NL 84C00037

INIS-mf--9027

LINEAR INTEGRAL EQUATIONS AND SOLITON SYSTEMS

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PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE
WISKUNDE EN NATUURWETENSCHAPPEN
AAN DE RIJKSUNIVERSITEIT TE LEIDEN,
OP GEZAG VAN DE RECTOR MAGNIFICUS DR. A.A.H. KASSENAAR,
HOOGLERAAR IN DE FACULTEIT DER GENEESKUNDE,
VOLGENS BESLUIT VAN HET
COLLEGE VAN DEKANEN TE VERDEDIGEN OP
WOENSDAG 2 NOVEMBER 1983 TE KLOKKE 16.15 UUR

door

GILLES REINOUT WILLEM QUISPEL

geboren te Bilthoven in 1953

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"Do not try to know the truth, for knowledge by the mind is not true knowledge. But you can know what is not true - which is enough to liberate you from the false. The idea that you know what is true is dangerous, for it keeps you imprisoned in the mind. It is when you do not know, that you are free to investigate. And there can be no salvation, without investigation, because non-investigation is the main cause of bondage."

Sri Nisargadatta Maharaj

voor Nel

voor mijn vader en moeder

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CHAPTER I

INTRODUCTION AND SUMMARY

Many phenomena in physics are of an essentially nonlinear nature. For centuries most of these phenomena were studied in linear approximation. Only in recent decades the mathematical methods have begun to be developed, to study certain classes of nonlinear systems exactly.

In 1955 Fermi, Pasta and Ulam [1)] published a paper in which they performed a numerical investigation of a one-dimensional system of anharmonic oscillators described by differential-difference equations of the type

$$\partial_t^2 u(n,t) = u(n+1,t) - 2u(n,t) + u(n-1,t) + \alpha\Big\{\big(u(n+1,t) - u(n,t)\big)^2 - \big(u(n,t) - u(n-1,t)\big)^2\Big\} , \tag{1}$$

where $u(n,t)$ denotes the displacement of the n'th mass point at time t. Contrary to what was expected at that time, they found no tendency toward equipartition of energy among the degrees of freedom of the system.

This discovery was an important stimulus for research on dynamical systems, which at present has developed two main branches, the study of integrable nonlinear systems, by which we mean nonlinear systems that can be solved exactly using only linear methods, and the study of nonintegrable systems.

In this thesis a study will be presented of classical integrable dynamical systems in one temporal and one spatial dimension; some general references on this subject are refs. 2-8. For results on integrable systems in higher spatial dimensions see e.g. refs. 9 and 10, and for reviews of integrable quantum systems see refs. 11 and 12. A review of nonintegrable systems can be found in ref. 13.

One of the interesting features of integrable nonlinear dynamical systems is the fact that, under appropriate boundary conditions, they give rise to solitons. Solitons are solitary waves that asymptotically preserve their energy, momentum and amplitude upon collision with other solitary waves. Some examples of integrable dynamical systems which will be discussed in this thesis are the Korteweg-de Vries equation, which describes e.g. waves in shallow water, the nonlinear Schrödinger equation, occurring e.g. in the theoretical description of plasma waves, the sine-Gordon equation, which describes a system of coupled pendula, the equations of motion for the isotropic and the anisotropic Heisenberg spin chain, etc. Of course these equations also have other physical applications, as different physical situations can often be described by the same mathematical model. For an application of soliton systems to field theory see ref. 14, and for an application to solid state physics see ref. 15.

Of course integrable dynamical systems form a small minority, as most systems turn out to be nonintegrable. One can, however, hope that soliton theory can in some sense be regarded as a zero'th order theory [8)], that may be used as a starting point for perturbation expansions [16)].

There are several approaches that have been used to study soliton systems, using e.g. the inverse scattering transform [2-7)], the Riemann-Hilbert method [8,9,17)], prolongation structures [18-20)], or bilinearization [21)]. One of the most successful of these methods uses the inverse scattering transform, also called the inverse spectral transform. This method for solving the initial value problem of integrable nonlinear evolution equations under suitable boundary conditions at infinity, can be regarded as the non-linear analogue of the Fourier transform; a thorough treatment of this analogy is given in ref. 2. A schematic representation of the method is given in the following diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\begin{array}{c}\text{initial datum at } t=0 \\ u(x,0)=u_0(x)\end{array}} \arrow[r, "\text{direct}", "\text{spectral problem}"'] \arrow[d, dashed, "\begin{array}{l}\text{time evolution}\\ \text{in configuration}\\ \text{space}\end{array}"] & \boxed{\begin{array}{c}\text{scattering data at } t=0 \\ S(0)\end{array}} \arrow[d, "\begin{array}{l}\text{time evolution}\\ \text{in spectral}\\ \text{space}\end{array}"'] \\
\boxed{\begin{array}{c}\text{solution at } t>0 \\ u(x,t)\end{array}} & \boxed{\begin{array}{c}\text{scattering data at } t>0 \\ S(t)\end{array}} \arrow[l, "\text{inverse}"', "\text{spectral problem}"]
\end{tikzcd}$$

The success of the inverse scattering method stems from the fact that the time evolution of the scattering data is governed by linear equations, in contrast to the time evolution in configuration space. One of the key steps in this method is the inverse spectral problem yielding the solution $u(x,t)$ of the nonlinear evolution equation, in terms of the scattering data $S(t)$. The solution of this inverse problem is expressed by a so-called Gel'fand-Levitan equation.

As an example, consider the case that $u(x,t)$ obeys the Korteweg-de Vries equation [22], i.e.

$$\partial_t u = \partial_x^3 u - 6u\partial_x u \ , \tag{2}$$

which first led to the discovery of the inverse scattering transform by Gardner, Greene, Kruskal and Miura [23]. The Gel'fand-Levitan equation for the Korteweg-de Vries equation reads as follows

$$K(x,y,t) + M(x+y,t) + \int_x^\infty dz\ K(x,z,t)M(z+y,t) = 0 \ , \qquad y>x \ , \tag{3}$$

where the kernel M is expressed in terms of the scattering data

$$\begin{aligned} R(k,t) &= R(k,0)\exp(-8ik^3t) \ , \\ p_n(t) &= p_n(0) \equiv p_n \ , \\ \rho_n(t) &= \rho_n(0)\exp(-8p_n^3t) \ , \end{aligned} \tag{4}$$

in the following way:

$$M(x,t) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} R(k,t)\exp(ikx)dk + \sum_{n=1}^{N} \rho_n(t)\exp(-p_n x) \ . \tag{5}$$

Here $R(k,0)$, p_n, and $\rho_n(0)$ play the role of a reflection coefficient, a discrete eigenvalue, and a normalization factor, respectively, associated with a Schrödinger equation in which $u_0(x)$ is the potential. A more detailed definition of these quantities is given in ref. 2. The Gel'fand-Levitan equation (3) is a linear equation, and from its solution $K(x,y,t)$, the solution $u(x,t)$ of the Korteweg-de Vries equation (2) is obtained as follows

$$u(x,t) = -2\partial_x \lim_{y \downarrow x} K(x,y,t) \ . \tag{6}$$

A convincing application of the inverse scattering transform to the analysis of experimental data has been given by A.R. Osborne et al. in ref. 24.

The treatment of soliton systems that will be presented in this thesis, however, is not based on the inverse scattering transform or on one of the other methods mentioned above, but on a new method that has been introduced by Fokas and Ablowitz, and which we will call the method of direct linearization.

In the method of direct linearization one starts from a singular linear integral equation, involving an arbitrary contour and measure in the complex plane. This singular linear integral equation yields not only very general solutions of the associated nonlinear evolution equations, but also many of the features of these evolution equations, e.g. Miura transformations, linear scattering problems, Bäcklund transformations, and integrable discretizations. The method has the advantage that different evolution equations can be treated in a comprehensive and unifying way. (For some general references on singular integral equations, their connection with Riemann-Hilbert problems and with inverse spectral problems see refs. 9, 21, 26, 27.)

Very briefly, the content of this thesis is the following. In chapter II the direct linearizations are given of several nonlinear partial differential equations, for example the Korteweg-de Vries equation, the modified Korteweg-de Vries equation, the sine-Gordon equation, the nonlinear Schrödinger equation, and the equation of motion for the isotropic Heisenberg spin chain; and we also discuss several relations between these equations. In chapter III the Bäcklund transformations of these partial differential equations are treated on the basis of a singular transformation of the measure (or equivalently of the plane-wave factor) occurring in the corresponding linear integral equations, and the Bäcklund transformations are used to derive the direct linearization of a chain of so-called modified partial differential equations; for example, from the Bäcklund transformation of the nonlinear Schrödinger equation the direct linearization of the equation of motion for the anisotropic Heisenberg spin chain is derived. Finally in chapter IV it is shown that the singular linear integral equations lead in a natural way to the direct linearizations of various nonlinear difference-difference equations. These equations for functions of two discrete variables n and m, reduce to the partial differential equations mentioned above, after two successive continuum limits. As an intermediate result we also present the direct linearizations of the differential-difference equations that obtain after one single continuum limit, e.g. the equation of motion for the Toda lattice, the discrete nonlinear Schrödinger equation, the discrete complex sine-Gordon equation, etcetera.

As an illustration, we here summarize some of the main steps of the treatment given in this thesis, using as an example the Korteweg-de Vries equation and its discrete analogues.

The singular integral equation providing the direct linearization of the Korteweg-de Vries equation is the following

$$u_k + i\rho_k \int_C d\lambda(\ell)\, \frac{u_\ell}{k+\ell} = \rho_k \ , \tag{7}$$

where C and $d\lambda(k)$ denote an arbitrary contour and measure in the complex k-plane, and where the plane-wave factor ρ_k is given by

$$\rho_k \equiv \rho_k(x,t) = e^{i(kx-k^3t)}\rho_k(0,0) \ . \tag{8}$$

This equation was introduced in ref. 25, where it was shown that if the solution u_k of equation (7) for a given measure $d\lambda(k)$ and contour C is unique, then the function u which is given by an integration of $u_k = u_k(x,t)$ over the same contour and measure, i.e.

$$u \equiv u(x,t) = \int_C d\lambda(k) u_k \tag{9}$$

obeys the potential Korteweg-de Vries equation

$$\partial_t u = \partial_x^3 u - 3(\partial_x u)^2 \ . \tag{10}$$

(Note that $v \equiv \partial_x u$ obeys the Korteweg-de Vries equation (2).) Explicit solutions describing N solitons, for example, are obtained by choosing a measure containing N simple poles, i.e.

$$d\lambda(k) = \frac{1}{2\pi i} \sum_{n=1}^{N} \frac{c_n}{k-k_n}\, dk \ , \tag{11}$$

where c_n and k_n are constants, and a contour C that surrounds the poles k_n. For $N=2$ we obtain e.g.

$$u = \frac{c_1\rho_{k_1} + c_2\rho_{k_2} + ic_1c_2\rho_{k_1}\rho_{k_2}\dfrac{(k_1-k_2)^2}{2k_1k_2(k_1+k_2)}}{1 + ic_1\dfrac{\rho_{k_1}}{2k_1} + ic_2\dfrac{\rho_{k_2}}{2k_2} - c_1c_2\rho_{k_1}\rho_{k_2}\dfrac{(k_1-k_2)^2}{4k_1k_2(k_1+k_2)}}, \tag{12}$$

where ρ_{k_1} and ρ_{k_2} are given by equation (8).

If we apply a singular transformation to the plane-wave factor ρ_k in equation (7) of the form

$$\rho_k \to \tilde{\rho}_k = \frac{q+k}{q-k}\rho_k , \tag{13}$$

and denote by $\tilde{u}_k$ the solution of equation (7) with $\tilde{\rho}_k$ instead of ρ_k, and define the function

$$\tilde{u} \equiv \tilde{u}(x,t) = \int_C d\lambda(k)\tilde{u}_k , \tag{14}$$

then we can derive the following Bäcklund transformation (cf. refs. 28 and 29)

$$\partial_x\tilde{u} = -\partial_x u + iq(\tilde{u}-u) + \tfrac{1}{2}(\tilde{u}-u)^2, \tag{15}$$

$$\partial_t\tilde{u} = -\partial_t u + iq\partial_x^2\tilde{u} - iq\partial_x^2 u + \tfrac{1}{2}\partial_x^2(\tilde{u}-u)^2 - 3(\partial_x u)^2 - 3(\partial_x\tilde{u})^2 . \tag{16}$$

Given an arbitrary solution u of equation (10), the Bäcklund transformation enables one to obtain another solution $\tilde{u}$ of equation (10) after solving the ordinary Riccati differential equation (15). (The integration constant is determined by eq. (16).) Using equations (13) and (11) it can be shown that the Bäcklund transformation transforms an N-soliton solution u into an (N+1)-soliton solution $\tilde{u}$. From equations (15) and (16) it also follows that $u^- \equiv \tilde{u}-u$ obeys the modified Korteweg-de Vries equation

$$\partial_t u^- = \partial_x^3 u^- - 3iqu^-\partial_x u^- - \frac{3}{2}(u^-)^2\partial_x u^- . \tag{17}$$

Introducing a second singular transformation of the plane-wave factor

$$\rho_k \to \hat{\rho}_k \equiv \frac{p+k}{p-k}\rho_k, \quad u_k \to \hat{u}_k , \quad u \to \hat{u} \equiv \int_C d\lambda(k)\hat{u}_k , \tag{18}$$

it is immediately clear that $\hat{\tilde{\rho}}_k = \tilde{\hat{\rho}}_k$, hence $\hat{\tilde{u}} = \tilde{\hat{u}}$. From this, one can derive the following Bianchi-identity [30,31]

$$\frac{\hat{\tilde{u}}-u}{ip+iq} = -\frac{\tilde{u}-\hat{u}}{\tilde{u}-\hat{u}-ip+iq} . \tag{19}$$

Alternatively, we can say that if ρ_k in equation (7) is given by

$$\rho_k \equiv \rho_k(n,m) = \left(\frac{p+k}{p-k}\right)^n \left(\frac{q+k}{q-k}\right)^m \rho_k(0,0) \quad , \tag{20}$$

instead of by equation (8), and if the function $u(n,m)$ is defined by

$$u(n,m) \equiv \int_C d\lambda(k) u_k \ , \tag{21}$$

where $u_k = u_k(n,m)$ is the solution of eq. (7) with ρ_k given by (20), then $u(n,m)$ obeys the following difference-difference equation

$$\frac{u(n+1,m+1) - u(n,m)}{ip+iq} = -\frac{u(n,m+1) - u(n+1,m)}{u(n,m+1) - u(n+1,m) - ip + iq} . \tag{22}$$

Hence the linear integral equation (7), with (20), provides a direct linearization of equation (22). The N-soliton solution of equation (22) can be obtained using the measure given in equation (11). The 2-soliton solution e.g. is again given by equation (12), where we should now insert equation (20) for ρ_{k_1} and ρ_{k_2}.

Taking a continuum limit of equation (22) with respect to m, we obtain

$$\partial_t u(n,t) = -\frac{u(n+1,t) - u(n-1,t)}{u(n+1,t) - u(n-1,t) + 2ip} , \tag{23}$$

and taking a second continuum limit, with respect to n, we recover, after some obvious transformations, the potential Korteweg-de Vries equation (10).

More details and results concerning these and other equations are given in the following chapters of this thesis.

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CHAPTER II

ON SOME LINEAR INTEGRAL EQUATIONS GENERATING SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

1. Introduction

The study of nonlinear partial differential equations (PDE's), that are solvable by means of the inverse-scattering transform (IST) formalism, has become of great interest during the last decade. For many integrable PDE's the various ingredients of this formalism have been established and exact solutions, such as e.g. soliton solutions, have been found in a systematic way[1]). One of the underlying difficulties for this method is the choice of boundary conditions, which have to be taken into account from the very beginning. This feature very often obscures the fact that the crucial step of the inverse problem, i.e., the Gel'fand–Levitan–Marchenko equation, can in many respects be formulated in a way, which is independent of the choice of boundary conditions. A common feature of the nonlinear PDE's solvable via the IST is that solutions of these equations can be mapped onto a linear inhomogeneous integral equation and it is possible to explore these integral equations from a more general point of view[2,3]).

Recently, however, a new type of linear integral equation for the linearization of the Korteweg–de Vries equation (KdV) was proposed by Fokas and Ablowitz[4]), which enables one to extract complete information on the essential nonlinearity of the corresponding PDE, without having to go through the details of the inverse-scattering formalism, such as e.g. the choice of boundary conditions. The important feature is that the integral equation has a singular kernel and that it contains an integration with an arbitrary measure over a complex variable k, over an arbitrary contour in the complex k-plane. By choosing appropriate measures and contours, the various solutions of the KdV can be obtained directly from the integral equation.

As a first extension to the approach of Fokas and Ablowitz, one can take into consideration a more general inhomogeneous term in the integral equation, leading to a matrix structure, which can be inferred from the solutions. Taking into account the relations between the matrix elements, it has been shown that the integral equation of Fokas and Ablowitz provides solutions of the modified Korteweg–de Vries equation (MKdV) as well[5]). Secondly, one can take into consideration different types of integral equations. In refs. 5, 6 we have shown that the solutions of both the nonlinear Schrödinger equation (NLS) and the equation of motion for the classical Isotropic Heisenberg Spin Chain (IHSC) in the continuum limit can be found from one and the same linear integral equation. This has also been shown to be the case for the Boussinesq equation (BSQ) and the modified Boussinesq equation (MBSQ)[7]).

A major advantage of this way of treating nonlinear PDE's is that many relations between different PDE's become apparent in this context. As a first example, the Miura transformation connecting e.g. the KdV and the MKdV[8]), the NLS and the IHSC[9,10]), and the BSQ and the MBSQ[11]), can be derived systematically as a corollary from the integral equations, without having to rely on rather ad-hoc methods. Another example is the gauge equivalence between the Lax representations of the NLS and the IHSC[12]), which has been derived in a systematic way from the integral equation, cf. ref. 6. Similar gauge equivalences have also been discovered in the context of nonlinear σ-models[13]). Thirdly, the integral equations lead in a direct way to the associated linear eigenvalue problems for the PDE's under consideration, in which the solutions of the integral equation can be identified with the eigenvectors. Eliminating the potentials, which are the solutions of the PDE's, one can derive a nonlinear (integrable) PDE for the eigenvectors, cf. ref. 14. In this way the solutions of the classical Heisenberg spin chain with uniaxial anisotropy (AHSC) can be found from the linear integral equation for the NLS as well[15]).

The integral equation, which we have proposed in ref. 6 for the linearization of the nonlinear Schrödinger equation, has the following form

$$\phi_k^{(n)}(x,t) + \int_C d\lambda(l) \int_{C^*} d\lambda^*(l') \frac{e^{i(kx-k^2t)}\, e^{-i(l'x-l'^2t)}}{(k-l')(l'-l)}\, \phi_l^{(n)}(x,t) = \frac{1}{k^n}\, e^{i(kx-k^2t)}, \quad (1.1)$$

where n is an integer labelling the different solutions. In eq. (1.1) C and C^* denote an arbitrary contour and its complex conjugate in the complex k-plane, and $d\lambda(k)$ and $d\lambda^*(k')$ are an arbitrary measure and its complex conjugate. By choosing different measures and contours we can find different solutions $\phi_k^{(n)}$.

The choice of measures and contours is restricted by two conditions:

(i) The contour C and the measure $d\lambda(k)$ are to be chosen such that the kernel of the integral equation is properly defined and *regular*, in the sense that the solution $\phi_k^{(n)}$, for given measure and contour, is unique. This means e.g. that the homogeneous integral equation, i.e. the integral equation with the right-hand side replaced by zero, has only the zero solution.

(ii) The contour and the measure must be such that the differentiations with respect to x and t can be shifted through the integrals.

Taking into account these conditions, it has been possible to derive PDE's for the functions

$$\phi_{n,m}(x,t) = \int_C d\lambda(k) \frac{1}{k^m} \phi_k^{(n)}(x,t), \tag{1.2}$$

defined with respect to the same measure and contour as in eq. (1.1). In particular we have shown that the function $\phi_{0,0}$ satisfies the NLS and that $\phi_{0,1} = \phi_{1,0}$ and $\phi_{1,1}$ satisfy the IHSC[6]). By making use of the factor $1/k^n$ in the inhomogeneous term of eq. (1.1), we have been able to obtain solutions of different PDE's from one and the same integral equation.

In eq. (1.1) we have chosen a very specific dispersion relation $\omega(k) = k^2$ for the time-dependence occurring in the exponentials. Many of the conclusions, however, which can be drawn from the integral equation, are independent of the choice of the dispersion relation. Thus an obvious generalization of the treatment in ref. 6 is provided, if we choose a more general dispersion. In this way it is possible to derive from the integral equation a broad class of nonlinear PDE's, which may have different dispersive behaviour. All the time-independent features, however, will remain the same for the equations of such a class.

The present paper is devoted to the study of two types of linear integral equations, namely

$$\text{I.} \quad \phi_k^{(n)}(x,t) \pm \int_C d\lambda(l) \int_{C^*} d\lambda^*(l') \frac{\rho_k(x,t)\rho_{l'}^*(x,t)}{(k-l')(l'-l)} \phi_l^{(n)}(x,t) = \frac{1}{k^n}\rho_k(x,t) \tag{1.3}$$

and

$$\text{II.} \quad v_k^{(n)}(x,t) + \int_C d\lambda(l) \int_C d\lambda(l') \frac{\rho_k(x,t)\rho_{l'}(x,t)}{(k+l')(l'+l)} v_l^{(n)}(x,t) = \frac{1}{k^n}\rho_k(x,t). \tag{1.4}$$

Eq. (1.3), which from now on will be denoted as type I, is an obvious generalization of eq. (1.1), the difference being that the exponentials in (1.1) are replaced by functions $\rho_k(x, t)$ proportional to $e^{i(kx-k^r t)}$ with the dispersion relation $\omega(k) = k^r$, r being an integer, and satisfying the linear differential equations

$$-i\partial_x\rho_k(x, t) = k\rho_k(x, t), \quad i\partial_t\rho_k(x, t) = k^r\rho_k(x, t), \quad (r \text{ integer}). \tag{1.5}$$

Eq. (1.4) is a new integral equation (type II), in which the integrations are performed twice over the same contour C. For both equations it is understood that the measure and the contour are chosen such that the conditions (i) and (ii), mentioned above, are satisfied.

The outline of this paper is as follows. In section 2 we derive a set of algebraic relations and differential equations for the quantities

$$\phi_{n,m}(x, t) = \int_C d\lambda(k) \frac{1}{k^m} \phi_k^{(n)}(x, t), \quad \psi_{n,m}(x, t) = \int_C d\lambda(k) \frac{1}{k^m} \psi_k^{(n)}(x, t), \tag{1.6}$$

where

$$\psi_k^{(n)}(x, t) = \int_{C^*} d\lambda^*(l') \frac{\rho_k(x, t)}{k - l'} \phi_{l'}^{(n)*}(x, t). \tag{1.7}$$

In section 3 the same is done for the integral equation of type II, for which we have

$$v_{n,m}(x, t) = \int_C d\lambda(k) \frac{1}{k^m} v_k^{(n)}(x, t), \quad w_{n,m}(x, t) = \int_C d\lambda(k) \frac{1}{k^m} w_k^{(n)}(x, t) \tag{1.8}$$

with

$$w_k^{(n)}(x, t) = \int_C d\lambda(l) \frac{\rho_k(x, t)}{k + l} v_l^{(n)}(x, t). \tag{1.9}$$

In section 4 these relations will be used, both for type I and type II, to give a general framework in order to derive, for all positive values of r, cf. (1.5), closed PDE's in terms of $\phi_{0,0}$ and $v_{0,0}$ only, and we present some explicit results for $r = 2, 3, 4, 5$. In section 5 also PDE's for some other values of (n, m) are derived, together with the Miura transformations, which connect the solutions for different values of n and m. For negative values of r, it is more difficult to derive closed PDE's; some explicit results for the case $r = -1$ will be presented in section 6. Finally, in section 7 it is discussed how a reduction to a class of integral equations with only a single integration can be performed, leading e.g. to a generalization of the integral equation given by Fokas and Ablowitz[4]).

The treatment we give here provides a unifying framework for many partial differential equations. From the variety of equations, that can be described by one and the same linear integral equation, many conclusions can be drawn. The recent results on the connection between the conserved densities[16,17]), and the Lie–Bäcklund symmetries[18,19]) for the MKdV on the one hand, and the ones for the sine-Gordon equation on the other hand, can be regarded as a direct consequence of the integral equation. Furthermore, in a forthcoming publication we will show how Bäcklund transformations can be derived immediately from the integral equation by a singular transformation of measures[20]). Some of the connections we present here have also been found by Hirota[21]), who showed by a different method that several PDE's can be transformed into one and the same bilinear form. The relation between Hirota's method and our approach, however, is yet to be clarified. Moreover, it is not obvious that bilinearization implies exact integrability, cf. ref. 22.

2. Integral equation of type I; constitutive relations

The integral equation (1.3) can be written as a system of two coupled integral equations, i.e.,

$$\phi_k^{(n)} + \eta \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \psi_{l'}^{(n)*}}{k - l'} = \frac{1}{k^n} \rho_k, \quad (\eta = \pm 1), \tag{2.1a}$$

$$\psi_k^{(n)} - \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \phi_{l'}^{(n)*}}{k - l'} = 0. \tag{2.1b}$$

By taking the complex conjugate of (1.3) and using (2.1b), it is not difficult to show that the function $\psi_k^{(n)}$ also obeys an integral equation of type I, but with a different source term

$$\psi_k^{(n)} + \eta \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \rho_{l'}^*}{(k - l')(l' - l)} \psi_l^{(n)} = \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \rho_{l'}^*}{(k - l') l'^n}. \tag{2.2}$$

The functions $\phi_{n,m}$ and $\psi_{n,m}$, which will be investigated in this paper, can be regarded as elements of the infinite-dimensional matrices $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ resp., i.e.

$$(\boldsymbol{\Phi})_{n,m} = \phi_{n,m}, \quad (\boldsymbol{\Psi})_{n,m} = \psi_{n,m}, \tag{2.3}$$

where n and m can have all integer values.

These matrices have simple symmetry properties, as can be seen as follows.

Eq. (1.6) together with (2.1a) can be rewritten as

$$\phi_{n,m} = \int_C d\lambda(k) \frac{\phi_k^{(n)}}{\rho_k} \frac{1}{k^m} \rho_k = \int_C d\lambda(k) \frac{\phi_k^{(n)}}{\rho_k} \left(\phi_k^{(m)} + \eta \int_{C^*} d\lambda^*(l') \frac{\rho_k \psi_{l'}^{(m)*}}{k - l'} \right). \quad (2.4)$$

Interchanging the integrations over C and C^* and using (2.1b), yields the bilinear expression

$$\phi_{n,m} = \int_C d\lambda(k) \frac{\phi_k^{(n)} \phi_k^{(m)}}{\rho_k} - \eta \int_{C^*} d\lambda^*(l') \frac{\psi_{l'}^{(n)*} \psi_{l'}^{(m)*}}{\rho_{l'}^*}, \quad (2.5)$$

and from (2.5) it is immediately clear that

$$\boldsymbol{\Phi} = \boldsymbol{\Phi}^{\mathrm{T}}, \quad (2.6)$$

where the superscript T denotes the transposed matrix. In a similar way the bilinear expression for $\psi_{n,m}$ can be derived. From (1.6) and (2.1a) we have

$$\psi_{n,m} = \int_C d\lambda(k) \frac{\psi_k^{(n)}}{\rho_k} \left(\phi_k^{(m)} + \eta \int_{C^*} d\lambda^*(l') \frac{\rho_k \psi_{l'}^{(m)*}}{k - l'} \right). \quad (2.7)$$

Inserting (2.1b) in the first term of the right-hand side yields

$$\psi_{n,m} = \int_C d\lambda(k) \int_{C^*} d\lambda^*(l') \frac{\eta \psi_k^{(n)} \psi_{l'}^{(m)*} + \phi_k^{(m)} \phi_{l'}^{(n)*}}{k - l'}, \quad (2.8)$$

from which it is obvious that

$$\boldsymbol{\Psi} = -\boldsymbol{\Psi}^{\dagger}, \quad (2.9)$$

where $\boldsymbol{\Psi}^{\dagger}$ is the hermitean conjugate of $\boldsymbol{\Psi}$.

From eqs. (1.3) and (2.2), or equivalently (2.1a) and (2.1b) one can derive a set of algebraic relations connecting the different functions $\phi_k^{(n)}$ and $\psi_k^{(n)}$. For that purpose we consider the functions $k^p \phi_k^{(n)}(x, t)$, where p is an integer. Multiplying (1.3) by k^p, we have

$$k^p \phi_k^{(n)} + \eta \int_{C^*} d\lambda(l) \int_{C^*} d\lambda^*(l') \frac{\rho_k \rho_{l'}^*}{(k - l')(l' - l)} l^p \phi_l^{(n)}$$

$$+ \eta \int_C d\lambda(l) \int_{C^*} d\lambda^*(l') \frac{\rho_k \rho_{l'}^*}{(k - l')(l' - l)} [(k^p - l'^p) + (l'^p - l^p)] \phi_l^{(n)} = \frac{1}{k^{n-p}} \rho_k.$$

(2.10)

For positive p use can be made of the identity

$$k^p - l'^p = (k - l')\sum_{j=0}^{p-1} k^{p-1-j} l'^j, \quad (p > 0), \tag{2.11}$$

in order to rewrite the last term on the left-hand side of (2.10) in the following way, using also eqs. (2.1b) and (1.6),

$$\eta \sum_{j=0}^{p-1} \int_{C^*} \mathrm{d}\lambda^*(l') \left(k^{p-1-j} l'^j \rho_k \psi_{l'}^{(n)*} + \int_C \mathrm{d}\lambda(l)\, l'^{p-1-j} l^j \frac{\rho_k \overset{*}{\rho}_{l'}}{k - l'} \phi_l^{(n)} \right)$$

$$= \eta \sum_{j=0}^{p-1} \left(\psi_{n,-j}^* k^{p-1-j} \rho_k + \phi_{n,-j} \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \overset{*}{\rho}_{l'}}{l'^{j+1-p}(k - l')} \right). \tag{2.12}$$

Inserting (2.12) in (2.10) we find

$$k^p \phi_k^{(n)} + \eta \int_C \mathrm{d}\lambda(l) \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \overset{*}{\rho}_{l'}}{(k - l')(l' - l)} l^p \phi_l^{(n)}$$

$$= \frac{1}{k^{n-p}} \rho_k - \eta \sum_{j=0}^{p-1} \psi_{n,-j}^* k^{p-1-j} \rho_k - \eta \sum_{j=0}^{p-1} \phi_{n,-j} \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \overset{*}{\rho}_{l'}}{l'^{j+1-p}(k - l')}. \tag{2.13}$$

Thus $k^p \phi_k^{(n)}$ obeys an integral equation of type I, but with a different source term. Comparing eq. (2.13) with eqs. (1.3) and (2.2) and using the regularity condition that the homogeneous integral equation has only the zero solution, we find the relation

$$k^p \phi_k^{(n)} = \phi_k^{(n-p)} - \eta \sum_{j=0}^{p-1} (\psi_{n,-j}^* \phi_k^{(j+1-p)} + \phi_{n,-j} \psi_k^{(j+1-p)}), \quad (p > 0). \tag{2.14}$$

A similar relation can be derived for $\psi_k^{(n)}$. Multiplying (2.1b) with k^p, we have

$$k^p \psi_k^{(n)} = \int_{C^*} \mathrm{d}\lambda^*(l')\,(k^p - l'^p) \frac{\rho_k \phi_{l'}^{(n)*}}{k - l'} + \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k l'^p \phi_{l'}^{(n)*}}{k - l'}. \tag{2.15}$$

Making use of (2.11) in the first term on the right-hand side of (2.15), and substituting (2.14) in the second term, eq. (2.15) can be rewritten as

$$k^p \psi_k^{(n)} = \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k \phi_{l'}^{(n-p)*}}{k - l'} + \sum_{j=0}^{p-1} \phi_{n,-j}^* k^{p-1-j} \rho_k$$

$$- \eta \sum_{j=0}^{p-1} \int_{C^*} \mathrm{d}\lambda^*(l') \frac{\rho_k}{k - l'} (\psi_{n,-j} \phi_{l'}^{(j+1-p)*} + \phi_{n,-j}^* \psi_{l'}^{(j+1-p)*}). \tag{2.16}$$

Using (2.1a) and (2.1b), we immediately have

$$k^p\psi_k^{(n)} = \psi_k^{(n-p)} + \sum_{j=0}^{p-1} (\phi^*_{n,-j}\phi_k^{(j+1-p)} - \eta\psi_{n,-j}\psi_k^{(j+1-p)}), \quad (p > 0). \tag{2.17}$$

For $p < 0$, we have, instead of (2.11)

$$k^p - l'^p = -(k - l') \sum_{j=0}^{-p-1} k^{-j-1} l'^{j+p}, \quad (p < 0), \tag{2.18}$$

and this yields, by the same line of reasoning

$$k^p\phi_k^{(n)} = \phi_k^{(n-p)} + \eta \sum_{j=0}^{-p-1} (\psi^*_{n,-j-p}\phi_k^{(j+1)} + \phi_{n,-j-p}\psi_k^{(j+1)}), \quad (p < 0), \tag{2.19}$$

and

$$k^p\psi_k^{(n)} = \psi_k^{(n-p)} - \sum_{j=0}^{-p-1} (\phi^*_{n,-j-p}\phi_k^{(j+1)} - \eta\psi_{n,-j-p}\psi_k^{(j+1)}), \quad (p < 0). \tag{2.20}$$

Eqs. (2.14), (2.17), (2.19) and (2.20) can be written in a more compact way, using the matrices $\mathbf{\Phi}$ and $\mathbf{\Psi}$, cf. (2.3), and introducing the vectors $\boldsymbol{\phi}_k$ and $\boldsymbol{\psi}_k$ with components $\phi_k^{(n)}$ and $\psi_k^{(n)}$. We have the algebraic relations

$$k^p\boldsymbol{\phi}_k = \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{\phi}_k - \eta\mathbf{\Psi}^* \cdot \mathbf{Q}_p \cdot \boldsymbol{\phi}_k - \eta\mathbf{\Phi} \cdot \mathbf{Q}_p \cdot \boldsymbol{\psi}_k, \tag{2.21a}$$

$$k^p\boldsymbol{\psi}_k = \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{\psi}_k + \mathbf{\Phi}^* \cdot \mathbf{Q}_p \cdot \boldsymbol{\phi}_k - \eta\mathbf{\Psi} \cdot \mathbf{Q}_p \cdot \boldsymbol{\psi}_k. \tag{2.21b}$$

Here $\mathbf{J}^{\mathrm{T}}$ denotes the matrix that lowers the superscript of the components $\phi_k^{(n)}$ and $\psi_k^{(n)}$ by 1, or equivalently it is the transposed of a matrix $\mathbf{J}$, which is defined by

$$(\mathbf{J})_{n,m} = \delta_{m,n+1}, \quad (\mathbf{J}^{\mathrm{T}})_{n,m} = \delta_{n,m+1}. \tag{2.22}$$

The matrix $\mathbf{Q}_p$ in (2.21) is given by

$$\mathbf{Q}_p = \sum_{j=0}^{p-1} \mathbf{J}^j \cdot \mathbf{O} \cdot \mathbf{J}^{\mathrm{T}p-1-j}, \quad (p \geq 0), \tag{2.23a}$$

$$\mathbf{Q}_p = -\sum_{j=0}^{-p-1} \mathbf{J}^{p+j} \cdot \mathbf{O} \cdot \mathbf{J}^{\mathrm{T}-j-1}, \quad (p < 0), \tag{2.23b}$$

where the matrix $\mathbf{O}$ is defined by

$$(\mathbf{O})_{n,m} = \delta_{n,0}\delta_{m,0}, \tag{2.24}$$

and where we have identified $\mathbf{J}^{-1}$ and $\mathbf{J}^{\mathrm{T}}$.

