_2\;\;\underline{m}_3^*\cdot\underline{m}_3^*) + 8\mathrm{Re}\,(\underline{m}_1\cdot\underline{m}_3^*\;\;\underline{m}_2\cdot\underline{m}_3^*)] \\
& + w_1(m_1^2+m_2^2+2m_3^2)^2 + w_2\,[(m_1^2+m_3^2)(m_2^2+m_3^2) - |\underline{m}_1\cdot\underline{m}_3^*+\underline{m}_3\cdot\underline{m}_2^*|^2] \\
& + w_3\,[m_1^2\,m_2^2 + |\underline{m}_1\cdot\underline{m}_2|^2 + |\underline{m}_1\cdot\underline{m}_2^*|^2 + 2m_3^4 + |\underline{m}_3\cdot\underline{m}_3|^2 \\
& - 2\mathrm{Re}\,(\underline{m}_1\cdot\underline{m}_2\;\;\underline{m}_3^*\cdot\underline{m}_3^*) - 4\mathrm{Re}\,(\underline{m}_1\cdot\underline{m}_3^*\;\;\underline{m}_2\cdot\underline{m}_3^*)]
\end{aligned} \tag{4.6}$$

where

$$w_0 = \frac{4}{15}(P_1+P_6)\,,\quad w_1 = \frac{4}{9}(P_5+P_7)\,,\quad w_2 = \frac{8}{9}(P_2-P_7)\,,$$

$$w_3 = \frac{4}{15}(2P_3+4P_4+2P_6)\,. \tag{4.7}$$

It can be noted that the term w_0 is proportional to the fourth order part of $\Phi_0\,(m_1,\, m_2,\, m_3) + \Phi_1\,(\underline{m}_1,\, \underline{m}_2,\, \underline{m}_3)$ for the special case b=0, cf. eq. (3.6), (2.7) and (3.21) of ref. 2. The terms involving w_1, w_2 and w_3 give rise to new terms in the Landau expansion arising from the Hubbard interaction.

The final result for the free energy can now be presented in the form

$$f = f_n + \min_{\{\underline{m}_a\}} \hat{\Phi}\,(\underline{m}_1,\, \underline{m}_2,\, \underline{m}_3)\ , \tag{4.8}$$

where f_n is the normal fluid part of the free energy per unit volume in luding the normal contribution f_n from the Hubbard interaction, cf. eq. (3.21), and

$$\hat{\Phi}\,(\underline{m}_1,\, \underline{m}_2,\, \underline{m}_3) = \hat{\Phi}_0\,(m_1,\, m_2,\, m_3) + \hat{\Phi}_1\,(\underline{m}_1,\, \underline{m}_2,\, \underline{m}_3)\ . \tag{4.9}$$

The explicit expressions for $\hat{\Phi}_0$ and $\hat{\Phi}_1$ are

$$\begin{aligned}\hat{\Phi}_0\,(m_1,\, m_2,\, m_3) &= \Phi_0\,(m_1,\, m_2,\, m_3) + \frac{2}{3}\, S\,(m_1^2 + m_2^2 + 2m_3^2)\\ &+ w_0\,[2(m_1^2 + m_3^2)^2 + 2(m_2^2 + m_3^2)^2] + w_1(m_1^2 + m_2^2 + 2m_3^2)^2\\ &+ w_2\,(m_1^2 + m_3^2)\,(m_2^2 + m_3^2) + w_3\,(m_1^2\, m_2^2 + 2m_3^4)\\ &= \hat{u}_1\, m_1^2 + \hat{u}_2\, m_2^2 + 2\hat{u}_3\, m_3^2 + 2\hat{v}_1\, m_1^4 + 2\hat{v}_2\, m_2^4 + 4\hat{v}_3\, m_3^4 +\\ &+ \hat{v}_4\, m_1^2\, m_3^2 + \hat{v}_5\, m_2^2\, m_3^2 + w_1(m_1^2 + m_2^2 + 2m_3^2)^2 + w_2(m_1^2 + m_3^2)\, \times\\ &\times (m_2^2 + m_3^2) + w_3(m_1^2\, m_2^2 + 2m_3^4)\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\hat{\Phi}_1\,(\underline{m}_1,\, \underline{m}_2,\, \underline{m}_3) &= \Phi_1\,(\underline{m}_1,\, \underline{m}_2,\, \underline{m}_3) + w_0\,[|\underline{m}_1\cdot\underline{m}_1|^2 + |\underline{m}_2\cdot\underline{m}_2|^2 + 2|\underline{m}_3\cdot\underline{m}_3|^2\\ &+ 4|\underline{m}_1\cdot\underline{m}_3|^2 + 4|\underline{m}_1\cdot\underline{m}_3^*|^2 + 4|\underline{m}_2\cdot\underline{m}_3|^2 + 4|\underline{m}_2\cdot\underline{m}_3^*|^2 + 4\mathrm{Re}\,(\underline{m}_1\cdot\underline{m}_2\ \underline{m}_3^*\cdot\underline{m}_3^*)\\ &+ 8\mathrm{Re}\,(\underline{m}_1\cdot\underline{m}_3^*\ \ \underline{m}_2\cdot\underline{m}_3^*)\,] - w_2|\underline{m}_1\cdot\underline{m}_3^* + \underline{m}_3\cdot\underline{m}_2^*|^2\\ &+ w_3[\ |\underline{m}_1\cdot\underline{m}_2|^2 + |\underline{m}_1\cdot\underline{m}_2^*|^2 + |\underline{m}_3\cdot\underline{m}_3|^2 - 2\mathrm{Re}(\underline{m}_1\cdot\underline{m}_2\ \underline{m}_3^*\cdot\underline{m}_3^*) - 4\mathrm{Re}(\underline{m}_1\cdot\underline{m}_3^*\ \underline{m}_2\cdot\underline{m}_3^*)]\end{aligned}$$

$$= \hat{v}_1\ |\underline{m}_1\cdot\underline{m}_1|^2 + \hat{v}_2\ |\underline{m}_2\cdot\underline{m}_2|^2 + 2(\hat{v}_3 + \frac{1}{2}w_3)\ |\underline{m}_3\cdot\underline{m}_3|^2 + \hat{v}_4\ |\underline{m}_1\cdot\underline{m}_3|^2$$

$$+ (\hat{v}_4 - w_2)\ |\underline{m}_1\cdot\underline{m}_3^*|^2 + \hat{v}_5\ |\underline{m}_2\cdot\underline{m}_3|^2 + (\hat{v}_5 - w_2)\ |\underline{m}_2\cdot\underline{m}_3^*|^2 + 2(\hat{v}_6 - w_3)\ \times$$

$$\times\ \mathrm{Re}(\underline{m}_1\cdot\underline{m}_2\ \ \underline{m}_3^*\cdot\underline{m}_3^*) + 4(\hat{v}_6 - \frac{1}{2}w_2 - w_3)\ \mathrm{Re}\,(\underline{m}_1\cdot\underline{m}_3^*\ \ \underline{m}_2\cdot\underline{m}_3^*)$$

$$+ w_3\ (|\underline{m}_1\cdot\underline{m}_2|^2 + |\underline{m}_1\cdot\underline{m}_2^*|^2) \tag{4.11}$$

The Landau expansion (4.8)-(4.11) contains a number of terms with coefficients $\hat{u}$ and $\hat{v}$ which are also present in the absence of a Hubbard interaction. The terms with w_1, w_2, w_3 are extra terms which are due to the Hubbard interaction. The coefficients $\hat{u}$ and $\hat{v}$ in (4.10) and (4.11) are given by the BCS-values plus a shift due to the Hubbard interaction, namely

$$\hat{u}_i = u_i + \frac{2}{3}S\ ,\ i=1,2,3\ ,\ \hat{v}_j = v_j + w_0\ ,\ j=1,2,3,\ \hat{v}_{4,5} = v_{4,5} + 4w_0,\ \hat{v}_6 = v_6 = w_0 \tag{4.12}$$

For reasons of selfcontainedness we list the weak coupling values up to second order in the magnetic field b, as they have been given in ref. 2

$$u_1 = \frac{1}{3}(t+A_c\eta b)\ ,\ u_2 = \frac{1}{3}(t-A_c\eta b)\ ,\ u_3 = \frac{1}{3}(t+2B_c b^2) \tag{4.13}$$

and

$$\begin{aligned} v_1 &= v\ (1+\eta b) & , \quad v_4 &= 4v\ (1+\tfrac{1}{2}\eta b - \tfrac{4}{3}\gamma b^2) \\ v_2 &= v\ (1-\eta b) & , \quad v_5 &= 4v\ (1-\tfrac{1}{2}\eta b - \tfrac{4}{3}\gamma b^2) \\ v_3 &= v\ (1-4\gamma b^2) & , \quad v_6 &= 2v\ (1-\tfrac{2}{3}\gamma b^2) \end{aligned} \tag{4.14}$$

with

$$\gamma = (93/56)\ \frac{\beta^2}{\pi^2}\frac{\zeta(5)}{\zeta(3)}\ ,\ v = \frac{1}{15}\ B_c\ ,\ B_c = \frac{1}{2}\mathcal{N}(0)\ \frac{7}{8}\ \frac{\beta^2}{\pi^2}\ \zeta(3)$$

$$A_c = \frac{1}{2}\mathcal{N}(0)\ \ln\ (1.14\ \beta\hbar\omega)\ , \tag{4.15}$$

and where η is the asymmetry parameter in the density of states $\eta = \mathcal{N}'(0)/\mathcal{N}(0)$ and $t = \frac{1}{2}\mathcal{N}(0)(T-T_c)/T_c$. The coefficients w_0, w_1, w_2, w_3 have been evaluated in the present paper up to second order in the coupling constant I of the Hubbard interaction and for b=0, cf. eqs. (4.7), (3.23)-(3.30), (3.17), (3.18), (3.22). In the general case the integrations over the energies in (3.23)-(3.30) are hard to perform. In the weak-coupling limit, when the integrations are restricted to a small interval $|\varepsilon| < \hbar\omega$ around the Fermi energy, substantial simplifications occur. First of all the functions H_{ab} (k_1, k_2, k_3, k_4) and H_{abcd} (k_1, k_2, k_3, k_4) can be taken to be independent of k_r and hence independent of $\varepsilon_1,\ldots,\varepsilon_4$ in that interval, and may therefore be taken outside the integrations over the energies. The numerical values of the quantities H can be inferred from (3.17), (3.18), and they are

$$
\begin{aligned}
H_{11} &= -\frac{3}{2}\frac{1}{\pi}\left(\frac{2\pi}{k_F}\right)^3 , \quad H_{12} = \frac{1}{2}\frac{1}{\pi}\left(\frac{2\pi}{k_F}\right)^3 \\
H_{1234} &= -\frac{7}{10}\frac{1}{\pi}\left(\frac{2\pi}{k_F}\right)^3 \\
H_{1123} &= \frac{3}{5}\frac{1}{\pi}\left(\frac{2\pi}{k_F}\right)^3 , \quad H'_{1123} = -\frac{1}{10}\frac{1}{\pi}\left(\frac{2\pi}{k_F}\right)^3 \\
H_{1122} &= -\frac{6}{5}\frac{1}{\pi}\left(\frac{2\pi}{k_F}\right)^3 , \quad H'_{1122} = -\frac{3}{10}\frac{1}{\pi}\left(\frac{2\pi}{k_F}\right)^3 \\
H_{1112} &= \frac{1}{2}\frac{1}{\pi}\left(\frac{2\pi}{k_F}\right)^3 \\
H_{1111} &= -\frac{3}{2}\frac{1}{\pi}\left(\frac{2\pi}{k_F}\right)^3 .
\end{aligned}
\tag{4.16}
$$

With regard to the coefficients of the terms in the Landau expension, we restrict ourselves to those which are related to the terms which may lead to qualitative changes in the phase diagram, i.e. w_1, w_2, w_3, or equivalently P_2, ..., P_7. Assuming the density of mass $\mathcal{N}(\varepsilon)$ to be constant and equal to $\mathcal{N}(0)$, (ignoring effects of the asymmetry parameter η in the Hubbard terms), we have

$$K_1\ (\beta) = K_2\ (\beta) = 0 \tag{4.17a}$$

$$P_2 = \tfrac{1}{2}\, I^2\ (2H_{1123}\ J_1 - H_{1122}\ J_2) \tag{4.17b}$$

$$P_3 = \tfrac{1}{2}\, I^2\ (2H'_{1123}\ J_1 - H'_{1122}\ J_2) \tag{4.17c}$$

$$P_4 = -\tfrac{1}{2} I^2\ \frac{1}{4}\ H_{1234}\ J_3 \tag{4.17d}$$

$$P_5 = -\frac{1}{2} I^2 H_{1122} J_2 \tag{4.17e}$$

$$P_6 = -\frac{1}{2} I^2 H'_{1122} J_2 \quad , \quad P_7 = 0 \ , \tag{4.17f}$$

where

$$J_1 = \int_0^\beta d\tau\, K(\tau)\, K_1(\tau)\, L(\tau)^2 \quad , \quad J_2 = \int_0^\beta d\tau\, K_1(\tau)^2\, K(\tau)^2$$

$$J_3 = \int_0^\beta d\tau\, L(\tau)^4 \quad , \quad L(\tau) \equiv \int d\varepsilon\, \tfrac{1}{2}\mathcal{N}(\varepsilon)\, \underline{D}(\varepsilon,\tau) \ , \tag{4.18}$$

cf. also eqs. (3.22). In the weak coupling limit $\beta\hbar\omega \to \infty$ the numerical values of J_1, J_2, J_3 are given by

$$J_1 = -5.1\, [\tfrac{1}{2}\mathcal{N}(0)]^4 \beta \ , \ J_2 = 1.7\, [\tfrac{1}{2}\mathcal{N}(0)]^4 \beta \ , \ J_3 = 30.5\, [\tfrac{1}{2}\mathcal{N}(0)]^4 \beta . \tag{4.19}$$

From (4.17), (4.7), (4.19) and (4.16) we obtain the following estimates

$$\begin{aligned} w_1 &= \frac{4}{15}(3.4) \ . \ \beta(2\pi I)^2 k_F^{-3} (\tfrac{1}{2}\mathcal{N}(0))^4 \\ w_2 &= -\frac{4}{15}(13.6) \ . \ \beta(2\pi I)^2 k_F^{-3} (\tfrac{1}{2}\mathcal{N}(0))^4 \\ w_3 &= \frac{4}{15}(25.4) \ . \ \beta(2\pi I)^2 k_F^{-3} (\tfrac{1}{2}\mathcal{N}(0))^4 \end{aligned} \tag{4.20}$$

for the coefficients of the new terms in the Landau expansion arising from the Hubbard interaction.

5. Conclusions

In this paper we have calculated the Landau expansion for a system described by the hamiltonian (1.1), (1.3) containing a $\ell=1$ pairing potential which is appropriate for liquid ^{3}He with an additional contact term (1.4) of the Hubbard type. The Landau expansion in terms of the 18 real order parameters, or in terms of 3 complex vectors $\underline{m}_1$, $\underline{m}_2$, $\underline{m}_3$ has been given in eqs. (4.8)-(4.11). The coefficients of the Landau expansion, cf. (4.12), consist of contributions with coefficients u_1, u_2, u_3, $v_1, \ldots v_6$, which are also present in the absence of the Hubbard term and which have been given explicity in (4.13), (4.14), as well as contributions S, w_0, w_1, w_2, w_3 arising from the Hubbard term. Formal expressions up to the order I^2 in the coupling constant I of the Hubbard hamiltonian, in terms of integrals over the energy in a small interval $|\varepsilon| < \hbar\omega$ around the Fermi energy can be inferred from (4.7), (4.22)-(3.30), (3.17), (3.18), (3.21), (3.2) and (3.8)-(3.11). The coefficients S and w_0 give only rise to a shift in the BCS-coefficient u and v, cf. eq. (4.12). The other coefficients w_1, w_2 and w_3 have been calculated explicitly in the weak coupling limit and the results have been given in eq. (4.20).

In connection with this remark it may be of interest to make a comparison with the results which have been obtained in the context of the spin fluctuation model. The generalised Landau expansion which is based on general symmetry considerations [29] but does not take into account specific features of the Hubbard hamiltonian has the form

$$f = \alpha\, \mathrm{tr}(\underline{\underline{A}}\cdot\underline{\underline{A}}^{\dagger}) + \beta_1\, |\mathrm{tr}(\underline{\underline{A}}\cdot\tilde{\underline{\underline{A}}})|^2 + \beta_2\, (\mathrm{tr}(\underline{\underline{A}}\cdot\underline{\underline{A}}^{\dagger}))^2$$
$$+ \beta_3\, \mathrm{tr}(\underline{\underline{A}}^{\dagger}\cdot\underline{\underline{A}}\cdot(\underline{\underline{A}}^{\dagger}\cdot\underline{\underline{A}})^{*}) + \beta_4\, \mathrm{tr}((\underline{\underline{A}}\cdot\underline{\underline{A}}^{\dagger})^2) + \beta_5\, \mathrm{tr}(\tilde{\underline{\underline{A}}}\cdot\underline{\underline{A}}\cdot(\tilde{\underline{\underline{A}}}\cdot\underline{\underline{A}})^{*}), \qquad (5.1)$$

in which the 3×3 order matrices $\underline{\underline{A}}$ are defined by

$$m^i = i \sum_{\alpha=x,y,z} A_{\alpha i}\, \underline{\underline{\sigma}}^{\alpha}\cdot\underline{\underline{\sigma}}^{y}, \quad \tilde{\underline{\underline{A}}} = \sqrt{2}\left(-\tfrac{1}{2}\,(\underline{m}_1 - \underline{m}_2),\ -\tfrac{i}{2}\,(\underline{m}_1 + \underline{m}_2),\ \underline{m}_3\right) \qquad (5.2)$$

eq. (5.1) can be rewritten as

$$f = \tfrac{1}{2}\alpha \sum_{i=1}^{3} \mathrm{tr}(\underline{\underline{m}}^{i}\cdot\underline{\underline{m}}^{i\dagger}) + \tfrac{1}{4}\beta_3 \sum_{i,j=1}^{3} \mathrm{tr}(\underline{\underline{m}}^{i}\cdot\underline{\underline{m}}^{i\dagger}\cdot\underline{\underline{m}}^{j}\cdot\underline{\underline{m}}^{j\dagger} + \underline{\underline{m}}^{i}\cdot\underline{\underline{m}}^{j\dagger}\cdot\underline{\underline{m}}^{i}\cdot\underline{\underline{m}}^{j\dagger}$$
$$+ \underline{\underline{m}}^{i}\cdot\underline{\underline{m}}^{j\dagger}\cdot\underline{\underline{m}}^{j}\cdot\underline{\underline{m}}^{i\dagger})$$
$$+ \tfrac{1}{4}(\beta_4 + \beta_5) \sum_{i,j=1}^{3} \mathrm{tr}(\underline{\underline{m}}^{i\dagger}\cdot\underline{\underline{m}}^{i}\cdot\underline{\underline{\sigma}}^{y}\cdot\underline{\underline{m}}^{j}\cdot\underline{\underline{m}}^{j\dagger}\cdot\underline{\underline{\sigma}}^{y} + \underline{\underline{m}}^{i\dagger}\cdot\underline{\underline{m}}^{j}\cdot\underline{\underline{\sigma}}^{y}\cdot\underline{\underline{m}}^{i}\cdot\underline{\underline{m}}^{j\dagger}\cdot\underline{\underline{\sigma}}^{y}$$
$$+ \underline{\underline{m}}^{i\dagger}\cdot\underline{\underline{m}}^{j}\cdot\underline{\underline{\sigma}}^{y}\cdot\underline{\underline{m}}^{j}\cdot\underline{\underline{m}}^{i\dagger}\cdot\underline{\underline{\sigma}}^{y})$$
$$+ \tfrac{1}{4}(\beta_2 - \beta_4) \sum_{i,j=1}^{3} \mathrm{tr}(\underline{\underline{m}}^{i\dagger}\cdot\underline{\underline{m}}^{i}\cdot\underline{\underline{\sigma}}^{y}\cdot\underline{\underline{m}}^{j}\cdot\underline{\underline{m}}^{j\dagger}\cdot\underline{\underline{\sigma}}^{y})$$
$$+ \tfrac{1}{4}(\beta_2 + \beta_4 - 2\beta_3) \sum_{i,j=1}^{3} \mathrm{tr}(\underline{\underline{m}}^{i}\cdot\underline{\underline{m}}^{i\dagger}\cdot\underline{\underline{m}}^{j}\cdot\underline{\underline{m}}^{j\dagger})$$
$$+ \tfrac{1}{4}(\beta_1 - \tfrac{1}{2}\beta_5 - \tfrac{1}{2}\beta_4 + \tfrac{1}{2}\beta_3) \left|\sum_{i=1}^{3} \mathrm{tr}(\underline{\underline{m}}^{i}\cdot\underline{\underline{\sigma}}^{y}\cdot\underline{\underline{m}}^{i}\cdot\underline{\underline{\sigma}}^{y})\right|^2 \qquad (5.3)$$

In comparing with eq. (5-3) with eq. (3.32) with the matrices $\underline{\underline{h}}^i$ replaced by $\underline{\underline{m}}^i$, cf. eq. (4.1), we can identify

$$\alpha = \frac{2}{3}\,S,\ \beta_3 = \frac{4}{15}\,(P_1+P_6) = w_0,\ \beta_4+\beta_5 = \frac{4}{15}\,(P_3+2P_4+P_6) = \frac{1}{2}\,w_3\ ,$$

$$2(\beta_3-\beta_4) = \frac{4}{9}\,(P_2-P_7) = \frac{1}{2}\,w_2,\ \ \beta_2+\beta_4-2\beta_3 = \frac{4}{9}\,(P_5+P_7) = w_1\ , \tag{5.4}$$

and due to the absence of the last term of eq. (5.3) we also obtain the relation

$$\beta_1 + \frac{1}{2}\,\beta_3 = \frac{1}{2}\,(\beta_4+\beta_5), \tag{5.5}$$

Eq.(5.5) implies in particular that the coefficients in front of the terms $|\underline{m}_1\cdot\underline{m}_2|^2$ and $|\underline{m}_1\cdot\underline{m}_2^*|^2$ in the Landau expansion are equal, so that the ABM phase and the planar phase have equal free energies, cf. ref. 37. The standard results for the coefficients $\beta_1,\ ..\ ,\ \beta_5$, as obtained within the spin fluctuation model, i.e. $\beta_2=-2.0\beta_1$, $\beta_3=0.5\beta_1$, $\beta_4=5.5\beta_1$, $\beta_5=7.0\beta_1$, where $\beta_1=-0.05\nu\delta$, are quite different from (5.4) and (4.20) and do not agree with eq. (5.5), and this will remain to be the case when only quadratic contributions in I are taken into account in the β's. In the absence of explicit results not much can be concluded about higher orders of perturbations, but preliminary estimates which we shall not give in more detail rais some doubts about the validty of (5.5) already in third order perturbation calculation.

In a following paper [43] the Landau expansion (4.8)-(4.11) will be used to investigate proporties of the phase diagram of liquid ^{3}He, as described by the hamiltonian (1.1), (1.3) including the Hubbard term (1.4). Although it is not really possible to do a complete and rigorous minimization with respect to the 3 complex vectors, $\underline{m}_1$, $\underline{m}_2$, $\underline{m}_3$ some interesting new features will be described, based on the results for the coefficients w. In particular it turns out that the ABM phase does not occur in zero magnetic field and that the phase trancitions at the polycritical point changes the BW phase in a more complicated phase. Moreover it turns out that the global features of the phase diagram are mainly determined by the signs of the parameters , but not so much by the detailed numerical values. In this respect the values given in (4.20) may serve as a guideline for qualitative considerations, as it does not seem likely that the higher orders of perturbation in I will change the signs of these coefficients.

APPENDIX A

In this appendix we derive eq. (2.16),(2.17) for the contractions between the Nambu operators $\underline{\Psi}(\underline{x},\tau)$. For the Nambu operators in $\underline{k}$-space (2.3) we have the following immaginary time evolution, cf. eqs. (2.15) and (2.2),

$$\frac{d}{d\tau}\, c_{\underline{k}p}(\tau) = [\mathcal{H}_{0,r}, c_{\underline{k}p}(\tau)] = -\sum_{\underline{k}>0} D_{\underline{k}pp'}\, c_{\underline{k}p'}(\tau), \tag{A1}$$

where we have used the anticommutation relations (2.4). From (A1) we immediately have

$$\underline{c}_{\underline{k}}(\tau) = e^{-\underline{\underline{D}}_{\underline{k}}\tau}\cdot \underline{c}_{\underline{k}}. \tag{A2}$$

From the anticommutation relations (2.4) together with the invariance of the trace under cyclic permutations, we have

$$\begin{aligned}\delta_{\underline{k},\underline{k}'}\,\delta_{p,p'} &= \langle c_{\underline{k}p}\, c^{\dagger}_{\underline{k}'p'}\rangle + \langle c^{\dagger}_{\underline{k}'p'}\, c_{\underline{k}p}\rangle \\ &= \langle c_{\underline{k}p}\, c^{\dagger}_{\underline{k}'p'}\rangle + \langle c_{\underline{k}p}(\beta)\, c^{\dagger}_{\underline{k}'p'}\rangle \\ &= [(\underline{\underline{1}}^{(4)} + e^{-\beta\,\underline{\underline{D}}_{\underline{k}}})\cdot\langle \underline{c}_{\underline{k}}\, \underline{c}^{\dagger}_{\underline{k}'}\rangle]_{pp'}\;,\quad \underline{k},\underline{k}'>0,\end{aligned}$$

leading to the following expression for the pair contractions

$$\langle \underline{c}_{\underline{k}}(\tau)\, \underline{c}^{\dagger}_{\underline{k}'}\rangle = \frac{e^{-\tau\,\underline{\underline{D}}_{\underline{k}}}}{\underline{\underline{1}}^{(4)} + e^{-\beta\,\underline{\underline{D}}_{\underline{k}}}}\,\delta_{\underline{k},\underline{k}'}\quad,\quad \underline{k},\underline{k}'>0. \tag{A3}$$

Similarly we have

$$\langle \underline{c}_{\underline{k}}(\tau)\, \tilde{\underline{c}}_{\underline{k}'}\rangle = \langle \tilde{\underline{c}}^{\dagger}_{\underline{k}}(\tau)\, \underline{c}^{\dagger}_{\underline{k}'}\rangle = 0\;,\; \underline{k},\underline{k}'>0. \tag{A4}$$

In order to calculate the contractions (2.16) we rewrite eq. (2.10) as follows, cf. (2.12),

$$\underline{\Psi}(\underline{x},\tau) = \frac{1}{\sqrt{\Omega}}\sum_{\underline{k}>0}\left(e^{i\,\underline{k}\cdot\underline{x}}\,\underline{c}_{\underline{k}}(\tau) + e^{-i\,\underline{k}\cdot\underline{x}}\,\underline{\underline{J}}\cdot\tilde{\underline{c}}^{\dagger}_{\underline{k}}(\tau)\right)\;, \tag{A5}$$

Using (A5) we find

$$<\underline{\Psi}(\underline{x},\tau)\ \underline{\Psi}^{\dagger}(\underline{x}')> = \frac{1}{\Omega}\sum_{\underline{k}>0}\ \sum_{\underline{k}'>0}\ [e^{i(\underline{k}\cdot\underline{x}\ -\ \underline{k}'\cdot\underline{x}')} <c_{\underline{k}}\ (\tau)\ c^{\dagger}_{\underline{k}'}> +$$

$$+\ e^{i(\underline{k}\cdot\underline{x}\ +\ \underline{k}'\cdot\underline{x}')}<c_{\underline{k}}\ (\tau)\ \tilde{c}_{\underline{k}'}>\cdot\underline{\underline{J}} + e^{-i(\underline{k}\cdot\underline{x}\ +\ \underline{k}'\cdot\underline{x}')}<\tilde{c}^{\dagger}_{\underline{k}}\ (\tau)\ c^{\dagger}_{\underline{k}'}> +$$

$$+\ e^{\ i(\underline{k}\cdot\underline{x}\ -\ \underline{k}'\cdot\underline{x}')}\ \underline{\underline{J}}\cdot<\tilde{c}^{\dagger}_{-\underline{k}}\ (\tau)\ \tilde{c}_{-\underline{k}'}>\cdot\underline{\underline{J}}] \qquad (A.6)$$

From (A4) we see that the terms with $<c_{\underline{k}}\ (\tau)\ \tilde{c}_{\underline{k}'}>$ and $<\tilde{c}^{\dagger}_{\underline{k}}\ (\tau)\ c^{\dagger}_{\underline{k}'}>$ vanish. For the last term on the right hand side in (A6) we use the identity

$$<\tilde{c}^{\dagger}_{-\underline{k}}\ (\tau)\ \tilde{c}_{-\underline{k}'}> = \widetilde{<c_{-\underline{k}'}\ (\beta-\tau)\ c_{-\underline{k}}>}$$

$$= \frac{e^{-(\beta-\tau)\ \tilde{\underline{\underline{D}}}_{\underline{k}'}}}{\underline{\underline{1}}^{(4)} + e^{-\beta\ \tilde{\underline{\underline{D}}}_{\underline{k}'}}}\ \delta_{\underline{k},\underline{k}'} = \underline{\underline{J}}\cdot \frac{e^{(\beta-\tau)\ \underline{\underline{D}}_{-\underline{k}}}}{\underline{\underline{1}}^{(4)} + e^{\beta\ \underline{\underline{D}}_{-\underline{k}}}}\cdot\underline{\underline{J}}\ \delta_{\underline{k},\underline{k}'}, \qquad (A7)$$

which follows form (2.20). Inserting eq. (A7) and (A4) in (A6) we find

$$<\underline{\Psi}(\underline{x},\tau)\ \underline{\Psi}^{\dagger}(\underline{x}')> \frac{1}{\Omega}\sum_{\underline{k}>0}\left[e^{i\cdot\underline{k}\cdot(\underline{x}-\underline{x}')}\frac{e^{-\tau\ \underline{\underline{D}}_{\underline{k}}}}{\underline{\underline{1}}^{(4)} + e^{-\beta\ \underline{\underline{D}}_{\underline{k}}}} + e^{-i\cdot\underline{k}\cdot(\underline{x}-\underline{x}')}\frac{e^{-\tau\underline{\underline{D}}_{-\underline{k}}}}{\underline{\underline{1}}^{(4)} + e^{-\beta\ \underline{\underline{D}}_{-\underline{k}}}}\right] \qquad (A8)$$

which is identical to eqs.(2.16), (2.17).