Next we derive differential relations for the $\phi_k^{(n)}$ and $\psi_k^{(n)}$. For that purpose we suppose a linear differential equation for ρ_k of the form

$$-\mathrm{i}\partial\rho_k = k^p\rho_k, \tag{2.25}$$

where ∂ can be either ∂_x or $-\partial_t$, corresponding to $p=1$ and $p=r$, respectively, cf. eq. (1.5). Applying $-\mathrm{i}\partial$ to the integral equation (1.3), we find

$$-\mathrm{i}\partial\phi_k^{(n)}+\eta\int\limits_C \mathrm{d}\lambda(l)\int\limits_{C^*}\mathrm{d}\lambda^*(l')\frac{\rho_k\rho_{l'}^*}{(k-l')(l'-l)}(-\mathrm{i}\partial\phi_l^{(n)})$$

$$+\eta\int\limits_C \mathrm{d}\lambda(l)\int\limits_{C^*}\mathrm{d}\lambda^*(l')\frac{\rho_k\rho_{l'}^*}{(k-l')(l'-l)}(k^p-l'^p)\phi_l^{(n)}=\frac{1}{k^{n-p}}\rho_k. \tag{2.26}$$

Using again (2.11) for $p\geqslant 0$, and also eq. (2.1b), eq. (2.26) can be rewritten as

$$-\mathrm{i}\partial\phi_k^{(n)}+\eta\int\limits_C \mathrm{d}\lambda(l)\int\limits_{C^*}\mathrm{d}\lambda^*(l')\frac{\rho_k\rho_{l'}^*}{(k-l')(l'-l)}(-\mathrm{i}\partial\phi_l^{(n)})$$

$$=\frac{1}{k^{n-p}}\rho_k-\eta\sum_{j=0}^{p-1}\psi_{n,-j}^*k^{p-1-j}\rho_k. \tag{2.27}$$

From (2.27) we see that $-\mathrm{i}\partial\phi_k^{(n)}$ obeys an integral equation of type I, and by comparing (2.27) with (1.3), and using the regularity condition, we have the relation

$$-\mathrm{i}\partial\phi_k^{(n)}=\phi_k^{(n-p)}-\eta\sum_{j=0}^{p-1}\psi_{n,-j}^*\phi_k^{(j+1-p)},\quad (p\geqslant 0). \tag{2.28}$$

For $p<0$ we can again use (2.18), from which it follows in the same way that

$$-\mathrm{i}\partial\phi_k^{(n)}=\phi_k^{(n-p)}+\eta\sum_{j=0}^{-p-1}\psi_{n,-j-p}^*\phi_k^{(j+1)},\quad (p<0). \tag{2.29}$$

The relations for $\psi_k^{(n)}$ can be obtained by applying $-\mathrm{i}\partial$ to (2.1b). We find

$$-\mathrm{i}\partial\psi_k^{(n)}=\int\limits_{C^*}\mathrm{d}\lambda^*(l')(k^p-l'^p)\frac{\rho_k\phi_{l'}^{(n)*}}{k-l'}-\int\limits_{C^*}\mathrm{d}\lambda^*(l')\frac{\rho_k}{k-l'}(-l'^p+\mathrm{i}\partial)\phi_{l'}^{(n)*}. \tag{2.30}$$

For $p\geqslant 0$, using (2.11) in the first term and (2.14) and (2.28) in the second term of the right-hand side of (2.30), we find

$$-\mathrm{i}\partial\psi_k^{(n)}=\sum_{j=0}^{p-1}\left(\phi_{n,-j}^*k^{p-1-j}\rho_k-\eta\int\limits_{C^*}\mathrm{d}\lambda^*(l')\frac{\rho_k}{k-l'}\phi_{n,-j}^*\psi_{l'}^{(j+1-p)*}\right)$$

$$=\sum_{j=0}^{p-1}\phi_{n,-j}^*\phi_k^{(j+1-p)},\quad (p\geqslant 0), \tag{2.31}$$

where the last step follows from eq. (2.1a). For $p<0$, by similar manipulations, we obtain the result

$$-\mathrm{i}\partial\psi_k^{(n)} = -\sum_{j=0}^{-p-1} \phi^*_{n,-j-p}\phi_k^{(j+1)}, \quad (p<0). \tag{2.32}$$

Using the matrices $\mathbf{J}^{\mathrm{T}}$ and $\mathbf{Q}_p$, cf. (2.22) and (2.23), eqs. (2.28), (2.29), (2.31) and (2.32) can be rewritten as

$$-\mathrm{i}\partial\boldsymbol{\phi}_k = \mathbf{J}^{\mathrm{T}p}\cdot\boldsymbol{\phi}_k - \eta\boldsymbol{\Psi}^*\cdot\mathbf{Q}_p\cdot\boldsymbol{\phi}_k, \tag{2.33a}$$

$$-\mathrm{i}\partial\boldsymbol{\psi}_k = \boldsymbol{\Phi}^*\cdot\mathbf{Q}_p\cdot\boldsymbol{\phi}_k. \tag{2.33b}$$

Taking into account eq. (1.5) with $\partial = \partial_x$, $p=1$, or $\partial = -\partial_t$, $p=r$, eqs. (2.33a) and (2.33b) lead to the relations

$$\mathrm{i}\partial_t\boldsymbol{\phi}_k = \mathbf{J}^{\mathrm{T}r}\cdot\boldsymbol{\phi}_k - \eta\boldsymbol{\Psi}^*\cdot\mathbf{Q}_r\cdot\boldsymbol{\phi}_k, \tag{2.34a}$$

$$\mathrm{i}\partial_t\boldsymbol{\psi}_k = \boldsymbol{\Phi}^*\cdot\mathbf{Q}_r\cdot\boldsymbol{\phi}_k, \tag{2.34b}$$

$$-\mathrm{i}\partial_x\boldsymbol{\phi}_k = \mathbf{J}^{\mathrm{T}}\cdot\boldsymbol{\phi}_k - \eta\boldsymbol{\Psi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k, \tag{2.34c}$$

$$-\mathrm{i}\partial_x\boldsymbol{\psi}_k = \boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k. \tag{2.34d}$$

Eqs. (2.34a)–(2.34d) in combination with the algebraic relations (2.21a) and (2.21b) form the constitutive relations arising from the linear integral equation (1.3) with (1.5). In fact, (2.21) and (2.34) form the linear problem for the matrices $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$, i.e. we have a set of linear equations for the wave functions $\boldsymbol{\phi}_k$ and $\boldsymbol{\psi}_k$, of which the coefficients contain the potentials $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$. Note that the potentials $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ can be obtained *explicitly* from the wave functions $\boldsymbol{\phi}_k$ and $\boldsymbol{\psi}_k$ by an integration over the contour C, cf. (1.6) and (2.3).

Multiplying (2.21) and (2.34) by k^{-m} and integrating over the contour C, we can derive the algebraic relations and the PDE's containing only the matrices $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$. The result is:

$$\boldsymbol{\Phi}\cdot\mathbf{J}^p = \mathbf{J}^{\mathrm{T}p}\cdot\boldsymbol{\Phi} - \eta\boldsymbol{\Psi}^*\cdot\mathbf{Q}_p\cdot\boldsymbol{\Phi} - \eta\boldsymbol{\Phi}\cdot\mathbf{Q}_p\cdot\boldsymbol{\Psi}, \tag{2.35a}$$

$$\boldsymbol{\Psi}\cdot\mathbf{J}^p = \mathbf{J}^{\mathrm{T}p}\cdot\boldsymbol{\Psi} + \boldsymbol{\Phi}^*\cdot\mathbf{Q}_p\cdot\boldsymbol{\Phi} - \eta\boldsymbol{\Psi}\cdot\mathbf{Q}_p\cdot\boldsymbol{\Psi}, \tag{2.35b}$$

$$-\mathrm{i}\partial_x\boldsymbol{\Phi} = \mathbf{J}^{\mathrm{T}}\cdot\boldsymbol{\Phi} - \eta\boldsymbol{\Psi}^*\cdot\mathbf{O}\cdot\boldsymbol{\Phi}, \tag{2.35c}$$

$$-\mathrm{i}\partial_x\boldsymbol{\Psi} = \boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\Phi}, \tag{2.35d}$$

$$\mathrm{i}\partial_t\boldsymbol{\Phi} = \mathbf{J}^{\mathrm{T}r}\cdot\boldsymbol{\Phi} - \eta\boldsymbol{\Psi}^*\cdot\mathbf{Q}_r\cdot\boldsymbol{\Phi}, \tag{2.35e}$$

$$\mathrm{i}\partial_t\boldsymbol{\Psi} = \boldsymbol{\Phi}^*\cdot\mathbf{Q}_r\cdot\boldsymbol{\Phi}. \tag{2.35f}$$

Eqs. (2.35c)–(2.35f) for fixed r form a system of coupled partial differential equations, which, in combination with the algebraic relations (2.35a), (2.35b) for integer p, is completely integrable in the sense that solutions can be found from a linear integral equation, cf. (1.3), (1.6), (1.7), (2.3). (The linear integral equation may be regarded as the spectral decomposition of the associated linear eigenvalue problem.)

Remark. Instead of eq. (1.5) one can also consider a more general dispersion relation

$$\omega(k) = \sum_r \lambda_r k^r, \tag{2.36}$$

leading to

$$\mathrm{i}\partial_t \rho_k = \omega(k)\rho_k. \tag{2.37}$$

In this case (2.35e) and (2.35f) can be rewritten as

$$\mathrm{i}\partial_t \mathbf{\Phi} = \omega(\mathbf{J}^{\mathrm{T}}) \cdot \mathbf{\Phi} - \eta \mathbf{\Psi}^* \cdot \mathbf{Q} \cdot \mathbf{\Phi}, \tag{2.38a}$$

$$\mathrm{i}\partial_t \mathbf{\Psi} = \mathbf{\Phi}^* \cdot \mathbf{Q} \cdot \mathbf{\Phi}, \tag{2.38b}$$

in which the matrix $\mathbf{Q}$ is given by

$$\mathbf{Q} = \sum_r \lambda_r \mathbf{Q}_r. \tag{2.39}$$

3. Integral equation of type II; constitutive relations

The analysis, given in the preceding section for the integral equation of type I, can be done with only minor modifications for the integral equation of type II, given in eq. (1.4). As in section 2 we rewrite eq. (1.4) as a system of two coupled integral equations

$$v_k^{(n)} + \int_C \mathrm{d}\lambda(l) \frac{\rho_k w_l^{(n)}}{k+l} = \frac{1}{k^n} \rho_k, \tag{3.1a}$$

$$w_k^{(n)} - \int_C \mathrm{d}\lambda(l) \frac{\rho_k v_l^{(n)}}{k+l} = 0. \tag{3.1b}$$

Here it is not useful to introduce a quantity $\eta = \pm 1$, since a factor η in front of the second term of the left-hand side of (1.4) can be included in the measure $\mathrm{d}\lambda(l)$. The integral equation for the quantity $w_k^{(n)}$ is again of type II, but has a different source term

$$w_k^{(n)} + \int_C \mathrm{d}\lambda(l) \int_C \mathrm{d}\lambda(l') \frac{\rho_k \rho_{l'}}{(k+l')(l'+l)} w_l^{(n)} = \int_C \mathrm{d}\lambda(l) \frac{\rho_k \rho_l}{(k+l)l^n}. \tag{3.2}$$

We define matrices $\mathbf{V}$ and $\mathbf{W}$ by

$$(\mathbf{V})_{n,m} = v_{n,m}, \quad (\mathbf{W})_{n,m} = w_{n,m}, \tag{3.3}$$

cf. (1.8). These matrices are symmetric, i.e.

$$\mathbf{V} = \mathbf{V}^{\mathrm{T}}, \quad \mathbf{W} = \mathbf{W}^{\mathrm{T}}, \tag{3.4}$$

as follows from the bilinear expressions

$$v_{n,m} = \int_C \mathrm{d}\lambda(k) \frac{1}{\rho_k} (v_k^{(n)} v_k^{(m)} + w_k^{(n)} w_k^{(m)}) \tag{3.5}$$

and

$$w_{n,m} = \int_C \mathrm{d}\lambda(k) \int_C \mathrm{d}\lambda(l) \frac{v_l^{(n)} v_k^{(m)} + w_k^{(n)} w_l^{(m)}}{k+l}, \tag{3.6}$$

which can be derived in an analogous way as (2.5) and (2.8).

The algebraic relations for $v_k^{(n)}$ and $w_k^{(n)}$ can be derived as follows. From (1.4) one has

$$k^p v_k^{(n)} + \int_C \mathrm{d}\lambda(l) \int_C \mathrm{d}\lambda(l') \frac{\rho_k \rho_{l'}}{(k+l')(l'+l)} l^p v_l^{(n)} + \int_C \mathrm{d}\lambda(l) \int_C \mathrm{d}\lambda(l') \frac{\rho_k \rho_{l'}}{(k+l')(l'+l)}$$

$$\times [(k^p - (-l')^p) + ((-l')^p - l^p)] v_l^{(n)} = \frac{1}{k^{n-p}} \rho_k. \tag{3.7}$$

We now use the relations

$$k^p - (-l')^p = (k+l') \sum_{j=0}^{p-1} k^{p-1-j} (-l')^j, \quad (p > 0), \tag{3.8}$$

and

$$k^p - (-l')^p = -(k+l') \sum_{j=0}^{-p-1} k^{-j-1} (-l')^{j+p}, \quad (p < 0), \tag{3.9}$$

Then eq. (3.7) can be rewritten as

$$\begin{aligned} k^p v_k^{(n)} &+ \int_C \mathrm{d}\lambda(l) \int_C \mathrm{d}\lambda(l') \frac{\rho_k \rho_{l'}}{(k+l')(l'+l)} l^p v_l^{(n)} \\ &= \frac{1}{k^{n-p}} \rho_k - \sum_{j=0}^{p-1} (-1)^j \Big(w_{n,-j} k^{p-1-j} \rho_k \\ &\quad + (-1)^p v_{n,-j} \int_C \mathrm{d}\lambda(l) \frac{\rho_k \rho_l}{(k+l) l^{j+1-p}} \Big), \end{aligned} \tag{3.10}$$

for $p \geqslant 0$, and

$$k^p v_k^{(n)} + \int_C d\lambda(l) \int_C d\lambda(l') \frac{\rho_k \rho_{l'}}{(k+l')(l'+l)} l^p v_l^{(n)}$$
$$= \frac{1}{k^{n-p}} \rho_k + (-1)^p \sum_{j=0}^{-p-1} (-1)^j \Big(w_{n,-j-p} k^{-j-1} \rho_k$$
$$+ (-1)^p v_{n,-j-p} \int_C d\lambda(l) \frac{\rho_k \rho_l}{(k+l) l^{j+1}} \Big), \tag{3.11}$$

for $p < 0$, cf. eq. (1.8). From eqs. (3.10) and (3.11), we can conclude, on account of the regularity condition, that

$$k^p v_k^{(n)} = v_k^{(n-p)} - \sum_{j=0}^{p-1} (-1)^j (w_{n,-j} v_k^{(j+1-p)} + (-1)^p v_{n,-j} w_k^{(j+1-p)}), \quad (p \geqslant 0), \tag{3.12}$$

and

$$k^p v_k^{(n)} = v_k^{(n-p)} + (-1)^p \sum_{j=0}^{-p-1} (-1)^j (w_{n,-j-p} v_k^{(j+1)} + (-1)^p v_{n,-j-p} w_k^{(j+1)}), \quad (p < 0). \tag{3.13}$$

Introducing the matrix $\mathbf{R}_p$ by

$$\mathbf{R}_p = \sum_{j=0}^{p-1} \mathbf{J}^j \cdot \mathbf{O} \cdot (-\mathbf{J}^{\mathrm{T}})^{p-1-j}, \quad (p \geqslant 0), \tag{3.14a}$$

and

$$\mathbf{R}_p = - \sum_{j=0}^{-p-1} \mathbf{J}^{j+p} \cdot \mathbf{O} \cdot (-\mathbf{J}^{\mathrm{T}})^{-j-1}, \quad (p < 0), \tag{3.14b}$$

eqs. (3.12) and (3.13) can be combined to

$$k^p \boldsymbol{v}_k = \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{v}_k + (-1)^p \mathbf{W} \cdot \mathbf{R}_p \cdot \boldsymbol{v}_k + \mathbf{V} \cdot \mathbf{R}_p \cdot \mathbf{w}_k, \tag{3.15}$$

for all integer p, where $\boldsymbol{v}_k$ and $\mathbf{w}_k$ are vectors with components $v_k^{(n)}$ and $w_k^{(n)}$.

The algebraic relations for $w_k^{(n)}$ can be derived multiplying (3.1b) by k^p, i.e.

$$k^p w_k^{(n)} = \int_C d\lambda(l) (k^p - (-l)^p) \frac{\rho_k v_l^{(n)}}{k+l} + (-1)^p \int_C d\lambda(l) \frac{\rho_k l^p v_l^{(n)}}{k+l}. \tag{3.16}$$

Using (3.8) and (3.12) for $p \geqslant 0$, and (3.9) and (3.13) for $p < 0$, we find

$$k^p w_k^{(n)} = (-1)^p w_k^{(n-p)} + \sum_{j=0}^{p-1} (-1)^j (v_{n,-j} v_k^{(j+1-p)} - (-1)^p w_{n,-j} w_k^{(j+1-p)}), \quad (p \geqslant 0), \tag{3.17}$$

and

$$k^p w_k^{(n)} = (-1)^p w_k^{(n-p)} + (-1)^p \sum_{j=0}^{-p-1} (-1)^j (-v_{n,-j-p} v_k^{(j+1)} + (-1)^p w_{n,-j-p} w_k^{(j+1)}), \quad (p < 0). \tag{3.18}$$

In matrix notation eqs. (3.17) and (3.18) yield

$$k^p w_k = (-\mathbf{J}^{\mathrm{T}})^p \cdot w_k + \mathbf{W} \cdot \mathbf{R}_p \cdot w_k - (-1)^p \mathbf{V} \cdot \mathbf{R}_p \cdot v_k, \tag{3.19}$$

for all integer p.

In order to derive differential relations for $v_k^{(n)}$ and $w_k^{(n)}$, we can take again (2.25), but now we must assume p to be odd, in order that $\mathrm{i}\partial(\rho_k \rho_{l'})$ contains a factor $(k + l')$. Applying $-\mathrm{i}\partial$ to the integral equation (1.4), we have

$$-\mathrm{i}\partial v_k^{(n)} + \int_C \mathrm{d}\lambda(l) \int_C \mathrm{d}\lambda(l') \frac{\rho_k \rho_{l'}}{(k + l')(l' + l)} (-\mathrm{i}\partial v_l^{(n)})$$
$$+ \int_C \mathrm{d}\lambda(l) \int_C \mathrm{d}\lambda(l') \frac{\rho_k \rho_{l'}}{(k + l')(l' + l)} (k^p + l'^p) v_l^{(n)} = \frac{1}{k^{n-p}} \rho_k. \tag{3.20}$$

For odd values of p, we can use eq. (3.8) or (3.9) in the last term of the left-hand side of (3.20). Applying (3.1b) and the regularity condition for the integral equation (1.4), we obtain

$$-\mathrm{i}\partial v_k^{(n)} = v_k^{(n-p)} - \sum_{j=0}^{p-1} (-1)^j w_{n,-j} v_k^{(j+1-p)}, \quad (p \text{ odd}, p > 0), \tag{3.21}$$

and

$$-\mathrm{i}\partial v_k^{(n)} = v_k^{(n-p)} - \sum_{j=0}^{-p-1} (-1)^j w_{n,-j-p} v_k^{(j+1)}, \quad (p \text{ odd}, p < 0), \tag{3.22}$$

and eqs. (3.21) and (3.22) can be combined to

$$-\mathrm{i}\partial v_k = \mathbf{J}^{\mathrm{T}p} \cdot v_k - \mathbf{W} \cdot \mathbf{R}_p \cdot v_k, \quad (p \text{ odd}), \tag{3.23}$$

for odd values of p. Applying $-\mathrm{i}\partial$ to eq. (3.1b) we can derive an equation for $-\mathrm{i}\partial w_k^{(n)}$, which yields

$$-\mathrm{i}\partial w_k = \mathbf{V} \cdot \mathbf{R}_p \cdot v_k, \quad (p \text{ odd}). \tag{3.24}$$

Taking into account eq. (1.5) for the factors $\rho_k(x, t)$, we have the relations

$$\mathrm{i}\partial_t v_k = \mathbf{J}^{\mathrm{T}r} \cdot v_k - \mathbf{W} \cdot \mathbf{R}_r \cdot v_k, \quad (r \text{ odd}), \tag{3.25a}$$

$$\mathrm{i}\partial_t w_k = \mathbf{V} \cdot \mathbf{R}_r \cdot v_k, \quad (r \text{ odd}), \tag{3.25b}$$

$$-\mathrm{i}\partial_x v_k = \mathbf{J}^{\mathrm{T}} \cdot v_k - \mathbf{W} \cdot \mathbf{O} \cdot v_k, \tag{3.25c}$$

$$-\mathrm{i}\partial_x w_k = \mathbf{V} \cdot \mathbf{O} \cdot v_k, \tag{3.25d}$$

which. in combination with the algebraic relations (3.15) and (3.19), form the constitutive relations of the linear problem associated with the integral equation (1.4).

Multiplying (3.15), (3.19) and (3.25) by k^{-m} and integrating over the contour C, we obtain the equations

$$\mathbf{V} \cdot \mathbf{J}^p = \mathbf{J}^{\mathrm{T}p} \cdot \mathbf{V} + (-1)^p \mathbf{W} \cdot \mathbf{R}_p \cdot \mathbf{V} + \mathbf{V} \cdot \mathbf{R}_p \cdot \mathbf{W}, \tag{3.26a}$$

$$\mathbf{W} \cdot \mathbf{J}^p = (-\mathbf{J}^{\mathrm{T}})^p \cdot \mathbf{W} + \mathbf{W} \cdot \mathbf{R}_p \cdot \mathbf{W} - (-1)^p \mathbf{V} \cdot \mathbf{R}_p \cdot \mathbf{V}, \tag{3.26b}$$

$$-\mathrm{i}\partial_x \mathbf{V} = \mathbf{J}^{\mathrm{T}} \cdot \mathbf{V} - \mathbf{W} \cdot \mathbf{O} \cdot \mathbf{V}, \tag{3.26c}$$

$$-\mathrm{i}\partial_x \mathbf{W} = \mathbf{V} \cdot \mathbf{O} \cdot \mathbf{V}, \tag{3.26d}$$

$$\mathrm{i}\partial_t \mathbf{V} = \mathbf{J}^{\mathrm{T}r} \cdot \mathbf{V} - \mathbf{W} \cdot \mathbf{R}_r \cdot \mathbf{V}, \quad (r \text{ odd}), \tag{3.26e}$$

$$\mathrm{i}\partial_t \mathbf{W} = \mathbf{V} \cdot \mathbf{R}_r \cdot \mathbf{V}, \quad (r \text{ odd}). \tag{3.26f}$$

Again, a more general dispersion relation can be taken into account. Choosing,

$$\mathrm{i}\partial_t \rho_k = \omega(k) \rho_k, \tag{3.27}$$

where $\omega(k)$ is an odd meromorphic function of k, i.e.

$$\omega(k) = \sum_r \lambda_r k^r, \quad (\lambda_r = 0, \text{ if } r \text{ even}), \tag{3.28}$$

we find

$$\mathrm{i}\partial_t \mathbf{V} = \omega(\mathbf{J}^{\mathrm{T}}) \cdot \mathbf{V} - \mathbf{W} \cdot \mathbf{R} \cdot \mathbf{V}, \tag{3.29a}$$

and

$$\mathrm{i}\partial_t \mathbf{W} = \mathbf{V} \cdot \mathbf{R} \cdot \mathbf{V}, \tag{3.29b}$$

where

$$\mathbf{R} = \sum_r \lambda_r \mathbf{R}_r, \quad (\lambda_r = 0, \text{ if } r \text{ even}). \tag{3.30}$$

4. Partial differential equations

In this section we present, for positive values of r, a method for deriving matrix partial differential equations in terms of only $\mathbf{\Phi}$, in the case of the integral equation of type I, and in terms of only $\mathbf{V}$, in the case of the integral equation of type II, without involving the matrices $\mathbf{J}$ or $\mathbf{J}^{\mathrm{T}}$. From the form of these matrix PDE's, one immediately has partial differential equations containing only the (0, 0) components of the matrices $\mathbf{\Phi}$ and $\mathbf{V}$, i.e. the functions $\phi_{0,0}$ and $v_{0,0}$.

4.1. *Matrix PDE's for type I*

In order to get rid of the matrices $\mathbf{J}^{\mathrm{T}}$, we first derive a recursive relation for the quantity

$$F_k^{(p)} \equiv \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{\phi}_k - \eta \boldsymbol{\Psi}^* \cdot \mathbf{Q}_p \cdot \boldsymbol{\phi}_k = k^p \boldsymbol{\phi}_k + \eta \boldsymbol{\Phi} \cdot \mathbf{Q}_p \cdot \boldsymbol{\psi}_k, \tag{4.1}$$

cf. (2.21a), occurring in the right-hand side of (2.33a).

Using (2.34c) and the recursion relation

$$\mathbf{Q}_p = \mathbf{J} \cdot \mathbf{Q}_{p-1} + \mathbf{O} \cdot \mathbf{J}^{\mathrm{T}p-1} = \mathbf{Q}_{p-1} \cdot \mathbf{J}^{\mathrm{T}} + \mathbf{J}^{p-1} \cdot \mathbf{O}, \tag{4.2}$$

cf. (2.23a), (2.23b), which is valid for positive, as well as negative values of p, we obtain

$$F_k^{(p)} = \mathbf{J}^{\mathrm{T}p-1} \cdot (-i\partial_x \boldsymbol{\phi}_k + \eta \boldsymbol{\Psi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k) - \eta \boldsymbol{\Psi}^* \cdot (\mathbf{J}^{p-1} \cdot \mathbf{O} + \mathbf{Q}_{p-1} \cdot \mathbf{J}^{\mathrm{T}}) \cdot \boldsymbol{\phi}_k. \tag{4.3}$$

We rewrite eq. (4.3), using also (2.35d), as

$$\begin{aligned} F_k^{(p)} = & -\mathrm{i}\partial_x(\mathbf{J}^{\mathrm{T}p-1} \cdot \boldsymbol{\phi}_k - \eta \boldsymbol{\Psi}^* \cdot \mathbf{Q}_{p-1} \cdot \boldsymbol{\phi}_k) - \mathrm{i}\eta\partial_x(\boldsymbol{\Psi}^* \cdot \mathbf{Q}_{p-1} \cdot \boldsymbol{\phi}_k) \\ & + \eta(\mathbf{J}^{\mathrm{T}p-1} \cdot \boldsymbol{\Psi}^* - \boldsymbol{\Psi}^* \cdot \mathbf{J}^{p-1}) \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k \\ & - \eta \boldsymbol{\Psi}^* \cdot \mathbf{Q}_{p-1} \cdot (-\mathrm{i}\partial_x \boldsymbol{\phi}_k + \eta \boldsymbol{\Psi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k) \\ = & -\mathrm{i}\partial_x F_k^{(p-1)} - \eta \mathbf{G}^{(p-1)*} \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k - \eta \boldsymbol{\Phi} \cdot \mathbf{O} \cdot G_k^{(p-1)}, \end{aligned} \tag{4.4}$$

where we have introduced the vector

$$G_k^{(p)} \equiv -\mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{\psi}_k + k^p \boldsymbol{\psi}_k + \eta \boldsymbol{\Psi} \cdot \mathbf{Q}_p \cdot \boldsymbol{\psi}_k = \boldsymbol{\Phi}^* \cdot \mathbf{Q}_p \cdot \boldsymbol{\phi}_k, \tag{4.5}$$

cf. (2.21b), which occurs in (2.34b), and the matrix

$$\mathbf{G}^{(p)} \equiv -\mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{\Psi} + \boldsymbol{\Psi} \cdot \mathbf{J}^p + \eta \boldsymbol{\Psi} \cdot \mathbf{Q}_p \cdot \boldsymbol{\Psi} = \boldsymbol{\Phi}^* \cdot \boldsymbol{Q}_p \cdot \boldsymbol{\Phi}, \tag{4.6}$$

cf. (2.35b), (2.35f), which can be obtained from (4.5), after an integration over the contour C.

Next we express $G_k^{(p)}$ in terms of $F_k^{(p)}$. Differentiating (4.5) with respect to x, we find, using (2.34d) and (2.35d),

$$-\mathrm{i}\partial_x G_k^{(p)} = (k^p - \mathbf{J}^{\mathrm{T}p} + \eta \boldsymbol{\Psi} \cdot \mathbf{Q}_p) \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k + \eta \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\Phi} \cdot \mathbf{Q}_p \cdot \boldsymbol{\Psi}_k. \tag{4.7}$$

Inserting in eq. (4.7), eq. (4.1), and the matrix

$$\mathbf{F}^{(p)} = \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{\Phi} - \eta \boldsymbol{\Psi}^* \cdot \mathbf{Q}_p \cdot \boldsymbol{\Phi}, \tag{4.8}$$

which can be obtained from (4.1), after an integration over the contour C, we find

$$-\mathrm{i}\partial_x G_k^{(p)} = \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot F_k^{(p)} - \mathbf{F}^{(p)*} \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k. \tag{4.9}$$

Eq. (4.9) can be formally solved and inserted in (4.4), which leads to

$$\boldsymbol{F}_k^{(p)} = \Omega[\boldsymbol{F}_k^{(p-1)}], \tag{4.10}$$

where the action of the operator Ω on an arbitrary vector $\boldsymbol{a}_k$, with components $a_k^{(n)}$, is defined as

$$\begin{aligned}\Omega[\boldsymbol{a}_k] = &-\mathrm{i}\partial_x \boldsymbol{a}_k - \mathrm{i}\eta \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \partial_x^{-1}(\boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{a}_k - \mathbf{A}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k) \\ &+ \mathrm{i}\eta\partial_x^{-1}(\boldsymbol{\Phi} \cdot \mathbf{O} \cdot \mathbf{A}^* - \mathbf{A} \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^*) \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k,\end{aligned} \tag{4.11}$$

and the elements of the associated matrix $\mathbf{A}$ are given by

$$(\mathbf{A})_{n,m} = a_{n,m}, \quad a_{n,m} = \int_C \mathrm{d}\lambda(k) \frac{1}{k^m} a_k^{(n)}. \tag{4.12}$$

In eq. (4.11) ∂_x^{-1} denotes an integration over x, in which the integration constant has yet to be determined. In appendix A, it will be shown that no undetermined integration constants appear in the right-hand side of (4.10). For every $p > 0$, the vector $\boldsymbol{F}_k^{(p)} = -\mathrm{i}\partial\boldsymbol{\phi}_k$ can be evaluated using

$$\boldsymbol{F}_k^{(p)} = \Omega^p[\boldsymbol{\phi}_k] \tag{4.13}$$

and no integration constant will occur in the final result.

The linear problem corresponding to (2.21) and (2.33), for positive r, i.e. $\omega(k) = k^r$, $(r > 0)$ and $\mathrm{i}\partial_t\rho_k = k^r\rho_k$ in (1.5), can be expressed as

$$\mathrm{i}\partial_t\boldsymbol{\phi}_k = \boldsymbol{F}_k^{(r)} = \Omega^r[\boldsymbol{\phi}_k], \tag{4.14a}$$

$$\mathrm{i}\partial_t\boldsymbol{\psi}_k = \boldsymbol{G}_k^{(r)} = \mathrm{i}\partial_x^{-1}(\boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \Omega^r[\boldsymbol{\phi}_k] - \Omega^r[\boldsymbol{\Phi}]^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k), \tag{4.14b}$$

$$-\mathrm{i}\partial_x\boldsymbol{\phi}_k = k\boldsymbol{\phi}_k + \eta\boldsymbol{\Phi} \cdot \mathbf{O} \cdot \boldsymbol{\psi}_k, \tag{4.14c}$$

$$-\mathrm{i}\partial_x\boldsymbol{\psi}_k = \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k, \tag{4.14d}$$

in which the action of the operator Ω on an arbitrary matrix $\mathbf{A}$ is defined as

$$\begin{aligned}\Omega[\mathbf{A}] = &-\mathrm{i}\partial_x\mathbf{A} - \mathrm{i}\eta\boldsymbol{\Phi} \cdot \mathbf{O} \cdot \partial_x^{-1}(\boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \mathbf{A} - \mathbf{A}^* \cdot \mathbf{O} \cdot \boldsymbol{\Phi}) \\ &+ \mathrm{i}\eta\partial_x^{-1}(\boldsymbol{\Phi} \cdot \mathbf{O} \cdot \mathbf{A}^* - \mathbf{A} \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^*) \cdot \mathbf{O} \cdot \boldsymbol{\Phi}\,.\end{aligned} \tag{4.15}$$

The corresponding PDE can be expressed as

$$\mathrm{i}\partial_t\boldsymbol{\Phi} = \Omega^r[\boldsymbol{\Phi}], \tag{4.16}$$

as follows by integrating (4.14a) over the contour C. Furthermore, from the form of the operator Ω or from the relations (A.5), (A.6) and (4.4) it is obvious that the (0, 0) element of the matrix relation (4.16) leads to a closed partial differential equation, containing only $\phi_{0,0}$. The corresponding linear problem is given by the $n = 0$ component of (4.14a)–(4.14d).

In order to obtain the PDE for $\phi_{0,0}$ for a specific value of r in an explicit form, one first has to evaluate the vector $\boldsymbol{F}_k^{(r)} = \Omega^r[\boldsymbol{\phi}_k]$. The vectors $\boldsymbol{F}_k^{(r)}$ and $\boldsymbol{G}_k^{(r)}$, for positive r, can be found, either by using (4.10) a number of times in such a way that the "integrations" ∂_x^{-1} can be cancelled in every step, or by applying the recursion relations (A.5) and (A.6), in combination with (4.4). The results for $r = 1, 2, 3, 4, 5$ are presented in appendix B. The PDE's for $\phi_{0,0}$ for $r = 2, 3, 4, 5$ can be found, integrating (B.2)–(B.5) over the contour C, and inserting the result in (4.16). We find

$$r = 2, \quad \mathrm{i}\partial_t\phi_{0,0} + \partial_x^2\phi_{0,0} + 2\eta|\phi_{0,0}|^2\phi_{0,0} = 0, \tag{4.17}$$

$$r = 3, \quad \partial_t\phi_{0,0} - \partial_x^3\phi_{0,0} - 6\eta|\phi_{0,0}|^2\partial_x\phi_{0,0} = 0, \tag{4.18}$$

$$\begin{aligned} r = 4, \quad & \mathrm{i}\partial_t\phi_{0,0} - \partial_x^4\phi_{0,0} - \eta(8|\phi_{0,0}|^2\partial_x^2\phi_{0,0} + 6\phi_{0,0}^*(\partial_x\phi_{0,0})^2 \\ & + 2\phi_{0,0}^2\partial_x^2\phi_{0,0}^* + 4|\partial_x\phi_{0,0}|^2\phi_{0,0}) + 6|\phi_{0,0}|^4\phi_{0,0} = 0, \end{aligned} \tag{4.19}$$

$$\begin{aligned} r = 5, \quad & \partial_t\phi_{0,0} + \partial_x^5\phi_{0,0} + 10\eta[|\phi_{0,0}|^2\partial_x^3\phi_{0,0} + (\partial_x^2\phi_{0,0})(\partial_x\phi_{0,0}^*)\phi_{0,0} \\ & + \phi_{0,0}(\partial_x^2\phi_{0,0}^*)(\partial_x\phi_{0,0}) + 2\phi_{0,0}^*(\partial_x\phi_{0,0})\partial_x^2\phi_{0,0} + |\partial_x\phi_{0,0}|^2\partial_x\phi_{0,0}] \\ & + 30|\phi_{0,0}|^4\partial_x\phi_{0,0} = 0. \end{aligned} \tag{4.20}$$

Eq. (4.17), for $r = 2$, is the nonlinear Schrödinger equation (NLS); eq. (4.18), for $r = 3$, may be called the complex modified Korteweg–de Vries equation, whereas eqs. (4.19) and (4.20), for $r = 4$ and $r = 5$, respectively, have to our knowledge not been given before in the literature.

Remarks

(i) By applying the operator Ω^p to the functions $\phi_{n,m}$ extra factors k^p and l'^p are introduced in the integrands of the bilinear expression (2.5). In fact, using the property

$$\int_C \mathrm{d}\lambda(k)\frac{1}{\rho_k}\phi_k^{(n)}\psi_k^{(m)} + \int_{C^*} \mathrm{d}\lambda^*(l')\frac{1}{\rho_{l'}^*}\psi_{l'}^{(n)*}\phi_{l'}^{(m)*} = 0, \tag{4.21}$$

and eqs. (2.33) and (2.21a), one can show that

$$-\mathrm{i}\partial\phi_{n,m} = \int_C \mathrm{d}\lambda(k)\,k^p\frac{1}{\rho_k}\phi_k^{(n)}\phi_k^{(m)} - \eta\int_{C^*}\mathrm{d}\lambda^*(l')l'^p\frac{1}{\rho_{l'}^*}\psi_{l'}^{(n)*}\psi_{l'}^{(m)*}, \tag{4.22}$$

and furthermore we have $-\mathrm{i}\partial\boldsymbol{\Phi} = \Omega^p[\boldsymbol{\Phi}]$. Operators Ω were already formulated in a different context in ref. 23, see also ref. 24.

(ii) As was noted at the end of section 2, it is also possible to choose a more general dispersion relation $\omega(k) = \Sigma_r\lambda_r k^r$ in the integral equation, cf. eq.

(2.38). Taking into account only positive powers of k, i.e. $\lambda_r = 0$ for $r < 0$, we obtain PDE's of the form

$$i\partial_t \boldsymbol{\Phi} = \sum_{r>0} \lambda_r \Omega^r[\boldsymbol{\Phi}]. \tag{4.23}$$

All PDE's of the form (4.23) are completely integrable in the sense that solutions can be found from the linear integral equation (1.3) with the dispersion relation $\omega(k) = \Sigma_r \lambda_r k^r$. The (0, 0) element of (4.23) leads to a closed PDE for $\phi_{0,0}$, the solutions of which can be obtained from (1.3) and (1.6) with $n = m = 0$. An example of such a PDE is Hirota's equation[25]), which can be obtained, taking $\lambda_r = 0$ for $r \neq 2, 3$, i.e.

$$i\partial_t \phi_{0,0} + \lambda_2 \partial_x^2 \phi_{0,0} - i\lambda_3 \partial_x^3 \phi_{0,0} = -2\eta\lambda_2 |\phi_{0,0}|^2 \phi_{0,0} + 6i\lambda_3 \eta |\phi_{0,0}|^2 \partial_x \phi_{0,0}. \tag{4.24}$$

4.2. *Matrix PDE's for type II*

The elimination of the matrices $\mathbf{J}^{\mathrm{T}}$ in the case of the integral equation of type II, proceeds in a similar way as in the previous subsection. We define the quantities

$$Y_k^{(p)} \equiv \mathbf{J}^{\mathrm{T}p} \cdot v_k + (-1)^p \mathbf{W} \cdot \mathbf{R}_p \cdot v_k = k^p v_k - \mathbf{V} \cdot \mathbf{R}_p \cdot w_k, \tag{4.25}$$

and

$$Z_k^{(p)} \equiv k^p w_k - (-\mathbf{J}^{\mathrm{T}})^p \cdot w_k - \mathbf{W} \cdot \mathbf{R}_p \cdot w_k = -(-1)^p \mathbf{V} \cdot \mathbf{R}_p \cdot v_k, \tag{4.26}$$

for all integer p, cf. (3.15) and (3.19). Then, for all odd r, we have the equations

$$i\partial_t v_k = Y_k^{(r)}, \tag{4.27a}$$

$$i\partial_t w_k = Z_k^{(r)}. \tag{4.27b}$$

In an analogous way as before we can derive the recursion relation

$$Y_k^{(p)} = -i\partial_x Y_k^{(p-1)} + \mathbf{V} \cdot \mathbf{O} \cdot Z_k^{(p-1)} - (-1)^{p-1} \mathbf{Z}^{(p-1)} \cdot \mathbf{O} \cdot v_k \tag{4.28}$$

and the relation

$$-i\partial_x Z_k^{(p)} = -(-1)^p \mathbf{Y}^{(p)} \cdot \mathbf{O} \cdot v_k + \mathbf{V} \cdot \mathbf{O} \cdot Y_k^{(p)}, \tag{4.29}$$

where

$$\mathbf{Y}^{(p)} = \mathbf{J}^{\mathrm{T}p} \cdot \mathbf{V} + (-1)^p \mathbf{W} \cdot \mathbf{R}_p \cdot \mathbf{V}, \tag{4.30}$$

and

$$\mathbf{Z}^{(p)} = \mathbf{W} \cdot \mathbf{J}^p - (-\mathbf{J}^{\mathrm{T}})^p \cdot \mathbf{W} - \mathbf{W} \cdot \mathbf{R}_p \cdot \mathbf{W}. \tag{4.31}$$

In the derivation of (4.28) and (4.29) we have made use of the recursion relation

$$\mathbf{R}_{p+1} = \mathbf{J} \cdot \mathbf{R}_p + \mathbf{O} \cdot (-\mathbf{J}^{\mathrm{T}})^p = -\mathbf{R}_p \cdot \mathbf{J}^{\mathrm{T}} + \mathbf{J}^p \cdot \mathbf{O}, \tag{4.32}$$

and eqs. (3.25c), (3.25d), (3.26d), (4.25), (4.26), (4.30) and (4.31).

From eqs. (4.28) and (4.29) it can be inferred that

$$\mathbf{Y}_k^{(p)} = \begin{cases} \Omega^+[\mathbf{Y}_k^{(p-1)}], & (p \text{ even}), \\ \Omega^-[\mathbf{Y}_k^{(p-1)}], & (p \text{ odd}), \end{cases} \tag{4.33}$$

where

$$\Omega^{\pm}[\boldsymbol{a}_k] = -\mathrm{i}\partial_x \boldsymbol{a}_k + \mathrm{i}\mathbf{V} \cdot \mathbf{O} \cdot \partial_x^{-1}(\mathbf{V} \cdot \mathbf{O} \cdot \boldsymbol{a}_k \pm \mathbf{A} \cdot \mathbf{O} \cdot \boldsymbol{v}_k) \\ \pm \mathrm{i}\partial_x^{-1}(\mathbf{V} \cdot \mathbf{O} \cdot \mathbf{A} \pm \mathbf{A} \cdot \mathbf{O} \cdot \mathbf{V}) \cdot \mathbf{O} \cdot \boldsymbol{v}_k, \tag{4.34}$$

and, in a similar way as before it can be shown that the integrations ∂_x^{-1} do not yield undetermined constants in (4.33), cf. appendix A.

The linear problem, corresponding to (3.15), (3.19) and (3.25), can be expressed as, cf. (4.27a) and (4.27b),

$$\mathrm{i}\partial_t \boldsymbol{v}_k = \mathbf{Y}_k^{(r)} = (\Omega^-\Omega^+)^{(r-1)/2}\Omega^-[\boldsymbol{v}_k], \quad (r \text{ odd}), \tag{4.35a}$$

$$\mathrm{i}\partial_t \boldsymbol{w}_k = \mathbf{Z}_k^{(r)} = \mathrm{i}\partial_x^{-1}(\mathbf{V} \cdot \mathbf{O} \cdot \mathbf{Y}_k^{(r)} + \mathbf{Y}^{(r)} \cdot \mathbf{O} \cdot \boldsymbol{v}_k), \quad (r \text{ odd}), \tag{4.35b}$$

$$-\mathrm{i}\partial_x \boldsymbol{v}_k = k\boldsymbol{v}_k - \mathbf{V} \cdot \mathbf{O} \cdot \boldsymbol{w}_k, \tag{4.35c}$$

$$-\mathrm{i}\partial_x \boldsymbol{w}_k = \mathbf{V} \cdot \mathbf{O} \cdot \boldsymbol{v}_k, \tag{4.35d}$$

and we have the matrix PDE

$$\mathrm{i}\partial_t \mathbf{V} = (\Omega^-\Omega^+)^{(r-1)/2}\Omega^-[\mathbf{V}], \quad (r \text{ odd}), \tag{4.36}$$

in which

$$\Omega^{\pm}[\mathbf{A}] = -\mathrm{i}\partial_x \mathbf{A} + \mathrm{i}\mathbf{V} \cdot \mathbf{O} \cdot \partial_x^{-1}(\mathbf{V} \cdot \mathbf{O} \cdot \mathbf{A} \pm \mathbf{A} \cdot \mathbf{O} \cdot \mathbf{V}) \\ \pm \mathrm{i}\partial_x^{-1}(\mathbf{V} \cdot \mathbf{O} \cdot \mathbf{A} \pm \mathbf{A} \cdot \mathbf{O} \cdot \mathbf{V}) \cdot \mathbf{O} \cdot \mathbf{V}, \tag{4.37}$$

for an arbitrary $\mathbf{A}$. From the form of the operators Ω^+, Ω^-, or from the recursion relations (A.8), (A.9) and (4.29), it is clear that the (0, 0) element of (4.36) yields a closed PDE containing only $v_{0,0}$, and the $n = 0$ component of (4.35a)–(4.35d) is the associated linear problem.

As an example we write down the PDE's obeyed by $v_{0,0}$, for the values $r = 3, 5$. Using the explicit expressions for $\mathbf{Y}_k^{(3)}$ and $\mathbf{Y}_k^{(5)}$, given in eqs. (C.2a) and (C.3a) of appendix C, we find

$$r = 3, \quad \partial_t v_{0,0} - \partial_x^3 v_{0,0} = -6v_{0,0}^2 \partial_x v_{0,0}, \tag{4.38}$$

$$r = 5, \quad \partial_t v_{0,0} + \partial_x^5 v_{0,0} = 10[v_{0,0}^2 \partial_x^3 v_{0,0} + 4v_{0,0}(\partial_x v_{0,0})\partial_x^2 v_{0,0} \\ + (\partial_x v_{0,0})^3] - 30v_{0,0}^4 \partial_x v_{0,0}. \tag{4.39}$$

Eq. (4.38) is the modified Korteweg–de Vries equation (MKdV) and (4.39) is a higher-order MKdV type of equation. (Note that (4.20) may be regarded as a complex version of (4.39)). As in the previous subsection one may also consider linear combinations of PDE's, corresponding to a dispersion relation $\omega(k) = \Sigma_r \lambda_r k^r$, $r > 0$, but now $\omega(k)$ must be an odd function of k.

5. Miura transformations

In section 4 matrix partial differential equations for positive r were derived, from which immediately PDE's for the (0, 0) element of $\mathbf{\Phi}$ and $\mathbf{V}$ could be deduced. In this section we will consider the possibility of deriving closed equations for other elements of the matrices $\mathbf{\Phi}$ and $\mathbf{V}$. One could attempt to derive these equations by applying $\mathbf{J}$ and $\mathbf{J}^{\mathrm{T}}$ a number of times to the integral equation, and solving the matrices $\mathbf{O}\cdot\mathbf{\Phi}\cdot\mathbf{O}$ and $\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}$ via the relations (2.35) and (3.26). For general $\phi_{n,m}$, cf. (1.6), or $v_{n,m}$, cf. (1.8), this procedure is quite intricate and it is not obvious that a closed PDE containing only one specific $\phi_{n,m}$ or $v_{n,m}$ can be derived. In this section we shall derive explicit PDE's for the (0, 1), (1, 0) and (1, 1) elements, restricting ourselves to the choice of sign $\eta = 1$ in eq. (1.3). This is relatively easy, because the algebraic relations (2.35a), (2.35b), or (3.26a), (3.26b), reduce in this case to relations containing only two matrix elements, rather than three, which is the usual situation.

5.1. *Partial differential equation for $\phi_{1,0}$ (type I)*

Taking the (1, 1) element of (2.35b) for $\eta = 1$, $p = 1$, we have

$$\psi_{1,0} = \psi_{0,1} + \phi^{\dagger}_{1,0}\phi_{0,1} - \psi_{1,0}\psi_{0,1}, \tag{5.1}$$

and hence, cf. (2.6) and (2.9),

$$|1 - \psi_{1,0}|^2 + |\phi_{1,0}|^2 = 1. \tag{5.2}$$

Taking the (1,0) element of eqs. (2.35c) and (2.35d), with $\eta = 1$, we can express $\phi_{0,0}$ in terms of $\phi_{1,0}$ and $\psi_{1,0}$, i.e.

$$-\mathrm{i}\partial_x\phi_{1,0} = (1 - \psi^{\dagger}_{1,0})\phi_{0,0}, \tag{5.3a}$$

$$-\mathrm{i}\partial_x\psi_{1,0} = \phi^{\dagger}_{1,0}\phi_{0,0}, \tag{5.3b}$$

and eliminating $\phi_{0,0}$, we find

$$(1 - \psi_{1,0})\partial_x\psi^{\dagger}_{1,0} = \phi_{1,0}\partial_x\phi^{\dagger}_{1,0}. \tag{5.4}$$

Differentiating (5.3a) with respect to x and using (5.2), we have

$$-\partial_x^2\phi_{1,0}=\frac{i(1-\psi_{1,0})\partial_x\psi^*_{1,0}}{1-|\phi_{1,0}|^2}(-i\partial_x\phi_{1,0})+(1-\psi^*_{1,0})\phi_{0,0}(-i\partial_x \ln \phi_{0,0}), \tag{5.5}$$

and from (5.3a) and (5.4) we obtain an explicit expression for $\partial_x \ln \phi_{0,0}$ in terms of $\phi_{1,0}$, i.e.

$$\partial_x \ln \phi_{0,0}=\frac{\partial_x^2\phi_{1,0}}{\partial_x\phi_{1,0}}+\frac{\phi_{1,0}\partial_x\phi^*_{1,0}}{1-|\phi_{1,0}|^2}. \tag{5.6}$$

An explicit expression for $\partial_t \ln \phi_{0,0}$ in terms of $\phi_{1,0}$, can be found using the PDE for $\phi_{0,0}$ and expressing the terms in the right-hand side in terms of $\partial_x \ln \phi_{0,0}$, as given by (5.6), and of

$$|\phi_{0,0}|^2=\frac{|\partial_x\phi_{1,0}|^2}{1-|\phi_{1,0}|^2}, \tag{5.7}$$

cf. (5.3a) and (5.2). For $\phi_{0,0}$ we may choose any PDE of the type (4.16), but as an example we shall restrict ourselves to Hirota's equation[25]), with $\eta=1$, given in eq. (4.24). From (4.24) $\partial_t \ln \phi_{0,0}$ can be expressed as

$$\begin{aligned} i\partial_t \ln \phi_{0,0} = &-\lambda_2(y''+y'^2)+i\lambda_3(y'''+3y''y'+y'^3)\\ &+\frac{|\partial_x\phi_{1,0}|^2}{1-|\phi_{1,0}|^2}(-2\lambda_2+6i\lambda_3 y'), \end{aligned} \tag{5.8}$$

where we have used the abbreviation $y=\ln \phi_{0,0}$, and the primes denote differentiations with respect to x.

The function $\phi_{0,0}$ can be expressed in terms of $\phi_{1,0}$, using (5.6)–(5.8) and the relation

$$\phi_{0,0}=\frac{|\partial_x\phi_{1,0}|}{(1-|\phi_{1,0}|^2)^{1/2}}\exp\left[i\int_\Gamma \left\{dl_x\,\partial_x \operatorname{Im}\ln \phi_{0,0}+dl_t\,\partial_t \operatorname{Im}\ln \phi_{0,0}\right\}\right], \tag{5.9}$$

where Γ is an arbitrary curve in the (x,t)-plane connecting the points $(0,0)$ and (x,t), and (dl_x, dl_t) is an infinitesimal two-dimensional vector tangent to Γ (According to Stokes' theorem the right-hand side of (5.9) is independent of the choice of Γ).

The PDE for $\phi_{1,0}$ can be derived, taking the (1,0) element of (4.23) with $\lambda_r=0$ for $r\neq 2,3$, and using (B.2a) and (B.3a) with $\eta=1$, given in appendix B. We have

$$\begin{aligned} i\partial_t\phi_{1,0}= &-\lambda_2\partial_x^2\phi_{1,0}+i\lambda_3\partial_x^3\phi_{1,0}-2\lambda_2|\phi_{0,0}|^2\phi_{1,0}+3i\lambda_3\phi_{1,0}\phi^*_{0,0}\partial_x\phi_{0,0}\\ &+3i\lambda_3|\phi_{0,0}|^2\partial_x\phi_{1,0}. \end{aligned} \tag{5.10}$$

Inserting (5.6) and (5.7), eq. (5.10) can be rewritten as

$$\begin{aligned} \mathrm{i}\partial_t\phi_{1,0} = & -\lambda_2\partial_x^2\phi_{1,0} + \mathrm{i}\lambda_3\partial_x^3\phi_{1,0} \\ & + \frac{-2\lambda_2\phi_{1,0}|\partial_x\phi_{1,0}|^2 + 3\mathrm{i}\lambda_3|\partial_x\phi_{1,0}|^2\partial_x\phi_{1,0} + 3\mathrm{i}\lambda_3\phi_{1,0}(\partial_x\phi^*_{1,0})\partial_x^2\phi_{1,0}}{1-|\phi_{1,0}|^2} \\ & + \frac{3\mathrm{i}\lambda_3\phi^2_{1,0}|\partial_x\phi_{1,0}|^2\partial_x\phi^*_{1,0}}{(1-|\phi_{1,0}|^2)^2}. \end{aligned} \tag{5.11}$$

Eq. (5.11) is an integrable PDE, since its solutions can be obtained from the linear integral equation (1.3) with the dispersion relation $\omega(k) = \lambda_2 k^2 + \lambda_3 k^3$, and eq. (5.9) provides the Miura transformation mapping an arbitrary solution of (5.11) on a solution of Hirota's equation (5.8), (4.24).

The special case $\lambda_3 = 0$ of eq. (5.11) has been studied extensively in ref. 6, where it was also proved that this equation is equivalent to the equation of motion for the classical Isotropic Heisenberg Spin Chain (IHSC). Eq. (5.11) for $\lambda_3 \neq 0$ has, to our knowledge, not been given before in the literature.

5.2. *Partial differential equation for $\phi_{1,1}$ (type I)*

We now proceed with the derivation of a PDE for $\phi_{1,1}$. From the relations (2.35c) and (2.35d) with $\eta = 1$, we have, taking the (1, 1) element

$$-\mathrm{i}\partial_x\phi_{1,1} = (1-\psi^*_{1,0})\phi_{1,0}, \tag{5.12a}$$

$$-\mathrm{i}\partial_x\psi_{1,1} = |\phi_{1,0}|^2. \tag{5.12b}$$

Eq. (5.12a), together with (5.2), implies that

$$|\phi_{1,0}|^2 = \tfrac{1}{2} \pm \tfrac{1}{2}(1-4|\partial_x\phi_{1,1}|^2)^{1/2}. \tag{5.13}$$

By differentiating (5.12a) with respect to x, and using (5.4), (5.2) and (5.12a), we have

$$\begin{aligned} -\partial_x^2\phi_{1,1} &= -\mathrm{i}(1-\psi^*_{1,0})\partial_x\phi_{1,0} + \frac{\mathrm{i}\phi_{1,0}\partial_x|\phi_{1,0}|^2 - \mathrm{i}|\phi_{1,0}|^2\partial_x\phi_{1,0}}{1-\psi_{1,0}} \\ &= \frac{\partial_x\phi_{1,1}}{1-|\phi_{1,0}|^2}(\partial_x|\phi_{1,0}|^2 - \partial_x \ln \phi_{1,0}), \end{aligned} \tag{5.14}$$

so that

$$\partial_x \ln \phi_{1,0} = \tfrac{1}{2}[1 \mp (1-4|\partial_x\phi_{1,1}|^2)^{1/2}]\frac{\partial_x^2\phi_{1,1}}{\partial_x\phi_{1,1}} \mp \frac{\partial_x|\partial_x\phi_{1,1}|^2}{(1-4|\partial_x\phi_{1,1}|^2)^{1/2}}. \tag{5.15}$$

The relation for $\partial_t \ln \phi_{1,0}$, in terms of $\phi_{1,1}$, can be inferred from (5.15), (5.13)

and

$$i\partial_t \ln \phi_{1,0} = -\lambda_2(z'' + z'^2) + i\lambda_3(z''' + 3z''z' + z'^3)$$
$$+ \frac{|\phi_{1,0}|^2}{1-|\phi_{1,0}|^2}(-2\lambda_2|z'|^2 + 3i\lambda_3 z'^*(z'' + 2z'^2)) + \frac{|\phi_{1,0}|^4}{(1-|\phi_{1,0}|^2)^2} 3i\lambda_3 z'^*|z'|^2, \quad (5.16)$$

with $z \equiv \ln \phi_{1,0}$, which follows directly from (5.11). The function $\phi_{1,0}$ can be expressed in terms of $\phi_{1,1}$, using (5.13), (5.15), (5.16) and the relation

$$\phi_{1,0} = [\tfrac{1}{2} \pm \tfrac{1}{2}(1 - |4\partial_x \phi_{1,1}|^2)^{1/2}]^{1/2}$$
$$\times \exp\left[i \int_\Gamma \{dl_x \operatorname{Im} \partial_x \ln \phi_{1,0} + dl_t \operatorname{Im} \partial_t \ln \phi_{1,0}\}\right], \quad (5.17)$$

cf. also (5.9) for the meaning of Γ, dl_x and dl_t.

The PDE for $\phi_{1,1}$ can be derived taking the (1,1) element of (4.23), and using the expressions (B2a), (B.3a), given in appendix B. We have

$$i\partial_t \phi_{1,1} = -\lambda_2(\partial_x^2 \phi_{1,1} + 2\phi_{1,0}^2 \phi_{0,0}^*) + i\lambda_3[\partial_x^3 \phi_{1,1} + 6(\partial_x \ln \phi_{1,0})\phi_{1,0}^2 \phi_{0,0}^*]. \quad (5.18)$$

Using (5.13), (5.15) and the relation

$$\phi_{1,0}^2 \phi_{0,0}^* = (\partial_x \phi_{1,1})\partial_x |\phi_{1,0}|^2 - |\phi_{1,0}|^2 \partial_x^2 \phi_{1,1}, \quad (5.19)$$

which can be inferred from (5.3a), (5.2), (5.12a) and (5.14), in (5.18), we obtain the PDE for $\phi_{1,1}$, i.e.

$$i\partial_t \phi_{1,1} = \mp\lambda_2[(\partial_x \phi_{1,1})\partial_x(1 - 4|\partial_x \phi_{1,1}|^2)^{1/2} - (\partial_x^2 \phi_{1,1})(1 - 4|\partial_x \phi_{1,1}|^2)^{1/2}]$$
$$+ i\lambda_3\left[\partial_x^3 \phi_{1,1} + 6|\partial_x^2 \phi_{1,1}|^2 \partial_x \phi_{1,1} + \frac{6(\partial_x |\partial_x \phi_{1,1}|^2)^2}{1 - 4|\partial_x \phi_{1,1}|^2} \partial_x \phi_{1,1}\right]. \quad (5.20)$$

Eq. (5.20) is an integrable PDE in the sense that its solutions can be obtained from the linear integral equation (1.3) with $\omega(k) = \lambda_2 k^2 + \lambda_3 k^3$ and $n = 1$.

Introducing a real 3-dimensional vector $\boldsymbol{S} = (S^x, S^y, S^z)$ with $\boldsymbol{S} \cdot \boldsymbol{S} = 1$, and identifying

$$S^+ \equiv S^x + iS^y = -2i\partial_x \phi_{1,1}, \text{ and } S^z = \mp(1 - 4|\partial_x \phi_{1,1}|^2)^{1/2}, \quad (5.21)$$

eq. (5.20) can be differentiated to give

$$\partial_t \boldsymbol{S} = \lambda_2 \boldsymbol{S} \times \partial_x^2 \boldsymbol{S} + \lambda_3 \partial_x[\partial_x^2 \boldsymbol{S} + \tfrac{3}{2}\boldsymbol{S}(\partial_x \boldsymbol{S}) \cdot \partial_x \boldsymbol{S}]. \quad (5.22)$$

Eq. (5.22) has been given before by Papanicolaou[26]) and reduces in the case $\lambda_3 = 0$ to the IHSC. The Miura transformations (5.17) and (5.9) provide a mapping of the solutions of the Papanicolaou equation (5.22) on solutions of the Hirota equation (4.24). (A related mapping was obtained in ref. 26, by extending the line of reasoning given in ref. 9 for the special case $\lambda_3 = 0$.)

5.3. *Integral equation of type II*

So far we only considered Miura transformations that could be derived from the relations (2.35) for the integral equation of type I. About the integral equation of type II we can be very short. In fact, taking the (1, 0) and (1, 1) elements of (3.26c) and (3.26d), we have

$$-\mathrm{i}\partial_x v_{1,0} = (1 - w_{1,0})v_{0,0}, \tag{5.23a}$$

$$-\mathrm{i}\partial_x w_{1,0} = v_{1,0}v_{0,0}, \tag{5.23b}$$

$$-\mathrm{i}\partial_x v_{1,1} = (1 - w_{1,0})v_{1,0}, \tag{5.23c}$$

$$-\mathrm{i}\partial_x w_{1,1} = v_{1,0}^2, \tag{5.23d}$$

together with the algebraic relation

$$(1 - w_{1,0})^2 + v_{1,0}^2 = 1, \tag{5.24}$$

cf. (3.26b). From (5.23a) and (5.24), $v_{0,0}$ can be solved, giving

$$v_{0,0} = \frac{-\mathrm{i}\partial_x v_{1,0}}{(1 - v_{1,0}^2)^{1/2}} = -\mathrm{i}\partial_x(\arcsin v_{1,0}), \tag{5.25}$$

and from (5.23c) and (5.24) we have

$$v_{1,0}^2 = \tfrac{1}{2} \pm \tfrac{1}{2}(1 + 4(\partial_x v_{1,1})^2)^{1/2}. \tag{5.26}$$

Eqs. (5.25) and (5.26) imply that in the case of the integral equation of type II, the transformations from $v_{1,0}$ to $v_{0,0}$ and from $v_{1,1}$ to $v_{1,0}$ become trivial, so that the PDE's for $v_{1,0}$ and $v_{1,1}$ are equivalent to the one for $v_{0,0}$.

6. Complex sine-Gordon equation

Although the relations (2.35), which were derived in section 2 for the integral equation of type I, and the relations (3.26), which were derived in section 3 for the integral equation of type II, are valid for all integer r, the method for finding partial differential equations containing only $\phi_{0,0}$, which we developed in section 4, is restricted to positive values of r. For negative r values the situation is more complicated. For the special case $r = -1$, however, it is possible to derive closed PDE's for the functions $\phi_{0,0}$, $\phi_{0,1} = \phi_{1,0}$ and $\phi_{1,1}$, as will be shown in this section. In appendix D, we shall give a method to derive three coupled equations in $\phi_{0,0}$, $\phi_{1,0}$ and $\phi_{1,1}$, which applies to all negative r values.

6.1. *Integral equation of type I*

In the case $r=-1$, taking into account that $\mathbf{Q}_{-1}=-\mathbf{J}^{\mathrm{T}}\cdot\mathbf{O}\cdot\mathbf{J}$, cf. eq. (2.23), we find from eqs. (2.34a) and (2.34c), taking $\eta=1$,

$$\partial_x\partial_t\boldsymbol{\phi}_k=(\mathbf{J}+\boldsymbol{\Psi}^*\cdot\mathbf{J}^{\mathrm{T}}\cdot\mathbf{O}\cdot\mathbf{J})\cdot(\mathbf{J}^{\mathrm{T}}\cdot\boldsymbol{\phi}_k-\boldsymbol{\Psi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k)\\ -\mathrm{i}(\partial_x\boldsymbol{\Psi}^*)\cdot\mathbf{J}^{\mathrm{T}}\cdot\mathbf{O}\cdot\mathbf{J}\cdot\boldsymbol{\phi}_k, \tag{6.1}$$

which, in view of (2.35b) and (2.35d), can be rewritten as

$$\partial_x\partial_t\boldsymbol{\phi}_k=\boldsymbol{\phi}_k+\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{Q}_{-1}\cdot\boldsymbol{\phi}_k+\boldsymbol{\Phi}\cdot\mathbf{Q}_{-1}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k. \tag{6.2}$$

From eq. (6.2) we immediately obtain three coupled PDE's containing only $\phi_{0,0}$, $\phi_{1,0}$ and $\phi_{1,1}$, viz.

$$\partial_x\partial_t\phi_{0,0}=(1-2|\phi_{1,0}|^2)\phi_{0,0}, \tag{6.3a}$$

$$\partial_x\partial_t\phi_{1,0}=(1-|\phi_{1,0}|^2)\phi_{1,0}-\phi_{0,0}\phi^*_{1,0}\phi_{1,1}, \tag{6.3b}$$

$$\partial_x\partial_t\phi_{1,1}=(1-2|\phi_{1,0}|^2)\phi_{1,1}. \tag{6.3c}$$

Also for other negative r, it is possible to derive three coupled equations, as will be shown in appendix *D*. For $r=-1$, it is rather easy to derive closed PDE's containing only $\phi_{0,0}$, $\phi_{1,0}$ and $\phi_{1,1}$.

In fact, taking the (0, 0) element of (2.35e) for $r=-1$, we have

$$\mathrm{i}\partial_t\phi_{0,0}=(1-\psi_{1,0})\phi_{1,0}, \tag{6.4}$$

which, in combination with the algebraic relation (5.2), yields

$$|\phi_{1,0}|^2=\tfrac{1}{2}\pm\tfrac{1}{2}(1-4|\partial_t\phi_{0,0}|^2)^{1/2}. \tag{6.5}$$

Inserting (6.5) in (6.3a) and differentiating the result with respect to t, we obtain

$$\chi=\mp\,\partial_t\frac{\partial_x\chi}{(1-|\chi|^2)^{1/2}}, \tag{6.6}$$

with

$$\chi\equiv 2\partial_t\phi_{0,0}. \tag{6.7}$$

Eq. (6.6) can be called the complex sine-Gordon equation, since for real $\chi\equiv\sin\theta$ it reduces to $\partial_x\partial_t\theta=\mp\sin\theta$. Eq. (6.6) has been given before in eq. (3.5) of ref. 27, where it has been inferred from the equations for the reduced nonlinear $O(4)$ σ-model, derived by Pohlmeyer[28]) and Lund and Regge[29]). A bilinearization of (6.6) was given in ref. 21, while the inverse scattering scheme was formulated in ref. 30. (In our approach, the Gel'fand–Levitan equation can be obtained directly from the linear integral equation, as shown in appendix A of ref. 6 by a line of reasoning, which is independent of the value of r.)

Using eq. (5.3a) for $\phi_{0,0}$ and the equation

$$\phi_{1,1} = \frac{\mathrm{i}\partial_t \phi_{1,0}}{1 - \psi_{1,0}}, \tag{6.8}$$

which follows from the (1, 0) element of (2.34e) for $r = -1$, in combination with the algebraic relation (5.2), eq. (6.3b) leads to

$$\partial_x \partial_t \phi_{1,0} + \frac{(\partial_x \phi_{1,0})(\partial_t \phi_{1,0})}{1 - |\phi_{1,0}|^2} \phi^*_{1,0} = (1 - |\phi_{1,0}|^2)\phi_{1,0}. \tag{6.9}$$

Eq. (6.9) was for the first time given by Getmanov[31]) and can be shown to be related to the reduced nonlinear $O(4)$ σ-model, cf. ref. 27.

Finally, inserting (5.13) into (6.3c), we obtain

$$\partial_x \partial_t \phi_{1,1} = \mp (1 - 4|\partial_x \phi_{1,1}|^2)^{1/2} \phi_{1,1}, \tag{6.10}$$

which, after differentiating with respect to x, leads to the complex sine-Gordon equation (6.6) with $\chi = 2\partial_x \phi_{1,1}$, and $\partial_x \leftrightarrow \partial_t$.

Remarks

(i) From eqs. (2.34) and (2.21), for $r = -1$, one can derive the Lax representations[27]) for the complex sine-Gordon equation. Some details of the derivation will be given in appendix E.

(ii) In appendix D, it will be shown, that for all negative values of r, one can derive three coupled equations in $\phi_{0,0}$, $\phi_{0,1} = \phi_{1,0}$ and $\phi_{1,1}$.

6.2. *Integral equation of type II*

From the integral equation of type II, using $\mathbf{R}_{-1} = \mathbf{J}^{\mathrm{T}} \cdot \mathbf{O} \cdot \mathbf{J}$, cf. (3.14), we find from (3.26a) and (3.26b)

$$\partial_x \partial_t \boldsymbol{v}_k = (\mathbf{J} - \mathbf{W} \cdot \mathbf{J}^{\mathrm{T}} \cdot \mathbf{O} \cdot \mathbf{J}) \cdot (\mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{v}_k - \mathbf{W} \cdot \mathbf{O} \cdot \boldsymbol{v}_k) + \mathrm{i}(\partial_x \mathbf{W}) \cdot (\mathbf{J}^{\mathrm{T}} \cdot \mathbf{O} \cdot \mathbf{J}) \cdot \boldsymbol{v}_k, \tag{6.11}$$

which, in view of (3.26b) and (3.26d) can be rewritten as

$$\partial_x \partial_t \boldsymbol{v}_k = \boldsymbol{v}_k - \mathbf{V} \cdot \mathbf{O} \cdot \mathbf{V} \cdot \mathbf{J}^{\mathrm{T}} \cdot \mathbf{O} \cdot \mathbf{J} \cdot \boldsymbol{v}_k - \mathbf{V} \cdot \mathbf{J}^{\mathrm{T}} \cdot \mathbf{O} \cdot \mathbf{J} \cdot \mathbf{V} \cdot \mathbf{O} \cdot \boldsymbol{v}_k. \tag{6.12}$$

Multiplying (6.12) by k^{-m}, and integrating over the contour C, and taking the (0, 0) element of the resulting matrix equation, we have

$$\partial_x \partial_t v_{0,0} = v_{0,0}(1 - 2v_{1,0}^2). \tag{6.13}$$

From the (0, 0) element of (3.26e) we have

$$i\partial_t v_{0,0} = v_{1,0}(1 - w_{1,0}), \tag{6.14}$$

which, in combination with the algebraic relation

$$v_{1,0}^2 + (1 - w_{1,0})^2 = 1, \tag{6.15}$$

yields

$$v_{1,0}^2 = \tfrac{1}{2} \pm \tfrac{1}{2}[1 + 4(\partial_t v_{0,0})^2]^{1/2}. \tag{6.16}$$

Inserting (6.16) in (6.13), we obtain a closed PDE for $v_{0,0}$, viz.

$$\partial_x \partial_t v_{0,0} = \mp v_{0,0}[1 + 4(\partial_t v_{0,0})^2]^{1/2}, \tag{6.17}$$

which, after substituting

$$u = \arcsin 2i\partial_t v_{0,0}, \tag{6.18}$$

becomes the sine-Gordon equation

$$\partial_x \partial_t u = \mp \sin u. \tag{6.19}$$

In a similar way as before one may derive closed PDE's for $v_{1,0}$ and $v_{1,1}$. We shall not write down the result, because the Miura transformations (5.25) and (5.26) are trivial, so that the PDE's for $v_{1,0}$ and $v_{1,1}$ are equivalent to the sine-Gordon equation.

7. The Korteweg–de Vries equation and the modified nonlinear Schrödinger equation

7.1. *Reduction of the integral equation of type II*

In this section we will show how the integral equation of type II reduces to another type of integral equation with only one integration, instead of a two-fold integration. For that purpose we consider eqs. (3.1), cf. eq. (1.4), and introduce the quantity

$$u_k^{(n)} = v_k^{(n)} - i w_k^{(n)}. \tag{7.1}$$

Then eqs. (3.1a) and (3.1b) can be combined into a single integral equation, i.e.

$$u_k^{(n)} + i\rho_k \int_C d\lambda(l) \frac{u_l^{(n)}}{k + l} = \frac{1}{k^n} \rho_k. \tag{7.2}$$

Eq. (7.2) can be regarded as a generalization of the integral equation, given by Fokas and Ablowitz[4]) for $n = 0$, cf. also ref. 5.

The relations for $u_k^{(n)}$ can be found by combining the relations for the quantities $v_k^{(n)}$ and $w_k^{(n)}$, which we have given in section 3. From eqs. (3.15) and (3.19), we find

$$k^p \boldsymbol{u}_k = \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{u}_k + \mathrm{i}\mathbf{U} \cdot \mathbf{R}_p \cdot \boldsymbol{u}_k, \quad (p \text{ even}), \tag{7.3}$$

for all even integers p, where $\boldsymbol{u}_k$ denotes the vector with components $u_k^{(n)}$ and the matrix $\mathbf{U}$ is defined by

$$(\mathbf{U})_{n,m} = u_{n,m}, \quad u_{n,m} = \int_C \mathrm{d}\lambda(k) \frac{1}{k^m} u_k^{(n)}, \tag{7.4}$$

and from (7.1) we find

$$\mathbf{U} = \mathbf{V} - \mathrm{i}\mathbf{W} = \mathbf{U}^{\mathrm{T}}, \tag{7.5}$$

cf. also (3.4). From eqs. (3.23) and (3.24) we obtain

$$-\mathrm{i}\partial \boldsymbol{u}_k = \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{v}_k - \mathrm{i}\mathbf{U} \cdot \mathbf{R}_p \cdot \boldsymbol{v}_k, \quad (p \text{ odd}), \tag{7.6}$$

for odd integer p. Making use of (3.15) and (3.19), but now for odd p-values, we have

$$k^p \boldsymbol{u}_k = 2\mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{v}_k - \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{u}_k + \mathrm{i}\mathbf{U} \cdot \mathbf{R}_p \cdot \boldsymbol{u}_k - 2\mathrm{i}\mathbf{U} \cdot \mathbf{R}_p \cdot \boldsymbol{v}_k, \quad (p \text{ odd}), \tag{7.7}$$

leading, in combination with (7.6) to

$$(k^p + 2\mathrm{i}\partial)\boldsymbol{u}_k = -\mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{u}_k + \mathrm{i}\mathbf{U} \cdot \mathbf{R}_p \cdot \boldsymbol{u}_k, \quad (p \text{ odd}). \tag{7.8}$$

Taking into account eq. (1.5) for the factors $\rho_k(x, t)$, eq. (7.8) can be expressed as

$$(k + 2\mathrm{i}\partial_x)\boldsymbol{u}_k = -\mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{u}_k + \mathrm{i}\mathbf{U} \cdot \mathbf{O} \cdot \boldsymbol{u}_k, \tag{7.9}$$

$$2\mathrm{i}\partial_t \boldsymbol{u}_k = k^r \boldsymbol{u}_k + \mathbf{J}^{\mathrm{T}r} \cdot \boldsymbol{u}_k - \mathrm{i}\mathbf{U} \cdot \mathbf{R}_r \cdot \boldsymbol{u}_k, \quad (r \text{ odd}). \tag{7.10}$$

Eqs. (7.3), (7.9) and (7.10) are the constitutive relations associated with the integral equation and may be also derived directly from (7.2). Eq. (7.3) for $p = 2$, and eq. (7.9), can be combined in order to eliminate $\mathbf{J}^{\mathrm{T}}$ and this gives

$$(k + \mathrm{i}\partial_x)\mathrm{i}\partial_x \boldsymbol{u}_k = -(\partial_x \mathbf{U}) \cdot \mathbf{O} \cdot \boldsymbol{u}_k, \tag{7.11}$$

where we have also used the relation

$$-2\mathrm{i}\partial_x \mathbf{U} = \mathbf{U} \cdot \mathbf{J} + \mathbf{J}^{\mathrm{T}} \cdot \mathbf{U} - \mathrm{i}\mathbf{U} \cdot \mathbf{O} \cdot \mathbf{U}, \tag{7.12}$$

which can be derived multiplying (7.9) by k^{-m} and integrating over the contour C.