APPENDIX B

Some details concerning the derivation of eq.(2.23) are given in this appendix. From eq. (2.13) and (2.8) we have

$$f_r(\{\underline{\underline{h}}^i\}) - f_{0,r}(\{\underline{\underline{h}}^i\}) = \lim_{\Omega\to\infty} \frac{1}{\Omega}\Big\{\frac{1}{4}\, I \int d\underline{x} \sum_{p_1,p_2,p_3,p_4} O_{p_1p_2}\, \tilde{O}_{p_3p_4} \langle p_1p_2p_3p_4\rangle$$
$$- \frac{1}{2}\left(\frac{1}{4}\, I\right)^2 \int d\underline{x} \int d\underline{x}' \int_0^\beta d\tau \sum_{p_1,\dots,p_8} O_{p_1p_2}\, \tilde{O}_{p_3p_4}\, O_{p_5p_6}\, \tilde{O}_{p_7p_8} \times$$
$$\left[\langle p_1p_2p_3p_4p_5p_6p_7p_8\rangle - \langle p_1p_2p_3p_4\rangle\langle p_5p_6p_7p_8\rangle\right]\Big\} \qquad \text{(B1)}$$

up to second order terms in I. Here p_i is a short hand notation for the 4 components of the ith Nambu operator with the proper arguments, counted form the left to de right as it occurrs in the brackets in (2.13). Using the thermodynamic Wick theorem [44-46] the brackets can be expressed as Pfaffians with pair contractions as elements. For the first order term we have

$$\langle p_1p_2p_3p_4\rangle = \langle p_1p_2\rangle\,\langle p_3p_4\rangle - \langle p_1p_3\rangle\,\langle p_2p_4\rangle + \langle p_1p_4\rangle\,\langle p_2p_3\rangle\ . \qquad \text{(B2)}$$

The bracket $\langle p_1 \dots p_8\rangle$ in the second order term in principle gives rise to 105 products of contractions, but there is a great deal of simplification due to the specific form of the Hubbard interaction.

Firstly, it is easy to show that all products containing a contraction of the form $\langle p_{2i-1}\, p_{2i}\rangle$ vanish. In fact, using the relation $\underline{\underline{J}}\cdot\underline{\underline{Q}}\cdot\underline{\underline{J}} = -\,\tilde{\underline{\underline{Q}}}$ for $\underline{\underline{Q}} = \underline{\underline{O}},\tilde{\underline{\underline{O}}}$ we have,

$$\sum_{p_{2i-1},p_{2i}} Q_{p_{2i-1}\,p_{2i}} \langle p_{2i-1}\, p_{2i}\rangle = \frac{1}{\Omega} \sum_{\underline{k}} \mathrm{tr}^{(4)}\, \{\underline{\underline{J}}\cdot\underline{\underline{G}}\,(\underline{k},0)\cdot\underline{\underline{J}}\cdot\tilde{\underline{\underline{Q}}}\}$$
$$= -\frac{1}{\Omega} \sum_{\underline{k}} \mathrm{tr}^{(4)}\, \{\underline{\underline{G}}\,(\underline{k},0)\cdot\underline{\underline{Q}}\} \qquad \text{(B3)}$$

From eq. (2.17) and (2.5) it follows that the off diagonal part of $\underline{\underline{G}}\,(\underline{k},0)$ contains only odd powers of $\underline{\underline{\Delta}},\underline{\underline{\Delta}}^+$, and hence,with eq. (2.6),we have in (B3) an odd function of the vector $\underline{k}$, which vanishes upon summation.

Secondly, in the Hubbard hamiltonian

$$\mathcal{H}_I = \frac{1}{4}\, I \int d\underline{x}\;\; A(\underline{x})\; B(\underline{x})\ ,$$

$$A(\underline{x}) = \underline{\Psi}^{\dagger}(\underline{x})\cdot\underline{\underline{Q}}\cdot\underline{\Psi}(\underline{x}) \quad , \qquad B(\underline{x}) = \underline{\Psi}^{\dagger}(\underline{x})\cdot\tilde{\underline{\underline{Q}}}\cdot\underline{\Psi}(\underline{x}) \; , \tag{B4}$$

We have the prperty that both $A(\underline{x})$ and $B(\underline{x})$ are invariant under the interchange of the two Nambu-operators. This follows as a special case from a symmetry property of the quantity $\underline{\Psi}^{\dagger}_{a}\cdot\underline{\underline{Q}}\cdot\underline{\Psi}_{b}$, where $\underline{\Psi}_a \equiv \underline{\Psi}(\underline{x}_a, \tau_a)$, $\underline{\Psi}_b \equiv \underline{\Psi}(\underline{x}_b, \tau_b)$ are two Nambu operators and $\underline{\underline{Q}} = \underline{\underline{Q}}, \tilde{\underline{\underline{Q}}}$. In fact, using eq. (2.12) in combination with the anticommuncation relations (2.11) and the property $\mathrm{tr}^{(4)}\,(\underline{\underline{J}}\cdot\underline{\underline{Q}}\cdot\underline{\underline{J}}) = 0$, we have

$$\underline{\Psi}_a^{\dagger}\cdot\underline{\underline{Q}}\cdot\underline{\Psi}_b = \tilde{\underline{\Psi}}_a\cdot\underline{\underline{J}}\cdot\underline{\underline{Q}}\cdot\underline{\underline{J}}\cdot\tilde{\underline{\Psi}}_b^{\dagger} = -\underline{\Psi}_b^{\dagger}\cdot\underline{\underline{J}}\cdot\underline{\underline{Q}}\cdot\underline{\underline{J}}\cdot\underline{\Psi}_a = \underline{\Psi}_b^{\dagger}\cdot\underline{\underline{Q}}\cdot\underline{\Psi}_a . \tag{B5}$$

In the special case that $\underline{\Psi}_a$ and $\underline{\Psi}_b$ are the two Nambu operators involved in a factor $A(\underline{x})$ or $B(\underline{x})$ we have a=b, and hence eq. (B5) implies that it does not matter which of the two Nambu-operators contained in an operator A or B is chosen in a confraction. If we omit the products of contractions which contain self contractions of the form

$$A \qquad \text{or} \qquad B \; . \tag{B6}$$

and which vanish according to eq. (B3), and also disconnected products of contractions, namely

$$A(\underline{x},\tau)\; B(\underline{x},\tau) \quad A(\underline{x}')\; B(\underline{x}') \; , \tag{B7}$$

which are cancelled by the second term $\langle p_1 \ldots p_4\rangle\langle p_5 \ldots p_8\rangle$ in eq. (B1), we are left with 56 products of contractions, which can be subdivided into 5 different classes of equal products of contractions. Schematically these 5 types of decompositions can be represented by

$$A(\underline{x},\tau)\; B(\underline{x},\tau)\; A(\underline{x}')\; B(\underline{x}') \qquad\qquad A(\underline{x},\tau)\; B(\underline{x},\tau)\; A(\underline{x}')\; B(\underline{x}')$$

$$A(\underline{x},\tau)\; B(\underline{x},\tau)\; A(\underline{x}')\; B(\underline{x}') \qquad\qquad A(\underline{x},\tau)\; B(\underline{x},\tau)\; A(\underline{x}')\; B(\underline{x}')$$

$$A(\underline{x},\tau)\; B(\underline{x},\tau)\; A(\underline{x}')\; B(\underline{x}'). \tag{B8}$$

Summarizing, we conclude that the 56 nonvanishing terms contributing to the second order term in eq. (B1) can be divided into

a) 16 equal terms $\langle p_{1+i}\, p_{3+j}\rangle\ \langle p_{2-i}\, p_{5+k}\rangle\ \langle p_{4-j}\, p_{7+l}\rangle\ \langle p_{6-k}\, p_{8-l}\rangle\ (-1)^{i+j+k+l+1}$

b) 16 equal terms $\langle p_{1+i}\, p_{3+j}\rangle\ \langle p_{2-i}\, p_{7+l}\rangle\ \langle p_{4-j}\, p_{5+k}\rangle\ \langle p_{6-k}\, p_{8-l}\rangle\ (-1)^{i+j+k+l}$

c) 16 equal terms $\langle p_{1+i}\, p_{5+k}\rangle\ \langle p_{2-i}\, p_{7+l}\rangle\ \langle p_{3+j}\, p_{6-k}\rangle\ \langle p_{4-j}\, p_{8-l}\rangle\ (-1)^{i+j+k+l+1}$

d) 4 equal terms $\langle p_1 p_{5+k}\rangle\ \langle p_2 p_{6-k}\rangle\ \langle p_3 p_{7+l}\rangle\ \langle p_4 p_{8-l}\rangle\ (-1)^{k+l}$

e) 4 equal terms $\langle p_1 p_{7+l}\rangle\ \langle p_2 p_{8-l}\rangle\ \langle p_3 p_{5+k}\rangle\ \langle p_4 p_{6-k}\rangle\ (-1)^{k+l}$,

$i,j,k,l = 0,1$. Hence we can choose from each of the classes (a)-(e) one specific product of contractions, so that eq. (B1) reduces to

$$f_r\,(\{\underline{\underline{h}}^i\}) - f_{0,r}\,(\{\underline{\underline{h}}^i\}) = \lim_{\Omega\to\infty} \frac{1}{\Omega}\Bigg[\frac{1}{2}\, I \int d\underline{x} \sum_{p_1,\dots,p_4} O_{p_1p_2}\, \tilde{O}_{p_3p_4}\, \langle p_1p_4\rangle\, \langle p_2p_3\rangle$$

$$- \frac{1}{2}\, I^2 \int_0^\beta d\tau \int d\underline{x} \int d\underline{x}' \sum_{p_1,\dots,p_8} O_{p_1p_2}\, \tilde{O}_{p_3p_4}\, O_{p_5p_6}\, \tilde{O}_{p_7p_8} \times$$

$$\times\, [\,-\langle p_1p_3\rangle\, \langle p_2p_5\rangle\, \langle p_4p_7\rangle\, \langle p_6p_8\rangle + \langle p_1p_3\rangle\, \langle p_2p_7\rangle\, \langle p_4p_5\rangle\, \langle p_6p_8\rangle$$

$$-\langle p_1p_5\rangle\, \langle p_2p_7\rangle\, \langle p_3p_6\rangle\, \langle p_4p_8\rangle + \frac{1}{4}\langle p_1p_5\rangle\, \langle p_2p_6\rangle\, \langle p_3p_7\rangle\, \langle p_4p_8\rangle$$

$$+\frac{1}{4}\langle p_1p_7\rangle\, \langle p_2p_8\rangle\, \langle p_3p_5\rangle\, \langle p_4p_6\rangle]\Bigg]\,. \tag{B9}$$

From (B9) taking into account eq. (2.13) and (2.8) to identify the different Nambu operators, one can derive eq. (2.23) using (2.16), (2.18) and (2.21) for the pair contractions and (2.22) for the integrations over $\underline{x}$ and $\underline{x}'$.

APPENDIX C

In this appendix we calculate the integrals as given in eqs. (3.14), (3.15) and the quantities H, as given in eqs. (3.16), (3.17). We use the integral

$$\int \frac{d\Omega}{4\pi} e^{i\underline{k}\cdot\underline{x}} = \frac{\sin kx}{kx} \quad , \tag{C1}$$

where $\underline{x}$ is an arbitrary 3-component vector with length x, and the integrations in (C1) is performed over the solid angle associated with the vector $\underline{k}$, and the representation of the delta function

$$\delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4) = \left(\frac{1}{2\pi}\right)^3 \int d\underline{x}\; e^{i(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4)\cdot\underline{x}} \tag{C2}$$

For an arbitrary function $F(\underline{n}_1, \underline{n}_2, \underline{n}_3, \underline{n}_4)$ of the unit vectors $\underline{n}_r$ in the directions of the vectors $\underline{k}_r$, r=1,2,3,4, we then have

$$\begin{aligned}\left[\prod_{r=1}^{4}\int \frac{d\Omega_r}{4\pi}\right] F(\underline{n}_1, \underline{n}_2, \underline{n}_3, \underline{n}_4)\, \delta(\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4) = \\ = \left(\frac{1}{2\pi}\right)^3 \int d\underline{x}\; F\left(\frac{1}{i}\frac{\partial}{\partial\underline{\xi}_1}, \frac{1}{i}\frac{\partial}{\partial\underline{\xi}_2}, \frac{1}{i}\frac{\partial}{\partial\underline{\xi}_3}, \frac{1}{i}\frac{\partial}{\partial\underline{\xi}_4}\right) \prod_{r=1}^{4} \frac{\sin \xi_r}{\xi_r} \quad ,\end{aligned} \tag{C3}$$

where $\underline{\xi}_r = k_r\underline{x}$, $\xi_r=k_r x$ and Ω_r is the solid angle associated with the vector $\underline{k}_r$. In eq. (C3) we have used the fact that differentations with respect to ξ_r^i give a factor n_r^i in the integrands of the integral over $d\Omega_r$. The integrals over $\underline{x}$ can again be decomposed into an integral over the length x and an integral over the solid angle $d\Omega_x$ associated with the direction of $\underline{x}$. We next apply the differentiations in (C3) to the functions F, occurring in the integrals in eqs.(3.14), (3.15), and express the results in terms of the lengths $\xi_1,\dots,\xi_4$ and the unit vectors $\underline{e} \equiv \underline{x}/x = \underline{\xi}_r/\xi_r$ in the direction of $\underline{x}$. The integration over the solid angle $d\Omega_x$ is then straightforward using the integrals

$$\int\frac{d\Omega_x}{4\pi} e^i e^j = \frac{1}{3}\delta_{ij}, \quad \int\frac{d\Omega_x}{4\pi} e^i e^j e^k e^l = \frac{1}{15}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{C4}$$

yielding combinations of Kronecker delta symbols.(We must take care that there are also combinations of Kronecker deltas arising from multiple differentations in eq. (C3)). The results (3.14), (3.15) are then obtained after a tedious but straightforward calculation, together with the following formulae for the functions H, i.e.

$$H_{11} = -4\pi \int_0^\infty dx\, x^2 \left(3D_1 + \xi_1^2 D_1^2\right) \prod_{r=1}^{4} \xi_r^{-1} \sin \xi_r \tag{C5}$$

$$H_{12} = -4\pi \int_0^\infty dx\, x^2\, \xi_1\xi_2 D_1 D_2 \prod_{r=1}^{4} \xi_r^{-1} \sin \xi_r \tag{C6}$$

$$H_{1234} = 4\pi \int_0^\infty dx\, x^2\, \xi_1\xi_2\xi_3\xi_4\, D_1D_2D_3D_4 \prod_{r=1}^{4} \xi_r^{-1} \sin \xi_r \tag{C7}$$

$$H'_{1123} = 4\pi \int_0^\infty dx\, x^2\, 3\, \xi_2\xi_3\, D_1D_2D_3 \prod_{r=1}^{4} \xi_r^{-1} \sin \xi_r \tag{C8}$$

$$H_{1123} = 4\pi \int_0^\infty dx\, x^2\, \xi_1^2\xi_2\xi_3\, D_1^2D_2D_3 \prod_{r=1}^{4} \xi_r^{-1} \sin \xi_r \tag{C9}$$

$$H_{1122} = 4\pi \int_0^\infty dx\, x^2 \left(9\, D_1D_2 + 3\, \xi_1^2\, D_1^2D_2 + 3\, \xi_2^2\, D_2^2D_1\right) \prod_{r=1}^{4} \xi_r^{-1} \sin \xi_r \tag{C10}$$

$$H'_{1122} = 4\pi \int_0^\infty dx\, x^2\, \xi_1^2\xi_2^2\, D_1^2D_2^2 \prod_{r=1}^{4} \xi_r^{-1} \sin \xi_r \tag{C11}$$

$$H_{1112} = 4\pi \int_0^\infty dx\, x^2 \left(5\, \xi_1\, D_1^2 + \xi_1^3\, D_1^3\right) \xi_2\, D_2 \prod_{r=1}^{4} \xi_r^{-1} \sin \xi_r \tag{C12}$$

$$H_{1111} = 4\pi \int_0^\infty dx\, x^2 \left\{15\, D_1^2 + 10\, \xi_1^2\, D_1^3 + \xi_1^4\, D_1^4\right\} \prod_{r=1}^{4} \xi_r^{-1} \sin \xi_r \tag{C13}$$

in which D_r is the differential operator $D_r = \xi_r^{-1}\,(\partial/\partial\xi_r)$. Using the relation

$$\frac{\partial}{\partial \xi_r}\, g(\xi_r) = \frac{1}{x}\, \partial_r\, g(k_r x)\ ,\ \partial_r \equiv \partial/\partial k_r\ . \tag{C14}$$

It is straightforward to work out the differentiations in (C5)-(C13) and to derive eqs. (3.16), (3.17) in which the functions $h_{2n}(k_1, k_2, k_3, k_4)$ are given by

$$h_{2n}\,(k_1, k_2, k_3, k_4) = 4\pi \int_0^\infty dx\, x^{2-n} \prod_{r=1}^{4} \frac{\sin k_r x}{k_r x}\ , \tag{C15}$$

which can be rewritten as

$$h_{2n}\,(k_1, k_2, k_3, k_4) = 4\pi \sum_{\nu_1,\nu_2,\nu_3,\nu_4=\pm 1} (-1)^{\nu_1+\nu_2+\nu_3+\nu_4} \int_0^\infty dx\, x^{-2-2n}$$

$$\times\ \frac{e^{i(k_1\nu_1+k_2\nu_2+k_3\nu_3+k_4\nu_4)x}}{16\, k_1k_2k_3k_4}\ ,$$

and we decompose the summation into two parts, one for which the quantity $\kappa \equiv \nu_1k_1+\nu_2k_2+\nu_3k_3+\nu_4k_4$ is positive, and one for which κ is negative. We then replace the integrations over x by integrations in the complex z-plane along contours C_+ and C_- for the terms with $\kappa>0$ and $\kappa<0$ respectively, as indicated in figure 1.

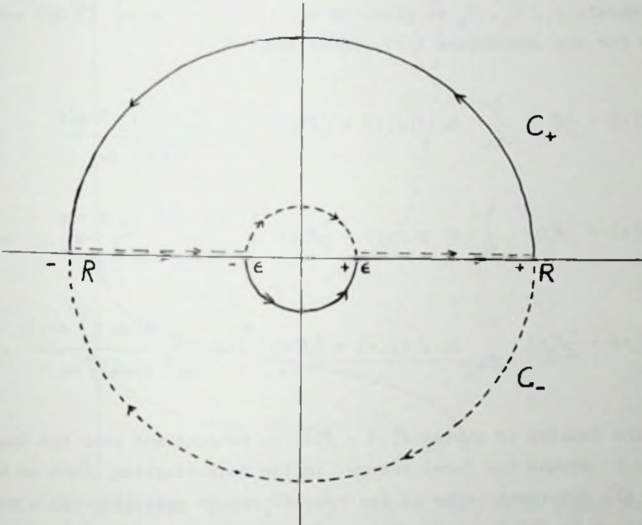

Figure 1. The contours C_+ and C_- in the complex z-plane, over which the integrations in eq. (C.16) are performed. The solid line belongs to C_+, the dashed line belongs to C_-.

Then

$$h_{2n} = 4\pi \sum_{\substack{\{\nu_r=\pm 1\} \\ \kappa>0}} \oint_{C_+} dz \, \frac{e^{i\kappa z} z^{-2-2n}}{16\, k_1k_2k_3k_4} + 4\pi \sum_{\substack{\{\nu_r=\pm 1\} \\ \kappa<0}} \oint_{C_-} dz \, \frac{e^{i\kappa z} z^{-2-2n}}{16\, k_1k_2k_3k_4} \qquad \text{(C16)}$$

in the limit $R\to\infty$, $\varepsilon\to 0$, cf. fig. 1. Here we ignore a singular contribution to h_{2n} arising from the half circle around the origin, since it does not contribute to the quantities H, as a consequence of the differentiations in (3.16), (3.17). Using the relation

$$\oint_{C_+} dz \, \frac{e^{i\kappa z}}{z^m} = -\oint_{C_-} dz \, \frac{e^{i\kappa z}}{z^m} = 2\pi i \, \frac{(i\kappa)^{m-1}}{(m-1)!}$$

eq. (3.19) follows in a straightforward way.

APPENDIX D

In this appendix we calculate the weak-coupling limit for the coefficients w_1, w_2, w_3 of the Hubbard terms in the Landau-expansion. This reduces to evaluating the integrals J_1, J_2, J_3 as given in eq. (4.18). From eq. (3.22) and eq. (4.18) we have for the quantities $K(\tau)$, $K_1(\tau)$ and $L(\tau)$

$$K(\tau) \simeq \frac{1}{2}\mathcal{N}(0) \int_{-\hbar\omega}^{\hbar\omega} d\varepsilon \; f(\varepsilon,\tau) = \frac{1}{2}\mathcal{N}(0) \int_{0}^{\hbar\omega} d\varepsilon \; \frac{\cosh\,(\frac{1}{2}\,\beta-\tau)\varepsilon}{\cosh\frac{1}{2}\,\beta\varepsilon} \quad , \tag{D1}$$

$$L(\tau) \simeq \frac{1}{2}\mathcal{N}(0) \int_{-\hbar\omega}^{\hbar\omega} d\tau \; p(\varepsilon,\tau) = \frac{1}{2}\mathcal{N}(0) \int_{0}^{\hbar\omega} d\varepsilon \; \frac{\sinh\,(\frac{1}{2}\,\beta-\tau)\varepsilon}{\varepsilon\,\cosh\frac{1}{2}\,\beta\varepsilon} \quad , \tag{D2}$$

$$K_1(\tau) \simeq \frac{1}{2}\mathcal{N}(0) \int_{-\hbar\omega}^{\hbar\omega} d\varepsilon \; f_1(\varepsilon,\tau) = \frac{1}{2}\mathcal{N}(0) \int_{0}^{\hbar\omega} d\varepsilon \; \frac{d}{d\varepsilon^2}\, \frac{\cosh\,(\frac{1}{2}\,\beta-\tau)\varepsilon}{\cosh\frac{1}{2}\,\beta\varepsilon} \quad , \tag{D3}$$

taking the density of states $\mathcal{N}(\varepsilon) = \mathcal{N}(0)$ to be constant over the small interval $(-\hbar\omega, \hbar\omega)$ around the fermi energy. In the weak-coupling limit we have to extract all divergent terms in the cut-off $\hbar\omega$ in eqs. (D1)-(D3), and take the limit $\beta\hbar\omega \to \infty$ in the remaining terms. Thus we have e.g.

$$K(\tau) = \frac{1}{2}\mathcal{N}(0)\,\beta \int_0^{\beta\hbar\omega} dx \left(e^{-(1-\frac{\tau}{\beta})x} + e^{-\frac{\tau}{\beta}x}\right)\left(1 + \sum_{n=1}^{\infty} (-1)^n\, e^{-nx}\right)$$

$$= \frac{1}{2}\mathcal{N}(0) \left[\frac{1-e^{-\tau\hbar\omega}}{\tau} + \frac{1-e^{-(\beta-\tau)\hbar\omega}}{\beta-\tau} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{\beta n+\tau} + \frac{1}{\beta(n+1)-\tau}\right)\right]$$

$$= \frac{1}{2}\mathcal{N}(0) \left[- \frac{e^{-\tau\hbar\omega}}{\tau} - \frac{e^{-(\beta-\tau)\hbar\omega}}{\beta-\tau} + \frac{\pi}{\beta}\,\mathrm{cosec}\,\frac{\pi\tau}{\beta}\right]. \tag{D4}$$

In a similar manner we have

$$L(\tau) = \int_{\tau}^{\frac{1}{2}\beta} d\tau'\; K(\tau') = -\frac{1}{2}\mathcal{N}(0)\,\ln\tan\frac{\pi\tau}{2\beta} + \mathrm{Ei}(-\tau\hbar\omega) - \mathrm{Ei}\big((\tau-\beta)\hbar\omega\big)$$

$$= \begin{cases} -\frac{1}{2}\mathcal{N}(0)\,\ln\tan\frac{\pi\tau}{2\beta}\;, & \text{for } \tau\hbar\omega \gg 1 \\[2ex] \frac{1}{2}\mathcal{N}(0)\,\ln\,(1.14\,\beta\hbar\omega)\;, & \text{for } \tau\hbar\omega \ll 1 \end{cases} \tag{D5}$$

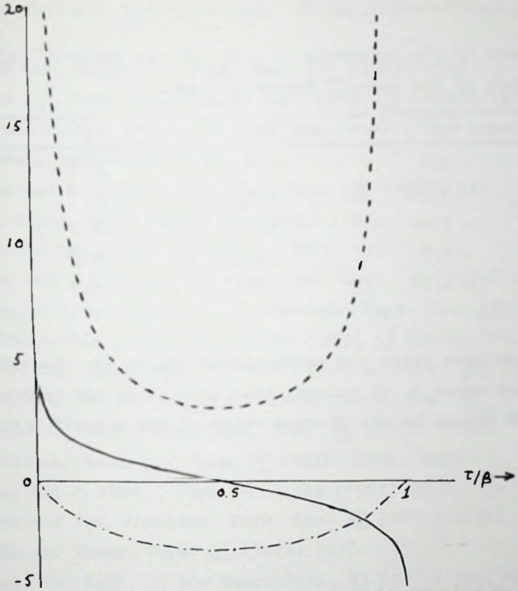

Figure 2. The functions $\beta K(\tau)$, $L(\tau)$ *and* $0.1\beta K_1(\tau)$ *as functions of* τ *in units* $\frac{1}{2}\mathcal{N}(0)$, *and for fixed values of the cut-off* $\beta\hbar\omega$, $\beta\hbar\omega = 10^3$. *The dashed line indicates* $K(\tau)$, *the solid line* $L(\tau)$ *and dashed-dotted line* $K_1(\tau)$.

and

$$K_1(\tau) = -\frac{1}{2}\int_0^\tau d\tau'\ L(\tau') + \frac{1}{2}\mathcal{N}(0)\left(\frac{\cosh\left(\frac{1}{2}\beta-\tau\right)\hbar\omega}{\cosh\frac{1}{2}\beta\hbar\omega} - 1\right)\frac{1}{2\hbar\omega} \qquad \text{(D6)}$$

The weak-coupling results (D4)-(D6) are plotted in fig. 2 for a fixed value of the cut-off $\hbar\omega$. Using these results for the functions $K(\tau)$, $K_1(\tau)$ and $L(\tau)$ the integrations in eq. (4.18) have been performed nummerically, and the results for several values of this cut-off $\beta\hbar\omega$ are given in table 1. There is only a slight dependence on the cut-off parameter in the integrals over τ, and we can extrapolate to the values given in eq. (4.19).

Table 1. Values of the integrals J_1, J_2, J_3 in units of $\beta(\frac{1}{2}\mathcal{N}(0))^4$ for different values of the cut-off parameter $\beta\hbar\omega$.

$\beta\hbar\omega$	J_1	J_2	J_3
10^6	- 5.1	1.7	30.4
10^5	- 5.1	1.7	30.3
10^4	- 5.0	1.7	29.9
10^3	- 4.8	1.7	28.0
10^2	- 4.1	1.7	21.1

In eq. (4.19) we have taken the extrapolated values for the weak-coupling limit $\beta\hbar\omega \to \infty$ but from table 1 it follows that especially the largest parameter J_3 depends to some extent on the precise value of the cut-off $\beta\hbar\omega$.

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CHAPTER V

ANALYSIS OF PHASES IN THE PRESENCE OF A HUBBARD INTERACTION

1. Introduction

The derivation of the phase diagram of superfluid ^{3}He in the presence of a magnetic field on the basis of a microscopic theory is in several respects an open problem. Many of the basic features are well understood [1,2], and a fair agreement with experimental results [3-5] has been obtained, cf. also refs. 6 and 7 for a general review. Nevertheless, there is as yet no first-principle derivation of the effective interactions based on the interatomic potential [8], see also refs. 9 and 10, and references therein, and furthermore a systematic investigation of the possible ordered phases [11-16] is lacking. It is the second problem to which we address ourselves in the present chapter.

There is a good deal of know-how on the description of liquid ^{3}He in terms of quasi particles. It has become a common practice to adopt a generalization of BCS theory, containing a dominant $\ell=1$ term of the expansion of the pairing potential into spherical harmonics, as a starting point for the description of the phases in ^{3}He [1]. The BCS-model has been extended to include spin fluctuation effects [17-22] in order to account for the experimentally observed phases, and in particular for the phase transition to the A-phase at high pressures. In the literature usually two phases are taken into consideration, namely the isotropic Balian-Werthamer (BW) phase [23], and the anisotropic Anderson-Brinkman-Morel (ABM) phase [24]. Other phases have also been discussed [15,16], but have not been considered as serious candidates for the absolute minimum of the Landau expansion for the free energy in terms of the order parameters of the problem.

In the presence of a magnetic field the situation is much more complicated. Not only do the values of the order parameters in the BW-phase and the ABM-phase differ to a large extent from the values in zero magnetic field, but the Landau expansion contains many other local minima which under suitable conditions might be candidates for the absolute minimum. In refs. 25 and 26 this problem has been treated in the absence of any spin fluctuation effects, and it has been shown under plausible assumptions that the problem of obtaining the absolute minimum of the Landau expansion in terms of the 18 real order parameters of the problem can be reduced to the minimization of an effective three-parameter function. This then leads to three possible phases, an extension of the BW-phase in non-zero magnetic field, with three non-vanishing order parameters, the ABM-phase with two order parameters, and the A1-phase with only one order parameter.