As an example we consider the special value $r = 3$, corresponding to the dispersion relation $\omega(k) = k^3$. In that case, eq. (7.10) can be further evaluated, using eqs. (C.2a) and (C.2b) of appendix C and the identification

$$\mathrm{i}\partial_t \boldsymbol{u}_k = \mathbf{Y}_k^{(3)} - \mathrm{i}\mathbf{Z}_k^{(3)}. \tag{7.13}$$

The result is

$$(\partial_t - \partial_x^3)\boldsymbol{u}_k = -3(\partial_x \mathbf{U}) \cdot \mathbf{O} \cdot \partial_x \boldsymbol{u}_k, \tag{7.14}$$

and integrating over C we obtain

$$(\partial_t - \partial_x^3)\mathbf{U} = -3(\partial_x \mathbf{U}) \cdot \mathbf{O} \cdot \partial_x \mathbf{U}. \tag{7.15}$$

The (0, 0) element of (7.15) is given by

$$(\partial_t - \partial_x^3)u_{0,0} + 3(\partial_x u_{0,0})^2 = 0, \tag{7.16}$$

which is the potential Korteweg–de Vries equation, i.e. $\partial_x u_{0,0}$ satisfies the KdV.

The Miura transformation, mapping the solutions $v_{0,0}$ of the MKdV on solutions $\partial_x u_{0,0}$ of the KdV, can be inferred from the (0, 0) element of the matrix relation

$$\partial_x \mathbf{U} = \partial_x \mathbf{V} + \mathbf{V} \cdot \mathbf{O} \cdot \mathbf{V}, \tag{7.17}$$

which follows from (7.5) and (3.26d).

A PDE for $u_{1,0}$ can be obtained, taking the (1, 0) element of (7.15). We have

$$(\partial_t - \partial_x^3)u_{1,0} = -3(\partial_x u_{1,0})\partial_x u_{0,0}. \tag{7.18}$$

The factor $\partial_x u_{0,0}$ can be eliminated from (7.18), using the (0, 1) element of the matrix relation

$$\partial_x^2 \mathbf{U} = \mathrm{i}\partial_x \mathbf{U} \cdot \mathbf{J} + (\partial_x \mathbf{U}) \cdot \mathbf{O} \cdot \mathbf{U}, \tag{7.19}$$

which follows by integrating (7.11) over the contour C. From (7.19) we have

$$\partial_x u_{0,0} = (\mathrm{i} + u_{1,0})^{-1}\partial_x^2 u_{1,0}, \tag{7.20}$$

and (7.18) can be expressed as

$$(\partial_t - \partial_x^3)u_{1,0} = \frac{-3(\partial_x u_{1,0})\partial_x^2 u_{1,0}}{\mathrm{i} + u_{1,0}}. \tag{7.21}$$

Eq. (7.21) is equivalent to (4.38), which is the MKdV, as a consequence of the relation

$$v_{0,0} = \partial_x \ln(\mathrm{i} + u_{1,0}), \tag{7.22}$$

which can be inferred from the (1, 0) element of (7.5), together with the (1, 0) element of (3.26d). As a result, the solutions of the MKdV can also be inferred from the integral equation of Fokas and Ablowitz, i.e. (7.2) with

$n = 0$, using

$$v_{0,0} = \partial_x \ln \left[\mathrm{i} + \int_C \mathrm{d}\lambda(k)\, u_k^{(0)} k^{-1} \right]. \tag{7.23}$$

Also for $u_{1,1}$, one can derive a PDE, which is equivalent to the MKdV.

Remark.

For the case $r = 5$, taking into account the $n = 0$ components of the vectors $\boldsymbol{Y}_k^{(5)}$ and $\boldsymbol{Z}_k^{(5)}$ given in eqs. (C.3a) and (C.3b) of appendix C and performing an integration over the contour C, it can be shown that

$$\begin{aligned} \partial_t u_{0,0} &= -\mathrm{i}(\mathbf{Y}^{(5)} - \mathrm{i}\mathbf{Z}^{(5)})_{0,0} \\ &= -\partial_x^5 u_{0,0} + 10(\partial_x^3 u_{0,0})\partial_x u_{0,0} + 5(\partial_x^2 u_{0,0})^2 - 10(\partial_x u_{0,0})^3. \end{aligned} \tag{7.24}$$

Eq. (7.24) is a (potential) higher-order Korteweg–de Vries equation, and the higher-order KdV for $v \equiv \partial_x u_{0,0}$ has already been given in ref. 32. The (0, 0) element of (7.17) provides again the Miura transformation mapping a solution of (4.39) on a solution of (7.24). Eq. (7.24) may also be derived using the constitutive relations associated with the integral equation (7.2).

7.2. *The modified nonlinear Schrödinger equation*

In this subsection we consider the integral equation of type I, (1.3) in the case that $\eta = -1$. In that case it is possible to express the constitutive relations in terms of the vector

$$\boldsymbol{p}_k \equiv \boldsymbol{\phi}_k + \mathrm{i}\boldsymbol{\psi}_k \tag{7.25a}$$

and the corresponding matrix

$$\mathbf{P} = \mathbf{\Phi} + \mathrm{i}\mathbf{\Psi}. \tag{7.25b}$$

From the algebraic relations (2.21a), (2.21b) and eqs. (2.34a)–(2.34d) with $\eta = -1$, we have

$$-\mathrm{i}\partial_x \boldsymbol{p}_k = \mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{\phi}_k + \mathrm{i}\mathbf{P}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k, \tag{7.26a}$$

$$(-k - \mathrm{i}\partial_x)\boldsymbol{p}_k = -\mathrm{i}\mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{\psi}_k - \mathbf{P} \cdot \mathbf{O} \cdot \boldsymbol{\psi}_k, \tag{7.26b}$$

and

$$\mathrm{i}\partial_t \boldsymbol{p}_k = \mathbf{J}^{\mathrm{T}'} \cdot \boldsymbol{\phi}_k + \mathrm{i}\mathbf{P}^* \cdot \mathbf{Q}_r \cdot \boldsymbol{\phi}_k. \tag{7.27}$$

Considering, as an example, the dispersion relation $\omega(k) = \lambda_2 k^2 + \lambda_3 k^3$, leading to

$$i\partial_t \rho_k = (\lambda_2 k^2 + \lambda_3 k^3)\rho_k, \quad D \equiv -2\lambda_2 + 3i\lambda_3\partial_x, \tag{7.28}$$

eq. (7.27) can be expressed as

$$(i\partial_t + \lambda_2\partial_x^2 - i\lambda_3\partial_x^3)p_k = (\partial_x \mathbf{P}^*)\cdot \mathbf{O} \cdot D\phi_k + 3\lambda_3(\partial_x \mathbf{P})\cdot \mathbf{O} \cdot \partial_x\psi_k, \tag{7.29}$$

which can be derived, using eqs. (B.2a), (B.2b), (B.3a) and (B.3b) of appendix B with $\eta = -1$.

From the $n = 1$ component of (7.26a), (7.26b) with $\eta = -1$, one can solve $\phi_k^{(0)}$ and $\psi_k^{(0)}$, in terms of $p_k^{(1)}$ and $p_{1,0}$. The result is

$$\phi_k^{(0)} = \frac{-i\partial_x p_k^{(1)}}{1 + ip_{1,0}^*}, \tag{7.30a}$$

and

$$\psi_k^{(0)} = \frac{i(-k - i\partial_x)p_k^{(1)}}{1 - ip_{1,0}}. \tag{7.30b}$$

Inserting (7.30a) in (7.29) we obtain

$$\begin{aligned}(i\partial_t + \lambda_2\partial_x^2 - i\lambda_3\partial_x^3)p_k^{(1)} &= 2i\lambda_2 \frac{(\partial_x p_{1,0}^*)\partial_x p_k^{(1)}}{1 + ip_{1,0}^*} \\ &+ 3i\lambda_3(\partial_x p_{1,0}^*)\left[-i\partial_x\left(\frac{\partial_x p_k^{(1)}}{1 + ip_{1,0}^*}\right) - \frac{\partial_x p_{1,0}}{1 - ip_{1,0}}\frac{\partial_x p_k^{(1)}}{1 + ip_{1,0}^*}\right],\end{aligned} \tag{7.31a}$$

and from (7.30a), (7.30b) and the $n = 0$ component of (2.34d) we have

$$\partial_x\left(\frac{(-k - i\partial_x)p_k^{(1)}}{1 - ip_{1,0}}\right) = \frac{(\partial_x p_{1,0}^*)\partial_x p_k^{(1)}}{|1 - ip_{1,0}|^2}. \tag{7.31b}$$

The relations (7.31a) and (7.31b) form the linear problem associated with the PDE for $p_{1,0}$, which after substituting

$$q \equiv 1 - ip_{1,0} \tag{7.32}$$

can be expressed as

$$(i\partial_t + \lambda_2\partial_x^2 - i\lambda_3\partial_x^3)q = \frac{2\lambda_2|\partial_x q|^2}{q^*} - 3i\lambda_3(\partial_x q^*)\left(\partial_x\left(\frac{\partial_x q}{q^*}\right) + \frac{(\partial_x q)^2}{|q|^2}\right). \tag{7.33}$$

The Miura transformation mapping solutions $p_{1,0}$ of (7.33), cf. (7.32), on solutions of equation (4.24) with $\eta = -1$, can be found integrating (7.30a) over the contour C.

For $\lambda_3 = 0$, $\lambda_2 = 1$, eq. (7.33) reduces to

$$(i\partial_t + \partial_x^2)q = \frac{2|\partial_x q|^2}{q^*}, \tag{7.34}$$

which is the modified nonlinear Schrödinger equation (MNLS). Note in this connection that the substitution $q=\sqrt{a}$ yields eq. (3.7) of ref. 15 in the limiting case $s\to 0$, $A\to 0$. In ref. 15 we have also shown that the corresponding potential equation, in terms of one real variable f, i.e.

$$\partial_t\left(\frac{\dot f}{f'}\right)+\partial_x\left(\frac{f'''}{f'}-\frac{3}{2}\frac{f''^2+\dot f^2}{f'^2}\right)=0, \tag{7.35}$$

has similarity solutions of the type $f=\delta t+F(x/\sqrt{t})$, satisfying Painlevé IV. For some details on the derivation of (7.35) from (7.34) we refer to appendix F.

The considerations given above imply in particular that the MNLS and the potential MNLS are completely integrable, since solutions can be obtained from the linear integral equations (2.1a), (2.1b). Furthermore, on substituting $s=\partial_x \ln q$, eq. (7.34) for the MNLS can be expressed as

$$\mathrm{i}\partial_t s+\partial_x^2 s=\{2\partial_x|s|^2-\partial_x s^2\}, \tag{7.36}$$

which may be regarded as a complex extension of the Burgers equation.

In the special case $\lambda_2=0$, $\lambda_3=1$, the substitution $s=\partial_x \ln q$ in (7.33) leads to

$$\partial_t s-\partial_x^3 s=\partial_x[3(\partial_x s)(s-s^*)+s^3-3s^*|s|^2+6|s|^2(s^*-s)], \tag{7.37}$$

which is an integrable complex version of the modified Korteweg–de Vries equation, which differs from the complex MKdV given in (4.18).

Remark.

The vector $\boldsymbol{p}_k$ can be solved directly from the integral equation

$$p_k^{(n)}-\mathrm{i}\rho_k\int_{C^*}\mathrm{d}\lambda^*(l')\frac{p_{l'}^{(n)*}}{k-l'}=\frac{1}{k^n}\rho_k, \tag{7.38a}$$

which can be inferred from (2.1a) and (2.1b), or equivalently from the integral equation

$$p_k^{(n)}-\int_C\mathrm{d}\lambda(l)\int_{C^*}\mathrm{d}\lambda^*(l')\frac{\rho_k\rho_{l'}^*}{(k-l')(l'-l)}p_l^{(n)}=\frac{1}{k^n}\rho_k+\mathrm{i}\int_{C^*}\mathrm{d}\lambda^*(k')\frac{\rho_k\rho_{k'}^*}{k'^n(k-k')}. \tag{7.38b}$$

Note that only eq. (7.38b) can be used for a direct derivation of the constitutive relations, since eq. (7.38a) is not of the right type for that purpose.

Appendix A

In order to show that the integration constants drop out in eq. (4.10), we consider the relation

$$G_k^{(p)} = -(\mathbf{J}^{\mathrm{T}} - \eta \mathbf{\Psi} \cdot \mathbf{O}) \cdot \mathbf{J}^{\mathrm{T}p-1} \cdot \boldsymbol{\psi}_k + k^p \boldsymbol{\psi}_k + \eta \mathbf{\Psi} \cdot \mathbf{J} \cdot \mathbf{Q}_{p-1} \cdot \boldsymbol{\psi}_k, \tag{A.1}$$

which can be found from (4.5), inserting (4.2). From the algebraic relations (2.21b) for $k\boldsymbol{\psi}_k$ and (2.35b) for $\mathbf{\Psi} \cdot \mathbf{J}$, we find

$$\begin{aligned} G_k^{(p)} &= (\mathbf{J}^{\mathrm{T}} - \eta \mathbf{\Psi} \cdot \mathbf{O}) \cdot (k^{p-1} \boldsymbol{\psi}_k - \mathbf{J}^{\mathrm{T}p-1} \cdot \boldsymbol{\psi}_k + \eta \mathbf{\Psi} \cdot \mathbf{Q}_{p-1} \cdot \boldsymbol{\psi}_k) \\ &\quad + \mathbf{\Phi}^* \cdot \mathbf{O} \cdot (k^{p-1} \boldsymbol{\phi}_k + \eta \mathbf{\Phi} \cdot \mathbf{Q}_{p-1} \cdot \mathbf{\Psi}_k) \\ &= (\mathbf{J}^{\mathrm{T}} - \eta \mathbf{\Psi} \cdot \mathbf{O}) \cdot G_k^{(p-1)} + \mathbf{\Phi}^* \cdot \mathbf{O} \cdot F_k^{(p-1)}. \end{aligned} \tag{A.2}$$

Assuming, by induction, that $G_k^{(p-1)}$ can be written in the form

$$G_k^{(p-1)} = \sum_s (\partial_x^s \mathbf{\Phi}^*) \cdot \mathbf{O} \cdot G_k^{(p-1,s)}, \tag{A.3}$$

we have from (A.2) and the complex conjugate of (2.35c)

$$\begin{aligned} G_k^{(p)} &= \mathrm{i} \sum_s (\partial_x^{s+1} \mathbf{\Phi}^*) \cdot \mathbf{O} \cdot G_k^{(p-1,s)} + \mathbf{\Phi}^* \cdot \mathbf{O} \cdot F_k^{(p-1)} \\ &\quad + \eta \sum_s \sum_{t=1}^{s} \binom{s}{t} (\partial_x^t \mathbf{\Psi}) \cdot \mathbf{O} \cdot (\partial_x^{s-t} \mathbf{\Phi}^*) \cdot \mathbf{O} \cdot G_k^{(p-1,s)}, \end{aligned} \tag{A.4}$$

implying with (2.35d) that indeed

$$G_k^{(p)} = \sum_s (\partial_x^s \mathbf{\Phi}^*) \cdot \mathbf{O} \cdot G_k^{(p,s)}, \tag{A.5}$$

with the recursion relation

$$\begin{aligned} G_k^{(p,s)} &= \mathrm{i} G_k^{(p-1,s-1)} + F_k^{(p-1)} \delta_{s,0} \\ &\quad + \mathrm{i}\eta \sum_{t \geq s+1} \sum_{u \geq t} \binom{u}{t} \binom{t-1}{s} (\partial_x^{t-1-s} \mathbf{\Phi}) \cdot \mathbf{O} \cdot (\partial_x^{u-t} \mathbf{\Phi}^*) \cdot \mathbf{O} \cdot G_k^{(p-1,u)}. \end{aligned} \tag{A.6}$$

Eqs. (A.5) and (A.6), in combination with (4.4), show that $F_k^{(p)}$ and $G_k^{(p)}$, for positive p, can be evaluated in a recursive way starting from $F_k^{(0)} = \boldsymbol{\phi}_k$ and

$G_k^{(0)} = 0$. This implies in particular that no undetermined integration constants will appear in the right-hand side of (4.10).

In a similar way, in the case of the integral equation of type II, we have the relation

$$Z_k^{(p)} = -(\mathbf{J}^{\mathrm{T}} - \mathbf{W}\cdot\mathbf{O})\cdot Z_k^{(p-1)} + \mathbf{V}\cdot\mathbf{O}\cdot Y_k^{(p-1)}, \tag{A.7}$$

from which it can be shown by induction that

$$Z_k^{(p)} = \sum_s (\partial_x^s \mathbf{V})\cdot\mathbf{O}\cdot Z_k^{(p,s)}, \tag{A.8}$$

where $Z_k^{(p,s)}$ satisfies the recursion relation

$$Z_k^{(p,s)} = \mathrm{i}Z_k^{(p-1,s-1)} + \mathbf{V}\cdot\mathbf{O}\cdot Y_k^{(p-1)} - \mathrm{i}\sum_{t\geq s+1}\sum_{u\geq t}\binom{u}{t}\binom{t-1}{s}(\partial_x^{t-1-s}\mathbf{V})\cdot\mathbf{O}\cdot(\partial_x^{u-t}\mathbf{V})\cdot\mathbf{O}\cdot Z_k^{(p-1,u)}. \tag{A.9}$$

Eq. (A.9), in combination with (A.8) and (4.28) shows that the vectors $Y_k^{(p)}$ and $Z_k^{(p)}$, for positive p, can be evaluated in a recursive way, so that the integration arising from ∂_x^{-1} will not produce an undetermined constant in (4.33).

Appendix B

In this appendix we give the explicit expressions for the vectors $F_k^{(p)}$ and $G_k^{(p)}$, for $p = 1, 2, 3, 4, 5$, defined by (4.1) and (4.5), in the case of the integral equation of type I. The results are

$$p = 1, \quad F_k^{(1)} = -\mathrm{i}\partial_x\boldsymbol{\phi}_k, \tag{B.1a}$$

$$G_k^{(1)} = \boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k, \tag{B.1b}$$

$$p = 2, \quad F_k^{(2)} = -\partial_x^2\boldsymbol{\phi}_k - 2\eta\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k, \tag{B.2a}$$

$$G_k^{(2)} = \mathrm{i}(\partial_x\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k - \mathrm{i}\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\partial_x\boldsymbol{\phi}_k, \tag{B.2b}$$

$$p = 3, \quad F_k^{(3)} = \mathrm{i}\partial_x^3\boldsymbol{\phi}_k + 3\mathrm{i}\eta(\partial_x\boldsymbol{\Phi})\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k + 3\mathrm{i}\eta\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\partial_x\boldsymbol{\phi}_k, \tag{B.3a}$$

$$G_k^{(3)} = -(\partial_x^2\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k - \boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\partial_x^2\boldsymbol{\phi}_k + (\partial_x\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\partial_x\boldsymbol{\phi}_k - 3\eta\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k, \tag{B.3b}$$

$$\begin{aligned} p = 4, \quad F_k^{(4)} = {} & \partial_x^4\boldsymbol{\phi}_k + \eta[4(\partial_x^2\boldsymbol{\Phi})\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k + 4\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\partial_x^2\boldsymbol{\phi}_k \\ & + 2\boldsymbol{\Phi}\cdot\mathbf{O}\cdot(\partial_x^2\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k + 2(\partial_x\boldsymbol{\Phi})\cdot\mathbf{O}\cdot(\partial_x\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k \\ & + 2\boldsymbol{\Phi}\cdot\mathbf{O}\cdot(\partial_x\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\partial_x\boldsymbol{\phi}_k + 6(\partial_x\boldsymbol{\Phi})\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\partial_x\boldsymbol{\phi}_k] \\ & + 6\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k, \end{aligned} \tag{B.4a}$$

$$
\begin{aligned}
\boldsymbol{G}_k^{(4)} = &-\mathrm{i}(\partial_x^3\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k + \mathrm{i}(\partial_x^2\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\partial_x\boldsymbol{\phi}_k \\
&-\mathrm{i}(\partial_x\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\partial_x^2\boldsymbol{\phi}_k + \mathrm{i}\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\partial_x^3\boldsymbol{\phi}_k \\
&+\mathrm{i}\eta[-4(\partial_x\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k \\
&+2\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot(\partial_x\boldsymbol{\Phi})\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k \\
&-2\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\Phi}\cdot\mathbf{O}\cdot(\partial_x\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k \\
&+4\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\partial_x\boldsymbol{\phi}_k],
\end{aligned} \tag{B.4b}
$$

$$
\begin{aligned}
p=5,\quad \boldsymbol{F}_k^{(5)} = &-\mathrm{i}\partial_x^5\boldsymbol{\phi}_k - 5\mathrm{i}\eta[(\partial_x^3\boldsymbol{\Phi})\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k + \boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\partial_x^3\boldsymbol{\phi}_k \\
&+(\partial_x^2\boldsymbol{\Phi})\cdot\mathbf{O}\cdot(\partial_x\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k + \boldsymbol{\Phi}\cdot\mathbf{O}\cdot(\partial_x\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\partial_x^2\boldsymbol{\phi}_k \\
&+(\partial_x\boldsymbol{\Phi})\cdot\mathbf{O}\cdot(\partial_x^2\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k + \boldsymbol{\Phi}\cdot\mathbf{O}\cdot(\partial_x^2\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\partial_x\boldsymbol{\phi}_k \\
&+2(\partial_x^2\boldsymbol{\Phi})\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\partial_x\boldsymbol{\phi}_k + 2(\partial_x\boldsymbol{\Phi})\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\partial_x^2\boldsymbol{\phi}_k \\
&+2(\partial_x\boldsymbol{\Phi})\cdot\mathbf{O}\cdot(\partial_x\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\partial_x\boldsymbol{\phi}_k] \\
&-10\mathrm{i}[(\partial_x\boldsymbol{\Phi})\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k \\
&+\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\partial_x\boldsymbol{\phi}_k \\
&+\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot(\partial_x\boldsymbol{\Phi})\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k],
\end{aligned} \tag{B.5a}
$$

$$
\begin{aligned}
\boldsymbol{G}_k^{(5)} = &(\partial_x^4\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k + \boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\partial_x^4\boldsymbol{\phi}_k - (\partial_x^3\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\partial_x\boldsymbol{\phi}_k \\
&-(\partial_x\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\partial_x^3\boldsymbol{\phi}_k + (\partial_x^2\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\partial_x^2\boldsymbol{\phi}_k \\
&+5\eta[(\partial_x^2\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k \\
&+\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\partial_x^2\boldsymbol{\phi}_k \\
&+\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot(\partial_x^2\boldsymbol{\Phi})\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k \\
&+\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\Phi}\cdot\mathbf{O}\cdot(\partial_x^2\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k \\
&+(\partial_x\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\boldsymbol{\Phi}\cdot\mathbf{O}\cdot(\partial_x\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k \\
&+\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot(\partial_x\boldsymbol{\Phi})\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\partial_x\boldsymbol{\phi}_k \\
&-(\partial_x\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\partial_x\boldsymbol{\phi}_k \\
&+\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot(\partial_x\boldsymbol{\Phi})\cdot\mathbf{O}\cdot(\partial_x\boldsymbol{\Phi}^*)\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k] \\
&+10[\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\Phi}\cdot\mathbf{O}\cdot\boldsymbol{\Phi}^*\cdot\mathbf{O}\cdot\boldsymbol{\phi}_k].
\end{aligned} \tag{B.5b}
$$

Appendix C

In this appendix we give the explicit expressions for the vectors $\boldsymbol{Y}_k^{(p)}$ and $\boldsymbol{Z}_k^{(p)}$, for $p = 1, 3, 5$, defined by (4.25) and (4.26) in the case of the integral equation of type II. The results are

$$
p=1,\quad \boldsymbol{Y}_k^{(1)} = -\mathrm{i}\partial_x\boldsymbol{v}_k, \tag{C.1a}
$$

$$
\boldsymbol{Z}_k^{(1)} = \mathbf{V}\cdot\mathbf{O}\cdot\boldsymbol{v}_k, \tag{C.1b}
$$

$$p=3,\quad \boldsymbol{Y}_k^{(3)} = \mathrm{i}\partial_x^3\boldsymbol{v}_k - 3\mathrm{i}(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\boldsymbol{v}_k - 3\mathrm{i}\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\partial_x\boldsymbol{v}_k, \tag{C.2a}$$

$$\begin{aligned}\boldsymbol{Z}_k^{(3)} = {}& -\mathbf{V}\cdot\mathbf{O}\cdot\partial_x^2\boldsymbol{v}_k - (\partial_x^2\mathbf{V})\cdot\mathbf{O}\cdot\boldsymbol{v}_k + (\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\partial_x\boldsymbol{v}_k \\ & + 3\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\boldsymbol{v}_k,\end{aligned} \tag{C.2b}$$

$$\begin{aligned}p=5,\quad \boldsymbol{Y}_k^{(5)} = {}& -\mathrm{i}\partial_x^5\boldsymbol{v}_k + 5\mathrm{i}[(\partial_x^3\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\boldsymbol{v}_k + \mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\partial_x^3\boldsymbol{v}_k \\ & + (\partial_x^2\mathbf{V})\cdot\mathbf{O}\cdot(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\boldsymbol{v}_k + \mathbf{V}\cdot\mathbf{O}\cdot(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\partial_x^2\boldsymbol{v}_k \\ & + (\partial_x\mathbf{V})\cdot\mathbf{O}\cdot(\partial_x^2\mathbf{V})\cdot\mathbf{O}\cdot\boldsymbol{v}_k + \mathbf{V}\cdot\mathbf{O}\cdot(\partial_x^2\mathbf{V})\cdot\mathbf{O}\cdot\partial_x\boldsymbol{v}_k \\ & + 2(\partial_x^2\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\partial_x\boldsymbol{v}_k + 2(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\partial_x^2\boldsymbol{v}_k \\ & + 2(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\partial_x\boldsymbol{v}_k] \\ & - 10\mathrm{i}[(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\boldsymbol{v}_k \\ & + \mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\boldsymbol{v}_k \\ & + \mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\partial_x\boldsymbol{v}_k],\end{aligned} \tag{C.3a}$$

$$\begin{aligned}\boldsymbol{Z}_k^{(5)} = {}& (\partial_x^4\mathbf{V})\cdot\mathbf{O}\cdot\boldsymbol{v}_k + \mathbf{V}\cdot\mathbf{O}\cdot\partial_x^4\boldsymbol{v}_k - (\partial_x^3\mathbf{V})\cdot\mathbf{O}\cdot\partial_x\boldsymbol{v}_k \\ & - (\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\partial_x^3\boldsymbol{v}_k + (\partial_x^2\mathbf{V})\cdot\mathbf{O}\cdot\partial_x^2\boldsymbol{v}_k \\ & - 5[(\partial_x^2\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\boldsymbol{v}_k + \mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\partial_x^2\boldsymbol{v}_k \\ & + \mathbf{V}\cdot\mathbf{O}\cdot(\partial_x^2\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\boldsymbol{v}_k + \mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot(\partial_x^2\mathbf{V})\cdot\mathbf{O}\cdot\boldsymbol{v}_k \\ & + (\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\boldsymbol{v}_k \\ & + \mathbf{V}\cdot\mathbf{O}\cdot(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\partial_x\boldsymbol{v}_k \\ & - (\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\partial_x\boldsymbol{v}_k \\ & + \mathbf{V}\cdot\mathbf{O}\cdot(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot(\partial_x\mathbf{V})\cdot\mathbf{O}\cdot\boldsymbol{v}_k] \\ & + 10\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\mathbf{V}\cdot\mathbf{O}\cdot\boldsymbol{v}_k.\end{aligned} \tag{C.3b}$$

The expressions for the vectors $\boldsymbol{Y}_k^{(2)}$, $\boldsymbol{Z}_k^{(2)}$, $\boldsymbol{Y}_k^{(4)}$, $\boldsymbol{Z}_k^{(4)}$, which have been used in the derivation of (C.2) and (C.3), respectively, have no direct meaning, in connection with a PDE, and are not given here.

Appendix D

In this appendix we shall show how three coupled equations for $\phi_{0,0}$, $\phi_{1,0}$ and $\phi_{1,1}$ can be derived for negative values of r, ($\eta = 1$).

In fact, using (4.2) for $p = -1$, we have

$$\mathbf{J}\cdot\tilde{\mathbf{O}} = \mathbf{O}\cdot\mathbf{J},\quad \mathbf{J}^{\mathrm{T}}\cdot\mathbf{O} = \tilde{\mathbf{O}}\cdot\mathbf{J}^{\mathrm{T}}, \tag{D.1}$$

where

$$\tilde{\mathbf{O}} = -\mathbf{Q}_{-1} = \mathbf{J}^{\mathrm{T}}\cdot\mathbf{O}\cdot\mathbf{J},\quad (\tilde{\mathbf{O}})_{m,n} = \delta_{m,1}\delta_{n,1}. \tag{D.2}$$

Inserting (D.2) in (4.2) we obtain a relation connecting p and $p+1$, viz.

$$\mathbf{Q}_p = \mathbf{Q}_{p+1} \cdot \mathbf{J} - \mathbf{J}^{p+1} \cdot \tilde{\mathbf{O}} = \mathbf{J}^{\mathrm{T}} \cdot \mathbf{Q}_{p+1} - \tilde{\mathbf{O}} \cdot \mathbf{J}^{\mathrm{T}p+1}. \tag{D.3}$$

Hence eq. (4.4), which has been derived using (4.2) and which therefore is valid for negative values of p as well, together with (4.5) and (4.6), leads to

$$\begin{aligned} \boldsymbol{F}_k^{(p)} = {} & -\mathrm{i}\partial_x \boldsymbol{F}_k^{(p-1)} - \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^* \cdot (\mathbf{J}^{\mathrm{T}} \cdot \mathbf{Q}_p - \tilde{\mathbf{O}} \cdot \mathbf{J}^{\mathrm{T}p}) \cdot \boldsymbol{\phi}_k \\ & - \boldsymbol{\Phi} \cdot (\mathbf{Q}_p \cdot \mathbf{J} - \mathbf{J}^p \cdot \tilde{\mathbf{O}}) \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k. \end{aligned} \tag{D.4}$$

Next we use (2.35a) for $p = -1$, i.e.

$$\boldsymbol{\Phi} \cdot \mathbf{J}^{\mathrm{T}} = \mathbf{J} \cdot \boldsymbol{\Phi} + \boldsymbol{\Psi}^* \cdot \tilde{\mathbf{O}} \cdot \boldsymbol{\Phi} + \boldsymbol{\Phi} \cdot \tilde{\mathbf{O}} \cdot \boldsymbol{\Psi}, \tag{D.5}$$

giving

$$\begin{aligned} \boldsymbol{F}_k^{(p)} = {} & -\mathrm{i}\partial_x \boldsymbol{F}_k^{(p-1)} - \boldsymbol{\Phi} \cdot \mathbf{O} \cdot (\mathbf{J} \cdot \boldsymbol{\Phi}^* + \boldsymbol{\Psi} \cdot \tilde{\mathbf{O}} \cdot \boldsymbol{\Phi}^* + \boldsymbol{\Phi}^* \cdot \tilde{\mathbf{O}} \cdot \boldsymbol{\Psi}^*) \cdot \mathbf{Q}_p \cdot \boldsymbol{\phi}_k \\ & - \boldsymbol{\Phi} \cdot \mathbf{Q}_p \cdot (\boldsymbol{\Phi}^* \cdot \mathbf{J}^{\mathrm{T}} - \boldsymbol{\Psi} \cdot \tilde{\mathbf{O}} \cdot \boldsymbol{\Phi}^* - \boldsymbol{\Phi}^* \cdot \tilde{\mathbf{O}} \cdot \boldsymbol{\Psi}^*) \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k \\ & + \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^* \cdot \tilde{\mathbf{O}} \cdot \mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{\phi}_k + \boldsymbol{\Phi} \cdot \mathbf{J}^p \cdot \tilde{\mathbf{O}} \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k \\ = {} & -\mathrm{i}\partial_x \boldsymbol{F}_k^{(p-1)} - (\boldsymbol{\Phi} \cdot \mathbf{J} + \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \boldsymbol{\Psi}) \cdot \tilde{\mathbf{O}} \cdot \boldsymbol{\Phi}^* \cdot \mathbf{Q}_p \cdot \boldsymbol{\phi}_k \\ & + \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^* \cdot \tilde{\mathbf{O}} \cdot (\mathbf{J}^{\mathrm{T}p} \cdot \boldsymbol{\phi}_k - \boldsymbol{\Psi}^* \cdot \mathbf{Q}_p \cdot \boldsymbol{\phi}_k) \\ & - \boldsymbol{\Phi} \cdot \mathbf{Q}_p \cdot \boldsymbol{\Phi}^* \cdot \tilde{\mathbf{O}} \cdot (\mathbf{J}^{\mathrm{T}} \cdot \boldsymbol{\phi}_k - \boldsymbol{\Psi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k) \\ & + (\boldsymbol{\Phi} \cdot \mathbf{J}^p + \boldsymbol{\Phi} \cdot \mathbf{Q}_p \cdot \boldsymbol{\Psi}) \cdot \tilde{\mathbf{O}} \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k. \end{aligned} \tag{D.6}$$

Inserting (2.34c), (2.35c) with the algebraic relation (2.35a), (4.1), (4.8), (4.5) and (4.6), we have

$$\begin{aligned} \boldsymbol{F}_k^{(p)} = {} & -\mathrm{i}\partial_x \boldsymbol{F}_k^{(p-1)} - (-\mathrm{i}\partial_x \boldsymbol{\Phi}) \cdot \tilde{\mathbf{O}} \cdot \boldsymbol{G}_k^{(p)} - \mathbf{G}^{(p)*} \cdot \tilde{\mathbf{O}} \cdot (-\mathrm{i}\partial_x \boldsymbol{\phi}_k) \\ & + \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^* \cdot \tilde{\mathbf{O}} \cdot \boldsymbol{F}_k^{(p)} + \mathbf{F}^{(p)} \cdot \tilde{\mathbf{O}} \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k. \end{aligned} \tag{D.7}$$

Hence, using also (4.9), which again is valid for negative values of p as well, we finally get

$$\boldsymbol{F}_k^{(p-1)} = \tilde{\Omega}[\boldsymbol{F}_k^{(p)}], \tag{D.8}$$

in which the action of the operator $\tilde{\Omega}$ on an arbitrary vector $\boldsymbol{a}_k$ is defined by, cf. (4.12),

$$\begin{aligned} \tilde{\Omega}[\boldsymbol{a}_k] = \mathrm{i}\partial_x^{-1} [& \boldsymbol{a}_k + (-\mathrm{i}\partial_x \boldsymbol{\Phi}) \cdot \tilde{\mathbf{O}} \cdot [\mathrm{i}\partial_x^{-1}(\boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{a}_k - \mathbf{A}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k)] \\ & + [-\mathrm{i}\partial_x^{-1}(\boldsymbol{\Phi} \cdot \mathbf{O} \cdot \mathbf{A}^* - \mathbf{A} \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^*)] \cdot \tilde{\mathbf{O}} \cdot (-\mathrm{i}\partial_x \boldsymbol{\phi}_k) \\ & - \boldsymbol{\Phi} \cdot \mathbf{O} \cdot \boldsymbol{\Phi}^* \cdot \tilde{\mathbf{O}} \cdot \boldsymbol{a}_k - \mathbf{A} \cdot \tilde{\mathbf{O}} \cdot \boldsymbol{\Phi}^* \cdot \mathbf{O} \cdot \boldsymbol{\phi}_k]. \end{aligned} \tag{D.9}$$

The equation for $\boldsymbol{\Phi}$, cf. (4.16), in the case of negative r, can be expressed in the form

$$\mathrm{i}\partial_t \boldsymbol{\Phi} = \tilde{\Omega}^{|r|}[\boldsymbol{\Phi}], \tag{D.10}$$

in which the action of the operator $\hat{\Omega}$ on an arbitrary matrix $\mathbf{A}$ is defined by

$$\begin{aligned}\hat{\Omega}[\mathbf{A}] = \mathrm{i}\partial_x^{-1}[\mathbf{A} &+ (-\mathrm{i}\partial_x\mathbf{\Phi})\cdot\bar{\mathbf{O}}\cdot[\mathrm{i}\partial_x^{-1}(\mathbf{\Phi}^*\cdot\mathbf{O}\cdot\mathbf{A}-\mathbf{A}^*\cdot\mathbf{O}\cdot\mathbf{\Phi})]\\ &+[-\mathrm{i}\partial_x^{-1}(\mathbf{\Phi}\cdot\mathbf{O}\cdot\mathbf{A}^*-\mathbf{A}\cdot\mathbf{O}\cdot\mathbf{\Phi}^*)]\cdot\bar{\mathbf{O}}\cdot(-\mathrm{i}\partial_x\mathbf{\Phi})\\ &-\mathbf{\Phi}\cdot\mathbf{O}\cdot\mathbf{\Phi}^*\cdot\bar{\mathbf{O}}\cdot\mathbf{A}-\mathbf{A}\cdot\bar{\mathbf{O}}\cdot\mathbf{\Phi}^*\cdot\mathbf{O}\cdot\mathbf{\Phi}]. \end{aligned}\tag{D.11}$$

Hence, the right-hand side of (D.10) can be, in principle, evaluated starting with $\mathbf{A}=\mathbf{\Phi}$, and from (D.2) in combination with (2.24), it is clear that (D.10) will lead to 3 coupled equations containing only $\phi_{0,0}$, $\phi_{1,0}$ and $\phi_{1,1}$.

Appendix E

The Lax representations for the complex sine-Gordon equation can be expressed as

$$-\mathrm{i}\partial_x\boldsymbol{\chi}_k^{(n)} = \mathbf{U}^{(n)}\cdot\boldsymbol{\chi}_k^{(n)}, \tag{E.1a}$$

$$\mathrm{i}\partial_t\boldsymbol{\chi}_k^{(n)} = \mathbf{V}^{(n)}\cdot\boldsymbol{\chi}_k^{(n)}, \quad (n=0,1), \tag{E.1b}$$

where $\boldsymbol{\chi}_k^{(n)}$, for $n=0,1$, is a two-dimensional vector with components

$$\boldsymbol{\chi}_k^{(n)} = \begin{pmatrix}\chi_{k1}^{(n)}\\ \chi_{k2}^{(n)}\end{pmatrix} = \mathrm{e}^{-\frac{1}{2}\mathrm{i}(kx-k^{-1}t)}\begin{pmatrix}\phi_k^{(n)}\\ \psi_k^{(n)}\end{pmatrix}. \tag{E.2}$$

We shall now evaluate the 2×2-matrices $\mathbf{U}^{(0)}$, $\mathbf{V}^{(0)}$, $\mathbf{U}^{(1)}$ and $\mathbf{V}^{(1)}$. From (2.34c) and (2.34d), in combination with the $n=0$ component of (2.21a), we have

$$\begin{aligned}-\mathrm{i}\partial_x\phi_k^{(0)} &= k\phi_k^{(0)}+\phi_{0,0}\psi_k^{(0)},\\ -\mathrm{i}\partial_x\psi_k^{(0)} &= \phi_{0,0}^*\phi_k^{(0)},\end{aligned}\tag{E.3}$$

leading with (E.2) to eq. (E.1a) with

$$\mathbf{U}^{(0)} = \begin{pmatrix}\frac{1}{2}k & \phi_{0,0}\\ \phi_{0,0}^* & -\frac{1}{2}k\end{pmatrix}. \tag{E.4}$$

In order to find $\mathbf{V}^{(0)}$, we consider the $n=1$ component of the algebraic relations (2.21a), (2.21b), which can be expressed as

$$k\phi_k^{(1)} = (1-\psi_{1,0}^*)\phi_k^{(0)}-\phi_{1,0}\psi_k^{(0)}, \tag{E.5a}$$

$$k\psi_k^{(1)} = \phi_{1,0}^*\phi_k^{(0)}+(1-\psi_{1,0})\psi_k^{(0)}. \tag{E.5b}$$

Eqs. (E.5a) and (E.5b) provide a direct relation between the vectors $\boldsymbol{\chi}_k^{(1)}$ and $\boldsymbol{\chi}_k^{(0)}$, viz.

$$k\boldsymbol{\chi}_k^{(1)} = \mathbf{g}\cdot\boldsymbol{\chi}_k^{(0)}, \tag{E.6}$$

where

$$\mathbf{g} = \begin{pmatrix} 1-\psi^\dagger_{1,0} & -\phi_{1,0} \\ \phi^\dagger_{1,0} & 1-\psi_{1,0} \end{pmatrix} \tag{E.7}$$

is a unitary matrix. Taking the $n=0$ component of (2.34a) and (2.34b), we have

$$\mathrm{i}\partial_t \phi_k^{(0)} = (1-\psi_{1,0})\phi_k^{(1)}, \tag{E.8a}$$

$$\mathrm{i}\partial_t \psi_k^{(0)} = -\phi^\dagger_{1,0}\phi_k^{(1)}, \tag{E.8b}$$

which, in combination with (E.2) and (E.7), lead to

$$\left(\frac{1}{2k}+\mathrm{i}\partial_t\right)\chi_k^{(0)} = \begin{pmatrix} 1-\psi_{1,0} & 0 \\ -\phi^\dagger_{1,0} & 0 \end{pmatrix}\chi_k^{(1)}, \tag{E.9}$$

or to (E.1b) with

$$\mathbf{V}^{(0)} = \frac{1}{2k}\left\{2\begin{pmatrix} 1-\psi_{1,0} & 0 \\ -\phi^\dagger_{1,0} & 0 \end{pmatrix}\cdot\mathbf{g}-\mathbf{1}\right\} = \frac{1}{2k}\mathbf{g}^{-1}\cdot\boldsymbol{\sigma}_3\cdot\mathbf{g}, \tag{E.10}$$

where $\boldsymbol{\sigma}_3$ is the Pauli matrix $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$.

Using (6.4) and (6.5), the right-hand side of (E.10) can be evaluated to be

$$\mathbf{V}^{(0)} = \frac{1}{2k}\begin{pmatrix} (1-4|\partial_t\phi_{0,0}|^2)^{1/2} & 2\mathrm{i}\partial_t\phi_{0,0} \\ -2\mathrm{i}\partial_t\phi^*_{0,0} & -(1-4|\partial_t\phi_{0,0}|^2)^{1/2} \end{pmatrix}. \tag{E.11}$$

From (2.34c) and (2.34d), taking the $n=1$ component we have

$$-\mathrm{i}\partial_x\phi_k^{(1)} = (1-\psi^\dagger_{1,0})\phi_k^{(0)}, \tag{E.12a}$$

$$-\mathrm{i}\partial_x\psi_k^{(1)} = \phi^\dagger_{1,0}\phi_k^{(0)}, \tag{E.12b}$$

or

$$(\tfrac{1}{2}k-\mathrm{i}\partial_x)\chi_k^{(1)} = \begin{pmatrix} 1-\psi^\dagger_{1,0} & 0 \\ \phi^\dagger_{1,0} & 0 \end{pmatrix}\chi_k^{(0)}. \tag{E.13}$$

Eq. (E.13) leads to the first equation of (E.1) with

$$\mathbf{U}^{(1)} = \tfrac{1}{2}k\left\{\begin{pmatrix} 2(1-\psi^\dagger_{1,0}) & 0 \\ 2\phi^\dagger_{1,0} & 0 \end{pmatrix}\cdot\mathbf{g}^{-1}-\mathbf{1}\right\} = \tfrac{1}{2}k\mathbf{g}\cdot\boldsymbol{\sigma}_3\cdot\mathbf{g}^{-1}, \tag{E.14}$$

which with (5.12a) and (5.13) can be rewritten as

$$\mathbf{U}^{(1)} = \tfrac{1}{2}k\begin{pmatrix} (1-4|\partial_x\phi_{1,1}|^2)^{1/2} & -2\mathrm{i}\partial_x\phi_{1,1} \\ 2\mathrm{i}\partial_x\phi^\dagger_{1,1} & -(1-4|\partial_x\phi_{1,1}|^2)^{1/2} \end{pmatrix}. \tag{E.15}$$

Finally, from (2.34a), (2.34b) and (2.21a) for $p=-1$ we have

$$\mathrm{i}\partial_t\phi_k^{(1)} = \frac{1}{k}\phi_k^{(1)} - \phi_{1,1}\psi_k^{(1)}, \tag{E.16a}$$

$$\mathrm{i}\partial_t\psi_k^{(1)} = -\phi^\dagger_{1,1}\phi_k^{(1)}, \tag{E.16b}$$

or

$$\left(\frac{1}{2k}+\mathrm{i}\partial_t\right)\chi_k^{(1)} = \begin{pmatrix} k^{-1} & -\phi_{1,1} \\ -\phi^\dagger_{1,1} & 0 \end{pmatrix}\chi_k^{(1)}, \tag{E.17}$$

leading to (E.1b) with

$$\mathbf{V}^{(1)} = \begin{pmatrix} \frac{1}{2}k^{-1} & -\phi_{1,1} \\ -\phi^\dagger_{1,1} & -\frac{1}{2}k^{-1} \end{pmatrix}. \tag{E.18}$$

The gauge equivalence[13]) between the Lax representations (E.1), for $n=0$ and $n=1$, can be formulated in a straightforward way, using (E.6). We have, e.g.

$$\mathrm{i}\partial_t\chi_k^{(1)} = k^{-1}\mathrm{i}\partial_t(\mathbf{g}\cdot\chi_k^{(0)}) = \mathrm{i}(\partial_t\mathbf{g})\cdot\mathbf{g}^{-1}\cdot\chi_k^{(1)} + \mathbf{g}\cdot\mathbf{V}^{(0)}\cdot\mathbf{g}^{-1}\cdot\chi_k^{(1)}, \tag{E.19}$$

leading to

$$\mathbf{V}^{(1)} = \mathrm{i}(\partial_t\mathbf{g})\cdot\mathbf{g}^{-1} + \mathbf{g}\cdot\mathbf{V}^{(0)}\cdot\mathbf{g}^{-1}, \tag{E.20}$$

and in a similar way it can be shown that

$$\mathbf{U}^{(1)} = -\mathrm{i}(\partial_x\mathbf{g})\cdot\mathbf{g}^{-1} + \mathbf{g}\cdot\mathbf{U}^{(0)}\cdot\mathbf{g}^{-1}. \tag{E.21}$$

Appendix F

In order to derive (7.35), one can insert $q=\kappa\,\mathrm{e}^{\mathrm{i}\gamma}$, ($\kappa>0$, γ real) in (7.34). After taking the imaginary and real part one has

$$\tfrac{1}{2}\mathrm{i}\partial_t\kappa^2 = -\mathrm{i}\partial_x(\kappa^2\gamma'), \tag{F.1}$$

and

$$\dot\gamma = \frac{\kappa''}{\kappa} - 2\frac{\kappa'^2}{\kappa^2} - 3\gamma'^2, \tag{F.2}$$

where also dots and primes have been used to denote the differentiations with respect to x and t, respectively.

Introducing a real function f, so that

$$\kappa^2 = 2f', \tag{F.3}$$

we have from (F.1)

$$\gamma' = \frac{-\dot f}{2f'}, \tag{F.4}$$

and from (F.2)

$$\dot{\gamma} = \frac{f'''}{2f'} - \frac{3}{4}\left(\frac{f''^2 + \dot{f}^2}{f'^2}\right). \tag{F.5}$$

The potential MNLS (7.35) follows immediately from the compatibility relation $\partial_x \dot{\gamma} = \partial_t \gamma'$.

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CHAPTER III

LINEAR INTEGRAL EQUATIONS AND BÄCKLUND TRANSFORMATIONS

1. Introduction

Bäcklund transformations [1] for integrable nonlinear partial differential equations (PDE's) were discovered in the investigation of the sine-Gordon equation in the context of differential geometry, a hundred years ago. For a review, see ref. 2. In the past decade many new Bäcklund transformations (BT's) for PDE's were discovered [3)-14)]. In particular in refs. 5-7 BT's were derived from symmetry properties of the Lax representation. More recently BT's have been investigated using a singular transformation of the reflection coefficient in the Inverse Scattering Transform [8)-11)].

Very recently a connection between the Korteweg-de Vries equation (KdV) and a linear singular integral equation with arbitrary measure and contour was discovered by Fokas and Ablowitz [15)]. Extending their treatment we have investigated the singular integral equations corresponding to the Nonlinear Schrödinger equation (NLS), the Isotropic Heisenberg Spin Chain, the modified Korteweg-de Vries equation, the sine-Gordon equation, the Boussinesq equation etc. [16)-19)].

In the present paper we give a systematic method to derive BT's connecting two solutions of a given integral equation related by a singular transformation of the measure, as introduced in ref. 20, see also ref. 21. From these BT's for the integral equations, it is straightforward to derive the BT's for the corresponding PDE's and also to derive new singular integral equations corresponding to so-called modified PDE's. Starting from the integral equation for the modified PDE, the procedure can be repeated again, in principle, to obtain multi-modified PDE's. (For the reflection coefficient in the Inverse Scattering Transform similar transformations have been used, but our treatment is more general, due to the fact that the integral equation contains an arbitrary measure and contour.) A different treatment in which the Bäcklund transformation is used to obtain the first and second modified KdV equation has been given within the context of Hirota's method [22].

The outline of the present paper is as follows. In section 2 we derive the BT for the integral equation corresponding to the class of PDE's containing the KdV equation and we derive the singular integral equation for the modified Korteweg-de Vries equation (MKdV). In section 3 the BT for the MKdV, the modified modified KdV equation and the modified modified modified KdV equation are discussed. In section 4 we treat the class of the NLS equation and the integral equation and BT for the Anisotropic Heisenberg Spin Chain (AHSC). In section 5 the real and complex versions of the modified sine-Gordon equation are derived from the BT's for the real and complex sine-Gordon equation. Finally in section 6 it is shown how we can derive BT's for the wave functions in the spectral problem, leading e.g. to an alternative form of the BT for the AHSC.

2. The KdV class

In this section we start from the integral equation defining the KdV class and derive a (matrix) Bäcklund transformation using a singular transformation of the measure in the integral equation. From the (matrix) BT we also obtain a (matrix) modified PDE with the associated singular integral equation and linear problem. The integral equation for the (matrix) modified PDE turns out to be a generalization of the integral equation of type II for the MKdV class proposed in ref. 18.

2.1. *Integral equation and constitutive relations*

The integral equation defining the KdV class is

$$\underline{u}_k(x,t) + i\rho_k(x,t)\int_C d\lambda(\ell)\,\frac{\underline{u}_\ell(x,t)}{k+\ell} = \rho_k(x,t)\underline{c}_k\ , \tag{2.1}$$

from which the vector function $\underline{u}_k(x,t)$ with components $u_k^{(n)}$, n integer, should be solved as a function of the complex variable k. In eq. (2.1) $\underline{c}_k$ is a vector with components $(\underline{c}_k)_n = 1/k^n$, n integer, C is an arbitrary contour in the complex k-plane and $d\lambda(\ell)$ is an arbitrary measure. $\rho_k(x,t)$ is a plane-wave factor satisfying the linear differential equations

$$\begin{aligned}&-i\partial_x\rho_k(x,t) = k\rho_k(x,t),\quad i\partial_t\rho_k(x,t) = \omega(k)\rho_k(x,t)\ ,\\ &\omega(k) = \sum_r \lambda_r k^r\ ,\quad \lambda_r = 0 \qquad \text{for r even}\ ,\end{aligned} \tag{2.2}$$

$\omega(k)$ being the dispersion. The measure and the contour are to be chosen in such a way that the solution $\underline{u}_k(x,t)$ of eq. (2.1) is unique, see ref. 18, cf. also ref. 15.

From eqs. (2.1) and (2.2) one can derive the constitutive relations [18]

$$2i\partial_x \underline{u}_k = -k\underline{u}_k - \underline{\underline{J}}^T \cdot \underline{u}_k + i\underline{\underline{U}} \cdot \underline{\underline{O}} \cdot \underline{u}_k \,, \tag{2.3}$$

$$2i\partial_t \underline{u}_k = \omega(k)\underline{u}_k + \omega(\underline{\underline{J}}^T) \cdot \underline{u}_k - i\underline{\underline{U}} \cdot \underline{\underline{R}} \cdot \underline{u}_k \,, \tag{2.4}$$

$$k^p \underline{u}_k = \underline{\underline{J}}^{T^p} \cdot \underline{u}_k + i\underline{\underline{U}} \cdot \underline{\underline{R}}_p \cdot \underline{u}_k \,, \qquad (p \text{ even}) \,, \tag{2.5}$$

in which the (symmetric) matrix $\underline{\underline{U}}$ can be obtained from the dyadic $\underline{u}_k \underline{c}_k$ by an integration over the same contour that occurs in (2.1):

$$\underline{\underline{U}} \equiv \int_C d\lambda(k) \underline{u}_k \underline{c}_k \,, \tag{2.6}$$

and in which we have used the following notations

$$(\underline{\underline{J}})_{n,m} \equiv \delta_{m,n+1} \,, \quad (\underline{\underline{J}}^T)_{n,m} \equiv \delta_{n,m+1} \,, \quad (\underline{\underline{O}})_{n,m} \equiv \delta_{n,0}\delta_{m,0} \,, \tag{2.7}$$

$$\underline{\underline{R}}_r \equiv (\mathrm{sgn}\, r) \sum_{j=0}^{|r|-1} \underline{\underline{J}}^{j+\frac{1}{2}r-\frac{1}{2}|r|} \cdot \underline{\underline{O}} \cdot (-\underline{\underline{J}}^T)^{-j-1+\frac{1}{2}r+\frac{1}{2}|r|} \,, \quad \underline{\underline{R}} \equiv \sum_{r \text{ odd}} \lambda_r \underline{\underline{R}}_r \,. \tag{2.8}$$

2.2. *Matrix PDE*

In this subsection we recapitulate some results from ref. 18. Differentiating (2.3) with respect to x and using (2.5) we can derive

$$(k+i\partial_x) i\partial_x \underline{u}_k = -(\partial_x \underline{\underline{U}}) \cdot \underline{\underline{O}} \cdot \underline{u}_k \,. \tag{2.9}$$

Integrating (2.3) over the contour C, cf. (2.6), we obtain

$$2i\partial_x \underline{\underline{U}} = -\underline{\underline{U}} \cdot \underline{\underline{J}} - \underline{\underline{J}}^T \cdot \underline{\underline{U}} + i\underline{\underline{U}} \cdot \underline{\underline{O}} \cdot \underline{\underline{U}} \,, \tag{2.10}$$

which will be used in subsection 2.4.

From (2.3) - (2.5) one may derive various PDE's for different choices of $\omega(k)$. Taking as an example $\omega(k) = k^3$, we have [18]

$$(\partial_t - \partial_x^3)\underline{u}_k = -3(\partial_x \underline{\underline{U}}) \cdot \underline{\underline{O}} \cdot \partial_x \underline{u}_k \,, \tag{2.11}$$

which upon integration over the contour C, cf. eq. (2.6), yields the following matrix PDE

$$(\partial_t - \partial_x^3)\underline{\underline{U}} = -3(\partial_x \underline{\underline{U}})\cdot \underline{\underline{O}} \cdot (\partial_x \underline{\underline{U}}) \ . \tag{2.12}$$

The (0,0), (1,0) and (1,1) elements of $\underline{\underline{U}}$ obey respectively

$$(\partial_t - \partial_x^3) u_{0,0} = -3(\partial_x u_{0,0})^2 \ , \tag{2.13}$$

$$(\partial_t - \partial_x^3) u_{1,0} = -3 \frac{(\partial_x u_{1,0})\partial_x^2 u_{1,0}}{i+u_{1,0}} \ , \tag{2.14}$$

$$(\partial_t - \partial_x^3) u_{1,1} = 3 \frac{(\partial_x^2 u_{1,1})^2}{1-2\partial_x u_{1,1}} \ , \tag{2.15}$$

and the relations between the different elements are

$$\partial_x u_{0,0} = (i+u_{1,0})^{-1} \ \partial_x^2 u_{1,0} \ , \tag{2.16}$$

$$u_{1,0} = -i \pm [2\partial_x u_{1,1} - 1]^{\frac{1}{2}} \ . \tag{2.17}$$

Eq. (2.13) is the potential Korteweg-de Vries equation, i.e. $\partial_x u_{0,0}$ satisfies the KdV; eq. (2.14) is equivalent to the potential modified Korteweg-de Vries equation, i.e. $v = \partial_x \ln (i+u_{1,0})$ satisfies $\partial_t v - \partial_x^3 v + 6v^2 \partial_x v = 0$, and (2.15) is equivalent to the MKdV.

The special case $\omega(k) = k^{-1}$ will be discussed in section 5.

2.3. *Singular transformation of the measure*

We introduce the singular transformation of the measure, cf. ref. 23,

$$d\lambda(k) \to d\tilde{\lambda}(k) = \frac{p-k}{p+k} \ d\lambda(k) \ , \tag{2.18}$$

where p is a complex parameter, and consider the corresponding solution $\tilde{\underline{u}}_k(x,t)$ of the integral equation (2.1) with $d\lambda(\ell)$ replaced by $d\tilde{\lambda}(\ell)$, i.e.

$$\tilde{\underline{u}}_k + i\rho_k \int_C d\tilde{\lambda}(\ell) \frac{\tilde{\underline{u}}_\ell}{k+\ell} = \rho_k \underline{c}_k \ , \tag{2.19}$$

$$\tilde{\underline{\underline{U}}} \equiv \int_C d\tilde{\lambda}(\ell) \tilde{\underline{u}}_\ell \underline{c}_\ell \ . \tag{2.20}$$

In eq. (2.18) it is understood that $d\tilde{\lambda}(k)$ is such that the solution of the integral equation (2.19) is also unique, and the contour should not pass through p and -p. In appendix A it will be argued that (2.18) increases the number of solitons by one, and another way of getting the BT will be presented.

Starting from (2.18) one can derive a relation between $\tilde{\underline{\underline{U}}}$ and $\underline{\underline{U}}$ which is a matrix generalization of the well-known BT for the KdV. In fact, using (2.18), (2.19) and the decomposition into partial fractions

$$\frac{p-k}{(p+\ell)(k+\ell)} = \frac{1}{k+\ell} - \frac{1}{p+\ell} , \tag{2.21}$$

we obtain

$$(p-k)\tilde{\underline{u}}_k + i\rho_k \int_C d\lambda(\ell)\, \frac{(p-\ell)\tilde{\underline{u}}_\ell}{k+\ell} = (p-k)\rho_k \underline{c}_k + i\rho_k \int_C d\lambda(\ell)\, \frac{(p-\ell)\tilde{\underline{u}}_\ell}{p+\ell} =$$

$$= (p-k)\rho_k \underline{c}_k + i\rho_k\, \tilde{\underline{\underline{U}}}\cdot\underline{\underline{O}}\cdot\underline{c}_k . \tag{2.22}$$

Taking into account that the homogeneous integral equation (2.1) has only the zero solution, we obtain the relation

$$(p-k)\tilde{\underline{u}}_k = p\underline{u}_k - \underline{\underline{J}}^T\cdot\underline{u}_k + i\tilde{\underline{\underline{U}}}\cdot\underline{\underline{O}}\cdot\underline{u}_k , \tag{2.23}$$

which may be regarded as the basic relation for the BT, and which will be used in the following subsections. Note that from (2.23) $\tilde{\underline{u}}_k$ can be expressed in terms of the vector $\underline{u}_k$, which is the solution of (2.1) with the measure $d\lambda(k)$, and various integrals of $\underline{u}_k$.

2.4. *Matrix Bäcklund transformation for $\underline{\underline{U}}$*

The inverse transformation of (2.18) can be obtained interchanging $p \leftrightarrow -p$, $d\lambda(k) \leftrightarrow d\tilde{\lambda}(k)$. From (2.23) we thus obtain the inverse relation

$$(-p-k)\underline{u}_k = -p\tilde{\underline{u}}_k - \underline{\underline{J}}^T\cdot\tilde{\underline{u}}_k + i\underline{\underline{U}}\cdot\underline{\underline{O}}\cdot\tilde{\underline{u}}_k , \tag{2.24}$$

cf. (2.23) with $p \leftrightarrow -p$, $\underline{u}_k \leftrightarrow \tilde{\underline{u}}_k$, $\underline{\underline{U}} \leftrightarrow \tilde{\underline{\underline{U}}}$. Multiplying (2.23) by the vector $\underline{c}_k$ and integrating over the contour C with the measure $d\lambda(k)$, we obtain, taking into account that $(p-k)d\lambda(k) = (p+k)d\tilde{\lambda}(k)$:

$$p(\tilde{\underline{\underline{U}}}-\underline{\underline{U}}) = -\tilde{\underline{\underline{U}}}\cdot\underline{\underline{J}} - \underline{\underline{J}}^T\cdot\underline{\underline{U}} + i\tilde{\underline{\underline{U}}}\cdot\underline{\underline{O}}\cdot\underline{\underline{U}} . \tag{2.25}$$

Adding (2.25) and its inverse with $p \leftrightarrow -p$, $\underline{\underline{U}} \leftrightarrow \underline{\underline{\tilde{U}}}$, we find, after eliminating $\underline{\underline{J}}$ and $\underline{\underline{J}}^T$ with eq. (2.10)

$$2i\partial_x(\underline{\underline{\tilde{U}}}+\underline{\underline{U}}) = 2p(\underline{\underline{\tilde{U}}}-\underline{\underline{U}}) + i(\underline{\underline{\tilde{U}}}-\underline{\underline{U}})\cdot\underline{\underline{O}}\cdot(\underline{\underline{\tilde{U}}}-\underline{\underline{U}}) \ . \tag{2.26}$$

Eq. (2.26) is the spatial part of the matrix BT associated with the integral equation (2.1), and is independent of the dispersion $\omega(k)$. The time-dependent part of the matrix BT can be inferred from the matrix PDE and (2.26). In the special case $\omega(k) = k^3$, cf. (2.12)

$$\partial_t(\underline{\underline{\tilde{U}}}+\underline{\underline{U}}) = \partial_x^2\left[-ip(\underline{\underline{\tilde{U}}}-\underline{\underline{U}}) + \tfrac{1}{2}(\underline{\underline{\tilde{U}}}-\underline{\underline{U}})\cdot\underline{\underline{O}}\cdot(\underline{\underline{\tilde{U}}}-\underline{\underline{U}})\right] - 3(\partial_x\underline{\underline{U}})\cdot\underline{\underline{O}}\cdot\partial_x\underline{\underline{U}} - 3(\partial_x\underline{\underline{\tilde{U}}})\cdot\underline{\underline{O}}\cdot\partial_x\underline{\underline{\tilde{U}}} \ . \tag{2.27}$$

The (0,0) element of (2.26) and (2.27) reduces to the well-known BT for the (potential) KdV [3)-5)].

2.5. *Modified matrix PDE*

Introducing

$$\underline{\underline{U}}^- \equiv \underline{\underline{\tilde{U}}}-\underline{\underline{U}} \ , \tag{2.28}$$

we have from (2.26)

$$\partial_x\underline{\underline{U}} = -\tfrac{1}{2}\partial_x\underline{\underline{U}}^- - \tfrac{1}{2}ip\underline{\underline{U}}^- + \tfrac{1}{4}\underline{\underline{U}}^-\cdot\underline{\underline{O}}\cdot\underline{\underline{U}}^- \ , \tag{2.29}$$

and the PDE for the matrix $\underline{\underline{U}}^-$ can be derived inserting (2.28) and (2.29) in eq. (2.12) and its counterpart with $\underline{\underline{U}}\to\underline{\underline{\tilde{U}}}$

$$(\partial_t-\partial_x^3)\underline{\underline{U}}^- = \left[(3ip\underline{\underline{U}}^- - \tfrac{3}{2}\,\underline{\underline{U}}^-\cdot\underline{\underline{O}}\cdot\underline{\underline{U}}^-)\cdot\underline{\underline{O}}\cdot\partial_x\underline{\underline{U}}^-\right]^s \ , \tag{2.30}$$

in which the superscript s denotes the symmetrical part of a matrix, i.e. $B^s_{n,m} = \tfrac{1}{2}(B_{n,m}+B_{m,n})$ for an arbitrary matrix $\underline{\underline{B}}$. Eq. (2.30) is a completely integrable matrix PDE, which we call the modified matrix PDE of (2.12). In this paper the term modified PDE will be used to denote a PDE, the solutions of which are obtained by combining a solution of another PDE and its Bäcklund transform. The relation mapping a solution of the modified PDE on a solution of the original PDE will be called a Miura transformation [24)]. In the case under consideration eq. (2.29) is a matrix Miura transformation, mapping a solution $\underline{\underline{U}}^-$ of (2.30) on a solution $\underline{\underline{S}} \equiv \partial_x \underline{\underline{U}}$ of the matrix KdV $(\partial_t - \partial_x^3)\underline{\underline{S}} = -6(\underline{\underline{S}} \cdot \underline{\underline{O}} \cdot \partial_x \underline{\underline{S}})^S$. In this connection it may be noted that the Miura transformation (2.29), mapping a solution of (2.30) on the matrix KdV, remains valid if all the matrices $\underline{\underline{O}}$ are replaced by arbitrary constant symmetric matrices $\underline{\underline{P}}$, as can be checked by explicit calculation.

Taking the (0,0) element of (2.30) we have immediately

$$(\partial_t - \partial_x^3)u^-_{0,0} = 3ipu^-_{0,0}\partial_x u^-_{0,0} - \frac{3}{2}(u^-_{0,0})^2 \partial_x u^-_{0,0} \quad , \tag{2.31}$$

which is equivalent to the MKdV, since the term with p can be transformed away. For the (1,1) element we have obtained the PDE

$$(\partial_t - \partial_x^3)z = -\tfrac{1}{2}(\partial_x z)^3 + \frac{3}{2}p^2(\sinh^2 z)\partial_x z \ ,$$

$$z \equiv \operatorname{arsinh}\left[-ip^{-1}\ \partial_x \ln(u^-_{1,1} - ip^{-1})\right] \ , \tag{2.32}$$

cf. appendix B for some details. Eq. (2.32) has been given in ref. 5 and is a special case of the second modified KdV, cf. eq. (3.21) and refs. 22, 25 and 26.

2.6. *Integral equation and constitutive relations for the matrix modified PDE*

In subsection 2.5 we have shown that the matrix $\underline{\underline{U}}^- = \tilde{\underline{\underline{U}}} - \underline{\underline{U}}$ obeys a PDE. In this subsection we will derive a linear integral equation for this modified PDE. (This integral equation will be used as a starting point in section 3 to repeat the procedure and to derive the BT for the modified PDE, as well as the second and third modified PDE.) For this purpose we need a wave function $\underline{u}^-_k$, which upon integration yields the potential $\underline{\underline{U}}^-$.

Defining

$$\underline{u}^{\pm}_k = (p-k)\tilde{\underline{u}}_k \pm (p+k)\underline{u}_k \ , \tag{2.33}$$

with inverse

$$\underline{u}_k = \frac{\underline{u}^+_k - \underline{u}^-_k}{2(p+k)} \quad , \quad \tilde{\underline{u}}_k = \frac{\underline{u}^+_k + \underline{u}^-_k}{2(p-k)} \quad , \tag{2.34}$$

and defining the new measure

$$d\lambda_1(k) = \frac{d\tilde{\lambda}(k)}{p-k} = \frac{d\lambda(k)}{p+k} \quad , \tag{2.35}$$

we have

$$\underline{\underline{U}}^{\pm} \equiv \int_C d\lambda_1(k) \underline{u}_k^{\pm} \underline{c}_k = \underline{\underline{\tilde{U}}} \pm \underline{\underline{U}} \ , \tag{2.36}$$

in agreement with (2.28) for $\underline{\underline{U}}^-$. Two integral equations for $\underline{u}_k^+ + \varepsilon \underline{u}_k^-$, $\varepsilon=\pm1$, follow quite directly from (2.1) and its counterpart with $u_k \to \tilde{u}_k$, $d\lambda(k) \to d\tilde{\lambda}(k)$, expressing the measures $d\lambda(k)$ and $d\tilde{\lambda}(k)$ in terms of $d\lambda_1(k)$. The result is

$$(\underline{u}_k^+ + \varepsilon \underline{u}_k^-) + i(p-\varepsilon k)\rho_k \int_C d\lambda_1(\ell) \frac{\underline{u}_\ell^+ + \varepsilon \underline{u}_\ell^-}{k+\ell} = 2(p-\varepsilon k)\rho_k \underline{c}_k \ , \quad \varepsilon=\pm1 \ . \tag{2.37}$$

Eqs. (2.37) can be rewritten as

$$\underline{u}_k^- + \frac{i(p^2-k^2)}{k} \rho_k \int_C d\lambda_1(\ell) \frac{\underline{u}_\ell^+}{k+\ell} = \frac{2(p^2-k^2)}{k} \rho_k \underline{c}_k - \frac{p}{k} \underline{u}_k^+ \quad , \tag{2.38}$$

$$\underline{u}_k^+ + \frac{i(p^2-k^2)}{k} \rho_k \int_C d\lambda_1(\ell) \frac{\underline{u}_\ell^-}{k+\ell} = -\frac{p}{k} \underline{u}_k^- \ . \tag{2.39}$$

From (2.38) and (2.39) one can also obtain one single integral equation containing only $\underline{u}_k^-$ which reads

$$\underline{u}_k^- + \int_C d\lambda_1(\ell) \int_C d\lambda_1(\ell') \frac{(p^2+k\ell')}{(k+\ell')(\ell'+\ell)} \rho_k \rho_{\ell'} \underline{u}_\ell^-$$

$$+ \rho_k \left[ip \int_C d\lambda_1(\ell) \frac{\underline{u}_\ell^-}{\ell} - p^2 \int_C d\lambda_1(\ell) \int_C d\lambda_1(\ell') \frac{\rho_{\ell'} \underline{u}_\ell^-}{\ell'(\ell'+\ell)} \right] = -2k\rho_k \underline{c}_k \ . \tag{2.40}$$

The linear integral equation (2.40), together with eq. (2.36), provides in a direct way solutions of the matrix PDE given by (2.30).

It is also straightforward to derive the constitutive relations in terms of the vectors $\underline{u}_k^+$ and $\underline{u}_k^-$. In order to do so, one could start from the integral equations for $\underline{u}_k^+$ and $\underline{u}_k^-$ with an appropriate uniqueness condition, but it is less laborious to use the constitutive relations corresponding to eq. (2.1), cf. (2.3) - (2.5), and the Bäcklund relation (2.23), in combination with (2.33).

Multiplying (2.23) by (p+k) and the inverse relation (2.24) by (p-k), and adding and subtracting the result, with the use of (2.33) one obtains

$$k\underline{u}_k^+ = -2p\underline{u}_k^- - \underline{\underline{J}}^T\cdot\underline{u}_k^+ + \tfrac{1}{2}i\underline{\underline{U}}^+\cdot\underline{\underline{O}}\cdot\underline{u}_k^+ - \tfrac{1}{2}i\underline{\underline{U}}^-\cdot\underline{\underline{O}}\cdot\underline{u}_k^- \quad , \tag{2.41}$$

$$k\underline{u}_k^- = \underline{\underline{J}}^T\cdot\underline{u}_k^- + \tfrac{1}{2}i\underline{\underline{U}}^-\cdot\underline{\underline{O}}\cdot\underline{u}_k^+ - \tfrac{1}{2}i\underline{\underline{U}}^+\cdot\underline{\underline{O}}\cdot\underline{u}_k^- \; , \tag{2.42}$$

which can be generalized using the algebraic relation

$$k^b\underline{u}_k = \underline{\underline{J}}^{T^b}\cdot\underline{u}_k + i\underline{\underline{U}}\cdot\underline{\underline{R}}_b\cdot\underline{u}_k \; , \qquad (b\ \text{even}) \; , \tag{2.43}$$

given in eq. (7.3) of ref. 18 and in eq. (2.5) of the present paper, and eqs. (2.33), (2.41) and (2.42). The result is

$$\begin{aligned} k^b\underline{u}_k^+ = {} & (-\underline{\underline{J}}^T)^b\cdot\underline{u}_k^+ + \tfrac{1}{2}i\underline{\underline{U}}^+\cdot\underline{\underline{R}}_b\cdot\underline{u}_k^+ + \tfrac{1}{2}i(-1)^b\underline{\underline{U}}^-\cdot\underline{\underline{R}}_b\cdot\underline{u}_k^- \\ & -p\left(1-(-1)^b\right)\left(k^{b-1}\underline{u}_k^- - \tfrac{1}{2}i\underline{\underline{U}}^-\cdot\underline{\underline{R}}_{b-1}\cdot\underline{u}_k^+\right) \quad , \end{aligned} \tag{2.44}$$

$$k^b\underline{u}_k^- = \underline{\underline{J}}^{T^b}\cdot\underline{u}_k^- + \tfrac{1}{2}i(-1)^b\underline{\underline{U}}^+\cdot\underline{\underline{R}}_b\cdot\underline{u}_k^- + \tfrac{1}{2}i\underline{\underline{U}}^-\cdot\underline{\underline{R}}_b\cdot\underline{u}_k^+ - \tfrac{1}{2}ip\left(1-(-1)^b\right)\underline{\underline{U}}^-\cdot\underline{\underline{R}}_{b-1}\cdot\underline{u}_k^- \; ,$$

$$(b\ \text{integer}). \tag{2.45}$$

From (2.3), (2.23), (2.33) and (2.36) one finds

$$2i\partial_x\underline{u}_k = \underline{u}_k^- - i\underline{\underline{U}}^-\cdot\underline{\underline{O}}\cdot\underline{u}_k \quad . \tag{2.46}$$

Multiplying (2.46) by (p+k), and the inverse relation with $\underline{u}_k \to \underline{\tilde{u}}_k$, $\underline{\underline{U}}^- \to -\underline{\underline{U}}^-$, $\underline{u}_k^- \to \underline{u}_k^-$ by (p-k), and adding and subtracting one obtains

$$i\partial_x\underline{u}_k^+ = p\underline{u}_k^- + \tfrac{1}{2}i\underline{\underline{U}}^-\cdot\underline{\underline{O}}\cdot\underline{u}_k^- \quad , \tag{2.47}$$

$$i\partial_x\underline{u}_k^- = -k\underline{u}_k^- + \tfrac{1}{2}i\underline{\underline{U}}^-\cdot\underline{\underline{O}}\cdot\underline{u}_k^+ \; . \tag{2.48}$$

In an analogous way from (2.4) one derives the relations for the time derivatives, i.e.

$$i\partial_t\underline{u}_k^\pm = \tfrac{1}{2}\left(\omega(k) + \omega(\underline{\underline{J}}^T)\right)\cdot\underline{u}_k^\pm - \tfrac{1}{4}i\left(\underline{\underline{U}}^+\cdot\underline{\underline{R}}\cdot\underline{u}_k^\pm + \underline{\underline{U}}^-\cdot\underline{\underline{R}}\cdot\underline{u}_k^\mp\right) \; , \tag{2.49}$$

which in the special case $\omega(k)=k^3$ reduce to, cf. (2.11), (2.33) and (2.29),

$$(\partial_t-\partial_x^3)\underline{u}_k^{\pm} = \tfrac{3}{2}(ip\underline{\underline{U}}^- - \tfrac{1}{2}\underline{\underline{U}}^-\cdot\underline{\underline{O}}\cdot\underline{\underline{U}}^-)\cdot\underline{\underline{O}}\cdot\partial_x\underline{u}_k^{\pm} - \tfrac{3}{2}(\partial_x\underline{\underline{U}}^-)\cdot\underline{\underline{O}}\cdot\partial_x\underline{u}_k^{\mp} \,. \tag{2.50}$$

Eqs. (2.44), (2.45), (2.47) and (2.48), which are independent of $\omega(k)$, together with the two eqs. (2.50), in the case $\omega(k) = k^3$, form the constitutive relations corresponding to the modified matrix PDE (2.30).