In the presence of spin fluctuations the Landau expansion contains more independent coefficients, and it may be anticipated that some modifications of the results of refs. 25, 26 will be necessary, even for small values of the spin fluctuation contributions. Since it is not possible to do the complete minimization of the Landau expansion for arbitrary values of these coefficients, one has to rely on certain input on their values. In the preceding chapter [27], therefore, we have performed an explicit model calculation to obtain the necessary information on the coefficients of the Landau expansion for the free energy, without relying on an a priori choice of certain classes of phases, in contrast to e.g. ref. 20. This model calculation was based on the hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I \,, \tag{1.1}$$

where

$$\mathcal{H}_0 = \sum_{\underline{k}} \sum_{\alpha,\beta} (\epsilon_k \mathbb{1}_{\alpha\beta} - b\sigma^z_{\alpha\beta}) a^\dagger_{\underline{k}\alpha} a_{\underline{k}\beta} + \tfrac{1}{2} \sum_{\underline{k},\underline{k}'} \sum_{\alpha,\beta} V(\underline{k}-\underline{k}') a^\dagger_{\underline{k}\alpha} a^\dagger_{-\underline{k}\beta} a_{-\underline{k}'\beta} a_{\underline{k}'\alpha}$$
$$(\alpha,\beta = \uparrow,\downarrow) \,, \tag{1.2}$$

in which $\underline{\underline{\mathbb{1}}}$ and $\underline{\underline{\sigma}}^z$ denote the 2×2 unit matrix and Pauli matrix, and in which

$$V(\underline{k}-\underline{k}') = \begin{cases} -3(V/\Omega)\underline{k}\cdot\underline{k}'/(kk'), & \text{if } |\epsilon_k|,|\epsilon_{k'}| < \hbar\omega \,, \\ 0 & , \text{ otherwise} \,, \end{cases} \tag{1.3}$$

is the $\ell=1$ term of the pairing interaction, and where

$$\mathcal{H}_I = I \int d\underline{x}\; \psi^\dagger_\uparrow(\underline{x})\psi_\uparrow(\underline{x})\psi^\dagger_\downarrow(\underline{x})\psi_\downarrow(\underline{x}) \,, \quad \psi_\alpha(\underline{x}) \equiv \frac{1}{\sqrt{\Omega}} \sum_{\underline{k}} e^{i\underline{k}\cdot\underline{x}} a_{\underline{k}\alpha} \tag{1.4}$$

is a Hubbard contact-interaction with coupling constant I. In (1.1)-(1.4) $\underline{k}$, $\underline{k}'$ denote wave vectors with lengths k and k', Ω is the volume of the system,

$V_1>0$ is the $\ell=1$ coefficient of the pairing-potential, ε_k the kinetic energy of the quasiparticles, b an external magnetic field in the z-direction, and $a^{\dagger}_{k\alpha}$ and $a_{k\alpha}$ are the (fermion) creation- and annihilation operators of the quasiparticles.

In spin fluctuation theory certain statistical approximations are used to express the contributions from the Hubbard term (1.4) to the free energy in terms of dynamical susceptibilities. In contrast to this approach we have used in ref. 27 a rigorous treatment based on a theorem of Bogoliubov Jr.[28-33]. As a consequence of this theorem the free energy can be obtained by a minimization procedure over the free energy of a reference system described by a reference hamiltonian containing a kinetic energy term, the Zeeman term, a bilinear term arising from the pairing interaction and the full Hubbard hamiltonian. Due to the presence of the Hubbard interaction it is not possible to give an exact evaluation of the free energy of the reference system, in contrast to the situation described in refs. 25, 26, but a perturbation calculation leads to correct expressions for the coefficients in the Landau expansion up to a certain order in the coupling constant I. In ref. 27 we have restricted ourselves to perturbations up to second order in I. The result of this calculation for the full Landau expansion with explicit coefficients up to the order I^2, is given by

$$f = \min_{\{\underline{m}_i\}} \Phi(\underline{m}_1,\underline{m}_2,\underline{m}_3) \ , \tag{1.5}$$

where

$$\Phi(\underline{m}_1,\underline{m}_2,\underline{m}_3) = \Phi_0(m_1,m_2,m_3) + \Phi_1(\underline{m}_1,\underline{m}_2,\underline{m}_3) \ , \tag{1.6}$$

in terms of the complex three component vectors $\underline{m}_1,\underline{m}_2,\underline{m}_3$, with lengths

$$m_i \equiv (\underline{m}_i\cdot\underline{m}_i^*)^{\frac{1}{2}} \ , \quad i=1,2,3 \ . \tag{1.7}$$

The three vectors $\underline{m}_1,\underline{m}_2,\underline{m}_3$ constitute the set of 18 real parameters of the system. In eq. (1.5) f denotes the free energy per unit volume of the system, and the functions Φ_0 and Φ_1 in eq. (1.6) are given by

$$\begin{aligned}\Phi_0(m_1,m_2,m_3) = {} & u_1m_1^2 + u_2m_2^2 + 2u_3m_3^2 + 2v_1m_1^4 + 2v_2m_2^4 + 4v_3m_3^4 \\ & + v_4m_1^2m_3^2 + v_5m_2^2m_3^2 + w_1(m_1^2+m_2^2+2m_3^2)^2 \\ & + w_2(m_1^2+m_3^2)(m_2^2+m_3^2) + w_3(m_1^2m_2^2+2m_3^4) \ ,\end{aligned} \tag{1.8}$$

depending only on the lengths m_i of the vectors $\underline{m}_i$, and

$$\Phi_1(\underline{m}_1,\underline{m}_2,\underline{m}_3) = v_1|\underline{m}_1\cdot\underline{m}_1|^2 + v_2|\underline{m}_2\cdot\underline{m}_2|^2 + (2v_3+w_3')|\underline{m}_3\cdot\underline{m}_3|^2$$
$$+ v_4|\underline{m}_1\cdot\underline{m}_3|^2 + (v_4-w_2)|\underline{m}_1\cdot\underline{m}_3^*|^2 + v_5|\underline{m}_2\cdot\underline{m}_3|^2 + (v_5-w_2)|\underline{m}_2\cdot\underline{m}_3^*|^2$$
$$+ 2(v_6-w_3')\mathrm{Re}(\underline{m}_1\cdot\underline{m}_2\ \underline{m}_3^*\cdot\underline{m}_3^*) + 2(2v_6-w_2-2w_3)\mathrm{Re}(\underline{m}_1\cdot\underline{m}_3^*\ \underline{m}_2\cdot\underline{m}_3^*)$$
$$+ w_3'|\underline{m}_1\cdot\underline{m}_2|^2 + w_3|\underline{m}_1\cdot\underline{m}_2^*|^2 . \tag{1.9}$$

The coefficients u and v in eqs. (1.8) and (1.9) are given by the BCS-values, as calculated in ref. 26, plus a shift due to the Hubbard interaction. In the weak-coupling approximation $\beta\hbar\omega\to\infty$, they are given by [27)]

$$u_1 = \frac{1}{3}(t + A_c\eta b), \quad u_2 = \frac{1}{3}(t - A_c\eta b), \quad u_3 = \frac{1}{3}(t + 2b^2B_c) , \tag{1.10}$$

where

$$t = \tfrac{1}{2}\mathcal{N}(0)(T-T_c')/T_c' , \quad A_c = \tfrac{1}{2}\mathcal{N}(0)\ln(1.14\beta_c\hbar\omega)$$

$$B_c = \tfrac{1}{2}\mathcal{N}(0)\ \frac{7}{8}\ \frac{\beta_c^2}{\pi^2}\ \zeta(3) . \tag{1.11}$$

In eq. (1.11) T_c' is the superfluid transition temperature in zero magnetic field, in which the spin fluctuation contribution S has been included, cf. eqs. (3.23) and (4.12) of ref. 27, $\mathcal{N}(0)$ is the density of states at the Fermi surface, and η is the asymmetry parameter in the density of states, i.e. we take $\mathcal{N}(\varepsilon) \simeq \mathcal{N}(0)(1+\eta\varepsilon)$ for $|\varepsilon| < \hbar\omega$. Furthermore, we have

$$\begin{aligned}
v_1 &= v + \frac{1}{15}B_c\eta b , & v_4 &= 4v + \frac{4}{15}B_c(\tfrac{1}{2}\eta b - \tfrac{4}{3}\gamma b^2)\\
v_2 &= v - \frac{1}{15}B_c\eta b , & v_5 &= 4v - \frac{4}{15}B_c(\tfrac{1}{2}\eta b + \tfrac{4}{3}\gamma b^2)\\
v_3 &= v - \frac{4}{15}B_c\gamma b^2, & v_6 &= 2v - \frac{4}{45}B_c\gamma b^2 ,
\end{aligned} \tag{1.12}$$

where $v \equiv \frac{1}{15}B_c+w_0$, in which w_0 is a shift due to the Hubbard interaction given in eq. (4.7) of ref. 27, and where $\gamma = (93/56)\ \beta_c^2\zeta(5)/\left(\pi^2\zeta(3)\right)$. The coefficients w_1,w_2,w_3,w_3' in eqs. (1.8) and (1.9) for the Hubbard terms have been calculated up to the order I^2 and for $b=0$ in the weak-coupling limit, and we have found the following values

$$\begin{aligned}
w_1 &= (3.4)\ \frac{4}{15}\ \beta(2\pi I)^2\ k_F^{-3}\left(\tfrac{1}{2}\mathcal{N}(0)\right)^4\\
w_2 &= (-13.6)\ \frac{4}{15}\ \beta(2\pi I)^2\ k_F^{-3}\left(\tfrac{1}{2}\mathcal{N}(0)\right)^4\\
w_3' = w_3 &= (25.4)\ \frac{4}{15}\ \beta(2\pi I)^2\ k_F^{-3}\left(\tfrac{1}{2}\mathcal{N}(0)\right)^4,
\end{aligned} \tag{1.13}$$

where k_F is the Fermi wavenumber. The relation $w_3 = w_3'$ is a consequence of our perturbation calculation and cannot be expressed to hold when (higher) odd orders of perturbation are taken into account. In the analysis of the phase diagram we shall not insist on the precise values of w_1, w_2, w_3 and w_3', but we shall assume only some global characteristics such as e.g. that the signs of w_1, w_2, w_3, w_3' are correctly given by (1.13). Also with regard to the global characteristics the values given by eq. (1.13) differ from the values $4w_1 = -0.25(\frac{2}{15} B_c \delta)$, $w_2 = 0.5(\frac{2}{15} B_c \delta)$, $2w_3 = -1.25(\frac{2}{15} B_c \delta)$, $2w_3' = -0.25(\frac{2}{15} B_c \delta)$, where δ is a parameter depending on the coupling I, as obtained in spin fluctuation theory, and derived in ref. 20 for unitary states only, and the signs of the spin fluctuation results are not affected by considering only terms up to the order I^2. Anyway, the values given in eq. (1.13) are not meant to be quantitatively correct, but will be used qualitatively as guideline for the specification of the regimes which will be taken into consideration. (In this connection it may be noted that eqs. (1.8) and (1.9) with $v_1 = v_2 = v_3 = v$, $v_4 = v_5 = 4v$, $v_6 = 2v$, and with arbitrary values of the coefficients w_1, w_2, w_3, w_3', are equivalent to the generalized Landau expansion [14], derived on the basis of symmetry properties taking $b=0$ in the fourth order terms).

In order to give a description of the possible phases, we have to determine the (local) extrema of (1.6), which are determined by the gap equations for the order vectors $\underline{m}_1, \underline{m}_2, \underline{m}_3$. They are given by

$$\alpha_1 \underline{m}_1 + 2v_1(\underline{m}_1 \cdot \underline{m}_1)\underline{m}_1^* + w_3(\underline{m}_1 \cdot \underline{m}_2^*)\underline{m}_2 + [(v_6 - w_3')\underline{m}_3 \cdot \underline{m}_3 + w_3' \, \underline{m}_1 \cdot \underline{m}_2]\underline{m}_2^*$$
$$+ [(v_4 - w_2)\underline{m}_1 \cdot \underline{m}_3^* + (2v_6 - w_2 - 2w_3)\underline{m}_2^* \cdot \underline{m}_3]\underline{m}_3 + v_4(\underline{m}_1 \cdot \underline{m}_3)\underline{m}_3^* = 0 \tag{1.14a}$$

$$\alpha_2 \underline{m}_2 + 2v_2(\underline{m}_2 \cdot \underline{m}_2)\underline{m}_2^* + w_3(\underline{m}_1^* \cdot \underline{m}_2)\underline{m}_1^* + [(v_6 - w_3')\underline{m}_3 \cdot \underline{m}_3 + w_3' \, \underline{m}_1 \cdot \underline{m}_1]\underline{m}_1^*$$
$$+ [(v_5 - w_2)\underline{m}_2 \cdot \underline{m}_3^* + (2v_6 - w_2 - 2w_3)\underline{m}_1^* \cdot \underline{m}_3]\underline{m}_3 + v_5(\underline{m}_2 \cdot \underline{m}_3)\underline{m}_3^* = 0 \tag{1.14b}$$

$$\alpha_3 \underline{m}_3 + [(2v_3 + w_3')\underline{m}_3 \cdot \underline{m}_3 + (v_6 - w_3')\underline{m}_1 \cdot \underline{m}_2]\underline{m}_3^* + \tfrac{1}{2}v_4(\underline{m}_1 \cdot \underline{m}_3)\underline{m}_1^*$$
$$+ \tfrac{1}{2}[(v_4 - w_2)\underline{m}_1^* \cdot \underline{m}_3 + (2v_6 - w_2 - 2w_3)\underline{m}_2 \cdot \underline{m}_3^*]\underline{m}_1 + \tfrac{1}{2}v_5(\underline{m}_2 \cdot \underline{m}_3)\underline{m}_2^*$$
$$+ \tfrac{1}{2}[(v_5 - w_2)\underline{m}_2^* \cdot \underline{m}_3 + (2v_6 - w_2 - w_3)\underline{m}_1 \cdot \underline{m}_3^*]\underline{m}_2 = 0 \,, \tag{1.14c}$$

where

$$\alpha_1 \equiv u_1 + 4v_1 m_1^2 + v_4 m_3^2 + 2w_1(m_1^2 + m_2^2 + 2m_3^2) + w_2(m_2^2 + m_3^2) + w_3 m_2^2 \tag{1.15a}$$

$$\alpha_2 \equiv u_2 + 4v_2 m_2^2 + v_5 m_3^2 + 2w_1(m_1^2 + m_2^2 + 2m_3^2) + w_2(m_1^2 + m_3^2) + w_3 m_1^2 \tag{1.15b}$$

$$\alpha_3 \equiv u_3 + 4v_3 m_3^2 + \tfrac{1}{2}v_4 m_1^2 + \tfrac{1}{2}v_5 m_2^2 + (2w_1 + \tfrac{1}{2}w_2)(m_1^2 + m_2^2 + 2m_3^2) + 2w_3 m_3^2 \,, \tag{1.15c}$$

depending only on the lengths m_1, m_2, m_3 of the vectors $\underline{m}_1, \underline{m}_2, \underline{m}_3$. The free energy per unit volume f can be evaluated using the relation

$$f_s \equiv \Phi(m_{1,s}, m_{2,s}, m_{3,s}) = \tfrac{1}{2}u_1 m_{1,s}^2 + \tfrac{1}{2}u_2 m_{2,s}^2 + u_3 m_{3,s}^2 , \tag{1.16}$$

for an arbitrary solution $\underline{m}_{1,s}$, $\underline{m}_{2,s}$, $\underline{m}_{3,s}$ of eqs. (1.14), and by investigating which solution leads to the lowest value of f_s. We will give solutions of the gap equations (1.14) for fixed lengths m_i, thereby determining the possible geometrical configurations of the vectors $\underline{m}_1, \underline{m}_2, \underline{m}_3$ that characterize the various phases. Having found these phases, we have reduced the 18 order parameter problem essentially to the minimization problem of certain families of 3 parameter functions in terms of m_1, m_2, m_3. On closer examination of the values of f_s associated with these phases a number of solutions can be ruled out as serious candidates for the global minimum of the Landau expansion, whereas other solutions, some of which have not been considered before in the literature, prevail. One of the most striking consequences of our analysis is that under very broad conditions, i.e. $w_3, w_3' > 0$, the ABM-phase will *not* occur, as other phases turn out to lead to lower values of f_s, cf. 13.

The outline of this chapter is as follows. In section 2 we will give a treatment of all the local minima of the Landau expansion in the case that either one or two of the vectors $\underline{m}_1, \underline{m}_2, \underline{m}_3$ are taken to be zero (the so-called one- or twodimensional solutions), under the assumption that the inner products involving $\underline{m}_1$ and $\underline{m}_2$ can be taken to be real. In section 3 we will present the threedimensional solutions with $\underline{m}_1, \underline{m}_2, \underline{m}_3 \neq 0$, under a so-called inertia condition. This assumption, which we already used in ref. 26, reduces the threedimensional problem essentially to a twodimensional one. From the extensive list of solutions obtained in this way, some solutions are immediately ruled out as being unstable, while other solutions can be ruled out in certain regions of values for the coefficients. Depending on the regime under consideration several special problems related to the phase diagram can then be treated explicitly. As an example we shall address ourselves in section 4 to the problem of the stability of the phase diagram in the absence of spin fluctuations, as presented in ref. 26, i.e. the problem of determining to which extent this phase diagram will undergo qualitative changes in the presence of small Hubbard interactions. Another problem that can be treated in detail, is the case that b=0. This will be done in section 5. Finally in section 6 some concluding remarks will be given, and we also pay attention to the so-called 'profound effect' [4].

2. One- and twodimensional solutions

The one- and twodimensional solutions of the gap equations (1.14) are those, for which one respectively two of the vectors $\underline{m}_1, \underline{m}_2, \underline{m}_3$ are different from zero. In contrast to the case that neither one of the order vectors vanishes, the one- and twodimensional cases can be treated to a full extent, taking the inner products involving $\underline{m}_1$ and $\underline{m}_2$ to be real.

2.1. *Onedimensional solutions*

The treatment of the onedimensional solutions is essentially the same as given in ref. 26. For $\underline{m}_2=\underline{m}_3=0$ we have from (1.15a) two solutions, one for which the vector $\underline{m}_1$ is real, i.e. $\underline{m}_1 \parallel \underline{m}_1^*$, leading to

$$\underline{\text{R1}}:\ \underline{m}_1\cdot\underline{m}_1,\quad |\underline{m}_1\cdot\underline{m}_1| = m_1^2 = -\tfrac{1}{2}u_1/(3v_1+w_1),\quad \underline{m}_2=\underline{m}_3=0,\quad f_s = -\tfrac{1}{4}u_1^2/(3v_1+w_1),$$
$$(u_1 < 0,\quad 3v_1+w_1 > 0), \qquad (2.1)$$

and the A1-solution characterized by

$$\underline{\text{A1}}:\ \underline{m}_1\cdot\underline{m}_1 = 0,\quad \underline{m}_2=\underline{m}_3=0,\quad m_1^2 = -\tfrac{1}{2}u_1/(2v_1+w_1),\quad f_{A1} = -\tfrac{1}{4}u_1^2/(2v_1+w_1),$$
$$(u_1 < 0,\quad 2v_1+w_1 > 0). \qquad (2.2)$$

It is obvious that the A1-solution is always favourable, in comparison with the solution (2.1). For $\underline{m}_1=\underline{m}_3=0$, $\underline{m}_2\neq 0$ we have two similar solutions, R2 and A2, from which the A2-solution, i.e. eq. (2.2) with $u_1\to u_2$, $v_1\to v_2$, $\underline{m}_1\to\underline{m}_2$, is most favourable. Furthermore taking $\eta b<0$, it is, however, immediately clear from eqs. (1.10) and (1.12) that $f_{A1} < f_{A2}$ implying that the A2-solution does not occur. We finally mention the A3-solution separately, i.e.

$$\underline{\text{A3}}:\ \underline{m}_3\cdot\underline{m}_3 = 0,\quad \underline{m}_1=\underline{m}_2=0,\quad m_3^2 = -\tfrac{1}{2}u_3/(2v_3+2w_1+\tfrac{1}{2}w_2+w_3),$$
$$f_{A3} = -\tfrac{1}{2}u_3^2/(2v_3+2w_1+\tfrac{1}{2}w_2+w_3),\quad (u_3<0,\quad 2v_3+2w_1+\tfrac{1}{2}w_2+w_3 > 0), \qquad (2.3)$$

2.2. *Twodimensional solutions with $\underline{m}_3=0$*

We now investigate the solutions for which $\underline{m}_3=0$. In the absence of the terms with coefficients w, the gap equations (1.14a) and (1.14b) are uncoupled for $\underline{m}_3=0$. The terms with coefficients w_3 and w_3' (1.9), however, yield some nontrivial couplings between $\underline{m}_1$ and $\underline{m}_2$. This makes the problem for $\underline{m}_3=0$ much more complicated in comparison with the situation in ref. 26. Nevertheless, under the assumption that the inner products

$\underline{m}_1\cdot\underline{m}_1$, $\underline{m}_1\cdot\underline{m}_2$, $\underline{m}_1\cdot\underline{m}_2^*$ and $\underline{m}_2\cdot\underline{m}_2$ can all be taken to be real, a complete treatment of the solutions for $\underline{m}_3=0$ can be given.

The minimization of $\Phi(\underline{m}_1,\underline{m}_2,0)$ with respect to the vectors $\underline{m}_1,\underline{m}_2$ will be carried out by first determining the *geometrical configurations*, i.e. the relative directions of the vectors $\underline{m}_1,\underline{m}_2$ and their complex conjugates, that minimize $\Phi(\underline{m}_1,\underline{m}_2,0)$ at fixed values of the lengths m_1,m_2. For each geometrical configuration (s) we can insert the expressions for the inner products $\underline{m}_1\cdot\underline{m}_1$, $\underline{m}_1\cdot\underline{m}_2$, $\underline{m}_1\cdot\underline{m}_2^*$ and $\underline{m}_2\cdot\underline{m}_2$ in terms of the lengths m_1,m_2 in eqs. (1.6), (1.8), (1.9) to obtain a two-parameter function of the form

$$\Phi_s(m_1,m_2) = u_1m_1^2 + u_2m_2^2 + (2v_1+w_1)m_1^4 + (2v_2+w_1)m_2^4 + \\ + (2w_1+w_2+w_3)m_1^2m_2^2 + a_{11}m_1^4 + a_{22}m_2^4 + a_{12}m_1^2m_2^2, \qquad (2.4)$$

in which a_{11}, a_{22} and a_{12} are constants depending not on u_1, u_2, m_1 and m_2, but possibly on the coefficients v and w, which can be taken to be constant in the neighbourhood of T_c. Minimizing $\Phi_s(m_1,m_2)$ with respect to the lengths m_1,m_2 we obtain the following solution with $m_1,m_2 \neq 0$

$$m_1^2 = \frac{(2w_1+w_2+w_3+a_{12})u_2 - 2(2v_2+w_1+a_{22})u_1}{4(2v_1+w_1+a_{11})(2v_2+w_1+a_{22}) - (2w_1+w_2+w_3+a_{12})^2}$$

$$m_2^2 = \frac{(2w_1+w_2+w_3+a_{12})u_1 - 2(2v_1+w_1+a_{11})u_2}{4(2v_1+w_1+a_{11})(2v_2+w_1+a_{22}) - (2w_1+w_2+w_3+a_{12})^2}, \qquad (2.5)$$

leading to a value $f_s = \frac{1}{2}u_1m_1^2 + \frac{1}{2}u_2m_2^2$ as a function of the parameters u_1,u_2, which is given by

$$f_s(u_1,u_2) = \frac{-(2v_1+w_1+a_{11})u_2^2 + (2w_1+w_2+w_3+a_{12})u_1u_2 - (2v_2+w_1+a_{22})u_1^2}{4(2v_1+w_1+a_{11})(2v_2+w_1+a_{22}) - (2w_1+w_2+w_3+a_{12})^2}. \qquad (2.6)$$

The solution (2.5) can exist and be a minimum of $\Phi_s(m_1,m_2)$ only under the following conditions

$$4(2v_1+w_1+a_{11})(2v_2+w_1+a_{22}) - (2w_1+w_2+w_3+a_{12})^2 > 0$$
$$(2w_1+w_2+w_3+a_{12})u_2 - 2(2v_2+w_1+a_{22})u_1 > 0$$
$$(2w_1+w_2+w_3+a_{12})u_1 - 2(2v_1+w_1+a_{11})u_2 > 0\,. \qquad (2.7)$$

We now turn to the specification of the possible configurations that may lead to (2.4). The details of the minimization procedure are worked out in appendix A, and here we only present a list of the various solutions

$s = \mathrm{I},\ldots,\mathrm{VI}$, together with the relevant features, i.e. the values of the inner products and of the constants a_{11}, a_{22}, a_{12}. (It is not necessary to give specific representations for these solutions, i.e. a special choice of a coordinate system in terms of which $\underline{m}_1$ and $\underline{m}_2$ are given, as we have done in ref. 26).

$$\textit{Phase I:}\quad \underline{m}_1\cdot\underline{m}_1 = \underline{m}_1\cdot\underline{m}_2 = \underline{m}_1\cdot\underline{m}_2^* = 0\ ,\quad \underline{m}_2 \parallel \underline{m}_2^*\ ,\quad |\underline{m}_2\cdot\underline{m}_2| = m_2^2$$
$$a_{11} = a_{12} = 0\ ,\quad a_{22} = v_2. \tag{2.8}$$

Note that phase I bifurcates with the A1-phase for $m_2 \to 0$, implying that for $(2w_1+w_2+w_3)u_1 = 2(2v_1+w_1)u_2$ there is a second order phase transition from phase I to A1. Furthermore since $a_{22} \neq a_{11}$, the numerator of eq. (2.6) contains a term $\sim \eta b$, implying that phase I would lead to a spontaneous magnetization, if it exists for $b=0$. Apart from I there is another solution I' which can be inferred from I interchanging $\underline{m}_1$ and $\underline{m}_2$, u_1 and u_2, v_1 and v_2, and a_{11} and a_{22}. Since for $\eta b<0$, $u_1^2 > u_2^2$, $v_1 < v_2$, cf. eqs. (1.10) and (1.12), it is clear that $f_{\mathrm{I'}} > f_{\mathrm{I}}$, so that I' can be ruled out as a candidate for the absolute minimum. In the special case $w_3 + w_3' = 0$, one can have another type of configuration I", with

$$\textit{Phase I'':}\quad \underline{m}_1\cdot\underline{m}_1 = 0,\quad \underline{m}_2 \parallel \underline{m}_2^*,\quad |\underline{m}_2\cdot\underline{m}_2| = m_2^2,\quad |\underline{m}_1\cdot\underline{m}_2| = |\underline{m}_1\cdot\underline{m}_2^*| \ \text{unspecified},$$
$$\Phi_{\mathrm{I''}}(m_1,m_2) = \Phi_{\mathrm{I}}(m_1,m_2),\quad (w_3+w_3'=0)\ , \tag{2.9}$$

and a similar one, I''', with $\underline{m}_1, \underline{m}_2$ interchanged.

$$\textit{Phase II:}\quad \underline{m}_1 \parallel \underline{m}_2^*,\quad \underline{m}_1\cdot\underline{m}_1 = \underline{m}_2\cdot\underline{m}_2 = \underline{m}_1\cdot\underline{m}_2^* = 0,\quad |\underline{m}_1\cdot\underline{m}_2| = m_1 m_2,$$
$$a_{11} = a_{22} = 0,\quad a_{12} = w_3'\ . \tag{2.10}$$

Phase II is the so-called planar or "twodimensional" solution, also treated in ref. 26, and for the first time given in ref. 16 for $b=0$, $m_1=m_2$. It bifurcates for $m_2 \to 0$, i.e. $(2w_1+w_2+w_3+w_3')u_1 = 2(2v_1+w_1)u_2$, with the A1-phase.

$$\textit{Phase III:}\quad \underline{m}_1 \parallel \underline{m}_2,\quad \underline{m}_1\cdot\underline{m}_1 = \underline{m}_2\cdot\underline{m}_2 = \underline{m}_1\cdot\underline{m}_2 = 0,\quad |\underline{m}_1\cdot\underline{m}_2^*| = m_1 m_2,$$
$$a_{11} = a_{22} = 0,\quad a_{12} = w_3. \tag{2.11}$$

Phase III is the axial or ABM-phase, which for $(2w_1+w_2+2w_3)u_1 = 2(2v_1+w_1)u_2$ bifurcates with the A1-phase. For $w_3 = w_3'$ phases II and III are degenerate [12] and yield the same value for the free energy. For $w_3'>w_3$ the ABM-phase is fabourable in comparison with the planar phase in the regions of existence.

$$\textit{Phase IV:}\quad \underline{m}_1 \parallel \underline{m}_1^*,\ \ \underline{m}_2 \parallel \underline{m}_2^*,\ \ |\underline{m}_1\cdot\underline{m}_1| = m_1^2,\ \ |\underline{m}_2\cdot\underline{m}_2| = m_2^2,\ \ \underline{m}_1\cdot\underline{m}_2 = \underline{m}_1\cdot\underline{m}_2^* = 0$$

$$a_{11} = v_1,\ \ a_{22} = v_2,\ \ a_{12} = 0\ . \tag{2.12}$$

From (2.11) it is immediately clear that $\Phi_{IV}(m_1,m_2) > \Phi_I(m_1,m_2)$, so that phase IV will not occur. Phase IV in the special case $b=0$, $m_1=m_2$, has been referred to as the bipolar phase [16]. In the special case that $w_3+w_3'=0$, one has another solution IV' with

$$\textit{Phase IV':}\quad \underline{m}_1 \parallel \underline{m}_1^*,\ \ \underline{m}_2 \parallel \underline{m}_2^*,\ \ |\underline{m}_1\cdot\underline{m}_1| = m_1^2,\ \ |\underline{m}_2\cdot\underline{m}_2| = m_2^2,\ \ |\underline{m}_1\cdot\underline{m}_2| = |\underline{m}_1\cdot\underline{m}_2^*|$$

$$\text{unspecified},\ \ \Phi_{IV'}(m_1,m_2) = \Phi_{IV}(m_1,m_2),\ \ w_3+w_3' = 0\ . \tag{2.13}$$

Solution IV' is given here for future reference, since it admits extensions to threedimensional solutions which will be treated in section 3.

$$\textit{Phase V:}\quad \underline{m}_1 \parallel \underline{m}_1^* \parallel \underline{m}_2 \parallel \underline{m}_2^*,\ \ |\underline{m}_1\cdot\underline{m}_1| = m_1^2,\ \ |\underline{m}_2\cdot\underline{m}_2| = m_2^2,$$

$$|\underline{m}_1\cdot\underline{m}_2| = |\underline{m}_1\cdot\underline{m}_2^*| = m_1 m_2\ ,$$

$$a_{11} = v_1,\ \ a_{22} = v_2,\ \ a_{12} = w_3+w_3'\ . \tag{2.14}$$

Phase V, in the special case $b=0$, $m_1=m_2$, has been referred to as the polar phase [16]. For $w_3+w_3'>0$ we always have $\Phi_V(m_1,m_2) > \Phi_I(m_1,m_2)$, implying that phase V does not occur under this condition.

$$\textit{Phase VI:}\quad \underline{m}_1\cdot\underline{m}_1 = \gamma\,\frac{2v_2m_2^2 - \gamma m_1^2}{4v_1v_2 - \gamma^2}\ ,\qquad \underline{m}_2\cdot\underline{m}_2 = \gamma\,\frac{2v_1m_1^2 - \gamma m_2^2}{4v_1v_2 - \gamma^2}\ ,$$

$$w_3'\ \underline{m}_1\cdot\underline{m}_2 = -w_3\ \underline{m}_1\cdot\underline{m}_2^* = \pm\sqrt{4v_1v_2(\underline{m}_1\cdot\underline{m}_1)(\underline{m}_2\cdot\underline{m}_2)}$$

$$a_{11} = \frac{-\gamma^2 v_1}{4v_1v_2 - \gamma^2}\ ,\quad a_{22} = \frac{-\gamma^2 v_2}{4v_1v_2 - \gamma^2}\ ,\quad a_{12} = \frac{4v_1v_2\gamma}{4v_1v_2 - \gamma^2}\ ,$$

$$\gamma \equiv w_3w_3'/(w_3+w_3')\ . \tag{2.15}$$

In the special case $b=0$, phase VI may be regarded as belonging to the so-called axiplanar phase, first considered by Mermin and Stare, as cited in ref. 15, but this phase has never been considered as a serious candidate for the absolute minimum of the Landau expansion. In section 4 we shall show, however, that phase VI will actually occur in the phase diagram for appropriate positive values of w_3,w_3'.