<u>Remark</u>: For p=0, eqs. (2.38), (2.39) with (2.35) are equivalent to eqs. (3.1a) and (3.1b) of ref. 18, with $v_k^{(n)} = -\tfrac{1}{2}u_k^{-(n)}/k$, $w_k^{(n)} = +\tfrac{1}{2}iu_k^{+(n)}/k$, which define the integral equation of type II, describing the MKdV class. The constitutive relations (2.44), (2.45), (2.47) - (2.49) are generalizations to the case $p\neq 0$ of the relations (3.15), (3.19) and (3.25a) - (3.25d) in ref. 18. (The matrices $\underline{\underline{V}}$ and $\underline{\underline{W}}$ in ref. 18 correspond to $-\tfrac{1}{2}\underline{\underline{U}}^-$ and $\tfrac{1}{2}i\underline{\underline{U}}^+$ respectively in the special case p=0.)

3. <u>The generalized MKdV class and beyond</u>

In the preceding section we applied a singular transformation of the measure (2.18) to the linear integral equation associated with the matrix PDE (2.14) for the KdV class. We have shown that the transformation of the measure leads in a natural way to a Bäcklund transformation for the matrix PDE, as well as to a <u>modified</u> matrix PDE (2.30) with corresponding linear integral equation (2.40) and constitutive relations (2.44), (2.45), (2.47) - (2.49). This integral equation, which is equivalent to (2.37) defines the generalized MKdV class. In the present section the procedure will be applied again, to derive the BT for the modified matrix PDE, as well as a <u>second modified</u> matrix PDE with its linear integral equation and linear problem. At the end of the section we shall apply the scheme for the third time to derive the BT for the (0,0) element of the second modified matrix PDE, as well as the PDE for the (0,0) element of the third modified matrix PDE.

3.1. *Bäcklund transformation for the modified matrix PDE*

We introduce the singular transformation of the measure

$$d\lambda_1(k) \to d\tilde{\lambda}_1(k) = \frac{q-k}{q+k}\, d\lambda_1(k) \;, \tag{3.1}$$

and consider the corresponding solutions $\tilde{\underline{u}}_k^+ + \varepsilon\tilde{\underline{u}}_k^-$ of (2.37) with $d\lambda_1(\ell)$

replaced by $d\tilde{\lambda}_1(\ell)$, i.e.

$$(\tilde{\underline{u}}_k^+ + \varepsilon\tilde{\underline{u}}_k^-) + i(p-\varepsilon k)\rho_k \int_C d\tilde{\lambda}_1(\ell)\, \frac{\tilde{\underline{u}}_\ell^+ + \varepsilon\tilde{\underline{u}}_\ell^-}{k+\ell} = 2(p-\varepsilon k)\rho_k \underline{c}_k \ , \tag{3.2}$$

leading to

$$\tilde{\underline{\underline{U}}}^{\pm} = \int_C d\tilde{\lambda}_1(k)\tilde{\underline{u}}_k^{\pm}\underline{c}_k \ . \tag{3.3}$$

Eq. (3.2) has exactly the same form as eq. (2.19) with $\tilde{\underline{u}}_k \to \frac{1}{2}(\tilde{\underline{u}}_k^+ + \varepsilon\tilde{\underline{u}}_k^-)$ and $\rho_k \to (p-\varepsilon k)\rho_k$. This means that the BT for $\underline{\underline{U}}^+$ and $\underline{\underline{U}}^-$ can be obtained immediately from (2.23) for the KdV with $p\to q$. We thus have

$$(q-k)\left(\frac{\tilde{\underline{u}}_k^+ + \varepsilon\tilde{\underline{u}}_k^-}{2}\right) = q\left(\frac{\underline{u}_k^+ + \varepsilon\underline{u}_k^-}{2}\right) - \underline{\underline{J}}^T\cdot\left(\frac{\underline{u}_k^+ + \varepsilon\underline{u}_k^-}{2}\right) + i\left(\frac{\tilde{\underline{\underline{U}}}^+ + \varepsilon\tilde{\underline{\underline{U}}}^-}{2}\right)\cdot\underline{\underline{O}}\cdot\left(\frac{\underline{u}_k^+ + \varepsilon\underline{u}_k^-}{2}\right), \tag{3.4}$$

which can be expressed as

$$(q-k)\tilde{\underline{u}}_k^{\mp} = q\underline{u}_k^{\mp} - \underline{\underline{J}}^T\cdot\underline{u}_k^{\mp} + \tfrac{1}{2}i\tilde{\underline{\underline{U}}}^+\cdot\underline{\underline{O}}\cdot\underline{u}_k^{\mp} + \tfrac{1}{2}i\tilde{\underline{\underline{U}}}^-\cdot\underline{\underline{O}}\cdot\underline{u}_k^{\pm} \ . \tag{3.5}$$

Eq. (3.5), which is independent of $\omega(k)$, may be regarded as the basic relation of the BT of the modified matrix PDE. Multiplying (3.5) by $\underline{c}_k$ and integrating over $d\lambda_1(k)$, with the use of (3.1) to evaluate the left-hand side, we obtain

$$q(\tilde{\underline{\underline{U}}}^{\mp} - \underline{\underline{U}}^{\mp}) = -(\underline{\underline{J}}^T\cdot\underline{\underline{U}}^{\mp} + \tilde{\underline{\underline{U}}}^{\mp}\cdot\underline{\underline{J}}) + \tfrac{1}{2}i\tilde{\underline{\underline{U}}}^+\cdot\underline{\underline{O}}\cdot\underline{\underline{U}}^{\mp} + \tfrac{1}{2}i\tilde{\underline{\underline{U}}}^-\cdot\underline{\underline{O}}\cdot\underline{\underline{U}}^{\pm} \ . \tag{3.6}$$

From (3.1) - (3.3), cf. (2.37) and (2.36), it is clear that the inverse Bäcklund transformation can be obtained substituting $q \leftrightarrow -q$, $\underline{u}_k^{\pm} \leftrightarrow \tilde{\underline{u}}_k^{\pm}$, $\underline{\underline{U}}^{\pm} \leftrightarrow \tilde{\underline{\underline{U}}}^{\pm}$ in (3.5) and (3.6). From (3.6) and the inverse relation, in combination with the expressions

$$\underline{\underline{U}}^-\cdot\underline{\underline{J}} + \underline{\underline{J}}^T\cdot\underline{\underline{U}}^- = -2i\partial_x\underline{\underline{U}}^- + \tfrac{1}{2}i\underline{\underline{U}}^-\cdot\underline{\underline{O}}\cdot\underline{\underline{U}}^+ + \tfrac{1}{2}i\underline{\underline{U}}^+\cdot\underline{\underline{O}}\cdot\underline{\underline{U}}^-, \tag{3.7}$$

$$\underline{\underline{U}}^+\cdot\underline{\underline{J}} + \underline{\underline{J}}^T\cdot\underline{\underline{U}}^+ = -2p\underline{\underline{U}}^- + \tfrac{1}{2}i\underline{\underline{U}}^+\cdot\underline{\underline{O}}\cdot\underline{\underline{U}}^+ - \tfrac{1}{2}i\underline{\underline{U}}^-\cdot\underline{\underline{O}}\cdot\underline{\underline{U}}^- \ , \tag{3.8}$$

which follow from (2.41), (2.42) and (2.48) after integration over the contour C, it can be shown that

$$2i\partial_x(\tilde{\underline{\underline{U}}}^-+\underline{\underline{U}}^-) = 2q(\tilde{\underline{\underline{U}}}^--\underline{\underline{U}}^-) + \tfrac{1}{2}i(\tilde{\underline{\underline{U}}}^--\underline{\underline{U}}^-)\cdot\underline{\underline{Q}}\cdot(\tilde{\underline{\underline{U}}}^+-\underline{\underline{U}}^+) + \tfrac{1}{2}i(\tilde{\underline{\underline{U}}}^+-\underline{\underline{U}}^+)\cdot\underline{\underline{Q}}\cdot(\tilde{\underline{\underline{U}}}^--\underline{\underline{U}}^-), \quad (3.9)$$

$$2p(\tilde{\underline{\underline{U}}}^-+\underline{\underline{U}}^-) = 2q(\tilde{\underline{\underline{U}}}^+-\underline{\underline{U}}^+) - \tfrac{1}{2}i(\tilde{\underline{\underline{U}}}^-+\underline{\underline{U}}^-)\cdot\underline{\underline{Q}}\cdot(\tilde{\underline{\underline{U}}}^-+\underline{\underline{U}}^-) + \tfrac{1}{2}i(\tilde{\underline{\underline{U}}}^+-\underline{\underline{U}}^+)\cdot\underline{\underline{Q}}\cdot(\tilde{\underline{\underline{U}}}^+-\underline{\underline{U}}^+) \quad . (3.10)$$

Taking the (0,0) element of (3.10) one can solve

$$\tilde{u}^+_{0,0} - u^+_{0,0} = 2qi \mp i[4(q^2-p^2) - (\tilde{u}^-_{0,0} + u^-_{0,0} - 2pi)^2]^{\frac{1}{2}} \,. \quad (3.11)$$

Next from (3.10) and (3.11) one can evaluate the matrices $\underline{\underline{Q}}\cdot(\tilde{\underline{\underline{U}}}^+-\underline{\underline{U}}^+)$ and $(\tilde{\underline{\underline{U}}}^+-\underline{\underline{U}}^+)\cdot\underline{\underline{Q}}$ in terms of $\underline{\underline{U}}^-$ and $\tilde{\underline{\underline{U}}}^-$. Inserting the result in (3.9) one obtains

$$\begin{aligned} 2i\partial_x(\tilde{\underline{\underline{U}}}^-+\underline{\underline{U}}^-) &= 2q(\tilde{\underline{\underline{U}}}^--\underline{\underline{U}}^-) \\ &- 2\Big(q \mp \tfrac{1}{2}[4(q^2-p^2) - (\tilde{u}^-_{0,0}+u^-_{0,0}-2pi)^2]^{\frac{1}{2}}\Big)(\tilde{u}^-_{0,0}+u^-_{0,0})^{-1}\Big[(\tilde{\underline{\underline{U}}}^--\underline{\underline{U}}^-)\cdot\underline{\underline{Q}}\cdot(\tilde{\underline{\underline{U}}}^-+\underline{\underline{U}}^-)\Big]^s . \end{aligned} \quad (3.12)$$

Eq. (3.12) is the spatial part of the matrix BT for $\underline{\underline{U}}^-$, associated with the integral equation (2.40). Eq. (3.12) again is common to all modified matrix PDE's which can be derived for various choices of the dispersion $\omega(k)$. (For the special case $\omega(k) = k^3$ the modified matrix PDE is given by (2.31)). Taking the (0,0) element of (3.12) we have the BT for (2.31)

$$2i\partial_x(\tilde{u}^-_{0,0} + u^-_{0,0}) = \pm(\tilde{u}^-_{0,0} - u^-_{0,0})\,[\,4(q^2-p^2) - (\tilde{u}^-_{0,0} + u^-_{0,0} - 2pi)^2\,]^{\frac{1}{2}} \,, \quad (3.13)$$

which for p=0 reduces to the well-known BT for the MKdV [5].

3.2. *Second modified PDE*

Introducing the matrices

$$\underline{\underline{U}}^{-+} \equiv \tilde{\underline{\underline{U}}}^- + \underline{\underline{U}}^- \,, \qquad \underline{\underline{U}}^{--} \equiv \tilde{\underline{\underline{U}}}^- - \underline{\underline{U}}^- \,, \quad (3.14)$$

we have from (3.12)

$$2i\partial_x\underline{\underline{U}}^{-+} = 2q\underline{\underline{U}}^{--} - \frac{2\big(q \mp \frac{1}{2}[4(q^2-p^2) - (u^{-+}_{0,0} - 2pi)^2]^{\frac{1}{2}}\big)}{u^{-+}_{0,0}}(\underline{\underline{U}}^{--}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{-+})^s \quad . \quad (3.15)$$

The (0,0) element of $\underline{\underline{U}}^{--}$ follows from (3.15), i.e.

$$u^{--}_{0,0} = \pm\, 2i[4(q^2-p^2) - (u^{-+}_{0,0} - 2pi)^2]^{-\frac{1}{2}}\, \partial_x u^{-+}_{0,0}\ . \tag{3.16}$$

From (3.15) and (3.16) one can solve the matrices $\underline{\underline{U}}^{--}\cdot\underline{\underline{Q}}$ and $\underline{\underline{Q}}\cdot\underline{\underline{U}}^{--}$ in terms of $\underline{\underline{U}}^{-+}$. Inserting the result in (3.15) it follows that

$$2q\underline{\underline{U}}^{--} = -4q\underline{\underline{U}}^{-} + 2q\underline{\underline{U}}^{-+} = 2i\partial_x\underline{\underline{U}}^{-+} + \frac{q \mp \frac{1}{2}[4(q^2-p^2) - (u^{-+}_{0,0} - 2pi)^2]^{\frac{1}{2}}}{q \pm \frac{1}{2}[4(q^2-p^2) - (u^{-+}_{0,0} - 2pi)^2]^{\frac{1}{2}}} \times$$

$$\times\left[\frac{2i\partial_x(\underline{\underline{U}}^{-+}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{-+})}{u^{-+}_{0,0}} \pm \frac{4i(\partial_x u^{-+}_{0,0})}{u^{-+\,2}_{0,0}}\,\frac{\left(q \mp \frac{1}{2}[4(q^2-p^2) - (u^{-+}_{0,0} - 2pi)^2]^{\frac{1}{2}}\right)}{[4(q^2-p^2) - (u^{-+}_{0,0} - 2pi)^2]^{\frac{1}{2}}}\ \underline{\underline{U}}^{-+}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{-+}\right]. \tag{3.17}$$

Adding the matrix PDE (2.30) for $\underline{\underline{U}}^{-}$ and its counterpart with $\underline{\underline{U}}^{-} \to \tilde{\underline{\underline{U}}}^{-}$ and using (3.17) one can derive a matrix PDE for $\underline{\underline{U}}^{-+}$. Eq. (3.17) which is independent of $\omega(k)$, is for $q\neq 0$ a Miura transformation mapping a solution of the PDE for $\underline{\underline{U}}^{-+}$ on a solution of the PDE for $\underline{\underline{U}}^{-}$. Taking as an example $\omega(k) = k^3$, we obtain from (2.30)

$$\begin{aligned}(\partial_t - \partial_x^3)\underline{\underline{U}}^{-+} &= \tfrac{3}{2}ip\,[(\underline{\underline{U}}^{-+}\cdot\underline{\underline{Q}}\cdot\partial_x\underline{\underline{U}}^{-+})^s + (\underline{\underline{U}}^{--}\cdot\underline{\underline{Q}}\cdot\partial_x\underline{\underline{U}}^{--})^s\,] \\ &- \tfrac{3}{8}\,[(\underline{\underline{U}}^{-+}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{--} + \underline{\underline{U}}^{--}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{-+})\cdot\underline{\underline{Q}}\cdot\partial_x\underline{\underline{U}}^{--}\,]^s \\ &- \tfrac{3}{8}\,[(\underline{\underline{U}}^{-+}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{-+} + \underline{\underline{U}}^{--}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{--})\cdot\underline{\underline{Q}}\cdot\partial_x\underline{\underline{U}}^{-+}\,]^s\ ,\end{aligned} \tag{3.18}$$

in which the explicit expression for $\underline{\underline{U}}^{--}\cdot\underline{\underline{Q}}$ must be inserted. The resulting matrix PDE,in terms of $\underline{\underline{U}}^{-+}$ only,can be regarded as the second modified matrix PDE of (2.12).

For the (0,0) element of (3.18) we obtain, using (3.16),

$$(\partial_t - \partial_x^3)u^{-+}_{0,0} = \frac{3}{2}\partial_x\left[\frac{(u^{-+}_{0,0} - 2ip)(\partial_x u^{-+}_{0,0})^2}{4(q^2-p^2) - (u^{-+}_{0,0} - 2ip)^2} - \frac{1}{12}(u^{-+}_{0,0} - 2ip)^3 - p^2 u^{-+}_{0,0}\right], \tag{3.19}$$

which in terms of the new variable z, defined by

$$u^{-+}_{0,0} - 2ip \equiv i(q+p)e^{z} - i(q-p)e^{-z}\ , \tag{3.20}$$

can be written as

$$(\partial_t-\partial_x^3)z = -\tfrac{1}{2}(\partial_x z)^3 + \frac{3}{8}\Big[\,(p+q)^2\, e^{2z} + (p-q)^2\, e^{-2z} - 2(p^2+q^2)\Big]\partial_x z. \quad (3.21)$$

Eq. (3.21) is a completely integrable PDE, which can be called the second modified KdV and which has been treated in the literature [22), 25),26)], see also ref. 27, and for $q^2 \neq p^2$, eq. (3.21) may be reduced to eq. (2.32). The Miura transformation [22),25),26)] mapping a solution of (3.21) on a solution of (2.31) is given by

$$u^-_{0,0} \;=\; \partial_x z + \tfrac{1}{2}i\{(q+p)e^{z} - (q-p)e^{-z} + 2p\}\;, \quad (3.22)$$

as follows from (3.16) and (3.20).

3.3. *Integral equation and linear problem for second modified PDE*

Defining

$$\underline{u}_k^{\alpha\pm} \equiv (q-k)\tilde{\underline{u}}_k^{\alpha} \pm (q+k)\underline{u}_k^{\alpha} \quad , \quad \alpha = \pm \quad , \quad (3.23)$$

with inverse

$$\tilde{\underline{u}}_k^{\alpha} = \frac{\underline{u}_k^{\alpha+}+\underline{u}_k^{\alpha-}}{2(q-k)} \quad , \quad \underline{u}_k^{\alpha} = \frac{\underline{u}_k^{\alpha+}-\underline{u}_k^{\alpha-}}{2(q+k)} \quad , \quad (3.24)$$

and introducing a new measure $d\lambda_2(k)$ by

$$d\lambda_2(k) = \frac{d\tilde{\lambda}_1(k)}{q-k} = \frac{d\lambda_1(k)}{q+k} \quad , \quad (3.25)$$

we have

$$\underline{\underline{U}}^{\alpha\pm} \equiv \int_C d\lambda_2(k)\underline{u}_k^{\alpha\pm}\underline{c}_k = \tilde{\underline{\underline{U}}}^{\alpha} \pm \underline{\underline{U}}^{\alpha}. \quad (3.26)$$

It is now straightforward to derive two coupled linear integral equations for $\underline{u}_k^{-+}$ and $\underline{u}_k^{--}$. Inserting (3.24) into (2.40) and its counterpart with $\underline{u}_k^- \to \tilde{\underline{u}}_k^-$, $d\lambda_1(k) \to d\tilde{\lambda}_1(k)$, and changing to the new measure (3.25) it is straightforward to show that

$$u_{\underline{k}}^{-\pm} + \int_C d\lambda_2(\ell) \int_C d\lambda_2(\ell') \frac{(p^2+k\ell')(q^2+k\ell')}{(k+\ell')(\ell'+\ell)} \rho_k \rho_{\ell'} u_{\underline{\ell}}^{-\pm}$$

$$+ \rho_k \Big[i p \int_C d\lambda_2(\ell) \ell^{-1} (q u_{\underline{\ell}}^{-\pm} - k u_{\underline{\ell}}^{-\mp}) - q \int_C d\lambda_2(\ell) \int_C d\lambda_2(\ell') \frac{(p^2+k\ell')\rho_{\ell'}}{(\ell'+\ell)} u_{\underline{\ell}}^{-\mp}$$

$$- p^2 \int_C d\lambda_2(\ell) \int_C d\lambda_2(\ell') \frac{\rho_{\ell'} \{(q^2+k\ell') u_{\underline{\ell}}^{-\pm} - q(k+\ell') u_{\underline{\ell}}^{-\mp}\}}{\ell'(\ell'+\ell)} \Big]$$

$$= -2\rho_k c_{\underline{k}} k \,[(q-k) \pm (q+k)] \,, \qquad (3.27)$$

which are two coupled linear integral equations for $u_{\underline{k}}^{-+}$ and $u_{\underline{k}}^{--}$. The solutions of the PDE for $\underline{\underline{U}}^{-+}$ can be obtained from $u_{\underline{k}}^{-+}$ using the integration (3.26) and for any choice of contours and measure the (0,0) element $u_{0,0}^{-+}$ is a solution of (3.19).

It is now straightforward to derive the linear problem for $u_{0,0}^{-+}$, or equivalently for the function z given by (3.20). From (3.5), using also (2.41), (2.42), (2.48) and the definitions (3.23) and (3.26) we have

$$u_{\underline{k}}^{--} = 2i\partial_x u_{\underline{k}}^{-} + \tfrac{1}{2}i\underline{\underline{U}}^{+-}\cdot\underline{\underline{O}}\cdot u_{\underline{k}}^{-} + \tfrac{1}{2}i\underline{\underline{U}}^{--}\cdot\underline{\underline{O}}\cdot u_{\underline{k}}^{+} \,, \qquad (3.28)$$

$$u_{\underline{k}}^{+-} = 2p u_{\underline{k}}^{-} + \tfrac{1}{2}i\underline{\underline{U}}^{+-}\cdot\underline{\underline{O}}\cdot u_{\underline{k}}^{+} + \tfrac{1}{2}i\underline{\underline{U}}^{-+}\cdot\underline{\underline{O}}\cdot u_{\underline{k}}^{-} \,. \qquad (3.29)$$

Multiplying (3.28) and (3.29) by $(q+k)$, and the inverse relations with $u_{\underline{k}}^{\pm} \leftrightarrow \tilde{u}_{\underline{k}}^{\pm}$, $\underline{\underline{U}}^{\pm} \leftrightarrow \tilde{\underline{\underline{U}}}^{\pm}$ by $(q-k)$, and using (3.23) and (3.26), we have the following relations between $u_{\underline{k}}^{--}$, $u_{\underline{k}}^{-+}$, $u_{\underline{k}}^{+-}$ and $u_{\underline{k}}^{++}$

$$\big((q-k) + \alpha(q+k)\big)u_{\underline{k}}^{--} = 2i\partial_x u_{\underline{k}}^{-\alpha} - \tfrac{1}{2}i\underline{\underline{U}}^{+-}\cdot\underline{\underline{O}}\cdot u_{\underline{k}}^{-\bar{\alpha}} - \tfrac{1}{2}i\underline{\underline{U}}^{--}\cdot\underline{\underline{O}}\cdot u_{\underline{k}}^{+\bar{\alpha}} \,, \qquad (3.30)$$

$$\big((q-k) + \alpha(q+k)\big)u_{\underline{k}}^{+-} = 2p u_{\underline{k}}^{-\alpha} - \tfrac{1}{2}i\underline{\underline{U}}^{+-}\cdot\underline{\underline{O}}\cdot u_{\underline{k}}^{+\bar{\alpha}} + \tfrac{1}{2}i\underline{\underline{U}}^{-+}\cdot\underline{\underline{O}}\cdot u_{\underline{k}}^{-\alpha} \,, \qquad (3.31)$$

where $\alpha = \pm$, and $\bar{\alpha} = +$ if $\alpha = -$, and $\bar{\alpha} = -$ if $\alpha = +$.

From (2.47) and (2.48) and their Bäcklund transforms it can be shown that

$$(-k-i\partial_x)\underline{u}_{-k}^{-\alpha} = -\tfrac{1}{4}i\underline{\underline{U}}^{-+}\cdot\underline{\underline{Q}}\cdot\underline{u}_{-k}^{+\alpha} - \tfrac{1}{4}i\underline{\underline{U}}^{--}\cdot\underline{\underline{Q}}\cdot\underline{u}_{-k}^{+\bar{\alpha}} \quad , \tag{3.32}$$

$$i\partial_x\underline{u}_{-k}^{+\alpha} = p\underline{u}_{-k}^{-\alpha} + \tfrac{1}{4}i\underline{\underline{U}}^{-+}\cdot\underline{\underline{Q}}\cdot\underline{u}_{-k}^{-\alpha} + \tfrac{1}{4}i\underline{\underline{U}}^{--}\cdot\underline{\underline{Q}}\cdot\underline{u}_{-k}^{-\bar{\alpha}} \quad . \tag{3.33}$$

Eqs. (3.30) - (3.33) can be simplified using (3.11) (cf. (3.26)) for $u_{0,0}^{+-}$. From (3.31) one can then express $\underline{\underline{Q}}\cdot\underline{u}_{-k}^{+-}$ and $\underline{\underline{Q}}\cdot\underline{u}_{-k}^{++}$ as linear combinations of $\underline{\underline{Q}}\cdot\underline{u}_{-k}^{-+}$ and $\underline{\underline{Q}}\cdot\underline{u}_{-k}^{--}$. Inserting these linear combinations in (3.32) one obtains

$$(-k-i\partial_x)\underline{u}_{-k}^{-\alpha} = -\frac{(\tfrac{1}{4}iu_{0,0}^{-+}+p)}{u_{0,0}^{+-}}\ \underline{\underline{U}}^{-\alpha}\cdot\underline{\underline{Q}}\cdot\underline{u}_{-k}^{--} - \frac{(\tfrac{1}{4}iu_{0,0}^{+-}\underline{\underline{U}}^{-\bar{\alpha}}+k\underline{\underline{U}}^{-\alpha})\cdot\underline{\underline{Q}}\cdot\underline{u}_{-k}^{-+}}{u_{0,0}^{-+}} \ , \tag{3.34}$$

with

$$u_{0,0}^{+-} = 2qi \mp i\left[\,4(q^2-p^2) - (u_{0,0}^{-+}-2pi)^2\,\right]^{\frac{1}{2}} \quad , \tag{3.35}$$

and $\underline{\underline{U}}^{--}\cdot\underline{\underline{Q}}$ given by (3.15) and (3.16). Eqs. (3.34) are constitutive relations belonging to the linear problem associated with the matrix PDE (3.18) for $\underline{\underline{U}}^{-+}$. The relations (3.34) are independent of the dispersion $\omega(k)$ and are valid for any matrix PDE for $\underline{\underline{U}}^{-+}$ which can be derived for various $\omega(k)$. In addition to (3.34) one may also derive algebraic relations involving the matrices $\underline{\underline{J}}$ and $\underline{\underline{J}}^T$, but we do not go in further details.

For the special case $\omega(k)=k^3$, eq. (2.50) leads to (cf. (2.47))

$$\begin{aligned}(\partial_t-\partial_x^3)\underline{u}_{-k}^{-\alpha} = {}& \tfrac{3}{4}ip\left[\,\underline{\underline{U}}^{-+}\cdot\underline{\underline{Q}}\cdot\partial_x\underline{u}_{-k}^{-\alpha}+\underline{\underline{U}}^{--}\cdot\underline{\underline{Q}}\cdot\partial_x\underline{u}_{-k}^{-\bar{\alpha}} + (\partial_x\underline{\underline{U}}^{-+})\cdot\underline{\underline{Q}}\cdot\underline{u}_{-k}^{-\alpha}+(\partial_x\underline{\underline{U}}^{--})\cdot\underline{\underline{Q}}\cdot\underline{u}_{-k}^{-\bar{\alpha}}\,\right]\\ &-\tfrac{3}{16}(\underline{\underline{U}}^{-+}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{-+}+\underline{\underline{U}}^{--}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{--})\cdot\underline{\underline{Q}}\cdot\partial_x\underline{u}_{-k}^{-\alpha} - \tfrac{3}{16}(\partial_x\underline{\underline{U}}^{-+})\cdot\underline{\underline{Q}}\cdot(\underline{\underline{U}}^{-+}\cdot\underline{\underline{Q}}\cdot\underline{u}_{-k}^{-\alpha}+\underline{\underline{U}}^{--}\cdot\underline{\underline{Q}}\cdot\underline{u}_{-k}^{-\bar{\alpha}})\\ &-\tfrac{3}{16}(\underline{\underline{U}}^{-+}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{--}+\underline{\underline{U}}^{--}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{-+})\cdot\underline{\underline{Q}}\cdot\partial_x\underline{u}_{-k}^{-\bar{\alpha}} - \tfrac{3}{16}(\partial_x\underline{\underline{U}}^{--})\cdot\underline{\underline{Q}}\cdot(\underline{\underline{U}}^{--}\cdot\underline{\underline{Q}}\cdot\underline{u}_{-k}^{-\alpha}+\underline{\underline{U}}^{-+}\cdot\underline{\underline{Q}}\cdot\underline{u}_{-k}^{-\bar{\alpha}}).\end{aligned} \tag{3.36}$$

The n=0 components of (3.34) and (3.36) form the linear problem associated with the PDE (3.19) for $u_{0,0}^{-+}$, when we insert (3.35) and (3.16).

3.4. *Third modified PDE*

Using (3.23) - (3.25) in (2.37) we have the integral equations

$$(\underline{u}_k^{++}+\varepsilon_1\underline{u}_k^{-+}) + \varepsilon_2(\underline{u}_k^{+-}+\varepsilon_1\underline{u}_k^{--})$$

$$+ i(p-\varepsilon_1 k)(q-\varepsilon_2 k)\rho_k \int_C d\lambda_2(\ell)\, \frac{\{(\underline{u}_\ell^{++}+\varepsilon_1\underline{u}_\ell^{-+}) + \varepsilon_2(\underline{u}_\ell^{+-}+\varepsilon_1\underline{u}_\ell^{--})\}}{k+\ell}$$

$$= 4(p-\varepsilon_1 k)(q-\varepsilon_2 k)\rho_k \underline{c}_k\ , \quad (\varepsilon_1,\varepsilon_2=\pm1)\ . \tag{3.37}$$

Eqs. (3.37) for $\varepsilon_1,\varepsilon_2=\pm1$ have the same form as eq. (2.1) with $\underline{u}_k \to \frac{1}{4}[(\underline{u}_k^{++}+\varepsilon_1\underline{u}_k^{-+}) + \varepsilon_2(\underline{u}_k^{+-}+\varepsilon_1\underline{u}_k^{--})]$, $\rho_k \to (p-\varepsilon_1 k)(q-\varepsilon_2 k)\rho_k$.

Applying the singular transformation of the measure

$$d\tilde{\lambda}_2(k) = \frac{r-k}{r+k}\, d\lambda_2(k)\ , \tag{3.38}$$

and introducing the solutions $\tilde{\underline{u}}_k^{++}+\varepsilon_1\tilde{\underline{u}}_k^{-+} + \varepsilon_2(\tilde{\underline{u}}_k^{+-}+\varepsilon_1\tilde{\underline{u}}_k^{--})$ of (3.37) with $d\lambda_2(k)$ replaced by $d\tilde{\lambda}_2(k)$, we have the relations

$$(r-k)\left[\frac{\tilde{\underline{u}}_k^{++} + \varepsilon_1\tilde{\underline{u}}_k^{-+} + \varepsilon_2(\tilde{\underline{u}}_k^{+-}+\varepsilon_1\tilde{\underline{u}}_k^{--})}{4}\right]$$

$$= r\left[\frac{\underline{u}_k^{++} + \varepsilon_1\underline{u}_k^{-+} + \varepsilon_2(\underline{u}_k^{+-}+\varepsilon_1\underline{u}_k^{--})}{4}\right] - \underline{\underline{J}}^T\cdot\left[\frac{\underline{u}_k^{++} + \varepsilon_1\underline{u}_k^{-+} + \varepsilon_2(\underline{u}_k^{+-}+\varepsilon_1\underline{u}_k^{--})}{4}\right]$$

$$+ i\left[\frac{\tilde{\underline{\underline{U}}}^{++} + \varepsilon_1\tilde{\underline{\underline{U}}}^{-+} + \varepsilon_2(\tilde{\underline{\underline{U}}}^{+-}+\varepsilon_1\tilde{\underline{\underline{U}}}^{--})}{4}\right]\cdot\underline{\underline{O}}\cdot\left[\frac{\underline{u}_k^{++} + \varepsilon_1\underline{u}_k^{-+} + \varepsilon_2(\underline{u}_k^{+-}+\varepsilon_1\underline{u}_k^{--})}{4}\right]\ , \tag{3.39}$$

in which the four matrices $\tilde{\underline{\underline{U}}}^{\alpha\pm}$ with $\alpha=\pm$ are given by

$$\tilde{\underline{\underline{U}}}^{\alpha\pm} = \int_C d\tilde{\lambda}_2(k)\tilde{\underline{u}}_k^{\alpha\pm}\,\underline{c}_k\ . \tag{3.40}$$

From (3.39) it is straightforward to show that (cf. appendix C for some details of the derivation)

$$-\left[(\tilde{u}^{--}_{0,0}+u^{--}_{0,0})(\tilde{u}^{-+}_{0,0}-u^{-+}_{0,0})^{-1}\left([4(q^2-p^2)-(\tilde{u}^{-+}_{0,0}-2pi)^2]^{\frac{1}{2}}+[4(q^2-p^2)-(u^{-+}_{0,0}-2pi)^2]^{\frac{1}{2}}\right)\right]^2$$

$$=\left[[4(q^2-p^2)-(\tilde{u}^{-+}_{0,0}-2pi)^2]^{\frac{1}{2}}-[4(q^2-p^2)-(u^{-+}_{0,0}-2pi)^2]^{\frac{1}{2}}\right]^2$$

$$+(\tilde{u}^{--}_{0,0}+u^{--}_{0,0})^2+(\tilde{u}^{-+}_{0,0}+u^{-+}_{0,0}-4ip)^2+16(p^2-r^2)\ , \tag{3.41}$$

which in combination with (3.16) and its inverse with $u^{--}_{0,0}\to\tilde{u}^{--}_{0,0}$, $u^{-+}_{0,0}\to\tilde{u}^{-+}_{0,0}$ gives the BT for the PDE (3.19). The result can be further simplified introducing the variable z defined by (3.20). From (3.41), taking into account the relations

$$u^{--}_{0,0}=-2\partial_x z,\quad [4(q^2-p^2)-(u^{-+}_{0,0}-2pi)^2]^{\frac{1}{2}}\ =\ (q+p)e^{z}+(q-p)e^{-z}\ , \tag{3.42}$$

(cf. (3.16)), one obtains after some straightforward algebra

$$\left(\partial_x(\tilde{z}+z)\right)^2=\left(\sinh^2\tfrac{1}{2}(\tilde{z}-z)\right)\left[16(p^2-r^2)-\left\{(q+p)e^{\frac{1}{2}(\tilde{z}+z)}-(q-p)e^{-\frac{1}{2}(\tilde{z}+z)}\right\}^2\right], \tag{3.43}$$

which is the BT for the second modified KdV (3.21). The BT for the special case (2.32) has been treated by Hirota's method in ref. 28 and in the context of prolongation structures in ref. 29.

The third modified KdV can be derived solving $(\tilde{z}-z)$ from (3.43) and inserting the result into the right-hand side of $(\partial_t-\partial_x^3)(z+\tilde{z})$, as given by (3.21) and its counterpart with $z\to\tilde{z}$.

The final result is in terms of the variable $y=\frac{1}{2}(\tilde{z}+z)$

$$(\partial_t-\partial_x^3)y=-\tfrac{1}{2}(\partial_x y)^3+\tfrac{3}{8}\left\{(q+p)^2e^{2y}+(q-p)^2e^{-2y}-2(q^2+p^2)\right\}\partial_x y$$

$$+\ \frac{3}{2}D\partial_x\left[\frac{[(q+p)^2e^{2y}-(q-p)^2e^{-2y}](\partial_x y)^2}{1-D[(q+p)e^{y}-(q-p)e^{-y}]^2}\right]$$

$$-\ \frac{3}{2}D(\partial_x y)\left[\partial_x\left[\frac{1}{\sqrt{D}}\operatorname{arsinh}\left(\frac{2\sqrt{D}\ \partial_x y}{[1-D\{(q+p)e^{y}-(q-p)e^{-y}\}^2]^{\frac{1}{2}}}\right)\right]\right]^2,\quad D\equiv\frac{1}{16(p^2-r^2)}. \tag{3.44}$$

Eq. (3.44) is an integrable equation, since its solutions can be obtained from a linear integral equation, such as e.g. (3.27), with measures $d\lambda_2(k)$

and $d\tilde{\lambda}_2(k)$, taking into account (3.26) and (3.20). In the limit $r\to\infty$, $(D\to 0)$, (3.44) reduces to the second modified KdV (3.21). A different version of the third modified KdV, in terms of the variable $\frac{1}{2}(\tilde{z}-z)$ has been given in ref. 28.

From this point one might continue to derive the integral equation for the third modified PDE, apply a singular transformation of measures and obtain the BT for the third modified PDE, as well as the PDE associated with the next level of modification. The procedure is systematic in terms of (matrix) functions $\underline{\underline{U}}$ with a sequence of superscripts + and - that can be defined in an analogous way as in the preceding steps, cf. e.g. (2.36) and (3.14). It is not obvious, however, that the coupled PDE's for the different matrices $\underline{\underline{U}}$ will lead to interesting closed PDE's in terms of only one function and we shall not go in further details here.

4. The NLS class

In the preceding sections we have considered the integral equation for the KdV class and by singular transformations of the measure we have obtained Bäcklund transformations and the first-, second-, and third modified KdV equation. In the present section we start from the integral equation for the NLS class and in subsection 4.1 we review some results obtained in ref. 18, which will be used in the following. By a singular transformation of the measure we shall derive the (matrix) BT for the NLS, as well as the matrix modified PDE, together with the associated linear integral equation. The PDE for the (0,0) element of the matrix modified PDE will turn out to be equivalent to the equation of motion for the classical Heisenberg spin chain with uniaxial anisotropy, and at the end of the section a BT for the AHSC will be derived.

4.1. *Integral equation and matrix PDE*

For the NLS class we have the linear integral equation

$$\underline{\phi}_k(x,t) + \int_C d\lambda(\ell) \int_{C^*} d\lambda^*(\ell') \frac{\rho_k(x,t)\rho^*_{\ell'}(x,t)}{(k-\ell')(\ell'-\ell)} \underline{\phi}_\ell(x,t) = \rho_k(x,t)\underline{c}_k \, , \qquad (4.1)$$

where $\underline{\phi}_k$ and $\underline{c}_k$ are vectors with components $(\underline{\phi}_k)_n = \phi_k^{(n)}$, $(\underline{c}_k)_n = 1/k^n$, n being an integer, C and C^* are an arbitrary contour and its complex conjugate in the complex k-plane, $d\lambda(\ell)$ and $d\lambda^*(\ell')$ are an arbitrary measure and its complex conjugate, $\rho_k(x,t)$ is a plane-wave factor satisfying the linear differential equations

$$-i\partial_x\rho_k(x,t) = k\rho_k(x,t)\ , \quad i\partial_t\rho_k(x,t) = \omega(k)\rho_k(x,t)\ , \tag{4.2}$$

where $\omega(k) = \sum_r\lambda_r k^r$ (r integer, λ_r real) is the dispersion.

Eq. (4.1) can be rewritten as

$$\underline{\phi}_k + \int_{C^*} d\lambda^*(\ell')\frac{\rho_k}{(k-\ell')}\underline{\psi}^*_{\ell'} = \rho_k\underline{c}_k \tag{4.3a}$$

in which the vector $\underline{\psi}_k$ is defined by

$$\underline{\psi}_k - \int_{C^*} d\lambda^*(\ell')\frac{\rho_k}{(k-\ell')}\underline{\phi}^*_{\ell'} = 0\ , \tag{4.3b}$$

and $\underline{\psi}_k$ satisfies the linear integral equation

$$\underline{\psi}_k + \int_C d\lambda(\ell)\int_{C^*} d\lambda^*(\ell')\frac{\rho_k\rho^*_{\ell'}}{(k-\ell')(\ell'-\ell)}\underline{\psi}_\ell = \int_{C^*} d\lambda^*(\ell')\frac{\rho_k\rho^*_{\ell'}}{(k-\ell')}\underline{c}_{\ell'}\ . \tag{4.4}$$

From now on the choice of measures and contours will be restricted by the condition that the solution $\underline{\phi}_k$ of the integral equation (4.1) is unique

From eqs. (4.1) and (4.2), or alternatively from (4.3), (4.4) and (4.2), taking into account the uniqueness condition one can derive the constitutive relations

$$k^p\underline{\phi}_k = \underline{\underline{J}}^{T^p}\cdot\underline{\phi}_k - \underline{\underline{\Psi}}^*\cdot\underline{\underline{Q}}_p\cdot\underline{\phi}_k - \underline{\underline{\Phi}}\cdot\underline{\underline{Q}}_p\cdot\underline{\psi}_k\ , \tag{4.5}$$

$$k^p\underline{\psi}_k = \underline{\underline{J}}^{T^p}\cdot\underline{\psi}_k + \underline{\underline{\Phi}}^*\cdot\underline{\underline{Q}}_p\cdot\underline{\phi}_k - \underline{\underline{\Psi}}\cdot\underline{\underline{Q}}_p\cdot\underline{\psi}_k\ , \quad \text{(p integer)}, \tag{4.6}$$

$$-i\partial_x\underline{\phi}_k = k\underline{\phi}_k + \underline{\underline{\Phi}}\cdot\underline{\underline{Q}}\cdot\underline{\psi}_k\ , \tag{4.7}$$

$$-i\partial_x\underline{\psi}_k = \underline{\underline{\Phi}}^*\cdot\underline{\underline{Q}}\cdot\underline{\phi}_k\ , \tag{4.8}$$

$$i\partial_t\underline{\phi}_k = \omega(\underline{\underline{J}}^T)\cdot\underline{\phi}_k - \underline{\underline{\Psi}}^*\cdot\underline{\underline{Q}}\cdot\underline{\phi}_k\ , \tag{4.9}$$

$$i\partial_t \underline{\psi}_k = \underline{\underline{\Phi}}^* \cdot \underline{\underline{Q}} \cdot \underline{\phi}_k \ , \tag{4.10}$$

in which the matrices $\underline{\underline{J}}^T$ and $\underline{\underline{Q}}$ have been defined by (2.7) ,

$$\underline{\underline{Q}}_p \equiv (\operatorname{sgn} p) \sum_{j=0}^{|p|-1} \underline{\underline{J}}^{j+\frac{1}{2}p-\frac{1}{2}|p|} \cdot \underline{\underline{Q}} \cdot \underline{\underline{J}}^{T^{-j-1+\frac{1}{2}p+\frac{1}{2}|p|}} \ , \quad \underline{\underline{Q}} \equiv \sum_r \lambda_r \underline{\underline{Q}}_r \ . \tag{4.11}$$

The symmetric matrix $\underline{\underline{\Phi}} = \underline{\underline{\Phi}}^T$ and the antihermitean matrix $\underline{\underline{\Psi}}$ can be obtained from the vectors $\underline{\phi}_k$ and $\underline{\psi}_k$ by an integration over the same contour C that appears in the integral equation ,

$$\underline{\underline{\Phi}} = \int_C d\lambda(k) \underline{\phi}_k \underline{c}_{-k} \ , \qquad \underline{\underline{\Psi}} = \int_C d\lambda(k) \underline{\psi}_k \underline{c}_{-k} \ . \tag{4.12}$$

For dispersion of the type $\omega(k) = \sum_r \lambda_r k^r$, $\lambda_r = 0$ for $r<0$, one can derive a closed PDE for the matrix $\underline{\underline{\Phi}}$, containing only $\underline{\underline{\Phi}}$ and its derivatives and $\underline{\underline{Q}}$, but not the matrices $\underline{\underline{J}}$ and $\underline{\underline{J}}^T$. Taking as an example

$$\omega(k) = \lambda_2 k^2 + \lambda_3 k^3 \ , \tag{4.13}$$

we have e.g.

$$\begin{aligned} i\partial_t \underline{\phi}_k = & - \lambda_2 \partial_x^2 \underline{\phi}_k - 2\lambda_2 \underline{\underline{\Phi}} \cdot \underline{\underline{Q}} \cdot \underline{\underline{\Phi}}^* \cdot \underline{\underline{Q}} \cdot \underline{\phi}_k + i\lambda_3 \partial_x^3 \underline{\phi}_k \\ & + 3i\lambda_3 (\partial_x \underline{\underline{\Phi}}) \cdot \underline{\underline{Q}} \cdot \underline{\underline{\Phi}}^* \cdot \underline{\underline{Q}} \cdot \underline{\phi}_k + 3i\lambda_3 \underline{\underline{\Phi}} \cdot \underline{\underline{Q}} \cdot \underline{\underline{\Phi}}^* \cdot \underline{\underline{Q}} \cdot \partial_x \underline{\phi}_k \ , \end{aligned} \tag{4.14}$$

$$\begin{aligned} i\partial_t \underline{\psi}_k = & \ i\lambda_2 (\partial_x \underline{\underline{\Phi}}^*) \cdot \underline{\underline{Q}} \cdot \underline{\phi}_k - i\lambda_2 \underline{\underline{\Phi}}^* \cdot \underline{\underline{Q}} \cdot \partial_x \underline{\phi}_k - \lambda_3 (\partial_x^2 \underline{\underline{\Phi}}^*) \cdot \underline{\underline{Q}} \cdot \underline{\phi}_k \\ & - \lambda_3 \underline{\underline{\Phi}}^* \cdot \underline{\underline{Q}} \cdot \partial_x^2 \underline{\phi}_k + \lambda_3 (\partial_x \underline{\underline{\Phi}}^*) \cdot \underline{\underline{Q}} \cdot \partial_x \underline{\phi}_k + 3\lambda_3 \underline{\underline{\Phi}}^* \cdot \underline{\underline{Q}} \cdot \underline{\underline{\Phi}} \cdot \underline{\underline{Q}} \cdot \underline{\underline{\Phi}}^* \cdot \underline{\underline{Q}} \cdot \underline{\phi}_k \ . \end{aligned} \tag{4.15}$$

For the (0,0) element of $\underline{\underline{\Phi}}$ we obtain

$$i\partial_t \phi_{0,0} + \lambda_2 \partial_x^2 \phi_{0,0} - i\lambda_3 \partial_x^3 \phi_{0,0} = -2\lambda_2 |\phi_{0,0}|^2 \phi_{0,0} + 6i\lambda_3 |\phi_{0,0}|^2 \partial_x \phi_{0,0}, \tag{4.16}$$

which is Hirota's equation.

4.2. *Singular transformation of the measure*

We introduce the singular transformation of the measure, cf. (A.10)-(A.11),

$$d\lambda(k) \to d\tilde{\lambda}(k) = \frac{p-k}{p^*-k}\, d\lambda(k) \;, \tag{4.17}$$

and consider the solutions of (4.3a) and (4.3b) with the new measure, i.e.

$$\tilde{\underline{\phi}}_k + \int_{C^*} d\tilde{\lambda}^*(\ell')\, \frac{\rho_k}{k-\ell'}\, \tilde{\underline{\psi}}^*_{\ell'} = \rho_k \underline{c}_k \;, \tag{4.18}$$

$$\tilde{\underline{\psi}}_k - \int_{C^*} d\tilde{\lambda}^*(\ell')\, \frac{\rho_k}{k-\ell'}\, \tilde{\underline{\phi}}^*_{\ell'} = 0 \;. \tag{4.19}$$

Using (4.18) and (4.19) and a relation similar to (2.21) we have

$$(p-k)\tilde{\underline{\phi}}_k + \int_{C^*} d\lambda^*(\ell')\, \frac{\rho_k}{k-\ell'}\, (p^*-\ell')\tilde{\underline{\psi}}^*_{\ell'} = (p-k)\rho_k \underline{c}_k + \rho_k \tilde{\underline{\underline{\Psi}}}^* \cdot \underline{\underline{O}} \cdot \underline{c}_k \;, \tag{4.20}$$

$$(p-k)\tilde{\underline{\psi}}_k - \int_{C^*} d\lambda^*(\ell')\, \frac{\rho_k}{k-\ell'}\, (p^*-\ell')\tilde{\underline{\phi}}^*_{\ell'} = -\rho_k \tilde{\underline{\underline{\Phi}}}^* \cdot \underline{\underline{O}} \cdot \underline{c}_k \;, \tag{4.21}$$

in which the matrices $\tilde{\underline{\underline{\Phi}}}$ and $\tilde{\underline{\underline{\Psi}}}$ can be obtained from $\tilde{\underline{\phi}}_k$, $\tilde{\underline{\psi}}_k$ by an integration with the new measure, i.e.

$$\tilde{\underline{\underline{\Phi}}} = \int_C d\tilde{\lambda}(k)\tilde{\underline{\phi}}_k \underline{c}_k \;, \qquad \tilde{\underline{\underline{\Psi}}} = \int_C d\tilde{\lambda}(k)\tilde{\underline{\psi}}_k \underline{c}_k \;, \tag{4.22}$$

From (4.20) and (4.21) we derive

$$(p-k)\tilde{\underline{\phi}}_k + \int_C d\lambda(\ell) \int_{C^*} d\lambda^*(\ell')\, \frac{\rho_k \rho^*_{\ell'}}{(k-\ell')(\ell'-\ell)}\, (p-\ell)\tilde{\underline{\phi}}_\ell$$

$$= \tilde{\underline{\underline{\Phi}}} \cdot \underline{\underline{O}} \cdot \int_{C^*} d\lambda^*(\ell')\, \frac{\rho_k \rho^*_{\ell'}}{k-\ell'}\, \underline{c}_{\ell'} + (p-k)\rho_k \underline{c}_k + \rho_k \tilde{\underline{\underline{\Psi}}}^* \cdot \underline{\underline{O}} \cdot \underline{c}_k \;, \tag{4.23}$$

$$(p-k)\tilde{\underline{\psi}}_k + \int_C d\lambda(\ell) \int_{C^*} d\lambda^*(\ell') \frac{\rho_k \rho^*_{\ell'}}{(k-\ell')(\ell'-\ell)} (p-\ell)\tilde{\underline{\psi}}_\ell$$

$$= \int_{C^*} d\lambda^*(\ell') \frac{\rho_k \rho^*_{\ell'}}{k-\ell'} (p^*-\ell')\underline{c}_{\ell'} + \tilde{\underline{\underline{\Psi}}}\cdot\underline{\underline{Q}}\cdot \int_{C^*} d\lambda^*(\ell') \frac{\rho_k \rho^*_{\ell'}}{k-\ell'} \underline{c}_{\ell'} - \rho_k \tilde{\underline{\underline{\Phi}}}^*\cdot\underline{\underline{Q}}\cdot\underline{c}_k \; . \tag{4.24}$$

Taking into account that the homogeneous integral equation corresponding to (4.1) has only the zero solution, we obtain immediately

$$(p-k)\tilde{\underline{\phi}}_k = p\underline{\phi}_k - \underline{\underline{J}}^T\cdot\underline{\phi}_k + \tilde{\underline{\underline{\Psi}}}^*\cdot\underline{\underline{Q}}\cdot\underline{\phi}_k + \tilde{\underline{\underline{\Phi}}}\cdot\underline{\underline{Q}}\cdot\underline{\psi}_k \; , \tag{4.25}$$

$$(p-k)\tilde{\underline{\psi}}_k = p^*\underline{\psi}_k - \underline{\underline{J}}^T\cdot\underline{\psi}_k + \tilde{\underline{\underline{\Psi}}}\cdot\underline{\underline{Q}}\cdot\underline{\psi}_k - \tilde{\underline{\underline{\Phi}}}^*\cdot\underline{\underline{Q}}\cdot\underline{\phi}_k \; , \tag{4.26}$$

which are the basic relations for the BT (for the NLS class).

4.3. *Matrix Bäcklund transformation*

The inverse transformation of (4.17) can be obtained interchanging $p \leftrightarrow p^*$, $d\lambda(k) \leftrightarrow d\tilde{\lambda}(k)$. From (4.25) and (4.26) we thus obtain the inverse relations

$$(p^*-k)\underline{\phi}_k = p^*\tilde{\underline{\phi}}_k - \underline{\underline{J}}^T\cdot\tilde{\underline{\phi}}_k + \underline{\underline{\Psi}}^*\cdot\underline{\underline{Q}}\cdot\tilde{\underline{\phi}}_k + \underline{\underline{\Phi}}\cdot\underline{\underline{Q}}\cdot\tilde{\underline{\psi}}_k \; , \tag{4.27}$$

$$(p^*-k)\underline{\psi}_k = p\tilde{\underline{\psi}}_k - \underline{\underline{J}}^T\cdot\tilde{\underline{\psi}}_k + \underline{\underline{\Psi}}\cdot\underline{\underline{Q}}\cdot\tilde{\underline{\psi}}_k - \underline{\underline{\Phi}}^*\cdot\underline{\underline{Q}}\cdot\tilde{\underline{\phi}}_k \; . \tag{4.28}$$

Integrating (4.25) and (4.26) over the contour C with the measure $d\lambda(k)$ and using the relation $(p^*-k)d\tilde{\lambda}(k) = (p-k)d\lambda(k)$ to evaluate the integrals on the left-hand side we have

$$p^*\tilde{\underline{\underline{\Phi}}} - p\underline{\underline{\Phi}} = \tilde{\underline{\underline{\Phi}}}\cdot\underline{\underline{J}} - \underline{\underline{J}}^T\cdot\underline{\underline{\Phi}} + \tilde{\underline{\underline{\Phi}}}\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Psi}} + \tilde{\underline{\underline{\Psi}}}^*\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Phi}} \; , \tag{4.29}$$

$$p^*(\tilde{\underline{\underline{\Psi}}}-\underline{\underline{\Psi}}) = \tilde{\underline{\underline{\Psi}}}\cdot\underline{\underline{J}} - \underline{\underline{J}}^T\cdot\underline{\underline{\Psi}} + \tilde{\underline{\underline{\Psi}}}\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Psi}} - \tilde{\underline{\underline{\Phi}}}^*\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Phi}} \; . \tag{4.30}$$

The matrices $\underline{\underline{J}}$ and $\underline{\underline{J}}^T$ in (4.29) and (4.30) can be eliminated with the relations

$$-2i\partial_x\underline{\underline{\Phi}} = \underline{\underline{J}}^T\cdot\underline{\underline{\Phi}} + \underline{\underline{\Phi}}\cdot\underline{\underline{J}} + \underline{\underline{\Phi}}\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Psi}} - \underline{\underline{\Psi}}^*\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Phi}} \; , \tag{4.31}$$

and

$$\underline{\underline{\Psi}}\cdot\underline{\underline{J}} - \underline{\underline{J}}^T\cdot\underline{\underline{\Psi}} + \underline{\underline{\Psi}}\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Psi}} - \underline{\underline{\Phi}}^*\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Phi}} = 0 \,, \tag{4.32}$$

which follow from (4.5), (4.7) and (4.6) resp. after integration over C. Using eqs. (4.29), (4.30) and the inverse equations with $p \leftrightarrow p^*$, $\underline{\underline{\Phi}} \leftrightarrow \underline{\underline{\tilde{\Phi}}}$, $\underline{\underline{\Psi}} \leftrightarrow \underline{\underline{\tilde{\Psi}}}$, in combination with (4.31) and (4.32), we obtain

$$2(p^*\underline{\underline{\tilde{\Phi}}} - p\underline{\underline{\Phi}}) = -2i\partial_x(\underline{\underline{\tilde{\Phi}}}-\underline{\underline{\Phi}}) + (\underline{\underline{\tilde{\Psi}}}^*-\underline{\underline{\Psi}}^*)\cdot\underline{\underline{Q}}\cdot(\underline{\underline{\tilde{\Phi}}}+\underline{\underline{\Phi}}) - (\underline{\underline{\tilde{\Phi}}}+\underline{\underline{\Phi}})\cdot\underline{\underline{Q}}\cdot(\underline{\underline{\tilde{\Psi}}}-\underline{\underline{\Psi}}) \,, \tag{4.33}$$

$$(p^*-p)(\underline{\underline{\tilde{\Psi}}}-\underline{\underline{\Psi}}) = (\underline{\underline{\tilde{\Phi}}}^*-\underline{\underline{\Phi}}^*)\cdot\underline{\underline{Q}}\cdot(\underline{\underline{\tilde{\Phi}}}-\underline{\underline{\Phi}}) - (\underline{\underline{\tilde{\Psi}}}-\underline{\underline{\Psi}})\cdot\underline{\underline{Q}}\cdot(\underline{\underline{\tilde{\Psi}}}-\underline{\underline{\Psi}}) \,. \tag{4.34}$$

From eqs. (4.33) and (4.34) it is straightforward to derive a relation containing only $\underline{\underline{\Phi}}$ and $\underline{\underline{\tilde{\Phi}}}$. In fact taking the (0,0) element of (4.34) we obtain a quadratic equation in $\tilde{\psi}_{0,0} - \psi_{0,0}$ which can be solved to give

$$\tilde{\psi}_{0,0} - \psi_{0,0} = \tfrac{1}{2}(p-p^*) \pm \frac{i}{2}\sqrt{|p-p^*|^2 - 4|\tilde{\phi}_{0,0} - \phi_{0,0}|^2} \,, \tag{4.35}$$

in which the left-hand side is imaginary as the matrix $\underline{\underline{\Psi}}$ is antihermitean. From (4.34) one can solve the matrix $\underline{\underline{Q}}\cdot(\underline{\underline{\tilde{\Psi}}}-\underline{\underline{\Psi}})$, and inserting the result in (4.33) we arrive at

$$\begin{aligned} 2i\partial_x(\underline{\underline{\tilde{\Phi}}}-\underline{\underline{\Phi}}) + (p^*+p)(\underline{\underline{\tilde{\Phi}}}-\underline{\underline{\Phi}}) \;&=\; -(p^*-p)(\underline{\underline{\tilde{\Phi}}}+\underline{\underline{\Phi}}) \\ &- 2\left(\tfrac{1}{2}(p-p^*) \pm \tfrac{1}{2}i\sqrt{|p-p^*|^2 - 4|\tilde{\phi}_{0,0}-\phi_{0,0}|^2}\right)(\tilde{\phi}_{0,0}-\phi_{0,0})^{-1}[(\underline{\underline{\tilde{\Phi}}}-\underline{\underline{\Phi}})\cdot\underline{\underline{Q}}\cdot(\underline{\underline{\tilde{\Phi}}}+\underline{\underline{\Phi}})]^S \,. \end{aligned} \tag{4.36}$$

Eq. (4.36) is the spatial part of the matrix Bäcklund transformation associated with the integral equation (4.1). It is independent of the dispersion $\omega(k)$ and is common to all matrix PDE's which can be derived for various $\omega(k)$. The time-dependent part of the BT can be inferred from the matrix PDE and (4.36).

4.4. *Modified matrix PDE*

Introducing

$$\underline{\underline{\Phi}}^{\pm} = \underline{\underline{\tilde{\Phi}}} \pm \underline{\underline{\Phi}} \,, \tag{4.37}$$

we have from the integrated version of (4.14) and its counterpart with $\underline{\underline{\Phi}} \to \underline{\underline{\tilde{\Phi}}}$

$$(i\partial_t \underline{\underline{\Phi}}^- + \lambda_2 \partial_x^2 \underline{\underline{\Phi}}^- - i\lambda_3 \partial_x^3 \underline{\underline{\Phi}}^-)$$

$$= -\tfrac{1}{2}\lambda_2(\underline{\underline{\Phi}}^-\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Phi}}^{-*} + \underline{\underline{\Phi}}^+\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Phi}}^{+*})\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Phi}}^- - \tfrac{1}{2}\lambda_2(\underline{\underline{\Phi}}^-\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Phi}}^{+*} + \underline{\underline{\Phi}}^+\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Phi}}^{-*})\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Phi}}^+$$

$$+ \frac{3}{2} i\lambda_3 \Big[(\underline{\underline{\Phi}}^-\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Phi}}^{-*} + \underline{\underline{\Phi}}^+\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Phi}}^{+*})\cdot\underline{\underline{Q}}\cdot\partial_x\underline{\underline{\Phi}}^- \Big]^s$$

$$+ \frac{3}{2} i\lambda_3 \Big[(\underline{\underline{\Phi}}^-\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Phi}}^{+*} + \underline{\underline{\Phi}}^+\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Phi}}^{-*})\cdot\underline{\underline{Q}}\cdot\partial_x\underline{\underline{\Phi}}^+ \Big]^s . \qquad (4.38)$$

From the matrix BT (4.36) it is straightforward to express $\underline{\underline{\Phi}}^+$ in terms of $\underline{\underline{\Phi}}^-$. From the (0,0) element we have

$$\phi^+_{0,0} = \mp 2\big(\partial_x\phi^-_{0,0} - \tfrac{1}{2}i(p+p^*)\phi^-_{0,0}\big)\Big/\sqrt{|p-p^*|^2 - 4|\phi^-_{0,0}|^2} \quad , \quad (p\neq p^*) . \quad (4.39)$$

Next one can solve the matrices $(\tilde{\underline{\underline{\Phi}}}+\underline{\underline{\Phi}})\cdot\underline{\underline{Q}}$ and $\underline{\underline{Q}}\cdot(\tilde{\underline{\underline{\Phi}}}+\underline{\underline{\Phi}})$ from (4.36). Inserting the result in (4.36) one obtains after some algebra

$$(p^*-p)\underline{\underline{\Phi}}^+ = (p^*-p)\underline{\underline{\Phi}}^- + 2(p^*-p)\underline{\underline{\Phi}} = -2i\partial_x\underline{\underline{\Phi}}^- - (p+p^*)\underline{\underline{\Phi}}^-$$

$$+2\{\phi^-_{0,0}|\phi^-_{0,0}|^2\}^{-1}\Big\{\tfrac{1}{2}(p-p^*) \pm \tfrac{1}{2}i\sqrt{|p-p^*|^2 - 4|\phi^-_{0,0}|^2}\Big\}^2 \times$$

$$\times \Big\{ i\partial_x(\underline{\underline{\Phi}}^-\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Phi}}^-) - i\big(\partial_x \ln \phi^-_{0,0} + \tfrac{1}{2}i(p+p^*)\big)\underline{\underline{\Phi}}^-\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Phi}}^-$$

$$\mp (p-p^*)\{|p-p^*|^2 - 4|\phi^-_{0,0}|^2\}^{-\frac{1}{2}}\big(\partial_x \ln \phi^-_{0,0} - \tfrac{1}{2}i(p+p^*)\big)\underline{\underline{\Phi}}^-\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Phi}}^- \Big\} . \qquad (4.40)$$

Inserting (4.40) in (4.38) we obtain the modified matrix PDE for $\underline{\underline{\Phi}}^-$, and if $p\neq p^*$ (4.40) is the Miura transformation mapping a solution of the PDE for $\underline{\underline{\Phi}}^-$ on a solution of the matrix PDE for $\underline{\underline{\Phi}}$. (In the special case (4.13) the PDE for $\underline{\underline{\Phi}}$ is the integrated version of (4.14), but (4.40) is independent of $\omega(k)$).

As an example we write down the (0,0) element of the matrix PDE in the case (4.13). The result is, cf. (4.38) and (4.39),

$$i\partial_t\phi^-_{0,0} + \lambda_2\partial^2_x\phi^-_{0,0} - i\lambda_3\partial^3_x\phi^-_{0,0}$$

$$= \left((-\tfrac{1}{2}\lambda_2 + \tfrac{3}{2}i\lambda_3\partial_x)\phi^-_{0,0}\right) \times \left(|\phi^-_{0,0}|^2 + \frac{|2\partial_x\phi^-_{0,0} - i(p+p^*)\phi^-_{0,0}|^2}{|p-p^*|^2 - 4|\phi^-_{0,0}|^2} \right)$$

$$+ \left\{ (-\tfrac{1}{2}\lambda_2 + \tfrac{3}{2}i\lambda_3\partial_x)\left(\frac{2\partial_x\phi^-_{0,0} - i(p+p^*)\phi^-_{0,0}}{\sqrt{|p-p^*|^2 - 4|\phi^-_{0,0}|^2}}\right)\right\} \times \frac{2\partial_x|\phi^-_{0,0}|^2}{\sqrt{|p-p^*|^2 - 4|\phi^-_{0,0}|^2}} , \tag{4.41}$$

and (4.39), with $\phi^+_{0,0} = \phi^-_{0,0} + 2\phi_{0,0}$ gives the Miura transformation mapping a solution of (4.41) on a solution of (4.16).

Taking $p^* \neq p$ and introducing the new variable

$$a(x,t) = 2|p-p^*|^{-1}\ \phi^-_{0,0}\left(x + \bar\lambda_2(p+p^*)t - \tfrac{3}{4}\lambda_3(p+p^*)^2 t, t\right) \times$$

$$\times \exp\left[-\tfrac{1}{2}i(p+p^*)x - \tfrac{i}{4}(p+p^*)^3\lambda_3 t + \tfrac{i}{4}(p+p^*)^2\bar\lambda_2 t\ \right] ,$$

$$\bar\lambda_2 \equiv \lambda_2 + \tfrac{3}{2}(p+p^*) , \tag{4.42}$$

eq. (4.41) can be simplified to

$$(i\partial_t + \bar\lambda_2\partial^2_x - i\lambda_3\partial^3_x)a = \{(-\tfrac{1}{2}\bar\lambda_2 + \tfrac{3}{2}i\lambda_3\partial_x)a\} \times \left\{\tfrac{1}{4}|p-p^*|^2\ |a|^2 + \frac{|\partial_x a|^2}{1-|a|^2}\right\}$$

$$+ \left\{(-\tfrac{1}{2}\bar\lambda_2 + \tfrac{3}{2}i\lambda_3\partial_x)\frac{\partial_x a}{\sqrt{1-|a|^2}}\right\} \times \frac{\partial_x|a|^2}{\sqrt{1-|a|^2}} , \tag{4.43}$$

In the special case $\lambda_3=0$, $\bar\lambda_2=1$ eq. (4.43) is equivalent to the AHSC

$$\partial_t\vec S = \vec S \times \partial^2_x\vec S + \vec S \times \left(\tfrac{1}{4}|p-p^*|^2 S^z + b\right)\vec e^{\,z} , \qquad \vec S\cdot\vec S = 1 , \tag{4.44}$$

in which the polar angles of the spin vector $\vec S$, i.e.

$$\vec S = (\sin\theta\ \cos\alpha,\ \ \sin\theta\ \sin\alpha,\ \ \cos\theta) , \tag{4.45}$$

can be expressed as, cf. eqs. (3.19) and (3.20) of ref. 17,

$$\theta = \arcsin|a| \ , \quad \partial_x\alpha = \frac{\mathrm{Im}\ \partial_x \ln a}{[1-|a|^2]^{\frac{1}{2}}} \ ,$$

$$\partial_t\alpha = \frac{\mathrm{Im}\ \partial_t \ln a}{[1-|a|^2]^{\frac{1}{2}}} - \frac{1}{2}\frac{|\partial_x a|^2}{[1-|a|^2]^{3/2}} - \frac{1}{8}|p-p^*|^2\frac{(2-|a|^2)}{[1-|a|^2]^{\frac{1}{2}}} - b \ . \qquad (4.46)$$

The Miura transformation mapping a solution of the AHSC on a solution $\phi_{0,0}$ of the NLS has been worked out in ref. 17 and can be regarded as a direct generalization of the Lakshmanan result [30] for the isotropic case. In the special case $\bar{\lambda}_2=0$, $\lambda_3=1$, eq. (4.43) may be regarded as a complex version of eq. (3.19) with p=0.

4.5. *Integral equation and linear problem*

Defining

$$\underline{\phi}_k^{\pm} = (p-k)\underline{\tilde{\phi}}_k \pm (p^*-k)\underline{\phi}_k \ ,$$

$$\underline{\psi}_k^{\pm} = (p-k)\underline{\tilde{\psi}}_k \pm (p^*-k)\underline{\psi}_k \ , \qquad (4.47)$$

with inverse

$$\underline{\tilde{\phi}}_k = \frac{\underline{\phi}_k^+ + \underline{\phi}_k^-}{2(p-k)} \ , \qquad \underline{\phi}_k = \frac{\underline{\phi}_k^+ - \underline{\phi}_k^-}{2(p^*-k)} \ ,$$

$$\underline{\tilde{\psi}}_k = \frac{\underline{\psi}_k^+ + \underline{\psi}_k^-}{2(p-k)} \ , \qquad \underline{\psi}_k = \frac{\underline{\psi}_k^+ - \underline{\psi}_k^-}{2(p^*-k)} \ , \qquad (4.48)$$

and introducing a new measure $d\lambda_1(k)$ by

$$d\lambda_1(k) = \frac{d\tilde{\lambda}(k)}{p-k} = \frac{d\lambda(k)}{p^*-k} \ , \qquad (4.49)$$

it is easy to show that

$$\underline{\underline{\Phi}}^{\pm} = \int_C d\lambda_1(k)\underline{\phi}_k^{\pm}\underline{c}_k = \underline{\underline{\tilde{\Phi}}} \pm \underline{\underline{\Phi}} \ ,$$

$$\underline{\underline{\Psi}}^{\pm} = \int_C d\lambda_1(k)\underline{\psi}_k^{\pm}\underline{c}_k = \underline{\underline{\tilde{\Psi}}} \pm \underline{\underline{\Psi}} \ . \qquad (4.50)$$

From (4.1) and the associated equation with $\underline{\phi}_k \to \tilde{\underline{\phi}}_k$, $d\lambda(\ell) \to d\tilde{\lambda}(\ell)$, it is straightforward to derive two coupled integral equations for $\underline{\phi}_k^+$ and $\underline{\phi}_k^-$. In fact, inserting (4.48) in (4.1) and the associated equation, and changing to the new measure (4.49), we have

$$(\underline{\phi}_k^+ + \underline{\phi}_k^-) + (p-k) \int_C d\lambda_1(\ell) \int_{C^*} d\lambda_1^*(\ell') \frac{(p^*-\ell')\rho_k \rho_{\ell'}^*}{(k-\ell')(\ell'-\ell)} (\underline{\phi}_\ell^+ + \underline{\phi}_\ell^-) = 2(p-k)\rho_k \underline{c}_k ,$$

$$(\underline{\phi}_k^+ - \underline{\phi}_k^-) + (p^*-k) \int_C d\lambda_1(\ell) \int_{C^*} d\lambda_1^*(\ell') \frac{(p-\ell')\rho_k \rho_{\ell'}^*}{(k-\ell')(\ell'-\ell)} (\underline{\phi}_\ell^+ - \underline{\phi}_\ell^-) = 2(p^*-k)\rho_k \underline{c}_k , \tag{4.51}$$

leading to

$$\underline{\phi}_k^\pm + \int_C d\lambda_1(\ell) \int_{C^*} d\lambda_1^*(\ell') \frac{\left(|p|^2 + (k-p-p^*)\ell'\right)\rho_k \rho_{\ell'}^*}{(k-\ell')(\ell'-\ell)} \underline{\phi}_\ell^\pm$$

$$- \int_C d\lambda_1(\ell) \int_{C^*} d\lambda_1^*(\ell') \frac{\rho_k \rho_{\ell'}^*}{\ell'-\ell} \left[\tfrac{1}{2}(p^*+p)\underline{\phi}_\ell^\pm + \tfrac{1}{2}(p-p^*)\underline{\phi}_\ell^\mp \right]$$

$$= \left((p-k) \pm (p^*-k)\right)\rho_k \underline{c}_k . \tag{4.52}$$

Eqs. (4.52) are two coupled linear integral equations and $\underline{\underline{\phi}}^-$, i.e. the solution of the modified matrix PDE, can be obtained from $\underline{\phi}_k^-$ integrating over the contour C, as in (4.50). These equations which were introduced in a slightly different notation in ref. 20, give a complete linearization of the AHSC (eq. (4.41) with $\lambda_3=0$, $\lambda_2=1$) as well as of other modified PDE's which may be derived for other choices of $\omega(k)$. (In this context we note that the bilinearization of the Heisenberg spin chain with orthorhombic anisotropy was given in ref. 31, and very recently the Riemann-Hilbert problem was settled by Mikhailov [32].)