Apart from the conditions (2.7) with (2.15) we also have the obvious condition

$$|\underline{m}_p \cdot \underline{m}_q| < m_p m_q \ , \quad p,q=1,2, \tag{2.16}$$

for the inner products in eq. (2.15). In order to investigate the possible bifurcations arising from (2.15) we note that the minimization of (2.4), with the values of a_{11}, a_{22}, a_{12} given in (2.15), with respect to m_1, m_2 leads to the expressions

$$\alpha_1 + \frac{2v_1}{\gamma}\alpha_2 + 2v_1 m_1^2 = 0 \ , \quad \alpha_2 + \frac{2v_2}{\gamma}\alpha_1 + 2v_2 m_2^2 = 0 \ , \tag{2.17}$$

in which α_1, α_2 are given by eqs. (1.16a) and (1.16b) with $m_3=0$. From eqs. (2.15) and (2.17) we obtain

$$\underline{m}_1\cdot\underline{m}_1 = -\tfrac{1}{2}\alpha_1/v_1 \ , \quad \underline{m}_2\cdot\underline{m}_2 = -\tfrac{1}{2}\alpha_2/v_2 \ , \quad w_3'^2(\underline{m}_1\cdot\underline{m}_2)^2 = w_3^2(\underline{m}_1\cdot\underline{m}_2)^2 = \alpha_1\alpha_2 \ , \tag{2.18}$$

so that in the case that $w_3, w_3' > 0$

$$m_1^2 m_2^2 = \alpha_1\alpha_2\left(\frac{1}{\gamma^2} + \frac{1}{4v_1v_2}\right) + \frac{\alpha_1^2}{2v_1\gamma} + \frac{\alpha_2^2}{2v_2\gamma} > \frac{\alpha_1\alpha_2}{w_3'^2} = (\underline{m}_1\cdot\underline{m}_2)^2 \ . \tag{2.19}$$

Eq. (2.19) and a similar inequality for $\underline{m}_1\cdot\underline{m}_2^*$ instead of $\underline{m}_1\cdot\underline{m}_2$ imply that the only physical bifurcation with another phase with $\underline{m}_3=0$ can occur under the condition

$$\underline{m}_1\cdot\underline{m}_1 = 0, \quad |\underline{m}_2\cdot\underline{m}_2| = m_2^2, \quad 2v_2 m_2^2 = \gamma m_1^2 \ . \tag{2.20}$$

Under the condition (2.20) we have a second-order phase transition from phase VI to phase I. Condition (2.20) will be worked out in section 4, where we show that this phase transition can actually occur for positive values of w_3, w_3'. Finally we note that for $w_3=0$, phase VI reduces to the ABM-phase, and for $w_3'=0$ to the planar phase, whereas for $w_3+w_3'=0$ it reduces to a special case of phase IV'.

2.3. *Twodimensional solutions with* $\underline{m}_2=0$

The analysis of the solutions with $\underline{m}_2=0$ (or $\underline{m}_1=0$) remains essentially the same as in section 3 of ref. 26. Again we have the property that the gap equation for $\underline{m}_2$, eq. (1.14b) with $\underline{m}_2=0$, leads to

$$\underline{m}_3\cdot\underline{m}_3 = \underline{m}_1^*\cdot\underline{m}_3 = 0\ . \tag{2.21}$$

Eqs. (1.14a) and (1.14c) then reduce to

$$\alpha_1\underline{m}_1 + 2v_1(\underline{m}_1\cdot\underline{m}_1)\underline{m}_1^* + v_4(\underline{m}_1\cdot\underline{m}_3)\underline{m}_3^* = 0 \tag{2.22a}$$

$$\alpha_3\underline{m}_3 + \tfrac{1}{2}v_4(\underline{m}_1\cdot\underline{m}_3)\underline{m}_1^* = 0\ . \tag{2.22b}$$

From (2.22b) we immediately have $\underline{m}_3 \parallel \underline{m}_1^*$, and hence $\underline{m}_1\cdot\underline{m}_1 = 0$, or $\underline{m}_1\cdot\underline{m}_3 = 0$, but not both these inner products can be zero, as $\underline{m}_1$ and $\underline{m}_3$ are inbedded in a threedimensional space. We end up with two solutions $s = \text{VII, VIII}$, for which $\Phi(\underline{m}_1,0,\underline{m}_3)$ can be expressed in terms of a function $\Phi_s(m_1,m_3)$ which has the form

$$\begin{aligned}\Phi_s(m_1,m_3) &= u_1m_1^2 + 2u_3m_3^2 + (2v_1+w_1)m_1^4 + (4v_3+4w_1+w_2+2w_3)m_3^4 \\ &\quad + (v_4+4w_1+w_2)m_1^2m_3^2 + a_{11}m_1^4 + a_{13}m_1^2m_3^2\ .\end{aligned} \tag{2.23}$$

Minimizing $\Phi_s(m_1,m_3)$ with respect to m_1 and m_3 we have the following solutions with $m_1,m_3 \neq 0$.

$$m_1^2 = \frac{-2(4v_3+4w_1+w_2+2w_3)u_1 + (v_4+4w_1+w_2+a_{13})2u_3}{4(4v_3+4w_1+w_2+2w_3)(2v_1+w_1+a_{11}) - (v_4+4w_1+w_2+a_{13})^2}$$

$$m_3^2 = \frac{-4(2v_1+w_1+a_{11})u_3 + (v_4+4w_1+w_2+a_{13})u_1}{4(4v_3+4w_1+w_2+2w_3)(2v_1+w_1+a_{11}) - (v_4+4w_1+w_2+a_{13})^2}\ , \tag{2.24}$$

leading to a free energy $f_s(u_1,u_3) = \tfrac{1}{2}u_1m_1^2 + u_3m_3^2$, which is given by

$$f_s = \frac{-(4v_3+4w_1+w_2+2w_3)u_1^2 + 2(v_4+4w_1+w_2+a_{13})u_1u_3 - 4(2v_1+w_1+a_{11})u_3^2}{4(4v_3+4w_1+w_2+2w_3)(2v_1+w_1+a_{11}) - (v_4+4w_1+w_2+a_{13})^2}\ . \tag{2.25}$$

The solution (2.24) can exist and be a minimum only under the following conditions

$$\begin{aligned}&4(4v_3+4w_1+w_2+2w_3)(2v_1+w_1+a_{11}) - (v_4+4w_1+w_2+a_{13})^2 > 0 \\ &(v_4+4w_1+w_2+a_{13})u_3 - (4v_3+4w_1+w_2+2w_3)u_1 > 0 \\ &(v_4+4w_1+w_2+a_{13})u_1 - 4(2v_1+w_1+a_{11})u_3 > 0\ .\end{aligned} \tag{2.26}$$

The phases VII and VIII are given by

Phase VII: $\underline{m}_1 \| \underline{m}_1^*$. $\underline{m}_3 \cdot \underline{m}_3 = \underline{m}_1 \cdot \underline{m}_3^* = \underline{m}_1 \cdot \underline{m}_3 = 0$, $|\underline{m}_1 \cdot \underline{m}_1| = m_1^2$,

$$a_{11} = v_1, \quad a_{13} = 0 . \tag{2.27}$$

In the special case $b=0$, phase VII reduces to the ε-solution given by Barton and moore [15]. As the numerator of (2.25) contains a term $\sim nb$, phase VII has a spontaneous magnetization, if it exists for $b=0$.

Phase VIII: $\underline{m}_1 \| \underline{m}_3^*$, $\underline{m}_1 \cdot \underline{m}_1 = \underline{m}_3 \cdot \underline{m}_3 = \underline{m}_1 \cdot \underline{m}_3^* = 0$, $|\underline{m}_1 \cdot \underline{m}_3| = m_1 m_3$,

$$a_{11} = 0 , \quad a_{13} = v_4. \tag{2.28}$$

Phase VIII has also a spontaneous magnetization. For $w_2 \geq 0$ phase VIII cannot occur, because in that case we can indicate a point with $\underline{m}_1 \cdot \underline{m}_1 = \underline{m}_3 \cdot \underline{m}_3 = \underline{m}_1 \cdot \underline{m}_3 = 0$, $|\underline{m}_1 \cdot \underline{m}_3^*| = m_1 m_3$ in the space of the order parameters, which leads to a lower value of $\Phi(\underline{m}_1, 0, \underline{m}_3)$, namely $\Phi(\underline{m}_1, 0, \underline{m}_3) = \Phi_{VIII}(m_1, m_3) - w_2 m_1^2 m_3^2$, but this point is not an extremum of $\Phi(\underline{m}_1, \underline{m}_2, \underline{m}_3)$. For $w_2 < 0$, however, phase VIII must be taken into account.

Remark. To conclude the list of twodimensional solutions, we mention briefly the twodimensional solutions with $\underline{m}_1 = 0$, which can be obtained from eqs. (2.26) and (2.27) by interchanging $\underline{m}_1$ and $\underline{m}_2$, u_1 and u_2, v_1 and v_2, v_4 and v_5, and a_{11} and a_{22}, a_{13} and a_{23}. These solutions which may be donoted by VII' and VIII' respectively are less favourable than the solutions VII and VIII and will not occur in practice.

3. Threedimensional solutions

3.1. *Inertia condition*

In the previous section we have considered the solutions of the gap equations, for which one of the order vectors $\underline{m}_1, \underline{m}_2, \underline{m}_3$ is equal to zero. We have given a complete list of possible (local) minima, under the assumption that the inner products can be chosen to be real.

The threedimensional solutions, for which neither of the vectors $\underline{m}_1, \underline{m}_2, \underline{m}_3$ vanishes, are very hard to analyze, because of the complicated couplings which occur in the Landau expansion. In order to have a grip on the problem, we have introduced in ref. 26 the inertia condition. This is an assumption, based on the physical picture that the only solutions of interest are those for which the geometrical configuration has a certain rigidity, in the sense that small changes of the parameters in the phase diagram (i.e. variations of the variables u_1, u_2, u_3) do not alter the configuration, but only the lengths of the order vectors. A stronger version of an inertia condition has been proposed in ref. 15, reducing the problem of minimizing the free energy essentially to a one-parameter problem. Here, however, we shall impose that we can minimize the Landau expansion after inserting the various possibilities for the inner products between $\underline{m}_1, \underline{m}_2$ and $\underline{m}_3$, which have to be determined in a self-consistent way from the gap equations. More explicitly we assume that we can minimize the Landau expansion under the condition that

$$\underline{m}_1 \cdot \underline{m}_3 = m_1 m_3 \ \lambda_{13}(m_1^2, m_2^2, m_3^2) \ , \qquad \underline{m}_1 \cdot \underline{m}_3^* = m_1 m_3 \ \mu_{13}(m_1^2, m_2^2, m_3^2) \ ,$$

$$\underline{m}_2 \cdot \underline{m}_3 = m_2 m_3 \ \lambda_{23}(m_1^2, m_2^2, m_3^2) \ , \qquad \underline{m}_2 \cdot \underline{m}_3^* = m_1 m_3 \ \mu_{23}(m_1^2, m_2^2, m_3^2) \ . \tag{3.1}$$

We shall show that eq. (3.1) reduces the minimization problem essentially to the $\underline{m}_1, \underline{m}_2$ problem, discussed in subsection 2.2. We shall not discuss the cases, in which eq. (3.1) is not satisfied. Although the gap equations may have more complicated solutions, which do not satisfy eq. (3.1), we have not found evidence that such solutions may be candidates for the absolute minimum of the Landau expansion.

Inserting eq. (3.1) into (1.5), with (1.6), (1.8) and (1.9) and minimizing with respect to the vectors $\underline{m}_1^*, \underline{m}_2^*, \underline{m}_3^*$ we find the following gap equations

$$\alpha_1' \underline{m}_1 + 2v_1(\underline{m}_1 \cdot \underline{m}_1)\underline{m}_1^* + [w_3' \ \underline{m}_1 \cdot \underline{m}_2 + (v_6 - w_3')\underline{m}_3 \cdot \underline{m}_3]\underline{m}_2^* + w_3(\underline{m}_1 \cdot \underline{m}_2^*)\underline{m}_2 = 0 \tag{3.2a}$$

$$\alpha_2' \underline{m}_2 + 2v_2(\underline{m}_2 \cdot \underline{m}_2)\underline{m}_2^* + [w_3' \ \underline{m}_1 \cdot \underline{m}_2 + (v_6 - w_3')\underline{m}_3 \cdot \underline{m}_3]\underline{m}_1^* + w_3(\underline{m}_1^* \cdot \underline{m}_2)\underline{m}_1 = 9 \tag{3.2b}$$

$$\alpha_3' \underline{m}_3 + [(2v_3 + w_3')\underline{m}_3 \cdot \underline{m}_3 + (v_6 - w_3')\underline{m}_1 \cdot \underline{m}_2]\underline{m}_3^* = 0 \ , \tag{3.2c}$$

where

$$\alpha_1' = \frac{\partial \Phi_0'(m_1, m_2, m_3)}{\partial m_1^2} \ , \qquad \alpha_2' = \frac{\partial \Phi_0'(m_1, m_2, m_3)}{\partial m_2^2} \ , \qquad \alpha_3' = \tfrac{1}{2} \frac{\partial \Phi_0'(m_1, m_2, m_3)}{\partial m_3^2} \ , \tag{3.3}$$

with

$$\Phi_0'(m_1, m_2, m_3) = \Phi_0(m_1, m_2, m_3) + v_4 m_1^2 m_3^2 |\lambda_{13}|^2 + (v_4 - w_2) m_1^2 m_3^2 |\mu_{13}|^2$$
$$+ v_5 \ m_2^2 m_3^2 |\lambda_{23}|^2 + (v_5 - w_2) m_2^2 m_3^2 |\mu_{23}|^2 + 2(2v_6 - w_2 - 2w_3)\mathrm{Re}(\mu_{13}\mu_{23}) \tag{3.4}$$

i.e. Φ_0' also contains the terms of $\Phi(\underline{m}_1, \underline{m}_2, \underline{m}_3)$ involving the inner products $\underline{m}_1 \cdot \underline{m}_3$, $\underline{m}_1 \cdot \underline{m}_3^*$, $\underline{m}_2 \cdot \underline{m}_3$ and $\underline{m}_2 \cdot \underline{m}_3^*$. In order that the solutions of eqs. (3.2) are also solutions of the original gap equations (1.14), eq. (3.2) has to be supplemented with the following equations

$$(\alpha_1-\alpha_1')\underline{m}_1 + [(v_4-w_2)\underline{m}_1\cdot\underline{m}_3^* + (2v_6-w_2-2w_3)\underline{m}_2^*\cdot\underline{m}_3]\underline{m}_3 + v_4(\underline{m}_1\cdot\underline{m}_3)\underline{m}_3^* = 0 \qquad (3.5a)$$

$$(\alpha_2-\alpha_2')\underline{m}_2 + [(v_5-w_2)\underline{m}_2\cdot\underline{m}_3^* + (2v_6-w_2-2w_3)\underline{m}_1^*\cdot\underline{m}_3]\underline{m}_3 + v_5(\underline{m}_2\cdot\underline{m}_3)\underline{m}_3^* = 0 \qquad (3.5b)$$

$$\begin{aligned}(\alpha_3-\alpha_3')\underline{m}_3 &+ \tfrac{1}{2}v_4(\underline{m}_1\cdot\underline{m}_3)\underline{m}_3^* + \tfrac{1}{2}[(v_4-w_2)\underline{m}_1^*\cdot\underline{m}_3 + (2v_6-w_2-2w_3)\underline{m}_2\cdot\underline{m}_3^*]\underline{m}_1\\ &+ \tfrac{1}{2}v_5(\underline{m}_2\cdot\underline{m}_3)\underline{m}_2^* + \tfrac{1}{2}[(v_5-w_2)\underline{m}_2^*\cdot\underline{m}_3 + (2v_6-w_2-2w_3)\underline{m}_1\cdot\underline{m}_3^*]\underline{m}_2 = 0. \qquad (3.5c)\end{aligned}$$

The analysis of eqs. (3.2) and (3.5) will consist of two steps. First we shall show that eqs. (3.2a), (3.2b), after taking into account (3.2c), reduce essentially to the $\underline{m}_1,\underline{m}_2$-problem as discussed in subsection 2.2. Secondly, the solutions that emerge, as generalizations of the ones given in section 2, are extended to the threedimensional situation by applying eqs. (3.5).

The solution of eq. (3.2c) is simple. In fact, for $m_3 \neq 0$ we have two possibilities

$$\text{i)}\quad \underline{m}_3 \parallel \underline{m}_3^* \qquad (3.6a)$$

$$\text{ii)}\quad \alpha_3' = 0, \quad (2v_3+w_3')\underline{m}_3\cdot\underline{m}_3 + (v_6-w_3')\underline{m}_1\cdot\underline{m}_2 = 0\ . \qquad (3.6b)$$

Taking into account (3.6) and assuming $\underline{m}_1\cdot\underline{m}_2 \neq 0$, eqs. (3.2a) and (3.2b) emerge as the gap equations of the following $\underline{m}_1,\underline{m}_2$-problem

$$\Phi(\underline{m}_1,\underline{m}_2) = \Phi_0'(m_1,m_2,m_3) + v_1|\underline{m}_1\cdot\underline{m}_1|^2 + v_2|\underline{m}_2\cdot\underline{m}_2|^2 + w_3|\underline{m}_1\cdot\underline{m}_2^*|^2 + w_3''|\underline{m}_1\cdot\underline{m}_2|^2, \qquad (3.7)$$

where

$$w_3'' = w_3' + (v_6-w_3')\underline{m}_3\cdot\underline{m}_3/(\underline{m}_1\cdot\underline{m}_2)\ . \qquad (3.8)$$

The minimization of eq. (3.7) at fixed w_3'' and w_3 leads, as far as $\underline{m}_1$ and $\underline{m}_2$ are concerned, to the phases I–VI, as given in section 2.2. These phases should be supplemented by the proper values of $\underline{m}_3$, together with the values of λ_{13}, λ_{23}, μ_{13}, μ_{23}. This will be done in subsections 3.2 and 3.3, where we also discuss the special case $\underline{m}_1\cdot\underline{m}_2 = 0$.

3.2. *Phases with real vector $\underline{m}_3$*

In this subsection we will treat the solutions of (3.2), (3.5) with real vector $\underline{m}_3$. We distinguish the following cases

i) $\underline{m}_3 \| \underline{m}_1$, $\underline{m}_3 \| \underline{m}_2$, (3.9a)

ii) $\underline{m}_3 \nparallel \underline{m}_1$, $\underline{m}_3 \| \underline{m}_2$, or equivalently $\underline{m}_3 \| \underline{m}_1$, $\underline{m}_3 \| \underline{m}_2$, (3.9b)

iii) $\underline{m}_3 \nparallel \underline{m}_1$, $\underline{m}_3 \nparallel \underline{m}_2$. (3.9c)

An explicit analysis, which will be carried out in appendix B, leads to four phases s=IX, X, XI, XII, for which we specify the geometrical configurations (s) by the values of the inner products $\underline{m}_p \cdot \underline{m}_q$, $\underline{m}_p \cdot \underline{m}_q^*$, p,q=1,2,3, in terms of the lengths m_1, m_2, m_3. Inserting the values of the inner products in the Landau expansion (1.6), (1.8), (1.9) we obtain functions $\Phi_s(m_1,m_2,m_3)$, which have to be minimized with respect to the lengths m_1,m_2,m_3, in order to obtain the free energies $f_s(u_1,u_2,u_3)$ corresponding to the various phases. In contrast to section 2, here we only present the functions Φ_s, but not the free energies f_s, as the minimization with respect to m_1,m_2,m_3 is rather complicated due to terms of the form $m_3^2 m_1 m_2$. The relevant information on the f_s of the threedimensional phases will be provided in sections 4 and 5.

Eq. (3.9a) leads immediately to the solution

Phase IX: $\underline{m}_3 \| \underline{m}_3^*$, $\underline{m}_1 \| \underline{m}_2 \| \underline{m}_3$, $|\underline{m}_p \cdot \underline{m}_q| = |\underline{m}_p \cdot \underline{m}_q^*| = m_p m_q$ (p,q = 1,2,3)

$$\begin{aligned}\Phi_{IX}(m_1,m_2,m_3) = {} & u_1 m_1^2 + u_2 m_2^2 + 2u_3 m_3^2 + (3v_1+w_1)m_1^4 + (3v_2+w_1)m_2^4 \\ & + (6v_3+4w_1+w_2+2w_3+w_3')m_3^4 + (3v_4+4w_1)m_1^2 m_3^2 + (3v_5+4w_1)m_2^2 m_3^2 \\ & + (2w_1+w_2+2w_3+w_3')m_1^2 m_2^2 - 2|3v_6-w_2-w_3'-2w_3|m_3^2 m_1 m_2 .\end{aligned} \tag{3.10}$$

Eq. (3.9b) does not lead to any solutions of (3.2), (3.5) as will be shown in appendix B. Furthermore eq. (3.9c) leads to

$$\underline{m}_1 \cdot \underline{m}_3 = \underline{m}_1 \cdot \underline{m}_3^* = \underline{m}_2 \cdot \underline{m}_3 = \underline{m}_2 \cdot \underline{m}_3^* = 0, \quad \alpha_1 = \alpha_1', \quad \alpha_2 = \alpha_2', \quad \alpha_3 = \alpha_3' , \tag{3.11}$$

as is also shown in appendix B. In the case (3.9c) one has to take into account the two possibilities $\underline{m}_1 \cdot \underline{m}_2 = 0$ and $\underline{m}_1 \cdot \underline{m}_2 \neq 0$. In appendix B, however, it is shown that $\underline{m}_1 \cdot \underline{m}_2 = 0$, combined with eq. (3.11) and $\underline{m}_3 \| \underline{m}_3^*$ does not lead to threedimensional solutions.

For $\underline{m}_1 \cdot \underline{m}_2 \neq 0$, eq. (3.11) can be used in combination with eqs. (3.2), (3.8) to obtain the possible phases. The twodimensional problem (3.7) yields only extensions of the phases II, IV' and V, since the phases III and IV have $\underline{m}_1 \cdot \underline{m}_2 = 0$, and since in the phases I and VI, the vectors $\underline{m}_1, \underline{m}_1^*, \underline{m}_2, \underline{m}_2^*$ already span a threedimensional space, which cannot be combined with eq. (3.11). The combination of $\underline{m}_3 \| \underline{m}_3^*$, eq. (3.11) and (3.7) with the twodimensional phases II, IV' and V leads to three threedimensional phases, which we denote at X, XI and XII respectively. For these phases we again specify the geometrical configurations by the inner products, as well as the functions $\Phi_s(m_1,m_2,m_3)$. We thus have

Phase X: $\underline{m}_3 \| \underline{m}_3^*$, $\underline{m}_1 \| \underline{m}_2^*$, $\underline{m}_1 \cdot \underline{m}_1 = \underline{m}_2 \cdot \underline{m}_2 = \underline{m}_1 \cdot \underline{m}_2^* = 0$, $|\underline{m}_1 \cdot \underline{m}_2| = m_1 m_2$,

$|\underline{m}_3 \cdot \underline{m}_3| = m_3^2$, $\underline{m}_1 \cdot \underline{m}_3 = \underline{m}_1 \cdot \underline{m}_3^* = \underline{m}_2 \cdot \underline{m}_3 = \underline{m}_2 \cdot \underline{m}_3^* = 0$

$$\begin{aligned}\Phi_X(m_1,m_2,m_3) &= u_1 m_1^2 + u_2 m_2^2 + 2u_3 m_3^2 + (2v_1+w_1)m_1^4 + (2v_2+w_1)m_2^4 \\ &+ (6v_3+4w_1+w_2+2w_3+w_3')m_3^4 + (v_4+4w_1+w_2)m_1^2 m_3^2 \\ &+ (v_5+4w_1+w_2)m_2^2 m_3^2 + (2w_1+w_2+w_3+w_3')m_1^2 m_2^2 - 2|v_6-w_3'|m_3^2 m_1 m_2,\end{aligned} \tag{3.12}$$

which for $v_6>w_3'$ is the BW-solution as presented in ref. 26. In the minimization of eq. (3.12) with respect to m_1,m_2 and m_3 one has to distinguish between $v_6>w_3'$ and $v_6<w_3'$. $v_6>w_3'$ leads in the case $b=0$ to the relation $m_1^2=m_2^2=2m_3^2$, which is characteristic for an isotropic superfluid. For arbitrary u_1,u_2,u_3, eq. (3.12) for $v_6>w_3'$ is therefore the generalization of the BW-phase to finite values of the magnetic field. For $v_6<w_3'$, however, the special case $b=0$ leads to $m_1^2=m_2^2 < 2m_3^2$, which describes an anisotropic phase different from the BW-phase. The BW-phase can have a bifurcation with the planar phase II in the limit $m_3 \to 0$.

Phase XI: $\underline{m}_1 \| \underline{m}_1^*$, $\underline{m}_2 \| \underline{m}_2^*$, $\underline{m}_3 \| \underline{m}_3^*$, $|\underline{m}_1 \cdot \underline{m}_1| = m_1^2$, $|\underline{m}_2 \cdot \underline{m}_2| = m_2^2$, $|\underline{m}_3 \cdot \underline{m}_3| = m_3^2$

$|\underline{m}_1 \cdot \underline{m}_2| = |\underline{m}_1 \cdot \underline{m}_2^*|$, $(w_3+w_3')\underline{m}_1 \cdot \underline{m}_2 + (v_6-w_3')m_3^2 = 0$,

$\underline{m}_1 \cdot \underline{m}_3 = \underline{m}_1 \cdot \underline{m}_3^* = \underline{m}_2 \cdot \underline{m}_3 = \underline{m}_2 \cdot \underline{m}_3^* = 0$,

$$\begin{aligned}\Phi_{XI}(m_1,m_2,m_3) &= u_1 m_1^2 + u_2 m_2^2 + 2u_3 m_3^2 + (3v_1+w_1)m_1^4 + (3v_2+w_1)m_2^4 \\ &+ \left(6v_3+4w_1+w_2+2w_3+w_3' - \frac{(v_6-w_3')^2}{w_3+w_3'}\right) m_3^4 \\ &+ (v_4+4w_1+w_2)m_1^2 m_3^2 + (v_5+4w_1+w_2)m_2^2 m_3^2 + (2w_1+w_2+w_3)m_1^2 m_2^2,\end{aligned} \tag{3.13}$$

where $\Phi_{XI}(m_1,m_2,m_3)$ must be minimized as a function of m_1,m_2,m_3 under the condition

$$m_3^2 \le \frac{|w_3+w_3'|}{|v_6-w_3'|} m_1 m_2 . \tag{3.14}$$

Phase XII: $\underline{m}_1 \| \underline{m}_1^* \| \underline{m}_2 \| \underline{m}_2^*$, $\underline{m}_3 \| \underline{m}_3^*$, $|\underline{m}_p \cdot \underline{m}_p| = m_p^2$, $p=1,2,3$,

$$|\underline{m}_1 \cdot \underline{m}_2| = |\underline{m}_1 \cdot \underline{m}_2^*| = m_1 m_2 , \quad \underline{m}_1 \cdot \underline{m}_3 = \underline{m}_1 \cdot \underline{m}_3^* = \underline{m}_2 \cdot \underline{m}_3 = \underline{m}_2 \cdot \underline{m}_3^* = 0 ,$$

$$\begin{aligned}\Phi_{XII}(m_1,m_2,m_3) &= u_1 m_1^2 + u_2 m_2^2 + 2u_3 m_3^2 + (3v_1+w_1)m_1^4 + (3v_2+w_1)m_2^4 \\ &+ (6v_3+4w_1+w_2+2w_3+w_3')m_3^4 + (v_4+4w_1+w_2)m_1^2 m_3^2 \\ &+ (v_5+4w_1+w_2)m_2^2 m_3^2 + (2w_1+w_2+2w_3+w_3')m_1^2 m_2^2 \\ &- 2|v_6-w_3'|m_3^2 m_1 m_2 . \end{aligned} \tag{3.15}$$

For $w_3 > -2(v_1 v_2)^{\frac{1}{2}}$, we have $\Phi_{XII}(m_1,m_2,m_3) - \Phi_X(m_1,m_2,m_3) = v_1 m_1^4 + v_2 m_2^4 + w_3 m_1^2 m_2^2 > 0$, so that phase XII cannot occur under this condition.

3.3. *Phases with complex vector* $\underline{m}_3$

For $\underline{m}_3 \nparallel \underline{m}_3^*$ we have eq. (3.66). In the analysis of this case we have again two possibilities

$$\underline{m}_1 \cdot \underline{m}_2 = 0 , \quad \text{or} \quad \underline{m}_1 \cdot \underline{m}_2 \neq 0 . \tag{3.16}$$

If $\underline{m}_1 \cdot \underline{m}_2 = 0$ the gap equations (3.2a) and (3.2b) are equivalent to the minimization of the twodimensional problem with

$$w_3'' = w_3' , \tag{3.17}$$

as is shown in appendix C. We then have three possible solutions of the two-dimensional problem, namely I(I')III and IV. As shown in appendix C, IV does not lead to threedimensional extensions. For III we have two three-dimensional extensions, with $\underline{m}_1 \| \underline{m}_2 \| \underline{m}_3^*$, and with $\underline{m}_1 \| \underline{m}_2 \| \underline{m}_3$, which we denote by XIII and XIV respectively. For phase I (and I') we have one three-dimensional extension, which is denoted by XV (and XV'). For these phases we specify again the geometrical configurations, the inner products and the function Φ_s. We have

Phase XIII: $\underline{m}_1 \| \underline{m}_2 \| \underline{m}_3^*$, $\underline{m}_1 \cdot \underline{m}_1 = \underline{m}_2 \cdot \underline{m}_2 = \underline{m}_3 \cdot \underline{m}_3 = \underline{m}_1 \cdot \underline{m}_2 = \underline{m}_1 \cdot \underline{m}_3^* = \underline{m}_2 \cdot \underline{m}_3^* = 0$

$$|\underline{m}_1 \cdot \underline{m}_3| = m_1 m_3 , \quad |\underline{m}_2 \cdot \underline{m}_3| = m_2 m_3 ,$$

$$\begin{aligned}\Phi_{XIII}(m_1,m_2,m_3) &= u_1 m_1^2 + u_2 m_2^2 + 2u_3 m_3^2 + (2v_1+w_1)m_1^4 + (2v_2+w_1)m_2^4 \\ &+ (4v_3+4w_1+w_2+2w_3)m_3^4 + (2v_4+4w_1+w_2)m_1^2 m_3^2 \\ &+ (2v_5+4w_1+w_2)m_2^2 m_3^2 + (2w_1+w_2+2w_3)m_1^2 m_2^2 . \end{aligned} \tag{3.18}$$

Phase XIV: $\underline{m}_1 \| \underline{m}_2 \| \underline{m}_3$, $\underline{m}_1 \cdot \underline{m}_1 = \underline{m}_2 \cdot \underline{m}_2 = \underline{m}_3 \cdot \underline{m}_3 = \underline{m}_1 \cdot \underline{m}_2 = \underline{m}_1 \cdot \underline{m}_3 = \underline{m}_2 \cdot \underline{m}_3 = 0$,

$|\underline{m}_1 \cdot \underline{m}_3^*| = m_1 m_3$, $|\underline{m}_2 \cdot \underline{m}_3^*| = m_2 m_3$,

$$\begin{aligned} \Phi_{XIV}(m_1,m_2,m_3) = {} & u_1 m_1^2 + u_2 m_2^2 + 2u_3 m_3^2 + (2v_1+w_1)m_1^4 + (2v_2+w_1)m_2^4 \\ & + (4v_3+4w_1+w_2+2w_3)m_3^4 + (2v_4+4w_1)m_1^2 m_3^2 + (2v_5+4w_1)m_2^2 m_3^2 \\ & + (2w_1+w_2+2w_3)m_1^2 m_2^2 - 2|2v_6-w_2-2w_3|m_3^2 m_1 m_2 \quad . \end{aligned} \tag{3.19}$$

Phases XIII and XIV can both be regarded as threedimensional extensions of the ABM-phase III. Phase XIV has already been introduced in ref. 26 in the special case $w_1=w_2=w_3=w_3'=0$, whereas XIII is unfavourable in comparison with XIV for $w_2 \geq 0$.

Phase XV: $\underline{m}_2 \| \underline{m}_2^*$, $\underline{m}_1 \| \underline{m}_3^*$, $|\underline{m}_2 \cdot \underline{m}_2| = m_2^2$, $|\underline{m}_1 \cdot \underline{m}_3| = m_1 m_3$,

$\underline{m}_1 \cdot \underline{m}_1 = \underline{m}_3 \cdot \underline{m}_3 = \underline{m}_1 \cdot \underline{m}_2 = \underline{m}_1 \cdot \underline{m}_2^* = \underline{m}_1 \cdot \underline{m}_3^* = \underline{m}_2 \cdot \underline{m}_3 = \underline{m}_2 \cdot \underline{m}_3^* = 0$,

$$\begin{aligned} \Phi_{XV}(m_1,m_2,m_3) = {} & u_1 m_1^2 + u_2 m_2^2 + 2u_3 m_3^2 + (2v_1+w_1)m_1^4 + (3v_2+w_1)m_2^4 \\ & + (4v_3+4w_1+w_2+2w_3)m_3^4 + (2v_4+4w_1+w_2)m_1^2 m_3^2 \\ & + (v_5+4w_1+w_2)m_2^2 m_3^2 + (2w_1+w_2+w_3)m_1^2 m_2^2 \quad . \end{aligned} \tag{3.20}$$

Phase XV can only be an absolute minimum for $w_2<0$, since, taking the configuration $\underline{m}_2 \| \underline{m}_2^*$, $\underline{m}_1 \| \underline{m}_3$, with $|\underline{m}_2 \cdot \underline{m}_2| = m_2^2$, $|\underline{m}_1 \cdot \underline{m}_3^*| = m_1 m_3$, $\underline{m}_1 \cdot \underline{m}_1 = \underline{m}_3 \cdot \underline{m}_3 = \underline{m}_1 \cdot \underline{m}_2 = \underline{m}_1 \cdot \underline{m}_2^* = \underline{m}_1 \cdot \underline{m}_3 = \underline{m}_2 \cdot \underline{m}_3 = \underline{m}_2 \cdot \underline{m}_3^* = 0$, which is *not* a solution of the gap equations (1.15), we have

$$\Phi(\underline{m}_1,\underline{m}_2,\underline{m}_3) = \Phi_{XV} - w_2(m_1^2+m_2^2)m_3^2 \quad . \tag{3.21}$$

By interchanging $\underline{m}_1$ and $\underline{m}_2$, u_1 and u_2, v_1 and v_2 and v_4 and v_5, we get another phase XV', given by

Phase XV': $\underline{m}_1 \| \underline{m}_1^*$, $\underline{m}_2 \| \underline{m}_3^*$, $|\underline{m}_1 \cdot \underline{m}_1| = m_1^2$, $|\underline{m}_2 \cdot \underline{m}_3| = m_2 m_3$,

$\underline{m}_2 \cdot \underline{m}_2 = \underline{m}_3 \cdot \underline{m}_3 = \underline{m}_1 \cdot \underline{m}_2 = \underline{m}_1 \cdot \underline{m}_2^* = \underline{m}_1 \cdot \underline{m}_3 = \underline{m}_1 \cdot \underline{m}_3^* = \underline{m}_2 \cdot \underline{m}_3^* = 0$,

$$\begin{aligned} \Phi_{XV'}(m_1^2,m_2^2,m_3^2) = {} & u_1 m_1^2 + u_2 m_2^2 + 2u_3 m_3^2 + (3v_1+w_1)m_1^4 + (2v_2+w_1)m_2^4 \\ & + (4v_3+2w_1+w_2+2w_3)m_3^4 + (v_4+4w_1+w_2)m_1^2 m_3^2 \\ & + (2v_5+4w_1+w_2)m_2^2 m_3^2 + (2w_1+w_2+w_3)m_1^2 m_2^2 \quad . \end{aligned} \tag{3.22}$$

Phases XV and XV' may occur actually in the phase diagram for sufficiently large values of the spin fluctuation coefficients w , as will be discussed in section 5

Finally, taking $\underline{m}_1 \cdot \underline{m}_2 \neq 0$, it is clear that the gap equations (3.2a) and (3.2b) are equivalent to the twodimensional problem (3.7) with

$$w_3'' = w_3' - (v_6 - w_3')/(2v_3 + w_3') . \tag{3.23}$$

In this case one can show that

$$\underline{m}_1 \cdot \underline{m}_3 = \underline{m}_1 \cdot \underline{m}_3^* = \underline{m}_2 \cdot \underline{m}_3 = \underline{m}_2 \cdot \underline{m}_3^* = 0 , \tag{3.24}$$

cf. appendix C. Therefore $\underline{m}_1$ and $\underline{m}_2$ must be real vectors perpendicular to the plane of $\underline{m}_3$ and $\underline{m}_3^*$. The only possible twodimensional solution then is V, and the threedimensional extension leads to a phase XVI given by

Phase XVI: $\underline{m}_1 \parallel \underline{m}_1^* \parallel \underline{m}_2 \parallel \underline{m}_2^*$, $|\underline{m}_1 \cdot \underline{m}_1| = m_1^2$, $|\underline{m}_2 \cdot \underline{m}_2| = m_2^2$,

$(2v_3+w_3')\underline{m}_3 \cdot \underline{m}_3 + (v_6 - w_3')m_1 m_2 = 0$, $|\underline{m}_1 \cdot \underline{m}_2| = |\underline{m}_1 \cdot \underline{m}_2^*| = m_1 m_2$,

$\underline{m}_1 \cdot \underline{m}_3 = \underline{m}_1 \cdot \underline{m}_3^* = \underline{m}_2 \cdot \underline{m}_3 = \underline{m}_2 \cdot \underline{m}_3^* = 0$,

$$\begin{aligned}\Phi_{XVI}(m_1,m_2,m_3) = {}& u_1 m_1^2 + u_2 m_2^2 + 2u_3 m_3^2 + (3v_1+w_1)m_1^4 + (3v_2+w_2)m_2^4 \\ & + (4v_3+4w_1+w_2+2w_3)m_3^4 + (v_4+4w_1+w_2)m_1^2 m_3^2 \\ & + (v_5+4w_1+w_2)m_2^2 m_3^2 + \left(2w_1+w_2+2w_3+w_3' - \frac{(v_6-w_3')^2}{2v_3+w_3'}\right) m_1^2 m_2^2 ,\end{aligned} \tag{3.25}$$

where $\varphi_{XVI}(m_1,m_2,m_3)$ must be minimized under the condition

$$m_3^2 \geq \frac{|v_6 - w_3'|}{|2v_3 + w_3'|} m_1 m_2 . \tag{3.26}$$

Phase XVI has been given before in ref. 26 and was called the threedimensional ε-solution, as an extension of solution VII given in section 2.

The threedimensional solutions XIII-XVI admit several bifurcations to the twodimensional solutions treated in section 2. These bifurcations have been indicated schematically in fig. 1, where we have also indicated the limiting behaviour of the relevant order parameter.

We have now completed the whole list of solutions which have real inner products, and which satisfy the inertia condition, and for which we assume that they are the physically relevant ones. Some solutions such as VI, VIII and XV are new, and can play an important role in certain regions of the spin fluctuation parameters.

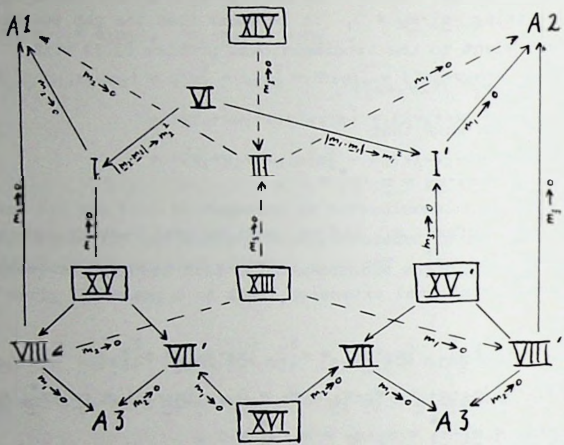

Figure 1. Scheme of possible bifurcations of the threedimensional phases XIII-XVI to phases of lower dimensionality.

4. Phase diagram under small perturbations of the Hubbard type

In the previous sections we have presented a large collection of solutions of the gap equations (1.15), namely A1-A3, I-XVI, which are the solutions with real inner products satisfying the inertia condition (3.1), and which we believe to be the physically relevant phases. In this section we investigate the possibility of changes in the phase diagram, as obtained in ref. 26, in the absence of spin fluctuation effects, under the influence of a perturbation of the Hubbard type.

For sufficiently small Hubbard interaction it can be shown that only four phases must be taken into account, i.e. phases

A1, I, VI, X:

We choose as before $\eta b<0$, so that phases

A2, I', I''', VII', VIII',

are ruled out immediately. Under rather general conditions for the coefficients w_2, w_3, w_3', such as e.g. the assumption that the signs of the coefficients are correctly given by the second order result in (1.13), one can show that the phases

A3, IV, IV', V, IX, XI, XII, XIII, XIV, XVI ,

cannot lead to an absolute minimum of the Landau expansion. For the phases IV, IV' and V this was already shown in subsection 2.2, and for phase XII in subsection 3.2. The phases A3, IX, XI, XIII, XIV and XVI are treated in appendix D. Furthermore, for sufficiently small values of w_1, w_2, w_3, w_3' one can also rule out the phases

VII, VIII, XV, XV' ,

as is shown in appendix E, so that we are left with

A1, I, II, III, VI, X .

Later on in this section it will also be shown that II and III will not occur for sufficiently small Hubbard interaction.

In order to investigate the qualitative changes in the phase diagram for small Hubbard coefficients, it is convenient to take $b=0$ in the coefficients of the fourth order terms in the Landau expansion, i.e.

$$v_1 = v_2 = v_3 = v, \quad v_4 = v_5 = 4v, \quad v_6 = 2v. \tag{4.1}$$

In ref. 26 we argued that the b-dependent terms in fourth order are important for determining the occurrence of certain phases. This is so, because in the absence of spin fluctuations there is a great deal of symmetry in the problem, implying a large degeneracy of phases. The b-dependence in the fourth order terms then leads to the symmetry breaking which is necessary to distinguish between the different phases. In the presence of the Hubbard interactions, the situation is different, and a large part of the degeneracies is lifted. Hence, we may assume that taking $b=0$ in the fourth order terms does not give any problem in determining the qualitative features of the phase diagram in the presence of a magnetic field.

Taking into account eq. (4.1) one can write down the following simplified expressions for the free energies of phases I, II, III, VI

$$f_I = -\tfrac{1}{4}\,\frac{(u_1+u_2)^2}{5v+4w_1+w_2+w_3-\dfrac{v^2}{5v-w_2-w_3}} - \tfrac{1}{4}\,\frac{(u_1-u_2)^2}{5v-w_2-w_3-\dfrac{v^2}{5v+4w_1+w_2+w_3}}$$

$$+\tfrac{1}{2}v\,\frac{(u_1+u_2)(u_1-u_2)}{(5v+4w_1+w_2+w_3)(5v-w_2-w_3)-v^2}\,, \tag{4.2}$$

$$f_{II} = -\tfrac{1}{4}\,\frac{(u_1+u_2)^2}{4v+4w_1+w_2+w_3+w_3'} - \tfrac{1}{4}\,\frac{(u_1-u_2)^2}{4v-w_2-w_3-w_3'}\,, \tag{4.3}$$

$$f_{III} = -\tfrac{1}{4}\,\frac{(u_1+u_2)^2}{4v+4w_1+w_2+2w_3} - \tfrac{1}{4}\,\frac{(u_1-u_2)^2}{4v-w_2-w_3-2w_3}\,, \tag{4.4}$$

$$f_{VI} = -\tfrac{1}{4}\,\frac{(u_1+u_2)^2}{4v+4w_1+w_2+w_3+\dfrac{2\gamma v}{2v+\gamma}} - \tfrac{1}{4}\,\frac{(u_1-u_2)^2}{4v-w_2-w_3-\dfrac{2\gamma v}{2v-\gamma}}\,,\quad \gamma = \frac{w_3 w_3'}{w_3+w_3'}\,. \tag{4.5}$$

The corresponding values of m_1^2 and m_2^2 at the minimum can be inferred from $m_1^2 = -\partial f_s/\partial u_1$, $m_2^2 = -\partial f_s/\partial u_2$. From the three conditions (2.7), only the last condition is not trivially satisfied for small w_1, w_2, w_3, w_3'. In the limit that the left-hand side of eq. (2.7) tends to zero, i.e.

$$(2w_1+w_2+w_3+a_{12})u_1 - 2(2v+w_1+a_{11})u_2 \to 0\,, \tag{4.6}$$

we have $m_2^2 \to 0$, and therefore eq. (4.6) describes the bifurcations of I, II, III and VI with the A1-phase.

Taking into account eq. (4.1) and the conditions

$$w_3>0,\quad w_3'>0\,,\quad 2\gamma < w_2+2w_3,\quad 2\gamma < w_2+w_3+w_3'\,, \tag{4.7}$$

in which the signs in the first two inequalities correspond to the ones in (1.13) and the two remaining inequalities for γ are well obeyed in a rather wide range of the parameter values, including e.g. the ones given in (1.13), it can be shown that the phases II and III do not occur. In fact, from eq. (2.6) with (2.11), (2.13) and (4.1) we have

$$f_{III} - f_{VI} = \frac{v\gamma^2}{4v^2-\gamma^2}\,(m_1^2-m_2^2)^2 + \left(w_3 - \frac{2v\gamma}{2v+\gamma}\right) m_1^2 m_2^2\,, \tag{4.8}$$

and this is positive, as

$$w_3 > \frac{2\gamma v}{2v+\gamma} \,, \tag{4.9a}$$

$$2v > w_3 \,. \tag{4.9b}$$

Eq. (4.9a) is trivially satisfied, and (4.9b) follows from (4.7) together with the first condition (2.7) for the ABM-phase, i.e.

$$4v > w_2+2w_3. \tag{4.10}$$

This argument only holds provided that VI exists, but when this is not the case, cf. eq. (2.20), phase I has a lower free energy than the right-hand side of eq. (4.5), so that $f_{III} - f_{VI} > 0$. A similar reasoning can be applied with regard to phase II.

Let us now turn to the specific features of the phase diagram involving A1, I, VI, X. For $u_1>0$ no ordering appears, i.e. we have the normal fluid phase (N). For $u_1=0$ we have a second order phase transition from N to the A1-phase. From A1 we can have second order transitions to II or III, or to I, but the first two possibilities are ruled out by the argument given above. Hence there is a second order transition from A1 to I, which occurs for

$$\text{A1} \to \text{I}: (2w_1+w_2+w_3)u_1 = 2(2v_1+w_1)u_2 \,, \tag{4.11}$$

cf. eqs. (4.6), (2.7) and (2.8). As we lower the temperature there will occur a second order phase transition from I to VI, as already indicated at the end of subsection 2.2. The condition for the phase transition follows from eq. (2.20). Inserting into eq. (2.20) the values for m_1^2, m_2^2 as obtained from eqs. (2.5) and (2.15), or equivalently eqs. (2.5) and (2.8), we find for the second order phase transition

$$\begin{aligned} \text{I} \to \text{VI}: 2v_2[-2(2v_1+w_1)u_2 + (2w_1+w_2+w_3)u_1] = \\ = \gamma[-2(3v_2+w_1)u_1 + (2w_1+w_2+w_3)u_2] \,. \end{aligned} \tag{4.12}$$

Taking $v_1=v_2=v$, as in eq. (4.1), it is easy to show that the transition takes place at a larger value of u_2/u_1, i.e. a smaller value of b, than the transition from I to A1. Hence, under the conditions (4.1) and (4.7) we get the following picture, when only onedimensional and twodimensional phases are taken into account

$$\frac{u_1-u_2}{u_1+u_2} < \frac{(2v-\gamma)\left(4v-w_2-w_3-\frac{2\gamma v}{2v-\gamma}\right)}{(2v+\gamma)\left(4v+4w_1+w_2+w_3+\frac{2\gamma v}{2v+\gamma}\right)} \Longrightarrow \text{Phase VI},$$

$$\frac{(2v-\gamma)\left(4v-w_2-w_3-\frac{2\gamma v}{2v-\gamma}\right)}{(2v+\gamma)\left(4v+4w_1+w_2+w_3+\frac{2\gamma v}{2v+\gamma}\right)} < \frac{u_1-u_2}{u_1+u_2} < \frac{4v-w_2-w_3}{4v+4w_1+w_2+w_3} \Longrightarrow \text{Phase I},$$

$$u_1<0\ ,\quad \frac{u_1-u_2}{u_1+u_2} > \frac{4v-w_2-w_3}{4v+4w_1+w_2+w_3} \Longrightarrow \text{Phase A1}. \tag{4.13}$$

To complete the treatment of the phase diagram under the influence of a small Hubbard term, we must take into account the BW-solution, given by eq. (3.12). Introducing

$$\lambda \equiv m_2/m_1\ ,\quad 0\leq\lambda\leq 1\ , \tag{4.14}$$

eq. (3.12) can be expressed as

$$\Phi_X(m_1,m_2,m_3) = \bar{\Phi}(m_1,m_3;\lambda) \equiv \bar{u}_1 m_1^2 + 2u_3 m_3^2 + (2v+w_1+a_{11})m_1^4$$
$$+ (4v+4w_1+w_2+2w_3+a_{33})m_3^4 + (4v+4w_1+w_2+a_{13})m_1^2 m_3^2\ , \tag{4.15}$$

with

$$\bar{u}_1 = u_1 + \lambda^2 u_2,$$
$$a_{11} = a_{11}(\lambda) = (2v+w_1)\lambda^4 + (2w_1+w_2+w_3+w_3')\lambda^2\ ,$$
$$a_{13} = a_{13}(\lambda) = (4v+4w_1+w_2)\lambda^2 - 2(2v-w_3')\lambda\ ,$$
$$a_{33} = 2v + w_3'\ . \tag{4.16}$$

The free energy f_X can be expressed as

$$f_X = \min_{\lambda} f(\lambda)\ ,\quad f(\lambda) = \min_{m_1,m_3} \bar{\Phi}(m_1,m_3;\lambda)\ . \tag{4.17}$$

From (4.15) it is clear that $f(\lambda)$ is given by eq. (2.25) with u_1 replaced by $\bar{u}_1$, and with the values of a_{11}, a_{13}, a_{33} as given in (4.16) inserted. The conditions of existence can be inferred from (2.26) with the same replacements. The first two conditions (2.26) are trivially satisfied, but the third condition, which is equivalent to $m_3^2\geq 0$, leads to

$$-4u_3(2v+w_1+a_{11}) + (4v+4w_1+w_2+a_{13})(u_1+\lambda^2 u_2) > 0\ . \tag{4.18}$$

The values of λ at the extremum is given by

$$\partial f/\partial\lambda = [2\lambda u_2 + m_1^2(\partial a_{11}/\partial\lambda) + m_3^2(\partial a_{13}/\partial\lambda)]m_1^2$$

$$= \Big\{2\lambda u_2 + N(\lambda)^{-1}\,\tilde{u}_1\left[-2(4v+4w_1+w_2+2w_3+a_{33})\frac{\partial a_{11}}{\partial\lambda} + (4v+4w_1+w_2+a_{13})\frac{\partial a_{13}}{\partial\lambda}\right]$$

$$N(\lambda)^{-1}\,u_3\left[2(4v+vw_1+w_3+a_{13})\frac{\partial a_{11}}{\partial\lambda} - 4(2v+w_1+a_{11})\frac{\partial a_{13}}{\partial\lambda}\right]\Big\}\,m_1^2 = 0\ .$$

$$N(\lambda) \equiv 4(4v+4w_1+w_2+2w_3+a_{33})(2v+w_1+a_{11}) - (4v+4w_1+w_2+a_{13})^2\ . \tag{4.19}$$

First order transitions from the BW-phase to the other phases VI, I and A1 can occur under the condition

$$f(\lambda) = \tfrac{1}{2}(f_{11}u_1^2 + f_{12}u_1u_2 + f_{22}u_2^2)\ , \tag{4.20}$$

in which f_{11}, f_{12} and f_{22} for the different phases are given by

$$f_{11} = \tfrac{1}{2}(2v+w_1)^{-1},\quad f_{12} = f_{22} = 0\ ,\qquad (A1)\ . \tag{4.21a}$$

$$f_{11} = \frac{2(3v+w_1)}{(4v+2w_1)(6v+2w_1) - (2w_1+w_2+w_3)^2}\ ,\quad f_{22} = \frac{2(2v+w_1)}{(4v+w_1)(6v+w_1) - (2w_1+w_2+w_3)^2}\ ,$$

$$f_{12} = \frac{-2(2w_1+w_2+w_3)}{(4v+2w_1)(6v+2w_1) - (2w_1+w_2+w_3)^2}\ ,\qquad (I)\ . \tag{4.21b}$$

$$f_{11} = f_{22} = \tfrac{1}{2}\left(4v+4w_1+w_2+w_3 + \frac{2\gamma v}{2v+\gamma}\right)^{-1} + \tfrac{1}{2}\left(4v-w_2-w_3 - \frac{2\gamma v}{2v-\gamma}\right)^{-1}$$

$$f_{12} = \left(4v+4w_1+w_2+w_3 + \frac{2\gamma v}{2v+\gamma}\right)^{-1} - \left(4v-w_2-w_3 - \frac{2\gamma v}{2v-\gamma}\right)^{-1}\ ,\qquad (VI)\ . \tag{4.21c}$$

Inserting the explicit formula (2.25) with $u_1 \to \tilde{u}_1$ for $f(\lambda)$, and $\tilde{u}_1 = u_1 + \lambda^2 u_2$ in (4.20), we obtain

$$\tfrac{1}{2}f_{11}\tilde{u}_1^2 + \tfrac{1}{2}(f_{12}-2\lambda^2 f_{11})\tilde{u}_1u_2 + \tfrac{1}{2}(f_{22}-f_{12}\lambda^2+f_{11}\lambda^4)u_2^2 =$$

$$= N(\lambda)^{-1}\left[(4v+4w_1+w_2+2w_3+a_{33})\tilde{u}_1^2 - 2(4v+4w_1+w_2+a_{13})\tilde{u}_1u_3 + 4(2v+w_1+a_{11})u_3^2\right]. \tag{4.22}$$

Inserting (4.19) in (4.22) one obtains a quadratic equation in $u_3/\tilde{u}_1$, from which we can solve

$$u_3/\tilde{u}_1 \equiv (u_3/\tilde{u}_1)(\lambda)\ , \tag{4.23}$$

in which the right-hand side depends on λ and on the parameters v, w_1, w_2, w_3, w_3'. Substituting (4.23) into (4.19), one obtains

$$u_2/\bar{u}_1 = (u_2/\bar{u}_1)(\lambda) \;, \tag{4.24}$$

and also

$$\frac{u_2}{u_1} = \frac{(u_2/\bar{u}_1)(\lambda)}{1-\lambda^2(u_2/\bar{u}_1)(\lambda)} = \frac{u_2}{u_1}(\lambda) \;, \quad \frac{u_3}{u_1} = \frac{(u_3/\bar{u}_1)(\lambda)}{1-\lambda^2(u_2/\bar{u}_1)(\lambda)} = \frac{u_3}{u_1}(\lambda) \;. \tag{4.25}$$

The functions (4.25) give the parameter representation in terms of λ of the first order lines between BW and the phases VI, I, A1. In comparing the three first order lines, it is easy to see that the phase transition corresponding to the lowest value of b, or to the largest value of u_2/u_1, will actually occur. Suppose e.g. that at the largest value of u_2/u_1 we have $f_X = f_I$, then we have $f_X = f_{A1}$ at a smaller value of u_2/u_1, implying that $f_I < f_{A1}$ at the largest value of u_2/u_1. Similarly, we have $f_I < f_{VI}$ at the largest value of u_2/u_1, so that the phase transition will occur from BW to I.

In the special case $\eta b=0$, we have

$$f_X = -\frac{u^2+\frac{2}{3}u(u_3-u)}{\frac{10}{3}v+4w_1+w_2+\frac{2}{3}w_3+w_3'} - \frac{1}{3}\frac{(4v+4w_1+w_2+w_3+w_3')(u_3-u)^2}{(2v+w_3)(\frac{10}{3}v+4w_1+w_2+\frac{2}{3}w_3+w_3')} \;, \tag{4.26}$$

($u_1=u_2=u$), implying with eq. (4.5) that b^2 on the transition line from BW to VI will increase proportional to $|t|$, but then $|\eta b|/|t|$ will decrease proportional to $|t|^{-\frac{1}{2}}$, so that the asymmetry in the density of states becomes unimportant far below T_c. For $T\to 0$, we therefore have $\lambda\to 1$ on the first order line, and λ will decrease to some finite nonnegative value λ_0, for $T\uparrow T_c$, which may be determined inserting $u_3 \simeq u_1+u_2 = u_1+u_2(1-\lambda^2)$ in eqs. (4.19), (4.22) and eliminating $u_2/\tilde{u}_1$ from both expressions.

There are no second order transitions between BW and VI, and between BW and I, but in principle a second order transition between BW and A1 might occur in the limit

$$\lambda\downarrow 0, \quad m_3^2\downarrow 0 \;, \tag{4.27a}$$

$$\left(-2u_3 + \frac{4v+4w_1+w_2}{4v+2w_1}u_1\right) \downarrow 0 \;. \tag{4.27b}$$

From (4.19) and $\partial a_{13}(\lambda)/\partial\lambda\big|_{\lambda=0} < 0$, we have, on the one hand

$$u_2 - \frac{2w_1+w_2+w_3+w_3'}{4v+2w_1} u_1 > 0 \tag{4.28}$$

at the transition, whereas on the other hand with $-2u_3+u_1+u_2 < 0$, we have from eq. (4.27b)

$$u_2 - \frac{2w_1+w_2}{4v+2w_1} u_1 < 0 \tag{4.29}$$

at the transition. In the absence of spin fluctuation contributions (4.28) and (4.29) cannot be satisfied at the same time, cf. ref. 26, so that a second order transition is not possible. In the case of a small Hubbard interaction, (4.28) and (4.29) can be satisfied, but even then there will be no second order transition between BW and A1. In fact, at an arbitrary extremum of $f(\lambda)$, taking the derivative at the minimum of $\bar{\Phi}$ as in (4.17), i.e.

$$\frac{d}{d\lambda} = \frac{dm_1}{d\lambda}\frac{\partial}{\partial m_1} + \frac{dm_3}{d\lambda}\frac{\partial}{\partial m_3} + \frac{\partial}{\partial \lambda}, \tag{4.30}$$

we have

$$\frac{d^2}{d\lambda^2} f(\lambda) = \frac{\partial^2\bar{\Phi}}{\partial\lambda^2} + \frac{2\dfrac{\partial^2\bar{\Phi}}{\partial m_1\partial\lambda}\dfrac{\partial^2\bar{\Phi}}{\partial m_3\partial\lambda}\dfrac{\partial^2\bar{\Phi}}{\partial m_1\partial m_3} - \left(\dfrac{\partial^2\bar{\Phi}}{\partial m_1\partial\lambda}\right)^2\dfrac{\partial^2\bar{\Phi}}{\partial m_3^2} - \left(\dfrac{\partial^2\bar{\Phi}}{\partial m_3\partial\lambda}\right)^2\dfrac{\partial^2\bar{\Phi}}{\partial m_1^2}}{\dfrac{\partial^2\bar{\Phi}}{\partial m_1^2}\dfrac{\partial^2\bar{\Phi}}{\partial m_3^2} - \left(\dfrac{\partial^2\bar{\Phi}}{\partial m_1\partial m_3}\right)^2} \tag{4.31}$$

Evaluating the second derivatives from (4.15) with the use of the gap equations and taking the limit $\lambda\downarrow 0$, it can be shown that (taking $m_3\to 0$)

$$\left.\frac{d^2}{d\lambda^2} f(\lambda)\right|_{\lambda=0} = 2u_2m_1^2 + 2(2w_1+w_2+w_3+w_3')m_1^4 - \frac{8(2v+w_1)(2v-w_3')^2m_1^4}{4(2v+w_1)(6v+4w_1+w_2+2w_3+w_3') - (4v+4w_1+w_2)^2} \tag{4.32}$$

With the use of (4.29) it is clear that for sufficiently small w_1, w_2, w_3, w_3'

$$\left.\frac{d^2}{d\lambda^2} f(\lambda)\right|_{\lambda=0} < 0, \tag{4.33}$$

implying that in that case there is a first order transition between BW and A1.

From the considerations given above we can draw a qualitative picture of

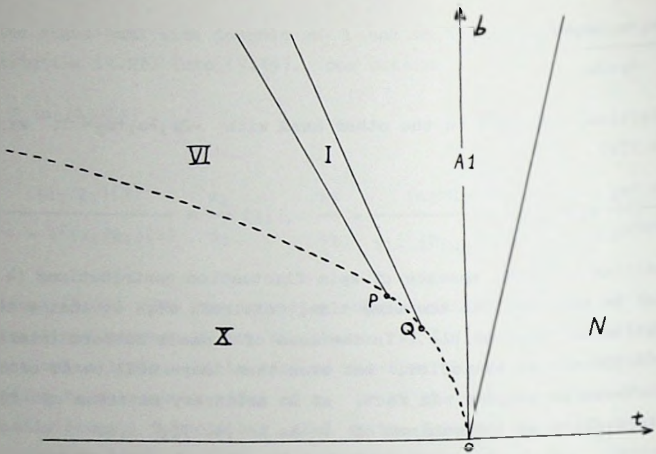

Figure 2. Qualitative picture of the phase diagram for ^3He *in the presence of a small Hubbard interaction. b is the magnetic field and* $t = \frac{1}{2}\mathcal{N}(0) \times (T-T_c)/T_c$ *the reduced temperature. N, A1, I, VI and X denote the phases that occur, as explained in the text. The solid lines are second order transitions, whereas the dashed lines are first order. P and Q denote critical endpoints.*

the phase diagram in the presence of a small Hubbard interaction, as given in figure 2. In this phase diagram the phase transitions between VI and I, between I and A1, and between A1 and N are second order, and the analytical expressions for these lines are given in eq. (4.13). The phase transitions between BW (X) and VI, I and A1 are first order, and the expressions for them can be inferred from (4.25) together with eqs. (4.22) and (4.19), cf. also (4.23) and (4.24). The points P and Q are critical endpoints at which two equal phases (namely VI and I, and I and A1, respectively) are in equilibrium with a third phase (the BW-phase). In comparing figure 2 with the phase diagram obtained in ref. 26 in the absence of a Hubbard interaction, the main difference is, that in the latter phase diagram there is no phase I and the critical endpoints P and Q are merged together. In that case ($w_2=w_3=w_3'=0$) phase VI is degenerate with several twodimensional phases, including the planar phase II and the ABM-phase III, but for $w_3>0$, $w_3'>0$ this degeneracy is lifted in favour of the phase VI. Also on the first order line of the phase diagram in ref. 26 there is a tricritical point, at the left of P, at which the first order transition between BW and II changes into a second order transition. In the present situation the transition at the left of P is always first order. Fig. 2 gives the situation for sufficiently small values of w_1, w_2, w_3, w_3', with the signs as given in (1.13). Numerical calculations indicate that for larger values of w_1, w_2, w_3, w_3' a tricritical point can appear, at which the transition between BW and A1 becomes of second order. This will be discussed in a future publication [34].

5. Phase diagram for b=0

In the previous section we have shown that in the presence of a sufficiently small Hubbard interaction many of the solutions A1-A3, I-XVI do not occur. In fact, we showed that one only needs to take into account the phases A1, I, VI and X, for a qualitative description of the phase diagram. For $b=0$, however, there are many simplifying features. In this section we study the phase diagram for $b=0$, also for larger values of w_1, w_2, w_3, w_3' under the restrictions

$$w_2<0, \quad w_3>0, \quad w_3'>0, \quad q \equiv -w_2/w_3 < 1 \ . \tag{5.1}$$

The three inequalities in eq. (5.1) express that the signs of w_2, w_3 and w_3' are the same as in eq. (1.13), and $q<1$ is well obeyed in a large range of parameter values including the ones in (1.13)

5.1. *Free energy of phases at b=0*

We now present a list of the free energies of the possible phases at $b=0$ together with their regions of existence in the space of spin fluctuation parameters satisfying (5.1). In section 4 we mentioned that the phases

A2, I', I''', VII', VIII'

cannot occur for $\eta b<0$. For $\eta b=0$, however, these phases are degenerate with the phases A1, I, I", VII and VIII respectively. Nevertheless, we only choose those phases that pertain for infinitesimal ηb. Furthermore it was shown that the phases

A3, IV, IV', V, IX, XI-XIV, XVI

do not lead to an absolute minimum of the Landau expansion. This is so for $b \neq 0$, but in the special case $b=0$ phase XVI is degenerate with VI, phase XI is degenerate with X for $2v<w_3'$, and phase XIV for $4v<w_2+2w_3$ is degenerate with A1. From the phases XV and XV', phase XV is favourable, cf. appendix E, with respect to XV' for $\eta b<0$, but for $b=0$ these two phases are degenerate. Finally comparing eqs. (4.3) and (4.4) with (4.5), it is clear that for $\eta b=0$, $(u_1=u_2)$,

$$f_{II} > f_{VI} \ , \quad f_{III} > f_{VI} \ . \tag{5.2}$$

Hence we are left with the phases

$$\text{A1, I, VI, VII, VIII, X, XV} \ . \tag{5.3}$$

All these phases exist under the condition $u<0$ $(u_1=u_2=u_3=u$, $b=0)$ together with (5.1) and some extra conditions denoted by $C_s>0$, depending on the type of solution. The free energy of the different phases can be expressed as

$$f_s = -u^2/N_s \ , \quad (N_s>0) \ , \tag{5.4}$$

and in table 1.we indicate N_s, C_s for the phases (5.3), and also the phases which are degenerate at $b=0$, but will not occur for $\eta b<0$. N_{A1} has been given given in eq. (2.2), and N_I and N_{VI} follow immediately from eqs. (4.2) and (4.5). $C_I=0$ describes the bifurcation at b=0 between the phases I and A1, whereas $C_{VI}=0$ describes the bifurcation at $b=0$ between the phases VI and I. N_{VII} and N_{VIII} can be inferred from eqs. (2.25) with (2.27) and (2.28), and $C_{VIII}=0$ is the bifurcation condition between VIII and A3, which, however, does not occur in practice.

N_X can be derived taking e.g. the minimum of eq. (4.15) as a function of m_1, m_3 at $\lambda=1$, with $a_{11}=2v+3w_1+w_2+w_3+w_3'$, $a_{13}=4v+w_1+w_2-2|2v-w_3'|$, $a_{33}=2v+w_3'$, and N_{XV} follows directly from (E.13) and (E.12) of appendix E. The condition $C_{XV}=0$ turns out to be important for determining the phase diagram.

Table 1. Possible phases s under the condition (5.1) at b=0, together with the degenerate phases, the denominators N_s and additional conditions C_s of existence.

Phase s	Degenerate phase	N_s	C_s
A1	A2, XIV ($4v < w_2+2w_3$)	$8v+4w_1$	-
I	I'	$5v+4w_1+w_2+w_3 - v^2/(5v-w_2-w_3)$	$4v-w_2-w_3$
VI	XVI	$4v+4w_1+w_2+w_3 + \frac{2\gamma v}{2v+\gamma}$	$4v-w_2-w_3-\frac{2\gamma v}{2v-\gamma}$
VII	VII'	$4v+4w_1+w_2+w_3 + \frac{8v-w_2-2w_3}{8v-w_2+2w_3} w_3$	-
VIII	VIII'	$8v+4w_1+w_2^2/(4v+w_2-2w_3)$	w_3-2v
X	XI (with X for $2v<w_3'$)	$4v+4w_1+w_2+w_3+w_3' - \frac{(w_3+w_3'+\|2v-w_3'\|)^2}{2v+3w_3+2w_3'+2\|2v-w_3'\|}$	-
XV	XV'	$\frac{16}{3}v+4w_1+w_2+\frac{2}{3}w_3$	$w_3-v)(5v-w_2-w_3$ $-3v^2$

Remark. For the sake of completeness we also list the denominators N_s, together with the degenerate solutions at b=0 for the solutions s which do not occur under the condition (5.1), cf. table 2. For the solutions in table 2 we have not investigated, however, how the degeneracy will be lifted for $\eta b<0$. Solution XIII does not exist for $w_3'>2v$, and tables 1 and 2 contain the complete information on the denominators N_s for all solutions s at b=0, independent of the values of the spin fluctuation parameters w.

Table 2. The remaining solutions s at b=0, together with the degeneracies and the denominators N_s for the free energies.

Solution s	Degenerate solutions	N_s
R1	R2, IX for $6v < w_2+2w_3+w_3'$	$12v+4w_1$
II	XII for $w_3'<2v$	$4v+4w_1+w_2+w_3+w_3'$
III	A3, XIII for $w_3<2v$, XIV for $w_2+2w_3 < 4v$	$4v+4w_1+w_2+2w_3$
IV	IV', XII for $w_3'>2v$, XIII	$6v+4w_1+w_2+w_3$
V	IX for $6v > w_2+2w_3+w_3'$	$6v+4w_1+w_2+2w_3+w_3'$

5.2. *Phase diagram*

In order to describe the phase diagram, we introduce the quantities

$$z \equiv w_3/2v, \quad z' \equiv w_3'/2v, \quad -qz \equiv w_2/2v \ , \tag{5.5}$$

where we take $z>0$, $z'>0$, $0<q<1$, in agreement with (5.1). We first show that phase A1 will not occur under these conditions. Next we investigate the phases I, VII, VIII and XV, for which N_s does not depend on z', and finally the two phases VI and X with denominators N_s depending on z,z' and q will be studied.

Phase A1: Comparing A1 with VIII and XV we have

$$f_{A1} < f_{VIII} \rightarrow 4v+w_2-2w_3 > 0 \rightarrow z < (1+\tfrac{1}{2}q)^{-1} < 1, \tag{5.6}$$

so that for $z>1$, where VIII exists, we cannot have A1. On the other hand, we have

$$f_{A1} < f_{XV} \rightarrow \tfrac{8}{3}v-w_2-\tfrac{2}{3}w_3 < 0 \rightarrow 2 < (1-\tfrac{3}{2}q)z \ , \tag{5.7}$$

which cannot be satisfied for $0<z<1$. This argument only holds when XV exists, i.e. for $z>\frac{1}{2}$, but if XV does not exist we have $f_{XV} > f_{VII}$ so that A1 does not occur for $b=0$.

Phases I, VII, VIII, XV: Using the results of table 1 it is straightforward to show that

$$(N_I - N_{XV})/2v = -\tfrac{1}{3}\left(\tfrac{5}{2} + (q-1)z\right)^{-1}F(z) , \qquad (5.8)$$

$$(N_{VII} - N_{XV})/2v = -\tfrac{4}{3}\left(4 + (q+2)z\right)^{-1}F(z) , \qquad (5.9)$$

$$(N_{VIII} - N_{XV})/2v = -\tfrac{4}{3}\left((q+2)z - 2\right)^{-1}F(z) , \qquad (5.10)$$

where

$$F(z) \equiv (1-q)z^2 - \tfrac{1}{2}(6-q)z + 2 . \qquad (5.11)$$

For every q with $0<q<1$, $F(z)$ has to roots $z_\pm$ which are explicitly given by

$$z_\pm = \tfrac{1}{4}\,\frac{6-q}{1-q} \pm \sqrt{\left(\tfrac{1}{4}\,\frac{6-q}{1-q}\right)^2 - \frac{2}{1-q}} , \qquad (5.12)$$

for which

$$-1 + \sqrt{3} < z_- < 1 , \quad z_+ > 2(1-q)^{-1} . \qquad (5.13)$$

It is clear that the condition $C_{XV}>0$ is equivalent to $F(z)<0$ which, together with the conditions $C_I>0$ and $C_{VIII}>0$, implies that $N_I>N_{XV}$, $N_{VII}>N_{XV}$, $N_{VIII}>N_{XV}$ in eqs. (5.8)-(5.10). Hence phase XV will be favourable in comparison with the phases I, VII and VIII for $z_-<z<z_+$. For $z>z_+$ the conditions $C_I>0$ and $C_{XV}>0$ do not hold, and we can have only solutions VII and VIII.

It is easy to see that $N_{VIII}<N_{VII}$, from eqs. (5.9) and (5.10), and therefore VIII will be the more favourable one, in that region. For $0<z<z_-$ only the solutions I and VII exist, and from (5.8) and (5.9) we have $N_{VII}<N_I \Rightarrow z<1$, so that for $z<z_-$ phase VII will be more favourable.

As far as these phases I, VII, VIII, XV is concerned we have the following picture. I does not occur, and

$$\begin{aligned} 0<z<z_- &: \text{Phase VII} \\ z_-<z<z_+ &: \text{Phase XV} \\ z>z_+ &: \text{Phase VIII.} \end{aligned} \qquad (5.14)$$

Phases VI and X: From table 1 it is straightforward to derive the following expressions

$$(N_X - N_{XV})/2v = z'-1 \ , \tag{5.15}$$

$$(N_X - N_{VIII})/2v = (z'-1) + \frac{4}{3}\,\frac{F(z)}{(2+q)z-2} \equiv \delta(z,z') \ , \tag{5.16}$$

$$(N_{VI} - N_{XV})/2v = \frac{1}{3}\,\frac{z'\left[(z+1)^2-3\right]+z(z-2)}{z+z'+zz'} \equiv \frac{1}{3}\,\frac{\alpha(z,z')}{z+z'+zz'} \ , \tag{5.17}$$

$$(N_{VI} - N_X)/2v = \frac{z+1}{z+z'+zz'}\left[(z'+1)(z+1)-(3z'^2+1)\right] \equiv \frac{(z+1)\beta(z,z')}{z+z'+zz'} \ , \tag{5.18}$$

$$(N_{VI} - N_{VII})/2v = \frac{z\left[2-(1-\frac{1}{2}q)z(z'+1)\right]}{\left[2+(1+\frac{1}{2}q)z\right](z+z'+zz')} \equiv \frac{z\gamma(z,z')}{z+z'+zz'} \ . \tag{5.19}$$

From (5.15) we see that X may only occur for $z'<1$, and XV for $z'>1$, as mentioned above. Using eqs. (5.17) and (5.18) we may have phase VI under the conditions $\beta(z,z')<0$ for $0<z'<1$, and $\alpha(z,z')<0$ for $z'>1$. Both conditions imply $z\leq 1$, and hence there is no phase transition between VI and VIII. From eq. (5.19) and $z<z_-$, we find that VII may only occur for $z'>1$, so that there is no phase transition between X and VII. We can now discuss the possible phase transitions. For $0<z<1$, $0<z'<1$, the only possible phases are X and VI, and there is a first order phase transition which is given by $\beta(z,z')=0$. $0<z'<1$. For $0<z<1$, $z'>1$, we can have the phases VI, VII and XV. VI is most favourable for a region containing $z'>\frac{1}{3}$, $z=0$ and $z'=1$, $z<1$, and XV is most favourable for a region containing $1<z<z_+$, $z'>1$. VII will actually occur, since $\alpha(z,z')>0$ for $z=z_-$ for sufficiently large z'. There is a second order transition between VII and XV which is described by $z=z_-$, a first order transition between VI and VII described by $\gamma(z,z')=0$, and a first order transition between VI and XV described by $\alpha(z,z')=0$. For $z>1$, $z'>1$ the only possible phases are XV and VIII, and there is a second order transition at $z=z_+$. Finally, for $z>1$, $0<z'<1$ we have only X and VIII, and there is a first order transition at $\delta(z,z')=0$. On the basis of these considerations the phase diagram at $b=0$, under the condition (5.1), is given in figure 3. The transition lines $\gamma=0$, $\alpha=0$, $\beta=0$ and $\delta=0$, $z=z_+$, $z=z_-$ are indicated in this figure. The transition line $z'=1$ between X and XV is a first order transition, since the vector $\underline{m}_3$ changes discontinuously from a real vector to a vector with $\underline{m}_3\cdot\underline{m}_3=0$, in spite of the fact that the

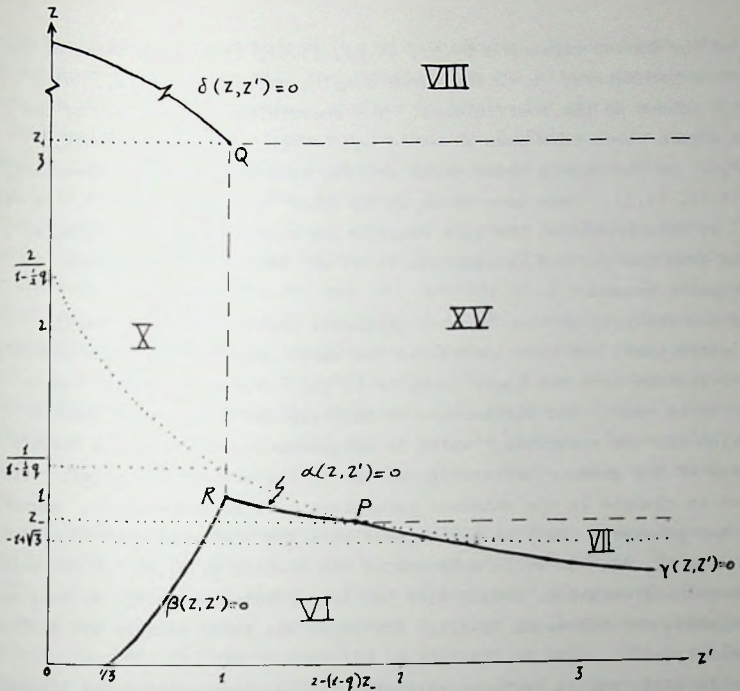

Figure 3. Phase diagram of liquid ^3He at b=0, for fixed q, 0<q<1 and positive z and z'. The solid lines are first order transitions, the dashed lines are second order transitions (or first order transitions without latent heat). P and Q are critical endpoints, R is a triple point. The dotted lines are auxiliary curves.

lengths m_1,m_2,m_3 of the order vectors are continuous at this transition. When an external magnetic field is taken into account the situation is more complicated. In fact, phase I is at $b=0$ for $z=z_-$ degenerate with VII and XV and may be anticipated to be important for $b\neq 0$ also outside the region of occurrence of phase VI.

6. Concluding remarks

In this chapter we have presented a systematic study of the phases that can occur in liquid ^3He in the presence of a magnetic field and including the contributions from the Hubbard interaction (1.4). This has been done on the basis of the Landau expansion (1.5), (1.6), (1.8), (1.9), in terms of the 18 real order parameters, i.e. the three complex vectors $\underline{m}_1, \underline{m}_2, \underline{m}_3$, with explicit values of the coefficients. The coefficients u_1, u_2, u_3 of the second degree terms have been given in eq. (1.10), the coefficients, $v_1, \ldots, v_6$ of the fourth order terms arising from the pairing interaction, cf. eqs. (1.2), (1.3), have been given up to order b^2 in eq. (1.12). In eq. (1.13) we have presented the spin fluctuation contributions w_1, w_2, w_3, w_3', arising from the Hubbard interaction (1.4) at $b=0$, up to the order I^2 in the coupling constant I.

As the analysis of the 18 order parameter problem is an extremely complicated task, we have introduced two basic ingredients in the analysis. We have assumed that the inner products of the vectors $\underline{m}_1, \underline{m}_2, \underline{m}_3$ can be chosen to be real, and furthermore we have applied a so-called inertia condition for the solutions, which is an assumption expressing a certain rigidity of the geometrical configurations of the vectors $\underline{m}_1, \underline{m}_2, \underline{m}_3$, with respect to changes in the external parameters such as temperature, magnetic field and pressure. In this way we have obtained onedimensional solutions R1, R2, A1, A2, A3, in which only one of the vectors $\underline{m}_1, \underline{m}_2, \underline{m}_3$ is nonvanishing, twodimensional solutions I-VIII with two nonvanishing vectors, as well as threedimensional solutions IX-XVI, for which all three vectors are different from zero

As it turns out to be too complicated to investigate the phase diagram on the basis of the complete Landau expansion for arbitrary values of the coefficients w_1, w_2, w_3, w_3', we have used the second order results for the coefficients, as given in (1.13), as a guideline for the global features of these coefficients, thereby assuming that higher order contributions will not affect the signs of the coefficients, but merely change the second order values by certain (positive) enhancement factors.

As a result of the present analysis we were able to deal with two separate problems. The first problem is concerned with the stability of the phase diagram in the absence of any spin fluctuations, as given in chapter III. For this purpose we have investigated the phase diagram in the presence of a magnetic field for small values of the coupling constant I of the Hubbard interaction. In this limit it is safe to assume that the coefficients w_1, w_2, w_3, w_3' are correctly given by (1.13). For sufficiently small values of these coefficients, taking $\eta b<0$, we have shown that only phases A1, I, VI and X can occur. The phase diagram has been presented in fig. 2. Here X is the extension of the BW-phase to finite values of the magnetic field, as already considered in chapter III, and there are two new phases I and VI, which have not been treated before in the literature. In the limit $w_3, w_3' \to 0$, phase VI becomes degenerate with the ABM-phase or the planar phase and the region, in which phase I occurs, shrinks to zero. For small values of the spin fluctuation parameters it may not be easy to distinguish the phase diagram in fig. 2 experimentally from the one given in chapter III, in which VI and I are replaced by the ABM- or planar phase.

The second problem we have dealt with is the phase diagram at zero magnetic field, also for larger values of the spin fluctuation parameters, under the restriction that w_2, w_3 and w_3' satisfy eq. (5.1), in which the signs of w_2, w_3, w_3' correspond to the ones in (1.13), and the inequality $q<1$ is obeyed for a large range of coefficient values, including the values (1.13). In that case only the phases VI, VII, VIII, X and XV play a role in the phase diagram. X is again the BW-phase, VII is the so-called ε-solution introduced in ref. 15, and VI, VIII and XV are new phases. The resulting phase diagram at a fixed value of $q=-w_2/w_3$, with $0<q<1$, has been presented in fig. 3.

On the basis of the two phase diagrams, given in figs. 2 and 3 one may attempt to give some discussion of the so-called profound effect [4,5], i.e. the effect of a small fixed magnetic field on the phase diagram of ^3He as a function of the temperature T amd pressure p. When p increases, the parameters w_3 and w_3' can be expected to increase as well. Starting from the BW-phase one could, at a certain pressure p_0 have a phase transition to one of the ordered phases VI, XV or VIII, as displayed in fig. 3. If we assume that w_3'/w_3 would be independent of $I \equiv I(p)$, and therefore be equal to 1 as in (1.13), this transition would take place at the point $(z,z') = (1,1)$ in fig. 3. However, for w_3 slightly larger than w_3' one would cross the line $z'=1$ and have a transition from X to XV, whereas for w_3' slightly larger than w_3 one would cross the curve $\beta(z,z')=0$ in fig. 3 and have a transition from X (BW) to VI.

At this stage one might speculate that the second possibility could occur in practice, since VI is a twodimensional phase with only ↑↑- and ↓↓-ordering, but in the presence of a magnetic field one may anticipate a more complicated behaviour than what is usually referred to as the splitting of the A-phase. Adhering to the second possibility, one may conjecture that only the phases BW, VI, I and A1 are important in the discussion of the profound effect. Let us, one the basis of fig. 2, see what happens at a fixed small value of the magnetic field. For p sufficiently small, i.e. $p<p_0$, the critical endpoint Q in fig. 2 will still be present and for decreasing T one will pass from the normal phase N via a second order transition to A1 and via a first order

transition to BW. For a value $p=p_2<p_0$, the critical endpoint Q will disappear, and for $p_2<p<p_0$ one has with increasing T first a first order transition from BW to I, than a second order transition from I to A1 and a second order transition from A1 to N. At a slightly larger value $p_1<p_0$, the critical endpoint P will also disappear, and for $p_1<p<p_0$ there will be a first order transtion from BW to VI, followed by second order transtions from VI to I, from I to A1, and from A1 to N. Finally for $p>p_0$ phase VI has the lowest free energy at $b=0$, and for finite b there will be second order transitions from VI to I, from I to A1 and from A1 to N if T increases. If we lower the temperature, however, a first order transition from VI to BW may be expected. In fact, the coefficient v increases proportional to β^2 cf. eq. (1.11). On the other hand the spin fluctuation parameters w_3, w_3', cf. eq. (1.13), may be expected to increase at a slower rate (in second order perturbation theory they are proportional to β), so that z and z' decrease as a function of T. Hence for fixed p one might expect to cross the curve $\beta(z,z')=0$ in fig. 3 again, going from phase VI to the BW-phase.

The considerations presented above lead to a phase diagram as drawn qualitatively in fig. 4. Again, phase VI is not necessarily very different from the ABM and the region, in which I occurs, can be very small, so that

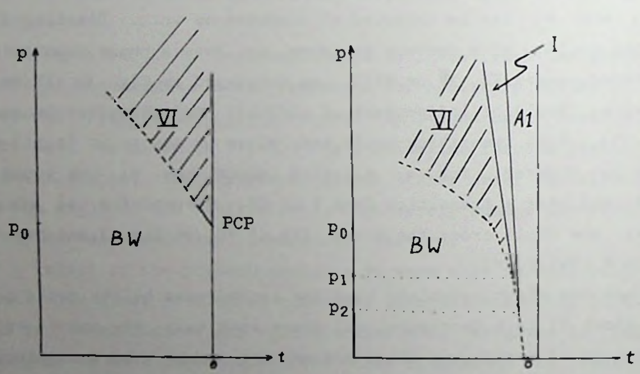

Figure 4. Possible phase diagram of ^{3}He as a function of pressure p and temperature $t=\frac{1}{2}\mathcal{N}(0)(T-T_c)/T_c$, at zero magnetic field and at a small fixed magnetic field respectively, as might be expected on the basis of a simple behaviour of w_3, w_3' as functions of p. The solid lines are second order, and the dashed lines are first order transitions. PCP is the polycritical point.

it might not be easy to distinguish experimentally between the situation drawn in fig. 4 and the usual picture of the profound effect [4].

The phase diagrams presented in figs. 2-4 are based on the assumption that the signs of the parameters w_3, w_3' are correctly given by the second order results of (1.13). In particular this assumption is sufficient to rule out the ABM-phase as a candidate for a possible ordering. From a theoretical point of view, taking into account the usual theories about enhancement, it will be hard to imagine that this assumption is not true. But it may be noted that the ABM indeed can occur, when at least one of the parameters w_3, w_3' is negative. It is not hard to extend the phase diagram of fig. 3 to negative values of z, and we can then have the ABM-phase in the region bounded by the lines $z'=-1$, $z<-1$, where we have a transition to VI, the line $1+4z-3z'=0$ for $-1<z'<\frac{2}{3}\sqrt{3}-1$, where we can have a transition to BW, and the curve $(1+z)(1+z')=1$, $z'>\frac{2}{3}\sqrt{3}-1$, where we have again a transition to VI. Apart from that, there is also a phase transition between VI and V at $(z+1)(z'+1)=1$, $z'<-1$, $z<-1$, between VI and X at $z=-1$, $z'<-1$, at $(z+1)(z'+1)=1$, $0<z<\frac{2}{3}\sqrt{3}-1$, and at $\beta(z,z')=0$, $0<z'<\frac{1}{3}$, and a transition between VI and V at $(z+1)(z'+1)=1$, $z<-1$, $z'<-1$. (The spin fluctuation results, reported in ref. 20, would indicate that a first order transition between BW and ABM at large pressures occurs on the line $1+4z-3z'=0$ at about $z'=-0.059$, $z=-0.295$). Future experiments could give more detailed information on the values of the spin fluctuation coefficients, and this information would enable one to give more concrete theoretical predictions on the regions in the phase diagram which apply to the situation in liquid ^{3}He.

APPENDIX A

In this appendix we will prove that the phases presented in section 2.2 are the only ones with $\underline{m}_3=0$. For this purpose we introduce the normalized inner products

$$\underline{m}_1\cdot\underline{m}_1 \equiv m_1^2x_1, \quad \underline{m}_2\cdot\underline{m}_2 \equiv m_2^2x_2, \quad \underline{m}_1\cdot\underline{m}_1 = m_1m_2y, \quad \underline{m}_1\cdot\underline{m}_2^* = m_1m_2z , \tag{A1}$$

and we assume that x_1,x_2,y,z can be chosen to be real. For $\underline{m}_3=0$ we then have to minimize the following expression

$$\Phi(\underline{m}_1,\underline{m}_2,0) = \Phi_0(m_1,m_2,0) + v_1m_1^4x_1^2 + v_2m_2^4x_2^2 + m_1^2m_2^2(w_3'y^2+w_3z^2) . \tag{A2}$$

In order to find the (local) minima of (A2) for fixed lengths m_1,m_2, we can minimize (A.2) with respect to x_1, x_2, y, z, which take values in the interval $[-1,1]$. However, as the vectors $\underline{m}_1, \underline{m}_2$ are embedded in a threedimensional space, not all the inner products are independent. In fact, there is always a set of numbers $(a_1, b_1, a_2, b_2) \neq (0,0,0,0)$, such that

$$a_1\underline{m}_1 + b_1\underline{m}_1^* + a_2\underline{m}_2 + b_2\underline{m}_2^* = 0 \ , \tag{A.3}$$

and this implies that the determinant of the linear system obtained from (A.3) by taking inner products with $\underline{m}_1^*, \underline{m}_1, \underline{m}_2^*, \underline{m}_2$, must vanish, i.e.

$$\begin{vmatrix} 1 & x_1 & z & y \\ x_1 & 1 & y & z \\ z & y & 1 & x_2 \\ y & z & x_2 & 1 \end{vmatrix} = 0 \ . \tag{A.4}$$

Eq. (A.4) is quadratic in the x_1 and x_2, and has the following two solutions

$$(1-x_1)(1-x_2) = (y-z)^2 \ , \tag{A.5}$$

and a similar solution with the minus signs replaced by plus signs. In the minimization of eq. (A.2), either eq. (A.5) or its counterpart with plus signs should be taken into account as a constraint.

In the minimization we can discriminate between two cases: (i) $x_1=\pm 1$, $x_2=\pm 1$, which implies $\underline{m}_1 \parallel \underline{m}_1^*$, $\underline{m}_2 \parallel \underline{m}_2^*$; (ii) either $x_1 \neq \pm 1$, or $x_2 \neq \pm 1$, in which case we can solve x_2 respectively x_1 from (A.5) or its counterpart.

(i) $\underline{m}_1 \parallel \underline{m}_1^*$, $\underline{m}_2 \parallel \underline{m}_2^*$ implies immediately that $|\underline{m}_1 \cdot \underline{m}_2| = |\underline{m}_1 \cdot \underline{m}_2^*|$ so that (A.5) is trivially satisfied. We then have from (A.2) three possibilities:

(a) $w_3 + w_3' > 0$. In this case the minimum is obtained for $|\underline{m}_1 \cdot \underline{m}_2| = |\underline{m}_1 \cdot \underline{m}_2^*| = 0$, with

$$\Phi(m_1, m_2, 0) = \Phi_0(m_1, m_2, 0) + v_1 m_1^4 + v_2 m_2^4 \ , \tag{A.6}$$

leading to phase IV, cf. eq. (2.12).

(b) $w_3 + w_3' < 0$. In this case the minimum is obtained for $|\underline{m}_1 \cdot \underline{m}_2| = |\underline{m}_1 \cdot \underline{m}_2^*| = m_1 m_2$, with

$$\Phi(\underline{m}_1, \underline{m}_2, 0) = \Phi_0(m_1, m_2, 0) + v_1 m_1^4 + v_2 m_2^4 + (w_3 + w_3') m_1^2 m_2^2 \ , \tag{A.7}$$

leading to phase V, cf. eq. (2.14).

(c) $w_3+w_3'=0$. In this case $|\underline{m}_1\cdot\underline{m}_2| = |\underline{m}_1\cdot\underline{m}_2^*|$ is unspecified, which leads to phase IV', cf. eq. (2.13).

ii) Let us assume that $x_1\neq\pm1$, i.e. $\underline{m}_1 \nparallel \underline{m}_1^*$. (The other case with $x_2\neq\pm1$ can be treated in a similar fashion). In this case, considering (A.5), we can rewrite (A.2) as follows

$$\Phi(\underline{m}_1,\underline{m}_2,0) = \Phi_0(m_1,m_2,0) + v_1 m_1^4 x_1^2 + v_2 m_2^4 \left(1 - \frac{(y-z)^2}{1-x_1}\right) + w_3' m_1^2 m_2^2 y^2 + w_3 m_1^2 m_2^2 z^2 , \tag{A.8}$$

(the other constraint with + signs may be treated similarly). In calculating the extrema of (A.8) we must take into account also the boundary extrema at the values ±1. Thus, in minimizing (A.8) over y and z, we have the following possibilities

(a) $|y| = |z| = 1$,
(b) $|y| = 1$, $|z| \neq 1$,
(c) $|y| \neq 1$, $|z| = 1$,
(d) $|y| \neq 1$, $|z| \neq 1$.

In general one can show that $\underline{m}_1\cdot\underline{m}_2 = \pm m_1 m_2$ implies $\underline{m}_1 \parallel \underline{m}_2^*$, and that $\underline{m}_1\cdot\underline{m}_2^* = \pm m_1 m_2$ implies $\underline{m}_1 \parallel \underline{m}_2$ (using the reality of the inner products). Hence case (a) immediately leads to $\underline{m}_1 \parallel \underline{m}_1^* \parallel \underline{m}_2 \parallel \underline{m}_2^*$, i.e. phase V given by eq. (2.14). In case (b) we have immediately $|x_1| = |x_2| = |z|$, which inserted in eq. (A.2) gives

$$\Phi(\underline{m}_1,\underline{m}_2,0) = \Phi_0(m_1,m_2,0) + w_3' m_1^2 m_2^2 + (v_1 m_1^4 + v_2 m_2^4 + w_3 m_1^2 m_2^2) z^2 . \tag{A.9}$$

Minimizing (A.9) with respect to z leads either to $z=\pm1$ (boundary extremum), which is again phase V, or to $z=0$. Hence we have the solution

$$\begin{aligned} &\underline{m}_1\cdot\underline{m}_1 = \underline{m}_2\cdot\underline{m}_2 = \underline{m}_1\cdot\underline{m}_2^* = 0, \quad \underline{m}_1\cdot\underline{m}_2 = \pm m_1 m_2 \\ &\Phi(\underline{m}_1,\underline{m}_2,0) = \Phi_0(m_1,m_2,0) + w_3' m_1^2 m_2^2 , \end{aligned} \tag{A.10}$$

which is phase II, given in eq. (2.10). We can treat case (c) in a similar way, leading to phase V, or to

$$\begin{aligned} &\underline{m}_1\cdot\underline{m}_1 = \underline{m}_2\cdot\underline{m}_2 = \underline{m}_1\cdot\underline{m}_2 = 0 , \quad \underline{m}_1\cdot\underline{m}_2^* = \pm m_1 m_2 \\ &\Phi(\underline{m}_1,\underline{m}_2,0) = \Phi_0(m_1,m_2,0) + w_3 m_1^2 m_2^2 , \end{aligned} \tag{A.11}$$

which is phase III, given in eq. (2.11).

We are left with the treatment of case (d). Minimizing (A.8) with respect to y and z, where $|y|\neq 1$, $|z|\neq 1$, leads to the following equations

$$-2v_2 m_2^4 \left(1 - \frac{(y-z)^2}{1-x_1}\right)\frac{y-z}{1-x_1} + w_3' m_1^2 m_2^2 y = 0 \ , \tag{A.12a}$$

$$2v_2 m_2^4 \left(1 - \frac{(y-z)^2}{1-x_1}\right)\frac{y-z}{1-x_1} + w_3 m_1^2 m_2^2 = 0 \ . \tag{A.12b}$$

By adding (A.12a) and (A.12b) we have immediately

$$w_3' y + w_3 z = 0 \ . \tag{A.13}$$

Furthermore, multiplying (A.12a) by w_3 and (A.12b) by w_3', one can show that either $y=z$, or

$$2v_2 m_2^2 \left(1 - \frac{(y-z)^2}{1-x_1}\right) = \gamma m_1^2(1-x_1), \quad \gamma \equiv \frac{w_3 w_3'}{w_3+w_3'} \ . \tag{A.14}$$

The case $y=z$, together with (A.13), leads to

$$y=z=0 \ , \quad \text{for arbitrary } w_3, w_3' \ , \tag{A.15}$$

$$y=z\neq 0 \ , \quad \text{for } w_3+w_3'=0 \ . \tag{A.16}$$

Eq. (A.15) leads to $\underline{m}_2 \| \underline{m}_2^*$, using eq. (A.5), and hence we have a minimum for $\underline{m}_1\cdot\underline{m}_1=0$. This yields the solution

$$\underline{m}_2 \| \underline{m}_2^*, \quad \underline{m}_1\cdot\underline{m}_1 = \underline{m}_1\cdot\underline{m}_2 = \underline{m}_1\cdot\underline{m}_2^* = 0 \ , \tag{A.17}$$
$$\Phi(\underline{m}_1,\underline{m}_2,0) = \Phi_0(m_1,m_2,0) + v_2 m_4^4 \ .$$

which is phase I, given in eq. (2.8). (Phase I' is obtained by consindering the case $x_2\neq\pm1$ instead of $x_1\neq\pm1$). Similarly eq. (A.16) yields the solution

$$\underline{m}_2 \| \underline{m}_2^*, \quad \underline{m}_1\cdot\underline{m}_1 = 0, \quad |\underline{m}_1\cdot\underline{m}_2| = |\underline{m}_1\cdot\underline{m}_2^*| \neq 0, \quad w_3+w_3'=0 \ ,$$
$$\Phi(\underline{m}_1,\underline{m}_2,0) = \Phi_0(m_1,m_2,0) + v_2 m_2^4 \ . \tag{A.18}$$

which is phase I", given in eq. (2.9). Finally, eq. (A.14) can be inserted into eq. (A.8), leading to

$$\Phi(\underline{m}_1,\underline{m}_2,0) = \Phi_0(m_1,m_2,0) + v_1m_1^4 + v_2m_2^4\left(1 - \frac{(y-z)^4}{(1-x_1)^2}\right)$$

$$= \Phi_0(m_1,m_2,0) + v_1m_1^4x_1^2 + \gamma m_1^2m_2^2(1-x_1) - \frac{\gamma^2}{4v_2}m_1^4(1-x_1)^2 . \qquad (A.19)$$

We can now minimize (A.19) with respect to x_1, leading either to $x_1=\pm1$ in contradiction with the assumption $x_1\neq\pm1$, or to

$$2v_1m_1^4x_1 - \gamma m_1^2m_2^2 + \frac{\gamma^2}{2v_2}m_1^4(1-x_1) = 0 . \qquad (A.20)$$

From (A.20), for $\gamma^2\neq4v_1v_2$, we can solve x_1 in terms of m_1 and m_2, and substitute the result in eqs. (A.13), (a.14) and (A.5), in order to find y, z and x_2. The result is

$$x_1 = \frac{2v_2(m_2^2/m_1^2) - \gamma}{4v_1v_2 - \gamma^2} , \quad x_2 = \frac{2v_1(m_1^2/m_2^2) - \gamma}{4v_1v_2 - \gamma^2}$$

$$w_3'y = -w_3z = \pm(4v_1v_2x_1x_2)^{\frac{1}{2}}$$

$$\Phi(\underline{m}_1,\underline{m}_2,0) = \Phi_0(m_1,m_2,0) + \frac{4v_1v_2m_1^2m_2^2 - \gamma(v_1m_1^4+v_2m_2^4)}{4v_1v_2 - \gamma^2} , \qquad (A.21)$$

which is phase VI, given in eqs. (2.11). This concludes the list of possible solutions for $\underline{m}_3=0$.

APPENDIX B

In this appendix we shall show that the solutions mentioned in subsection 3.2 are the only ones that can occur as $\underline{m}_3 \parallel \underline{m}_3^*$. We can rewrite eqs. (3.5) as follows:

$$(\alpha_1-\alpha_1')\underline{m}_1 + A\underline{m}_3 + B\underline{m}_3^* = 0 , \qquad (B.1a)$$

$$(\alpha_2-\alpha_2')\underline{m}_2 + C\underline{m}_3 + D\underline{m}_3^* = 0 , \qquad (B.1b)$$

$$(\alpha_3-\alpha_3')\underline{m}_3 + \tfrac{1}{2}A\underline{m}_1 + \tfrac{1}{2}B\underline{m}_1^* + \tfrac{1}{2}C\underline{m}_2 + \tfrac{1}{2}D\underline{m}_2^* = 0 , \qquad (B.1c)$$

where

$$A = (v_4-w_2)\underline{m}_1\cdot\underline{m}_3^* + (2v_6-w_2-2w_3)\underline{m}_3\cdot\underline{m}_2^*, \quad B = v_4(\underline{m}_1\cdot\underline{m}_3) \ , \tag{B.2a}$$

$$C = (v_5-w_2)\underline{m}_2\cdot\underline{m}_3^* + (2v_6-w_2-2w_3)\underline{m}_3\cdot\underline{m}_1^*, \quad D = v_5(\underline{m}_2\cdot\underline{m}_3) \ , \tag{B.2b}$$

Case (i) with $\underline{m}_1 \parallel \underline{m}_2 \parallel \underline{m}_3$, cf. eq. (3.9a), has been treated in subsection 3.2. In case (ii) with $\underline{m}_1 \nparallel \underline{m}_3$, $\underline{m}_3 \parallel \underline{m}_2$ we have from (B.1a), taking $\underline{m}_3 = \underline{m}_3^*$,

$$\alpha_1 = \alpha_1' \ , \quad A+B = 0 \ , \tag{B.3}$$

and hence

$$(2v_4-w_2)\underline{m}_1\cdot\underline{m}_3^* + (2v_6-w_2-2w_3)\underline{m}_3\cdot\underline{m}_2^* = 0 \ . \tag{B.4}$$

As $\underline{m}_2 \parallel \underline{m}_3 \parallel \underline{m}_3^*$ we have $\underline{m}_2\cdot\underline{m}_3^* \neq 0$, implying that $\underline{m}_1\cdot\underline{m}_3^* \neq 0$, hence $\underline{m}_1\cdot\underline{m}_2 \neq 0$, $\underline{m}_1 \nparallel \underline{m}_2$, $\underline{m}_2 \parallel \underline{m}_2^*$. So we may use the results of subsection 2.2. The only solutions that obey these conditions are I" and IV', but for these solutions we have $w_3+w_3''=0$, i.e. with eq. (3.8)

$$(w_3+w_3')\underline{m}_1\cdot\underline{m}_2 + (v_6-w_3')m_2m_3 = 0 \ . \tag{B.5}$$

But on the other hand we have from (B.4), using $\underline{m}_1\cdot\underline{m}_3^* = \dfrac{m_3}{m_2}\underline{m}_1\cdot\underline{m}_2$,

$$(2v_4-w_2)\frac{m_3}{m_2}\underline{m}_1\cdot\underline{m}_2 + (2v_6-w_2-2w_3)m_2m_3 = 0 \ . \tag{B.6}$$

Eqs. (B.5) and (B.6) yield a relation between m_2 and m_3 independent of u_1, u_2, u_3, from which we conclude that case (ii) does not lead to three-dimensional solutions.

In case (iii) with $\underline{m}_1 \nparallel \underline{m}_3$, $\underline{m}_2 \nparallel \underline{m}_3$, we have again eq. (B.3), but in addition

$$\alpha_2 = \alpha_2', \quad C+D = 0 \ . \tag{B.7}$$

From (B.7) and (B.3) we have immediately, noting that a meaningful solution should also exist when the field dependence in v_4 and v_5, leading to $v_4 \neq v_5$ as in eq. (1.12), is taken into account ,

$$\underline{m}_1\cdot\underline{m}_3 = \underline{m}_2\cdot\underline{m}_3 = \underline{m}_1\cdot\underline{m}_3^* = \underline{m}_2\cdot\underline{m}_3^* = 0, \quad \alpha_1=\alpha_1', \quad \alpha_2=\alpha_2', \quad \alpha_3=\alpha_3', \tag{B.8}$$

and hence $\underline{m}_3 \perp \underline{m}_1, \underline{m}_2$. We can now show that $\underline{m}_1\cdot\underline{m}_2 \neq 0$ for a solution of eqs. (3.2a) and (3.2b). In fact, $\underline{m}_1\cdot\underline{m}_2=0$ would lead to

(B.9)

$$(2v_1\, \underline{m}_1\cdot\underline{m}_1 + w_3\, \underline{m}_2\cdot\underline{m}_2)\underline{m}_1\cdot\underline{m}_2^* + (v_6-w_3')m_3^2m_2^2 = 0 \;,$$
$$(2v_2\, \underline{m}_2\cdot\underline{m}_2 + w_3\, \underline{m}_1\cdot\underline{m}_1)\underline{m}_1\cdot\underline{m}_2^* + (v_6-w_3')m_3^2m_1^2 = 0 \;, \qquad \text{(B.9)}$$

which yields

$$\underline{m}_1\cdot\underline{m}_2^* = 0 \;, \quad \text{or}$$
$$(2v_1m_1^2-w_3m_2^2)\underline{m}_1\cdot\underline{m}_1 = (2v_2m_2^2-w_3m_1^2)\underline{m}_2\cdot\underline{m}_2 \;. \qquad \text{(B.10)}$$

The first possibility $\underline{m}_1\cdot\underline{m}_2^*=0$ leads to $m_1m_2m_3=0$, which is not a threedimensional solution. The second possibility, with $\underline{m}_1\cdot\underline{m}_2=0$, cannot be embedded in a twodimensional subspace, so that eq. (B.8) does not allow for solutions in a threedimensional space. So the only possible solutions compatible with (B.8) are those with $\underline{m}_1\cdot\underline{m}_2\neq 0$, hence we can use the solutions of the $\underline{m}_1,\underline{m}_2$-problem. These are: II, IV' and V, leading to the three phases X, XI and XII respectively, given in subsection 3.2.

APPENDIX C

We show that the solutions of eqs. (3.2), (3.5) presented in section 3.3 are the only ones with $\underline{m}_3 \nparallel \underline{m}_3^*$. We first show that also in the case $\underline{m}_1\cdot\underline{m}_2=0$ a reduction to the $\underline{m}_1,\underline{m}_2$-problem (3.7) is possible. In fact, from (3.2a) and (3.2b) with $\underline{m}_1\cdot\underline{m}_2 = \underline{m}_3\cdot\underline{m}_3 = 0$ we have

$$(2v_1\, \underline{m}_1\cdot\underline{m}_1 + w_3\, \underline{m}_2\cdot\underline{m}_2)\underline{m}_1^*\cdot\underline{m}_2 = 0 \;,$$
$$(2v_2\, \underline{m}_2\cdot\underline{m}_2 + w_3\, \underline{m}_1\cdot\underline{m}_1)\underline{m}_1\cdot\underline{m}_2^* = 0 \;, \qquad \text{(C.1)}$$

from which we have either $\underline{m}_1\cdot\underline{m}_2^*=0$, or $\underline{m}_1\cdot\underline{m}_1 = \underline{m}_2\cdot\underline{m}_2 = 0$ $(v_1\neq v_2)$. From (3.2a) and (3.2b) and $\underline{m}_1\cdot\underline{m}_2^*=0$ we have $\underline{m}_1 \parallel \underline{m}_1^*$ or $\underline{m}_1\cdot\underline{m}_1=0$, and $\underline{m}_2 \parallel \underline{m}_2^*$ or $\underline{m}_2\cdot\underline{m}_2=0$. Taking into account now both solutions of eq. (C.1), we find that $\underline{m}_1 \parallel \underline{m}_1^*$, $\underline{m}_2 \parallel \underline{m}_2^*$ and $\underline{m}_1\cdot\underline{m}_2=0$ corresponds to phase IV, $\underline{m}_1\cdot\underline{m}_1 = \underline{m}_1\cdot\underline{m}_2 = 0$, $\underline{m}_2 \parallel \underline{m}_2^*$, to phase I, $\underline{m}_2\cdot\underline{m}_2 = \underline{m}_1\cdot\underline{m}_2 = 0$, $\underline{m}_1 \parallel \underline{m}_1^*$ to phase I', and in the $\underline{m}_1\cdot\underline{m}_2^*\neq 0$, $\underline{m}_1\cdot\underline{m}_1 = \underline{m}_2\cdot\underline{m}_2 = \underline{m}_1\cdot\underline{m}_2 = 0$, we obtain phase III. All these phases can be obtained as the ones with $\underline{m}_1\cdot\underline{m}_2=0$ from the twodimensional problem (3.7) with $w_3''=w_3'$.

In the analysis of the possible solutions of eqs. (3.5) we shall distinguish between three types of geometrical configurations, i.e.

i) Solutions with $\underline{m}_1$ and $\underline{m}_2$ in the plane of $\underline{m}_3, \underline{m}_3^*$.

ii) Solutions with only $\underline{m}_1$ (or only $\underline{m}_2$) having a component perpendicular to the plane $\underline{m}_3, \underline{m}_3^*$.

iii) Solutions with both $\underline{m}_1$ and $\underline{m}_2$ having a component perpendicular to the plane of $\underline{m}_3, \underline{m}_3^*$.

We first investigate the case i). If $\underline{m}_1 \| \underline{m}_1^*$, we then must have either solutions IV, IV' or V. But then we have also $\underline{m}_2 \| \underline{m}_2^*$, and hence from (B.1a) and (B.1b) we find

$$A=B, \quad C=D \Rightarrow \underline{m}_1 \cdot \underline{m}_3 = \underline{m}_2 \cdot \underline{m}_3 = \underline{m}_1 \cdot \underline{m}_3^* = \underline{m}_2 \cdot \underline{m}_3^* = 0 \ , \tag{C.2}$$

which is in contradiction with the assumption that $\underline{m}_1, \underline{m}_2, \underline{m}_3$ lie in the same plane. So we must have $\underline{m}_1 \nparallel \underline{m}_1^*$ and $\underline{m}_2 \nparallel \underline{m}_2^*$. The only solution of eqs. (3.2) for which $\underline{m}_1, \underline{m}_1^*, \underline{m}_2, \underline{m}_2^*$ lie in the same plane are then II and III. For solution III we have

$$\underline{m}_1 \cdot \underline{m}_1 = \underline{m}_2 \cdot \underline{m}_2 = \underline{m}_1 \cdot \underline{m}_2 = 0 \ , \tag{C.3}$$

which in combination with (B.1a) and (B.1b) gives

$$A\ \underline{m}_3 \cdot \underline{m}_1 + B\ \underline{m}_3^* \cdot \underline{m}_1 = 0 \ , \tag{C.4a}$$

$$A\ \underline{m}_3 \cdot \underline{m}_2 + B\ \underline{m}_3^* \cdot \underline{m}_2 = 0 \ , \tag{C.4b}$$

$$C\ \underline{m}_3 \cdot \underline{m}_1 + D\ \underline{m}_3^* \cdot \underline{m}_1 = 0 \ , \tag{C.4c}$$

$$C\ \underline{m}_3 \cdot \underline{m}_2 + D\ \underline{m}_3^* \cdot \underline{m}_2 = 0 \ . \tag{C.4d}$$

Using (B.2a) and (B.2b) we have two possibilities

$$\text{a)} \quad \underline{m}_1 \cdot \underline{m}_3 = \underline{m}_2 \cdot \underline{m}_3 = 0 \ , \tag{C.5}$$

$$\text{b)} \quad \underline{m}_1 \cdot \underline{m}_3^* = \underline{m}_2 \cdot \underline{m}_3^* = 0 \ . \tag{C.6}$$

From (B.1a), (B.1b) with (C.5) we have

$$\underline{m}_1 \| \underline{m}_3 , \quad \underline{m}_2 \| \underline{m}_3 \ , \tag{C.7}$$

and from (B.1a), (B.1b) with (C.6) we have

$$\underline{m}_1 \| \underline{m}_3^* , \quad \underline{m}_2 \| \underline{m}_3^* \ . \tag{C.8}$$

Eq. (C.7) immediately yields solution XIV, cf. eq. (3.19), and (C.8) yields solution XIII, cf. eq. (3.18). We now show that extensions of solution II do not exist. In fact, II has the inner products

$$\underline{m}_1\cdot\underline{m}_1 = \underline{m}_2\cdot\underline{m}_2 = \underline{m}_1\cdot\underline{m}_2^* = 0 \ , \tag{C.9}$$

leading to (C.4a), (C.4d), and

$$A\ \underline{m}_3\cdot\underline{m}_2^* + B\ \underline{m}_3^*\cdot\underline{m}_2^* = 0 \ , \tag{C.10a}$$

$$C\ \underline{m}_3\cdot\underline{m}_1^* + D\ \underline{m}_3^*\cdot\underline{m}_1^* = 0 \ , \tag{C.10b}$$

instead of (C.4b) and (C.4c). Eqs. (C.4a), (C.4d), (C.10a) and (C.10b) lead to the conclusion that

$$\underline{m}_1\cdot\underline{m}_3^* = \underline{m}_2\cdot\underline{m}_3^* = 0 \ , \quad (\underline{m}_1\cdot\underline{m}_3)(\underline{m}_2\cdot\underline{m}_3) = 0 \ , \tag{C.11}$$

which is in contradiction to the fact that $\underline{m}_1, \underline{m}_2, \underline{m}_3$ lie in the same plane.

Secondly we investigate case ii), in which $\underline{m}_1$ has a component perpendicular to the plane $\underline{m}_3, \underline{m}_3^*$, and $\underline{m}_2$ lies in that plane. Then from (B.1a) we have

$$\alpha_1 = \alpha_1' , \quad A\ \underline{m}_3 + B\ \underline{m}_3^* = 0 \Rightarrow A = B = 0 \ , \tag{C.12}$$

since $\underline{m}_3 \nparallel \underline{m}_3^*$. From (C.12) we have that (B.1b) and (B.1c) reduce to

$$(\alpha_2-\alpha_2')\underline{m}_2 + C\ \underline{m}_3 + D\ \underline{m}_3^* = 0 \ , \tag{C.13a}$$

$$(\alpha_3-\alpha_3')\underline{m}_3 + \tfrac{1}{2}C\ \underline{m}_2 + \tfrac{1}{2}D\ \underline{m}_2^* = 0 \ . \tag{C.13b}$$

If $\alpha_2=\alpha_2'$ we have $C=D=0$ from (C.13a), and hence, together with (C.12), $\underline{m}_1\cdot\underline{m}_3 = \underline{m}_1\cdot\underline{m}_3^* = \underline{m}_2\cdot\underline{m}_3 = \underline{m}_2\cdot\underline{m}_3^* = 0$, in contracdiction with the assumption that $\underline{m}_2$ lies in the plane of $\underline{m}_3, \underline{m}_3^*$. If $\alpha_2 \neq \alpha_2$ we have from (C.13a), (C.13b),

$$\left[(\alpha_3-\alpha_3')(\alpha_2-\alpha_2') + \tfrac{1}{2}(C^2+D^2)\right]\underline{m}_3 + CD\ \underline{m}_3^* = 0 \ , \tag{C.14}$$

which yields

$$CD = 0 \ . \tag{C.15}$$

Eqs. (C.12) and (C.15) lead to the following possibilities

$$\text{a)}\quad A = B = C = 0 \qquad \underline{m}_1\cdot\underline{m}_3 = \underline{m}_1\cdot\underline{m}_3^* = \underline{m}_2\cdot\underline{m}_3^* = 0 \ , \quad \underline{m}_2 \parallel \underline{m}_3^* \ , \tag{C.16}$$

$$\text{b)}\quad A = B = D = 0 \qquad \underline{m}_1\cdot\underline{m}_3 = \underline{m}_2\cdot\underline{m}_3 = 0 \ , \quad \underline{m}_2 \parallel \underline{m}_3 \ . \tag{C.17}$$

Eq. (C.16) yields $\underline{m}_1 \parallel \underline{m}_1^* \perp \underline{m}_3, \underline{m}_3^*$, $\underline{m}_2 \parallel \underline{m}_3^*$, which is an extension of solution I', and leads to phase XV', cf. eq. (3.22). Eq. (C.17) leads to $\underline{m}_2\cdot\underline{m}_2=0$ in eq. (C.13a), and from (3.2a) we have, with $\underline{m}_2 \nparallel \underline{m}_1$, $\underline{m}_1\cdot\underline{m}_2^*=0$. From (C.17) we then have also $\underline{m}_1\cdot\underline{m}_3^*=0$ and again $\underline{m}_1 \perp \underline{m}_3, \underline{m}_3^*$. But $A=0$ together with

$\underline{m}_1 \cdot \underline{m}_3^* = 0$ gives also $\underline{m}_2 \cdot \underline{m}_3^* = 0$, and this leads to a contradiction. Hence (C.17) does not lead to a solution.

Similarly, we can investigate the case that only $\underline{m}_2$ instead of $\underline{m}_1$ has a component perpendicular to the plane $\underline{m}_3, \underline{m}_3^*$. This leads to phase XV, cf. eq. (3.23).

Finally, in case iii), in which both $\underline{m}_1$ and $\underline{m}_2$ have a component perpendicular to the plane $\underline{m}_3, \underline{m}_3^*$, we have immediately from (B.1a), (B.1b),

$$\begin{aligned} &\alpha_1 = \alpha_1' , \quad \alpha_2 = \alpha_2' , \quad A = B = C = D = 0 , \\ &\underline{m}_1 \cdot \underline{m}_3 = \underline{m}_2 \cdot \underline{m}_3 = \underline{m}_1 \cdot \underline{m}_3^* = \underline{m}_2 \cdot \underline{m}_3^* = 0 , \end{aligned} \tag{C.18}$$

and because $\underline{m}_3 \| \underline{m}_3^*$ we have

$$\underline{m}_1 \| \underline{m}_1^* \| \underline{m}_2 \| \underline{m}_2^* , \tag{C.19}$$

leading to phase XVI, cf. eq. (3.25). The value of $\underline{m}_3 \cdot \underline{m}_3$ is determined by eq. (3.19).

APPENDIX D

In this appendix we show that the solutions A3, IX, XI, XIII, XIV and XVI do not occur for $w_3, w_3' > 0$, $w_2 < 0$. Here we restrict ourselves to the values of $v_1, \ldots, v_6$ for the case $b=0$ as given in eq. (4.1).

i) *Solution A3.* In order to prove that A3 does not occur for $w_3 > 0$ we regard A3 as the special case $m_1 = m_2 = 0$ of solution XVI, cf. eq. (3.25). As a consequence we

$$\tfrac{1}{2}\left(\frac{\partial^2 \Phi_{XVI}}{\partial m_1^2} + \frac{\partial^2 \Phi_{XVI}}{\partial m_2^2}\right)\Bigg|_{m_1=m_2=0} = u_1 + u_2 + 2(4v + 4w_1 + w_2) m_3^2 > 0 . \tag{D.1}$$

Subtracting from (D.1) the gap equation for A3, i.e.

$$2u_3 + (8v + 8w_1 + 2w_2 + 4w_3) m_3^2 = 0 , \tag{D.2}$$

we obtain

$$u_1 + u_2 - 2u_3 - 4w_3 m_3^2 > 0 , \tag{D.3}$$

which cannot be satisfied for $w_3 > 0$, cf. eq. (1.10).

ii) *Solution IX*. For $6v-w_2-2w_3-w_3' > 0$, and comparing IX with XIII, we have

$$\Phi_{IX}-\Phi_{XIII} = -w_2\, m_3^2(m_1-m_2)^2 + w_3'(m_3^2+m_1m_2)^2 + 4w_3\, m_3^2 m_1 m_2$$
$$+ v[m_1^4 + m_2^4 + 2m_3^4 + 4(m_1^2+m_2^2-3m_1m_2)m_3^2] > 0 \;, \tag{D.4}$$

for $w_2<0$, $w_3,w_3' > 0$. For $6v-w_2-2w_3-w_3' < 0$, we can compare IX with XIV, leading to

$$\Phi_{IX}-\Phi_{XIV} = v(m_1^4+m_2^4+2m_3^4) + 4v(m_1^2+m_2^2)m_3^2 + 4v\, m_3^2 m_1 m_2$$
$$+ w_3'(m_3^2-m_1m_2)^2 > 0 \;. \tag{D.5}$$

Hence IX does not occur for $w_2<0$, $w_3,w_3' > 0$.

iii) *Solution XI*. For $w_3'<2v$ we compare IX with X. Using the relation

$$(w_3+w_3')\underline{m}_1\cdot\underline{m}_2 + (v_6-w_3')m_3^2 = 0 \;,$$

cf. eq. (3.13), we have

$$\Phi_{XI} - \Phi_X = v(m_1^4+m_2^4) - w_3'\, m_1^2 m_2^2 + (2v-w_3')\, m_3^2(2m_1m_2 + \underline{m}_1\cdot\underline{m}_2) > 0 \tag{D.6}$$

for $w_3'<2v$. For $w_3'>2v$ we compare XI and XV, leading to

$$\Phi_{XI} - \Phi_{XV} = v(m_1^2-2m_3^2)^2 + (w_3'-2v)\left(1 - \frac{w_3'-2v}{w_3+w_3'}\right)m_3^4 > 0 \;, \tag{D.7}$$

so that phase XI does not occur for $w_3,w_3'>0$.

iv) *Solution XIII*. For $w_3<2v$ we rewrite Φ_{XIII} as follows

$$\Phi_{XIII} = u_1m_1^2 + u_2m_2^2 + 2u_3m_3^2 + (v+w_1+\tfrac{1}{4}w_2+\tfrac{1}{2}w_3)(m_1^2+m_2^2)^2$$
$$+ (4v+4w_1+w_2+2w_3)m_3^4 + (8v+4w_1+w_2)m_3^2(m_1^2+m_2^2)$$
$$+ (v-\tfrac{1}{4}w_2-\tfrac{1}{2}w_3)(m_1^2-m_2^2)^2 \;, \tag{D.8}$$

and for fixed $m_1^2-m_2^2$, there can only be an absolute minimum of (D.8) corresponding to a mixed phase with $m_1^2+m_2^2 > 0$, $m_3^2 > 0$, under the condition

$$p \equiv \frac{8v+4w_1+w_2}{2(v+w_1+\tfrac{1}{4}w_2+\tfrac{1}{2}w_3)} < 2 \Rightarrow w_3 > 2v \;. \tag{D.9}$$

For $w_3>2v$ we can use another argument. From (D.8) it is clear that XIII can only exist for $2v > \frac{1}{2}w_2+w_3$, so that

$$m_1^2-m_2^2 = \frac{u_2-u_1}{(4v-w_2-2w_3)} > 0 \ , \tag{D.10}$$

and hence, cf. eqs. (3.18), (3.20),

$$\Phi_{XIII} - \Phi_{XV} = -vm_2^4 + 4vm_2^2m_3^2 + w_3m_1^2m_2^2 > 0 \ , \tag{D.11}$$

for $w_3>2v$, $m_1^2>m_2^2$.

v) *Solution XIV.* For $4v > w_2+2w_3$ we have from the gap equations belonging to eq. (3.20)

$$u_1 + u_2 + (4v+4w_1+w_2+2w_3)(m_1^2+m_2^2+2m_3^2) + (4v-w_2-2w_3)m_3^2\left(2 - \frac{m_1^2+m_2^2}{m_1m_2}\right) = 0 \ , \tag{D.12a}$$

$$2u_3 + 2(4v+4w_1+w_2+2w_3)m_3^2 + (8v+4w_1\ )\ (m_1^2+m_2^2) - (8v-w_2-4w_3)m_1m_2 = 0\ . \tag{D.12b}$$

Eqs. (D.12) imply

$$u_1 + u_2 - 2u_3 = (4v-w_2-2w_3)\left(1 + \frac{m_3^2}{m_1m_2}\right)(m_1-m_2)^2 \ , \tag{D.13}$$

which is a contradiction, because the left-hand side of (D.13) is negative, whereas the right-hand side is a positive quantity. For $4v-w_2-2w_3 < 0$ one has from the gap equations

$$u_1 + u_2 - 2u_3 + (4v-w_2-2w_3)\left(\frac{m_3^2}{m_1m_2} - 1\right)(m_1+m_2)^2 = 0 \ , \tag{D.14a}$$

$$u_1 - u_2 + (4v-w_2-2w_3)\left(1 - \frac{m_3^2}{m_1m_2}\right)(m_1^2-m_2^2) = 0 \ , \tag{D.14b}$$

$$u_1 + u_2 + 2u_3 + (16v+8w_1)(m_1^2+m_2^2+2m_3^2) + (4v-w_2-2w_3)\left(\frac{m_3^2}{m_1m_2} - 1\right)(m_1-m_2)^2 . \tag{D.14c}$$

Eq. (D.14a) implies $(m_3^2/m_1m_2) \leq 1$, and from (D.14b) and (D.14c) we obtain

$$m_1^2-m_2^2 \leq 0 \ , \quad m_1^2 + m_2^2 + 2m_3^2 \leq -\frac{u_1+u_2+2u_3}{16v+8w_1} \ . \tag{D.15}$$

Using eq. (D.15) it is clear that

$$f_{XIV} \geq \tfrac{1}{4}(u_1-u_2)(m_1^2-m_2^2) - \tfrac{1}{4}(u_1+u_2)(m_1^2+m_2^2+2m_3^2)$$

$$\geq -\tfrac{1}{4}\,\frac{(u_1+u_2)(u_1+u_2+2u_3)}{16v+8w_1} \geq -\frac{u_1^2}{8v+4w_1} = f_{A1}\ , \tag{D.16}$$

so that XIV is less favourable than the A1 solution. It is easy to check that for $b=0$ the equality signs hold in (D.16), so that in that case XIV and A1 are degenerate.

vi) *Solution XVI.* From the gap equations for phase XVI, cf. eq. (3.25), with regard to m_1, m_2, m_3, one can show that

$$m_1^2+m_2^2 = N^{-1}\left[-2w_3(u_1+u_2) - (4v+4w_1+w_2)(u_1+u_2-2u_3)\right]\ , \tag{D.17a}$$

$$2m_3^2 = N^{-1}\left[-2u_3\left(2w_3+(\alpha+1)w_3'\right) + (4v+4w_1+w_2)(u_1+u_2-2u_3)\right]\ , \tag{D.17b}$$

where

$$N \equiv (4v+4w_1+w_2)\left(4w_3+(\alpha+1)w_3'\right) + 2w_3\left(2w_3+(\alpha+1)w_3'\right), \tag{D.18}$$

in which α is defined by

$$\frac{(v_6-w_3')^2}{2v_3+w_3'} \equiv 2v - \alpha w_3'\ , \qquad \alpha+1 = \frac{8v}{2v+w_3'}\ . \tag{D.19}$$

Using the inequalities

$$m_1^2 - m_2^2 < m_1^2 + m_2^2, \quad (4v+4w_1+w_2)(u_1+u_2-2u_3) > 2u_3\left(2w_3+(\alpha+1)w_3'\right)\ , \tag{D.20}$$

it can be shown that

$$f_{XVI} = \tfrac{1}{4}(u_1-u_2)(m_1^2-m_2^2) + \tfrac{1}{4}(u_1+u_2)(m_1^2+m_2^2) + u_3m_3^2$$

$$= \tfrac{1}{4}(u_1-u_2)(m_1^2-m_2^2) + \frac{1}{N}\Big[-\tfrac{1}{2}w_3(u_1+u_2)^2$$

$$- \tfrac{1}{4}(4v+4w_1+w_2)(u_1+u_2-2u_3)^2 - \left(2w_3+(\alpha+1)w_3'\right)u_3^2\Big]$$

$$\geq -\tfrac{1}{4}(u_1+u_2)^2\,\frac{1}{N}\left(4w_3+(\alpha+1)w_3'\right)\ . \tag{D.21}$$

Comparing (D.21) with f_{VI} given in eq. (4.5), we have

$$f_{XVI} \geq f_{VI}\,(b=0) > f_{VI}\ , \tag{D.22}$$

which implies that phase XVI does not occur for $w_3, w_3'>0$.

APPENDIX E

In this appendix we discuss the solutions VII, VIII and XV, XV'.

i) *Solution VII.* For sufficiently small values of w_1, w_2, w_3, w_3' the solution VII does not occur. In fact, we can write f_{VII} as follows, cf. eq. (2.25) together with (2.27),

$$f_{VII} = -u_3^2 \Big/ \left(4v + 4w_1 + w_2 + w_3 + \frac{8v-w_2-2w_3}{8v-w_2+2w_3} w_3 \right)$$

$$+ \frac{-\frac{1}{2}(u_1-u_3)^2(4v+4w_1+w_2) + w_3(u_3^2-u_1^2)}{(4v+4w_1+w_2+w_3)(4v-\frac{1}{2}w_2+w_3) + w_3(4v-\frac{1}{2}w_2-w_3)} . \tag{E.1}$$

If we consider VII as the special case $m_2=0$ of XVI, then we have for its stability, using the gap equation for m_1,

$$\frac{1}{2} \left. \frac{\partial^2 \Phi_{XVII}}{\partial m_1^2} \right|_{m_2=0} = u_2 - u_1 - \left(8v - w_2 - 2w_3 - (\alpha+1)w_3'\right)m_1^2 > 0 , \tag{E.2}$$

$\alpha \equiv (v_6 - w_3')/(2v_3 + w_3')$. On the other hand, using the gap equation for m_3, we have

$$\frac{1}{2} \left. \frac{\partial^2 \Phi_{XVI}}{\partial m_2^2} \right|_{m_2=0} = 2(u_1-u_3) + (u_2-u_1) + \left(2w_3 + (\alpha+1)w_3'\right)m_1^2 - 4w_3 m_3^2 > 0 . \tag{E.3}$$

Combination of (E.2) and (E.3) (without the term $-4w_3m_3^2$) yields

$$u_3 - u_1 < \tfrac{1}{2}(u_2-u_1) \left[1 + \frac{2w_3 + (\alpha+1)w_3'}{8v - w_2 - 2w_3 - (\alpha+1)w_3'} \right] . \tag{E.4}$$

On the other hand one has from (E.3), using the fact that $u_1 + u_2 - 2u_3 < 0$,

$$\left(2w_3 + (\alpha+1)w_3'\right)m_1^2 > 4w_3 m_3^2 , \tag{E.5}$$

which in combination with the eqs. (2.24) and (2.27) for m_1^2 and m_3^2 yields

$$u_3^2 - u_1^2 > - \frac{\frac{1}{4}\left(2w_3 + (\alpha+1)w_3'\right)(4v+4w_1+w_2+w_3)\ w_3^{-1} + (4v+2w_1+\frac{1}{4}w_2)}{8v - w_2 - 2w_3 - (\alpha+1)w_3'} \times$$

$$\times \left[1 + \frac{2w_3 + (\alpha+1)w_3'}{8v - w_2 - 2w_3 - (\alpha+1)w_3'} \right]^2 (u_2-u_1)^2 . \tag{E.6}$$

From (E.6), using the relation

$$\frac{8v-w_2-2w_3}{8v-w_2+2w_3} > \frac{1}{w_3}\frac{2\gamma v}{2v+\gamma},$$

which holds for $w_3'<2v$, $w_3<2v-\frac{1}{2}w_2$, one can show that from (E.4), (E.6) and (E.1), we have

$$f_{VII} > f_{VI}, \tag{E.7}$$

for sufficiently small values of w_1, w_2, w_3, w_3'.

ii) *Solution VIII.* Assuming that (2.25) in the case that (2.28) has an absolute minimum with $m_1m_3\neq 0$, we have the condition

$$p \equiv \frac{8v+4w_1+w_2}{(2v+w_1)^{\frac{1}{2}}(4v+4w_1+w_2+2w_3)^{\frac{1}{2}}} < 2, \tag{E.8}$$

which cannot be satisfied for sufficiently small values of w_1, w_2, w_3. Hence the mixed phase with $m_1, m_3\neq 0$ does not occur.

iii) *Solutions XV and XV'.* The condition XV at fixed m_2 has an absolute minimum with $m_1m_3\neq 0$, i.e.

$$p \equiv \frac{8v+4w_1+w_3}{(2v+w_1)^{\frac{1}{2}}(4v+4w_1+w_2+2w_3)^{\frac{1}{2}}} < 2, \tag{E.9}$$

cannot be satisfied for sufficiently small w_1, w_2, w_3. (Note that eqs. (E.8) and (E.9) coïncide).

Remark: In order to investigate the relative stability of the solutions XV and XV', we introduce the function

$$\begin{aligned}\Phi = {}& u_1m_1^2 + u_2m_2^2 + 2u_3m_3^2 + (\tfrac{5}{2}v+\tfrac{1}{2}\varepsilon v+w_1)m_1^4 + (\tfrac{5}{2}v-\tfrac{1}{2}\varepsilon v+w_1)m_2^4 \\ &+ (4v+4w_1+w_2+2w_3)m_3^4 + (6v-2\varepsilon v+4w_1+w_2)m_1^2m_2^2 \\ &+ (6v+2\varepsilon v+4w_1+w_2)m_2^2m_3^2 + (2w_1+w_2+w_3)m_1^2m_2^2,\end{aligned} \tag{E.10}$$

such that $\Phi=\Phi_{XV}$ for $\varepsilon=-1$, $\Phi=\Phi_{XV'}$ for $\varepsilon=+1$. Rewrite Φ as

$$\Phi = u_1 m_1^2 + u_2 m_2^2 + 2u_3 m_3^2 + \tfrac{1}{4}(5v-w_2-w_3)(m_1^2-m_2^2)^2$$

$$+ (\tfrac{4}{3}v+w_1+\tfrac{1}{4}w_2+\tfrac{1}{6}w_3)(m_1^2+m_2^2+2m_3^2)^2 + \tfrac{1}{12}(w_3-v)(m_1^2+m_2^2-4m_3^2)^2$$

$$+ \tfrac{1}{2}\varepsilon v(m_1^2-m_2^2)(m_1^2+m_2^2-4m_3^2) \ . \qquad (E.11)$$

From the condition that ϕ at fixed $m_1^2+m_2^2+2m_3^2$ has an absolute minimum for $(m_1^2-m_2^2)(m_1^2+m_2^2-4m_3^2) \neq 0$, we obtain the condition

$$(w_3-v)(5v-w_2-w_3) > 3v^2 \ . \qquad (E.12)$$

On the other hand from the gap equations for $m_1^2-m_2^2$, $m_1^2+m_2^2+2m_3^2$ and $m_1^2+m_2^2-4m_3^2$, it is easy to show that the absolute minimum f of Φ is given by

$$f = -\frac{\tfrac{1}{3}(u_1+u_2)(u_1+u_2+u_3)}{\tfrac{32}{3}v+8w_1+2w_2+\tfrac{4}{3}w_3}$$

$$-\frac{\tfrac{1}{4}(w_3-v)(u_1-u_2)^2 - \tfrac{1}{8}(5v-w_3-w_3)(u_1+u_2-2u_3)^2 + \tfrac{5}{8}\varepsilon v(u_1-u_2)(u_1+u_2-2u_3)}{(w_3-v)(5v-w_2-w_3) - 3v^2} \ . \qquad (E.13)$$

From (E.13) it is clear that XV with $\varepsilon=-1$ has a lower free energy than XV' with $\varepsilon=+1$.

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SAMENVATTING

De ontdekking van de superfluïde fasen bij temperaturen van 2.6 mK en lager in ^3He in 1972 heeft geleid tot een groot aantal theoretische onderzoekingen. Vanuit fundamenteel theoretisch oogpunt is ^3He van groot belang, omdat het een typisch voorbeeld is van een Fermi-vloeistof, waarin als gevolg van de kleine atoommassa's de quantumeffecten een grote rol spelen. De theoretische beschrijving van het fasediagram van ^3He is gebaseerd op een effectieve hamiltoniaan in termen van quasi-deeltjes met fermion (anti)commutatierelaties. De hamiltoniaan bevat naast de gebruikelijke kinetische energieterm ook een attractieve paarinteractie. Een dergelijke interactie leidt tot de vorming van zogenaamde Cooper-paren, dat zijn gebonden toestanden van paren Fermi-deeltjes met energieën in de buurt van het Fermi-oppervlak. In tegenstelling tot de situatie bij de BCS-theorie voor supergeleiding, waarin de $\ell=0$ term van de ontwikkeling van de paarpotentiaal in Legendre polynomen het belangrijkst is, is in het geval van ^3He de $\ell=1$ term dominant. Dit heeft tot gevolg dat in ^3He een Cooper-paar een totale spin 1 heeft met de bijbehorende triplettoestanden $|\uparrow\uparrow\rangle$, $\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$, $|\downarrow\downarrow\rangle$, in termen van de spin $\uparrow,\downarrow$ van de beide fermionen. Afgezien van de paarpotentiaal treden in ^3He ook polarisatie-effecten op, die beschreven kunnen worden door een repulsieve contactinteractie van het Hubbard-type. Voor de beschrijving van het fasediagram in een uitwendig magneetveld dient tevens een Zeeman-term in de hamiltoniaan te worden opgenomen.

In de literatuur is op grond van het hierboven beschreven model een interpretatie gegeven van de types van ordening, die overeenkomen met de experimenteel gevonden superfluïde fasen A en B. In afwezigheid van een uitwendig magneetveld ($b=0$) wordt de B fase beschreven door een isotrope fase waarin alle drie spin-triplettoestanden geordend zijn, de zogenaamde BW (Balian-Werthamer) fase, en de A fase wordt geïnterpreteerd in termen van een anisotrope fase, de zogenaamde ABM (Anderson-Brinkman-Morel) fase met alleen $\uparrow\uparrow$ en $\downarrow\downarrow$ ordening. In een uitwendig magneetveld ($b>0$) wordt de A fase opgesplitst in een fase met $\uparrow\uparrow$ en $\downarrow\downarrow$ ordening en een A1 fase, waarin alleen de toestand $\uparrow\uparrow$ geordend is. De argumenten, die aan deze identificaties ten grondslag liggen, zijn gebaseerd op bepaalde veronderstellingen, die vooral voor $b\neq0$ nogal een ad-hoc karakter hebben.

Voor een systematische beschrijving van het fasediagram kan uitgegaan worden van de Landau-ontwikkeling, waarbij rekening gehouden moet worden met alle mogelijke ordeparameters van het probleem. (De mogelijke spintoestanden voor een Cooper-paar, tezamen met de drie mogelijke ruimtelijke oriëntaties van een $\ell=1$ toestand geven aanleiding tot 9 complexe, of 18 reële ordeparameters.) Aangezien het bepalen van het absolute minimum voor een vierdegraadspolynoom in 18 reële variabelen met onbepaalde coëfficiënten ondoenlijk is, zal eerst van de effectieve hamiltoniaan gebruik gemaakt moeten worden om voldoende informatie te verkrijgen over de coëfficiënten voor de Landau-ontwikkeling.

Bij het berekenen van deze coëfficiënten wordt in dit proefschrift gebruik gemaakt van de methode van de separabele interacties. Deze methode maakt het mogelijk om een exacte uitdrukking voor de vrije energie te geven in het geval dat de Hamiltoniaan een eindig aantal separabele interacties bevat, d.w.z. interacties die te schrijven zijn als produkten van sommen van 1-deeltjesoperatoren. In de zwakke koppelingslimiet, waarin de paarinteracties alleen optreden in een klein schilletje om de Fermi-energie, kan de paarinteractie direct worden uitgedrukt in termen van separabele interacties. In hoofdstuk II wordt deze methode toegepast op het model beschreven door de bovengenoemde effectieve hamiltoniaan in het speciale geval dat er geen Hubbard-interactie is. In de paarpotentiaal kunnen verschillende ℓ-waarden van de ontwikkeling in Legendre polynomen worden meegenomen. Voor $b=0$ wordt bewezen dat alleen de BW-ordening behorend bij $\ell=1$ in deze ontwikkeling groter is dan de coëfficiënten van andere ℓ-waarden. Aannemend dat ook voor $b\neq0$ de coëfficiënten van andere ℓ-waarden niet belangrijk zijn, worden de coëfficiënten van de Landau-ontwikkeling berekend voor willekeurige waarden van het magneetveld b. Uitgaande van deze Landau-ontwikkeling wordt in hoofdstuk III het fasediagram voor ^{3}He voor $b\neq0$ in afwezigheid van een Hubbard-interactie bestudeerd. Op grond van een gedetailleerde behandeling van de oplossingen van het 18-ordeparameterprobleem wordt aangetoond dat het fasediagram verkregen kan worden door minimalisatie van een geschikte 3-parameterfunctie. Dit leidt tot de conclusie dat de volgende fasen kunnen optreden:
i) een generalisatie van de BW-fase voor $b\neq0$ met 3 onafhankelijke ordeparameters, ii) de ABM- of een daarmee ontaarde fase met 2 ordeparameters, iii) de A1-fase met 1 ordeparameter verschillend van 0.

In de twee laatste hoofdstukken van dit proefschrift wordt de Hubbard-term in rekening gebracht. Ook in dit geval kan men een exacte uitdrukking geven voor de vrije energie in termen van de vrije energie van een referentiesysteem.

Het probleem is dat de hamiltoniaan van het referentiesysteem, behalve de kinetische energie, de Zeeman-term, en een 1-deeltjesbenadering van de paar-interactie, ook de volledige Hubbard-interactie bevat. Bijgevolg ken men de vrije energie van het referentiesysteem niet exact berekenen. Met behulp van storingsrekening kan echter een Landau-ontwikkeling worden gevonden, die goed is tot op zekere orde in de koppelingsconstante van de Hubbard-interactie. In hoofdstuk IV wordt op deze wijze de Landau-ontwikkeling voor het systeem met Hubbard-interactie berekend in tweede orde storingsrekening. Het 18-orde-parameterprobleem in aanwezigheid van de Hubbard-interactie wordt onderzocht in hoofdstuk V. De minimalisering van de Landau-ontwikkeling leidt tot een groot aantal fasen en de analyse van het fasediagram is veel gecompliceerder dan in hoofdstuk III. Twee problemen worden in meer detail behandeld. Enerzijds wordt het fasediagram voor $b\neq 0$ bestudeerd voor kleine waarden van de koppelingsconstante van de Hubbard-interactie, waarvoor de tweede orde storingsrekening een correcte beschrijving geeft. Anderzijds wordt het fasediagram voor $b=0$ ook bestudeerd voor grotere waarden van deze koppelingsconstante, aannemend dat de tekens voor de coëfficiënten van de Hubbard-bijdragen tot de Landau-ontwikkeling correct gegeven worden door het tweede orde resultaat. De conclusie is dat onder deze omstandigheden andere dan de hierboven genoemde fasen kunnen optreden, leidend tot een alternatief voorstel voor de interpretatie van de A fase.

CURRICULUM VITAE

De auteur van dit proefschrift werd geboren op 19 augustus 1956 te Amsterdam. In 1974 behaalde hij het einddiploma Gymnasium β aan de Coornhert Scholengemeenschap te Haarlem. In datzelfde jaar begon hij de studie natuurkunde aan de Rijksuniversiteit te Leiden. In februari 1977 behaalde hij het kandidaatsexamen natuurkunde, met bijvakken sterrenkunde en wiskunde, en vervolgens op 9 oktober 1979 het doctoraalexamen natuurkunde met wiskunde als bijvak. Tijdens de studie werd een aanvang gemaakt met het onderzoek aan de theorie van ^{3}He, gedurende de experimentele stage onder leiding van Prof.dr. W.J. Huiskamp, en als theoretisch doctoraal onderzoek onder leiding van Prof.dr. H.W. Capel. Op 15 oktober 1979 trad hij in dienst van de Stichting voor Fundamenteel Onderzoek der Materie in de werkgroep VS-th-L. Onder leiding van Prof.dr. H.W. Capel verrichtte hij op het Instituut-Lorentz voor Theoretische Natuurkunde te Leiden onderzoek aan de theorie van quantumvloeistoffen, met name de theorie van superfluïde ^{3}He. De resultaten van dit onderzoek zijn in dit proefschrift neergelegd. Bovendien heeft hij onderzoek verricht aan integreerbare niet-lineaire systemen in twee dimensies. De resultaten uit dit onderzoek zijn gepubliceerd in een aantal overige artikelen. Voorts nam hij deel aan enkele zomerscholen, te weten de zomerschool "Fundamental Problems in Statistical Mechanics V" in Enschede (1980), het internationale zomerinstituut "Structural Elements in Particle Physics and Statistical Mechanics" in Freiburg (1981) en de zomerschool "Développements Récents en Mécanique Statistique et Théorie Quantique des Champs" in Les Houches (1982).

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Phys.Lett. 98A (1983) 83.

Apart from minor modifications chapters II and III of this thesis are contained in the publications 2 and 3 respectively.

NAWOORD

Gedurende de tijd dat ik aan dit proefschrift heb gewerkt was voor mij de steun en toewijding van Jeroen Wertenbroek van grote betekenis. Drs. Arie den Breems heeft in belangrijke mate bijgedragen aan de berekeningen van hoofdstukken IV en V.

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De promotie vindt plaats in de Senaatskamer van het Academiegebouw, Rapenburg 73, Leiden.

Toegang tot de promotie wordt in verband met de beperkte ruimte in de Senaatskamer uitsluitend verleend op vertoon van een toegangskaart.

NA AFLOOP VAN DE PROMOTIE IS ER EEN RECEPTIE IN HET ACADEMIEGEBOUW.

N.B. Met parkeerproblemen bij het Academiegebouw moet rekening worden gehouden.

STELLINGEN

1. Door gebruik te maken van het formalisme van Dirac voor systemen met intrinsieke restricties kan de Hamilton-beschrijving van de isotrope Heisenberg-spinketen verkregen worden uit een Lagrangiaan.

2. De conclusie van Campbell, Arvanitis en Fert, dat hun meetresultaten voor het magnetisch koppel uitgeoefend door een Cu(2 at% Mn)-preparaat het bestaan van een Almeida-Thoulesslijn in het (T,H)-fasediagram ondersteunen is niet gerechtvaardigd.

   L.A. Campbell, D. Arvanitis and A. Fert, Phys. Rev. Lett. 51 (1983) 57.
   J.R.L. de Almeida and D.J. Thouless, J. Phys. A11 (1978) 983.

3. Het door Santini et al. gepresenteerde formalisme voor de behandeling van een algemeen N×N spectraalprobleem is niet zonder meer toepasbaar op de behandeling van specifieke partiële differentiaalvergelijkingen.

   P. Santini, M.J. Ablowitz and A.S. Fokas, Clarkson Preprint (1983).

4. Voor een correcte beschrijving van het fasediagram van supervloeibaar ^{3}He in aanwezigheid van een magneetveld in de zwakke-koppelingslimiet, is het niet gerechtvaardigd ab initio de veldafhankelijkheid in de vierde orde coëfficiënten van de Landau-ontwikkeling te verwaarlozen.

   Dit proefschrift, hoofdstuk III.

5. In een ruim gebied van waarden voor de spinfluctuatiecoëfficiënten in de Landau-ontwikkeling voor ^{3}He, treedt de Anderson-Brinkman-Morelfase niet op.

   Dit proefschrift, hoofdstuk V.

6. Het door Boissonade in onderstaand artikel gepostuleerde effect bestaat niet.

   J. Boissonade, *"The screening effect in suspensions of freely moving spheres"*, J. de Physique 43 (1982) L-371.

7. Het door Stanley et al. op basis van computersimulaties gesuggereerde 'universele' gedrag in zogenaamde attractieve stochastische roosterwandelingen, in het geval dat de attractieparameter de kritische puntpercolatiewaarde van het rooster aanneemt, is onjuist.

H.E. Stanley, K. Kang, S. Redner and R.L. Blumberg, Phys. Rev. Lett. 51 (1983) 1223.

8. Riemann-Hilberttransformaties, die oplossingen van een lineair spectraalprobleem afbeelden op andere oplossingen van hetzelfde spectraalprobleem, commuteren, indien de bijbehorende integratiecontouren in het complexe vlak van de spectraalparameter elkaar niet snijden.

9. De niet-lineaire partiële differentievergelijking

$$[z(n+1,m) + z(n,m+1) + z^*(n+1,m) + z^*(n,m+1)]\; z(n,m)z(n+1,m+1) =$$

$$[z(n,m+1) + z^*(n,m+1)]\; |z(n+1,m)|^2 + [z(n+1,m) + z^*(n+1,m)]\; |z(n,m+1)|^2$$

$$+ [z(n+1,m)z(n,m+1) - z^*(n+1,m)z^*(n,m+1)]\; [z(n+1,m+1) + z(n,m)]$$

is exact integreerbaar. In een geschikte continuumlimiet gaat deze vergelijking over in de vergelijking voor het niet-lineaire sigma-model met niet-compacte SO(2,1) symmetrie.

10. Bij geschikte waarden van de spinfluctuatiecoëfficiënten in de Landau-ontwikkeling voor ^{3}He kan op de fase-overgangslijn tussen de Balian-Werthamerfase en de A1-fase een trikritisch punt optreden.

11. Het fasediagram dat door Bailin en Love is voorgesteld voor de beschrijving van een ultradichte materie van quarks, waarin Cooperparen met totale spin S=0 gevormd kunnen worden, is niet correct.

D. Bailin and A. Love, J. Phys. A12 (1979) L283,
Nucl. Phys. B190 [FS 3] (1981) 175.

12. De neurobiologische interpretatie van het bewustzijn wordt gerelativeerd door de neurobiologische structuur zelf van het bewustzijnsbeeld van de realiteit.

13. Op eenvoudige wijze kan een klasse van integreerbare differentiaal-differentievergelijkingen worden verkregen, met coëfficiënten die van de discrete variabele n afhangen. In tegenstelling tot de resultaten gevonden door Bruschi en Ragnisco, hangen deze vergelijkingen wél samen met een isospectrale deformatie van een bijbehorend lineair eigenwaardenprobleem.

M. Bruschi and O. Ragnisco, Lett. al Nuovo Cimento <u>31</u> (1981) 492.

14. De uitleg van de werking van de menselijke begeerte door Deleuze en Guattari in termen van "Machines Désirantes" wordt ontkracht door hun ontkenning, dat het in de aard van het begrip machine ligt dat "les machines ne se produisent pas, ou ne se reproduisent que par l'intermédiaire de l'homme, ..." .

G. Deleuze et F. Guattari, l'Anti-Œdipe, Capitalisme et Schizophrénie, (Eds. de Minuit, Paris, 1972), p. 338.

F.W. Nijhoff

Leiden, 2 mei 1984.