It is straightforward to derive the linear problem for $\phi_{0,0}^-$. From (4.5) - (4.7), (4.25), (4.26), (4.47) and (4.50) we have

$$\underline{\phi}_k^+ = 2i\partial_x \underline{\phi}_k + (p+p^*)\underline{\phi}_k + \underline{\underline{\Psi}}^{-*} \cdot \underline{\underline{Q}} \cdot \underline{\phi}_k + \underline{\underline{\phi}}^+ \cdot \underline{\underline{Q}} \cdot \underline{\psi}_k , \tag{4.53}$$

$$\underline{\psi}_k^- = \underline{\underline{\Psi}}^- \cdot \underline{\underline{Q}} \cdot \underline{\psi}_k - \underline{\underline{\phi}}^{-*} \cdot \underline{\underline{Q}} \cdot \underline{\phi}_k . \tag{4.54}$$

Multiplying (4.53) and (4.54) by (p^*-k) and using also the inverse relations with $p \leftrightarrow p^*$, $\underline{\phi}_k \leftrightarrow \tilde{\underline{\phi}}_k$, $\underline{\psi}_k \leftrightarrow \tilde{\underline{\psi}}_k$, $\underline{\underline{\Phi}} \leftrightarrow \tilde{\underline{\underline{\Phi}}}$. $\underline{\underline{\Psi}} \leftrightarrow \tilde{\underline{\underline{\Psi}}}$, it can easily be shown that

$$(p-p^*)\underline{\phi}_k^+ = 2i\partial_x \underline{\phi}_k^- + (p+p^*)\underline{\phi}_k^- - \underline{\underline{\Psi}}^{-*}\cdot\underline{\underline{Q}}\cdot\underline{\phi}_k^+ + \underline{\underline{\Phi}}^+\cdot\underline{\underline{Q}}\cdot\underline{\psi}_k^- ,$$

$$-2k\underline{\phi}_k^+ = 2i\partial_x \underline{\phi}_k^+ - \underline{\underline{\Psi}}^{-*}\cdot\underline{\underline{Q}}\cdot\underline{\phi}_k^- + \underline{\underline{\Phi}}^+\cdot\underline{\underline{Q}}\cdot\underline{\psi}_k^+ \quad , \tag{4.55}$$

and

$$(p-p^*)\underline{\psi}_k^- = \underline{\underline{\Psi}}^-\cdot\underline{\underline{Q}}\cdot\underline{\psi}_k^- - \underline{\underline{\Phi}}^{-*}\cdot\underline{\underline{Q}}\cdot\underline{\phi}_k^- \quad ,$$

$$-2k\underline{\psi}_k^- = \underline{\underline{\Psi}}^-\cdot\underline{\underline{Q}}\cdot\underline{\psi}_k^+ - \underline{\underline{\Phi}}^{-*}\cdot\underline{\underline{Q}}\cdot\underline{\phi}_k^+ - (p+p^*)\underline{\psi}_k^- \quad . \tag{4.56}$$

From (4.7) and (4.8) and their counterparts with $\underline{\phi}_k \rightarrow \tilde{\underline{\phi}}_k$ etc., it can be shown that

$$(-k-i\partial_x)\underline{\phi}_k^\mp = \tfrac{1}{2}\underline{\underline{\Phi}}^+\cdot\underline{\underline{Q}}\cdot\underline{\psi}_k^\mp + \tfrac{1}{2}\underline{\underline{\Phi}}^-\cdot\underline{\underline{Q}}\cdot\underline{\psi}_k^\pm \quad , \tag{4.57}$$

$$-i\partial_x \underline{\psi}_k^\mp = \tfrac{1}{2}\underline{\underline{\Phi}}^{+*}\cdot\underline{\underline{Q}}\cdot\underline{\phi}_k^\mp + \tfrac{1}{2}\underline{\underline{\Phi}}^{-*}\cdot\underline{\underline{Q}}\cdot\underline{\phi}_k^\pm \quad . \tag{4.58}$$

Using (4.56) one can solve the vectors $\underline{\underline{Q}}\cdot\underline{\psi}_k^\mp$ as linear combinations of $\underline{\underline{Q}}\cdot\underline{\phi}_k^-$ and $\underline{\underline{Q}}\cdot\underline{\phi}_k^+$. Inserting the result into (4.57) one obtains after some straightforward algebra

$$(-k-i\partial_x)\underline{\phi}_k^\mp = \tfrac{1}{2}\left[\frac{\psi_{0,0}^- - (p-p^*)}{\phi_{0,0}^-}\right]\underline{\underline{\Phi}}^\mp\cdot\underline{\underline{Q}}\cdot\underline{\phi}_k^+$$

$$+ \tfrac{1}{2}\left[\frac{\psi_{0,0}^-}{\phi_{0,0}^-}\,\underline{\underline{\Phi}}^\pm + \left(\frac{p+p^*-2k}{\phi_{0,0}^-}\right)\underline{\underline{\Phi}}^\mp\right]\cdot\underline{\underline{Q}}\cdot\underline{\phi}_k^- \quad , \tag{4.59}$$

with $\psi_{0,0}^- = \tilde{\psi}_{0,0} - \psi_{0,0}$ given by (4.35) with $\tilde{\phi}_{0,0} - \phi_{0,0} \equiv \phi_{0,0}^-$.

Eq. (4.59) is independent of $\omega(k)$. For the special case (4.13) the time-dependent part of the linear problem of the AHSC can be inferred from (4.14) and its counterpart with $\underline{\underline{\Phi}} \rightarrow \tilde{\underline{\underline{\Phi}}}$, $\underline{\phi}_k \rightarrow \tilde{\underline{\phi}}_k$.

We have

$$(i\partial_t + \lambda_2\partial_x^2 - i\lambda_3\partial_x^3)\underline{\phi}_k^{\mp}$$

$$= -\tfrac{1}{2}\lambda_2\Big[\,(\underline{\underline{\Phi}}^{+}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Phi}}^{+*} + \underline{\underline{\Phi}}^{-}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Phi}}^{-*})\cdot\underline{\underline{O}}\cdot\underline{\phi}_k^{\mp} + (\underline{\underline{\Phi}}^{+}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Phi}}^{-*} + \underline{\underline{\Phi}}^{-}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Phi}}^{+*})\cdot\underline{\underline{O}}\cdot\underline{\phi}_k^{\pm}\,\Big]$$

$$+ \tfrac{3}{4}i\lambda_3\Big[(\partial_x\underline{\underline{\Phi}}^{+})\cdot\underline{\underline{O}}\cdot(\underline{\underline{\Phi}}^{+*}\cdot\underline{\underline{O}}\cdot\underline{\phi}_k^{\mp} + \underline{\underline{\Phi}}^{-*}\cdot\underline{\underline{O}}\cdot\underline{\phi}_k^{\pm}) + (\partial_x\underline{\underline{\Phi}}^{-})\cdot\underline{\underline{O}}\cdot(\underline{\underline{\Phi}}^{+*}\cdot\underline{\underline{O}}\cdot\underline{\phi}_k^{\pm} + \underline{\underline{\Phi}}^{-*}\cdot\underline{\underline{O}}\cdot\underline{\phi}_k^{\mp})$$

$$+ (\underline{\underline{\Phi}}^{+}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Phi}}^{+*} + \underline{\underline{\Phi}}^{-}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Phi}}^{-*})\cdot\underline{\underline{O}}\cdot\partial_x\underline{\phi}_k^{\mp} + (\underline{\underline{\Phi}}^{+}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Phi}}^{-*} + \underline{\underline{\Phi}}^{-}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Phi}}^{+*})\cdot\underline{\underline{O}}\cdot\partial_x\underline{\phi}_k^{\pm}\Big] \; . \quad (4.60)$$

It is also straightforward to derive the remaining constitutive relations for the matrix $\underline{\underline{\Phi}}^{-}$, but the results will not be presented here.

4.6. *Bäcklund transformation for the AHSC*

From (4.51) it is clear that $\frac{1}{2}(\underline{\phi}_k^{+} + \underline{\phi}_k^{-})$ and $\frac{1}{2}(\underline{\phi}_k^{+} - \underline{\phi}_k^{-})$ satisfy an integral equation of the form (4.1) with $d\lambda(\ell)$ replaced by $d\lambda_1(\ell)$ and ρ_k replaced by $(p-k)\rho_k$ and $(p^*-k)\rho_k$ respectively. Similarly it can be shown that $\frac{1}{2}(\underline{\psi}_k^{+} + \underline{\psi}_k^{-})$ and $\frac{1}{2}(\underline{\psi}_k^{+} - \underline{\psi}_k^{-})$ satisfy (4.4) with $d\lambda(\ell)$ replaced by $d\lambda_1(\ell)$ and ρ_k by $(p-k)\rho_k$ and $(p^*-k)\rho_k$.

We now apply the singular transformation of measures

$$d\lambda_1(k) \to d\tilde{\lambda}_1(k) = \frac{q-k}{q^*-k}\, d\lambda_1(k) \quad , \qquad (4.61)$$

and introduce the functions $\frac{1}{2}(\tilde{\underline{\phi}}_k^{+} \pm \tilde{\underline{\phi}}_k^{-})$, $\frac{1}{2}(\tilde{\underline{\psi}}_k^{+} \pm \tilde{\underline{\psi}}_k^{-})$ that are the solutions of the integral equations mentioned above with $d\lambda_1(\ell)$ replaced by $d\tilde{\lambda}_1(\ell)$.

From (4.25) and (4.26) we immediately obtain the basic relations of the BT viz.

$$\tfrac{1}{2}(q-k)(\tilde{\underline{\phi}}_k^{+} + \varepsilon\tilde{\underline{\phi}}_k^{-}) = \tfrac{1}{2}q(\underline{\phi}_k^{+} + \varepsilon\underline{\phi}_k^{-}) - \tfrac{1}{2}\underline{\underline{J}}^{T}\cdot(\underline{\phi}_k^{+} + \varepsilon\underline{\phi}_k^{-})$$

$$+ \tfrac{1}{4}(\tilde{\underline{\underline{\Psi}}}^{+*} + \varepsilon\tilde{\underline{\underline{\Psi}}}^{-*})\cdot\underline{\underline{O}}\cdot(\underline{\phi}_k^{+} + \varepsilon\underline{\phi}_k^{-}) + \tfrac{1}{4}(\tilde{\underline{\underline{\Phi}}}^{+} + \varepsilon\tilde{\underline{\underline{\Phi}}}^{-})\cdot\underline{\underline{O}}\cdot(\underline{\psi}_k^{+} + \varepsilon\underline{\psi}_k^{-}) \, , \quad (4.62)$$

$$\tfrac{1}{2}(q-k)(\tilde{\underline{\psi}}_k^{+} + \varepsilon\tilde{\underline{\psi}}_k^{-}) = \tfrac{1}{2}q^*(\underline{\psi}_k^{+} + \varepsilon\underline{\psi}_k^{-}) - \tfrac{1}{2}\underline{\underline{J}}^{T}\cdot(\underline{\psi}_k^{+} + \varepsilon\underline{\psi}_k^{-})$$

$$+ \tfrac{1}{4}(\tilde{\underline{\underline{\Psi}}}^{+} + \varepsilon\tilde{\underline{\underline{\Psi}}}^{-})\cdot\underline{\underline{O}}\cdot(\underline{\psi}_k^{+} + \varepsilon\underline{\psi}_k^{-}) - \tfrac{1}{4}(\tilde{\underline{\underline{\Phi}}}^{+*} + \varepsilon\tilde{\underline{\underline{\Phi}}}^{-*})\cdot\underline{\underline{O}}\cdot(\underline{\phi}_k^{+} + \varepsilon\underline{\phi}_k^{-}) \, ,$$

$$\varepsilon = \pm 1, \quad (4.63)$$

in which the matrices $\tilde{\underline{\underline{\Phi}}}^{\pm}$ and $\tilde{\underline{\underline{\Psi}}}^{\pm}$ are defined by

$$\tilde{\underline{\underline{\Phi}}}^{\pm} = \int_C d\tilde{\lambda}_1(k)\tilde{\underline{\phi}}_k^{\pm}\underline{c}_k \ , \qquad \tilde{\underline{\underline{\Psi}}}^{\pm} = \int_C d\tilde{\lambda}_1(k)\tilde{\underline{\psi}}_k^{\pm}\underline{c}_k \ . \tag{4.64}$$

From (4.62) and (4.63) one can derive the spatial part of the BT for the PDE (4.41).

The result is given by

$$- \left\{ \left(\frac{\partial_x\tilde{\phi}^-_{0,0} - \frac{1}{2}i(p+p^*)\tilde{\phi}^-_{0,0}}{[|p-p^*|^2 - 4|\tilde{\phi}^-_{0,0}|^2]^{\frac{1}{2}}} - \frac{\partial_x\phi^-_{0,0} - \frac{1}{2}i(p+p^*)\phi^-_{0,0}}{[|p-p^*|^2 - 4|\phi^-_{0,0}|^2]^{\frac{1}{2}}} \right) \left(\frac{1}{\tilde{\phi}^-_{0,0} + \phi^-_{0,0}} \right) \times \right.$$

$$\left. \times \left([|p-p^*|^2 - 4|\tilde{\phi}^-_{0,0}|^2]^{\frac{1}{2}} + [|p-p^*|^2 - 4|\phi^-_{0,0}|^2]^{\frac{1}{2}} \right) + i(p+p^*-q-q^*) \left(\frac{\tilde{\phi}^-_{0,0} - \phi^-_{0,0}}{\tilde{\phi}^-_{0,0} + \phi^-_{0,0}} \right) \right\}^2$$

$$= (q^*-q)^2 + |\tilde{\phi}^-_{0,0} - \phi^-_{0,0}|^2 + \tfrac{1}{4}\left([|p-p^*|^2 - 4|\tilde{\phi}^-_{0,0}|^2]^{\frac{1}{2}} - [|p-p^*|^2 - 4|\phi^-_{0,0}|^2]^{\frac{1}{2}} \right)^2$$

$$+ 4 \left| \frac{\partial_x\tilde{\phi}^-_{0,0} - \frac{1}{2}i(p+p^*)\tilde{\phi}^-_{0,0}}{[|p-p^*|^2 - 4|\tilde{\phi}^-_{0,0}|^2]^{\frac{1}{2}}} - \frac{\partial_x\phi^-_{0,0} - \frac{1}{2}i(p+p^*)\phi^-_{0,0}}{[|p-p^*|^2 - 4|\phi_{0,0}|^2]^{\frac{1}{2}}} \right|^2 \ . \tag{4.65}$$

Some details of the derivation are presented in appendix D. The BT for the AHSC is rather complicated and it does not seem to be easy to derive a second modified NLS equation in terms of only one function.

5. The sine-Gordon equation

In the preceding section we have described a general scheme, independent of the dispersion $\omega(k)$, of deriving Bäcklund transformations and modified PDE's, on the basis of a singular transformation of the measure in the integral equation associated with the original PDE's. As specific examples we have treated $\omega(k) = k^3$ in the case of the integral equation (2.1) for the KdV class and $\omega(k) = \lambda_2 k^2 + \lambda_3 k^3$ in the case of the integral equation (4.1) for the NLS class. In this section we present some results on Bäcklund transformations and modified equations for $\omega(k) = k^{-1}$ starting from the integral equations (4.1) and (2.40) with $p=0$, for the NLS class and the MKdV class respectively.

5.1. *The NLS class*

In the special case $\omega(k)=k^{-1}$ the function $\phi_{0,0}$, defined by (4.1) and by the (0,0) element of (4.12) satisfies the PDE

$$\partial_x\partial_t\phi_{0,0} = \phi_{0,0}\,[\,1-4|\partial_t\phi_{0,0}|^2\,]^{\frac{1}{2}}\ , \tag{5.1}$$

cf. eqs. (6.3a) and (6.5) of ref. 18. In terms of the variable $\chi=2\partial_t\phi_{0,0}$, eq. (5.1) can be expressed as

$$\chi = \partial_t\left(\frac{\partial_x\chi}{[1-|\chi|^2]^{\frac{1}{2}}}\right)\ , \tag{5.2}$$

which may be regarded as the complex sine-Gordon equation.

The spatial part of the BT of (5.1), cf. also refs. 33 and 34, is given by the (0,0) element of (4.36), i.e.

$$\begin{aligned}2i\partial_x(\tilde{\phi}_{0,0}-\phi_{0,0}) &= -(p^*+p)(\tilde{\phi}_{0,0}-\phi_{0,0})\\ &\qquad \mp i(\tilde{\phi}_{0,0}+\phi_{0,0})\,[\,|p-p^*|^2-4|\tilde{\phi}_{0,0}-\phi_{0,0}|^2]^{\frac{1}{2}}\ .\end{aligned} \tag{5.3}$$

The time-dependent part of the BT can be inferred from (5.1), inserting (5.3) for $\partial_x(\tilde{\phi}_{0,0}-\phi_{0,0})$ in order to evaluate $\partial_x\partial_t(\tilde{\phi}_{0,0}-\phi_{0,0})$.

The modified PDE corresponding to (5.1) can be found inserting (4.39) into

$$\begin{aligned}\partial_x\partial_t\phi^-_{0,0} &= \tfrac{1}{2}(\phi^+_{0,0}+\phi^-_{0,0})\,[1-|\partial_t(\phi^+_{0,0}+\phi^-_{0,0})|^2]^{\frac{1}{2}}\\ &\quad - \tfrac{1}{2}(\phi^+_{0,0}-\phi^-_{0,0})\,[1-|\partial_t(\phi^+_{0,0}-\phi^-_{0,0})|^2]^{\frac{1}{2}}\ .\end{aligned} \tag{5.4}$$

The result in terms of the variable

$$a(x,t) \equiv 2|p-p^*|^{-1}\phi^-_{0,0}\,e^{-\frac{1}{2}i(p+p^*)x}\ , \qquad (p^*\neq p)\ , \tag{5.5}$$

can be expressed as

$$\partial_x\partial_t a + \frac{i}{2}(p+p^*)\partial_t a =$$

$$= \left[\tfrac{1}{2}a \mp \frac{|p-p^*|^{-1}\partial_x a}{[1-|a|^2]^{\frac{1}{2}}}\right]\left[1 - \left|\tfrac{1}{2}(p-p^*)\partial_t a \mp \partial_t\left(\frac{\partial_x a}{[1-|a|^2]^{\frac{1}{2}}}\right)\right|^2\right]^{\frac{1}{2}}$$

$$+ \left[\tfrac{1}{2}a \pm \frac{|p-p^*|^{-1}\partial_x a}{[1-|a|^2]^{\frac{1}{2}}}\right]\left[1 - \left|\tfrac{1}{2}(p-p^*)\partial_t a \pm \partial_t\left(\frac{\partial_x a}{[1-|a|^2]^{\frac{1}{2}}}\right)\right|^2\right]^{\frac{1}{2}} \tag{5.6}$$

Eq. (4.65) is again the spatial part of the BT for the PDE for $\phi^-_{0,0}$.

5.2. *The MKdV class*

From (2.40), in the special case $p=0$, we have the integral equation

$$\underline{v}_k + \int_C d\lambda(\ell)\int_C d\lambda(\ell')\,\frac{\rho_k\rho_{\ell'}}{(k+\ell')(\ell'+\ell)}\,\underline{v}_\ell = \rho_k\underline{c}_k \quad , \quad \underline{v}_k \equiv -\frac{\underline{u}^-_k}{2k} \quad , \tag{5.7}$$

which is the integral equation of type II of ref. 18. For the (0,0) element of the matrix

$$\underline{\underline{V}} = \int_C d\lambda(\ell)\underline{v}_\ell\underline{c}_\ell \quad , \tag{5.8}$$

we have, in the special case $\omega(k)=k^{-1}$, the PDE , cf. eq. (6.17) of ref. 18,

$$\partial_x\partial_t v_{0,0} = v_{0,0}\,[1 + 4(\partial_t v_{0,0})^2]^{\frac{1}{2}} \quad . \tag{5.9}$$

Eq. (5.9) can be regarded as the potential sine-Gordon equation, since the substitution

$$\theta = \arcsin 2i\partial_t v_{0,0} \tag{5.10}$$

gives the sine-Gordon equation

$$\partial_x\partial_t\theta = \sin\theta \quad . \tag{5.11}$$

For the quantity $v_{0,0} = -\tfrac{1}{2}u^-_{0,0}$ we have the BT

$$i\partial_x(\tilde{v}_{0,0}+v_{0,0}) = \mp(\tilde{v}_{0,0}-v_{0,0})[p^2 - (\tilde{v}_{0,0}+v_{0,0})^2]^{\frac{1}{2}} , \tag{5.12}$$

cf. (3.13) with $p=0$ and $q\to p$. For the quantity

$$w = \frac{1}{p}(\tilde{v}_{0,0}+v_{0,0}) \tag{5.13}$$

one can derive the modified PDE

$$\partial_x\partial_t w = \left[\tfrac{1}{2}w \mp \frac{\frac{1}{2}ip^{-1}\partial_x w}{[1-w^2]^{\frac{1}{2}}}\right]\left[1 + \left[p\partial_t w \mp i\partial_t\left(\frac{\partial_x w}{[1-w^2]^{\frac{1}{2}}}\right)\right]^2\right]^{\frac{1}{2}}$$

$$+ \left[\tfrac{1}{2}w \pm \frac{\frac{1}{2}ip^{-1}\partial_x w}{[1-w^2]^{\frac{1}{2}}}\right]\left[1 + \left[p\partial_t w \pm i\partial_t\left(\frac{\partial_x w}{[1-w^2]^{\frac{1}{2}}}\right)\right]^2\right]^{\frac{1}{2}} , \tag{5.14}$$

which, for imaginary p, may be regarded as a real counterpart of (5.6). Solutions of (5.14) can be found from the linear integral equation (3.27) with $p=0$ and $q\to p$, noting that $w = -\frac{1}{2}p^{-1}u^{-+}_{0,0}$.

Instead of the BT for the potential sine-Gordon equation one can consider the BT for the SG equation (5.11) itself. From (5.9) and (5.12) one can show that

$$iv_{0,0}[1 + 4(\partial_t v_{0,0})^2]^{\frac{1}{2}} + i\tilde{v}_{0,0}[1 + 4(\partial_t\tilde{v}_{0,0})^2]^{\frac{1}{2}}$$

$$= \mp\left[(\partial_t(\tilde{v}_{0,0}-v_{0,0}))[p^2-(\tilde{v}_{0,0}+v_{0,0})^2]^{\frac{1}{2}} - \frac{\tilde{v}^2_{0,0}-v^2_{0,0}}{[p^2-(\tilde{v}_{0,0}+v_{0,0})^2]^{\frac{1}{2}}}\partial_t(\tilde{v}_{0,0}+v_{0,0})\right]. \tag{5.15}$$

In terms of the variable θ, defined by (5.10), eq. (5.15) can be rewritten using (5.11) as

$$\cos\theta\,(\partial_x\theta) + \cos\tilde{\theta}\,(\partial_x\tilde{\theta}) = \pm i\left[(\sin\tilde{\theta} - \sin\theta)\,[p^2 + \tfrac{1}{4}(\partial_x(\tilde{\theta}+\theta))^2]^{\frac{1}{2}}\right.$$

$$\left. + \frac{(\sin\tilde{\theta} + \sin\theta)\,[\,(\partial_x\tilde{\theta})^2 - (\partial_x\theta)^2\,]}{4[p^2 + \frac{1}{4}(\partial_x(\tilde{\theta}+\theta))^2]^{\frac{1}{2}}}\right] . \tag{5.16}$$

Writing $2ip\sin\alpha \equiv \partial_x(\tilde\theta+\theta)$, eq. (5.16) can be worked out to give

$$[\tan\alpha \mp \tan\tfrac{1}{2}(\tilde\theta-\theta)][ip\cos\alpha \pm \tfrac{1}{2}\tan\tfrac{1}{2}(\tilde\theta+\theta)\partial_x(\tilde\theta-\theta)]\cos\tfrac{1}{2}(\tilde\theta+\theta) \times$$
$$\times \cos\tfrac{1}{2}(\tilde\theta-\theta) = 0 \,, \qquad (5.17)$$

whereas eq. (5.12) with (5.10) and (5.9) leads to

$$\partial_x^2(\tilde\theta+\theta) = \pm ip(\cos\alpha)\,\partial_x(\tilde\theta-\theta) \,. \qquad (5.18)$$

The only consistent solution of (5.17) and (5.18) is $\alpha = \pm\frac{1}{2}(\tilde\theta-\theta)$, leading to

$$\partial_x(\tilde\theta+\theta) = \pm 2ip\,\sin\tfrac{1}{2}(\tilde\theta-\theta) \quad . \qquad (5.19)$$

From (5.19) together with (5.11) we have

$$\partial_t(\tilde\theta-\theta) = \mp 2ip^{-1}\,\sin\tfrac{1}{2}(\tilde\theta+\theta) \,. \qquad (5.20)$$

Eqs. (5.19) and (5.20) form the well-known BT [1] for the SG.

In terms of the variable $\alpha = \frac{1}{2}(\tilde\theta-\theta)$ eqs. (5.19) and (5.20) yield the PDE

$$\partial_x\partial_t\alpha = [1 + p^2(\partial_t\alpha)^2]^{\frac{1}{2}}\,\sin\alpha \quad , \qquad (5.21)$$

which is the modified sine-Gordon equation (MSG). The solutions of (5.21) can be inferred from the solutions of the linear integral equation (3.27) with $p=0$ and $q \to p$. From (3.26) one obtains the function $u^{-+}_{0,0}$ leading with (3.16) and the 0,0 element of (3.14) to $u^{-}_{0,0}$ $(=-2v_{0,0})$ and $\tilde u^{-}_{0,0}$ $(=-2\tilde v_{0,0})$, and $\alpha = \pm\frac{1}{2}(\tilde\theta-\theta)$ is determined by (5.10). Multisoliton solutions were obtained by bilinearization in ref. 35.

Remark

From eq. (5.7) it can be shown that the integral equation (2.1) with dispersion $\omega(k) = k^{-1}$ yields solutions of the sine-Gordon equation as well as of a shallow water wave equation. Some details are presented in appendix E.

6. Bäcklund transformations for the wave functions

In the preceding sections we have given a general method to derive Bäcklund transformations for integrable PDE's using a singular transformation of measures in the corresponding linear integral equation. From this transformation one immediately obtains the basic Bäcklund relations containing both potentials and wave functions. Here the term wave functions is used to denote the solutions of the linear integral equations which depend on the (spectral) parameter k and which appear as eigenfunctions in the associated linear problems. The potentials denote the functions, obtained through integration of the wave functions over the contour C in the complex k-plane.

So far we have considered PDE's and BT's containing only the potentials, and which have been obtained after eliminating the wave functions by integration over the contour C. In the present section we will investigate PDE's and BT's in terms of only the wave functions, which in a number of cases can be derived by eliminating the potentials from the linear problem and the basic relations of the BT. Examples include e.g. the BT for the potential MKdV and the AHSC.

6.1. *The KdV class*

The linear problem for the KdV is given by the $n=0$ components of eqs. (2.9) and (2.11). From eq. (2.9), writing $a \equiv \ln u_k^{(0)}$, we have the relation

$$\partial_x u_{0,0} = \partial_x^2 a + (\partial_x a)^2 - ik\partial_x a \quad , \tag{6.1}$$

mapping the wave function $u_k^{(0)}$ on the potential $\partial_x u_{0,0}$ that satisfies the Korteweg-de Vries equation. Inserting (6.1) in (2.11) it is clear that $a = \ln u_k^{(0)}$ satisfies the PDE

$$(\partial_t - \partial_x^3)a = 3ik(\partial_x a)^2 - 2(\partial_x a)^3 \quad , \tag{6.2}$$

which is equivalent to the potential MKdV, i.e. the PDE for $\partial_x a$ is equivalent to the MKdV.

From the basic relation of the BT (2.23), using also (2.3), it follows that

$$(p-k)\underline{\tilde{u}}_k = (p+k)\underline{u}_k + 2i\partial_x \underline{u}_k + i(\underline{\underline{\tilde{U}}}-\underline{\underline{U}})\cdot\underline{\underline{O}}\cdot\underline{u}_k \quad , \tag{6.3}$$

implying that

$$(p-k)e^{(\tilde{a}-a)} = p+k+2i\partial_x a + i(\tilde{u}_{0,0} - u_{0,0}) \ . \tag{6.4}$$

From (6.4) and the inverse relation with $p \leftrightarrow -p$, $a \leftrightarrow \tilde{a}$, $u_{0,0} \leftrightarrow \tilde{u}_{0,0}$, it can be shown that

$$2i\partial_x(a+\tilde{a}) = -2k + (p-k)e^{(\tilde{a}-a)} - (p+k)e^{-(\tilde{a}-a)} , \tag{6.5}$$

which is the spatial part of the BT for the potential MKdV (6.2).

Substituting $z = \tilde{a}-a$ in (6.5), we have

$$\partial_x a = -\tfrac{1}{2}\partial_x z + \tfrac{1}{2}ik - \tfrac{1}{4}i(p-k)e^{z} + \tfrac{1}{4}i(p+k)e^{-z} . \tag{6.6}$$

From (6.2) and its counterpart with $a \to \tilde{a}$, and using (6.6), we obtain again the second modified KdV (3.21), i.e.

$$(\partial_t - \partial_x^3)z = -\tfrac{1}{2}(\partial_x z)^3 + \tfrac{3}{8}[(p-k)^2e^{2z} + (p+k)^2e^{-2z} - 2(p^2+k^2)]\partial_x z . \tag{6.7}$$

6.2. *The NLS class*

For the function

$$u(x,t) \equiv -\phi_k^{(0)}(x,t)/\psi_k^{(0)}(x,t) , \tag{6.8}$$

defined in terms of the wave functions $\phi_k^{(0)}$ and $\psi_k^{(0)}$ of the linear problem of the NLS, cf. the n=0 components of (4.7), (4.8), (4.14) and (4.15), in the special case $\lambda_2=1$, $\lambda_3=0$, one derives the PDE

$$\begin{aligned}\partial_t u &= i\partial_x\left(\frac{\partial_x u - u^2\partial_x u^*}{1-|u|^4}\right) - iu^2\partial_x\left(\frac{\partial_x u^* - u^{*2}\partial_x u}{1-|u|^4}\right)\\ &+ \frac{2iu}{(1-|u|^4)^2}(\partial_x u - u^2\partial_x u^*)(\partial_x u^* - u^{*2}\partial_x u)\\ &+ \frac{2(k+k^*)u^2\partial_x u^*}{(1-|u|^2)^2} + \frac{2i(k+k^*)^2u^3u^{*2}}{(1-|u|^4)^2} + \frac{2ikk^*u^2u^*}{(1+|u|^2)^2} ,\end{aligned} \tag{6.9}$$

cf. eq. (2.17) of ref. 17 with $\alpha=2$, apart from a misprint. In ref. 5 the invariance of (6.9) under $k \leftrightarrow k^*$ was used to obtain the BT for the NLS.

From the basic relations (4.25), (4.26) of the BT for the NLS, together with eqs. (4.5) - (4.8) and (6.8) we have, noting that $\psi_{0,0}$ is imaginary,

$$i\partial_x u = (p-k)\tilde{u}\,\frac{\tilde{\psi}_k^{(0)}}{\psi_k^{(0)}} + (p-k)u\,\frac{\tilde{\psi}_k^{(0)}}{\psi_k^{(0)}} - (p^*-k)u - pu + \tilde{\phi}_{0,0} - \tilde{\phi}_{0,0}^*u^2 . \tag{6.10}$$

Solving $\tilde{\psi}_k^{(0)}/\psi_k^{(0)}$ from (6.10) and inserting the result in the inverse relation with $p \leftrightarrow p^*$, $u \leftrightarrow \tilde{u}$, $\psi_k^{(0)} \leftrightarrow \tilde{\psi}_k^{(0)}$, $\phi_{0,0} \leftrightarrow \tilde{\phi}_{0,0}$, we find

$$\left(i\partial_x \tilde{u} + (p-k)\tilde{u} - \phi_{0,0} + \phi_{0,0}^* \tilde{u}^2 + p^*\tilde{u}\right) \left(i\partial_x u + (p^*-k)u - \tilde{\phi}_{0,0} + \tilde{\phi}_{0,0}^* u^2 + pu\right)$$
$$= (p^*-k)(p-k)(\tilde{u}+u)^2 \; . \qquad (6.11)$$

From eqs. (4.7) and (4.8) one also has the relation

$$i\partial_x u = -ku + \phi_{0,0} - \phi_{0,0}^* u^2 \; , \qquad (6.12)$$

from which one can express $\phi_{0,0}$ in terms of u:

$$\phi_{0,0} = \frac{i\partial_x u + ku - u^2(i\partial_x u^* - k^* u^*)}{1 - |u|^4} \; . \qquad (6.13)$$

Inserting (6.13), its complex conjugate, and the inverse relations with $\phi_{0,0} \to \tilde{\phi}_{0,0}$, $u \to \tilde{u}$, in (6.11), we have

$$\left[i\partial_x \tilde{u} + (p-k)\tilde{u} + p^*\tilde{u} + \frac{(i\partial_x u + ku)(-1+\tilde{u}^2 u^{*2}) - (i\partial_x u^* - k^* u^*)(\tilde{u}^2 - u^2)}{1 - |u|^4} \right]$$

$$\times \left[i\partial_x u + (p^*-k)u + pu + \frac{(i\partial_x \tilde{u} + k\tilde{u})(-1+u^2 \tilde{u}^{*2}) - (i\partial_x \tilde{u}^* - k^* \tilde{u}^*)(u^2 - \tilde{u}^2)}{1 - |u|^4} \right]$$

$$= (p^*-k)(p-k)(\tilde{u}+u)^2 \; , \qquad (6.14)$$

which is the BT for the PDE (6.9).

In the special case $p^*=k$, eq. (6.11) leads to

$$i\partial_x u = \tilde{\phi}_{0,0} - \tilde{\phi}_{0,0}^* u^2 - k^* u \; , \qquad (6.15)$$

cf. eq. (2.19) of ref. 17. Eqs. (6.12) and (6.15) lead to

$$\phi_{0,0}^- = \frac{(p-p^*)u}{1 + |u|^2} \; , \qquad (6.16)$$

showing that the PDE (6.9) with $k \to p^*$ is equivalent to eq. (4.41) for $\lambda_2=1$, $\lambda_3=0$, and thus to the AHSC (4.44), cf. ref. 17. Eq. (6.14) (cf. eq. (4.65)) is thus an alternative expression for the BT of the AHSC.

Appendix A

In this appendix we argue that the transformation (2.18) can increase the number of solitons by 1, and we also discuss another way of obtaining the Bäcklund transformation.

N-soliton solutions can be obtained from the integral equation (2.1) together with (2.6), choosing as a measure a linear combination of N simple poles, i.e.

$$d\lambda(k) = \frac{1}{2\pi i} \sum_{\alpha=1}^{N} \frac{g_\alpha}{k-k_\alpha} dk \,, \tag{A.1}$$

and a contour C enclosing $k_1, k_2, \ldots, k_N$. For an analytic function $f(k)$ we then have

$$\int_C f(k) d\lambda(k) = \sum_{\alpha=1}^{N} g_\alpha f(k_\alpha) \,, \tag{A.2}$$

so that (2.1) reduces to a set of N linear algebraic equations. (In view of (A.2) $d\lambda(k)$ may also be regarded as a linear combination of delta functions $d\lambda(k) = \sum_{\alpha=1}^{N} g_\alpha \delta(k-k_\alpha) dk$, cf. ref. 15).

Consider now the transformation (2.18) with $-p \neq k_\alpha$, then

$$d\tilde{\lambda}(k) = \frac{1}{2\pi i} \frac{\tilde{g}_p}{k+p} dk + \frac{1}{2\pi i} \sum_{\alpha=1}^{N} \frac{\tilde{g}_\alpha}{k-k_\alpha} dk \,, \tag{A.3}$$

$$\tilde{g}_p = -2p \sum_{\alpha=1}^{N} \frac{g_\alpha}{p+k_\alpha} \,, \quad \tilde{g}_\alpha = \frac{g_\alpha (p-k_\alpha)}{p+k_\alpha} \,. \tag{A.4}$$

Eq. (A.3) is a linear combination of N+1 simple poles, so that the new function $\tilde{\underline{\underline{U}}}$, which can be found from (2.6) with the measure (A.3) is an (N+1)-soliton solution.

In order to give a different, but equivalent, way of getting the BT, let us consider the singular transformation of the plane-wave factor

$$\rho_k \rightarrow \hat{\rho}_k = \frac{p-k}{p+k} \rho_k \,. \tag{A.5}$$

It is clear that $\hat{\rho}_k$ obeys the differential relations (2.2) as well and by defining a solution $\hat{\underline{u}}_k(x,t)$ of (2.1) with $\hat{\rho}_k$ instead of ρ_k, i.e.

$$\hat{\underline{u}}_k + i\hat{\rho}_k \int_C d\lambda(\ell)\, \frac{\hat{\underline{u}}_\ell}{k+\ell} = \hat{\rho}_k \underline{c}_k \,, \tag{A.6}$$

we obtain a solution

$$\hat{\underline{\underline{U}}} = \int_C d\lambda(k) \hat{\underline{u}}_k \underline{c}_k \tag{A.7}$$

of the matrix PDE (2.12). On the other hand, multiplying (A.6) by $(p+k)(p-k)^{-1}$, it is easy to show that $\hat{\underline{u}}_k(p+k)(p-k)^{-1}$ satisfies the integral equation (2.19). As a consequence we have

$$\hat{\underline{u}}_k = \frac{(p-k)}{(p+k)}\, \tilde{\underline{u}}_k \,. \tag{A.8}$$

Inserting (A.8) in (A.7) and using (2.18) and (2.20) we immediately obtain

$$\hat{\underline{\underline{U}}} = \tilde{\underline{\underline{U}}} \,, \tag{A.9}$$

implying that the Bäcklund transformation derived in subsection 2.4 can also be obtained as a result of the transformation (A.5) of the plane-wave factor ρ_k.

A similar result can be derived for the integral equation of the NLS type treated in section 4. One has the relations

$$\tilde{\underline{\underline{\Phi}}} = \int_C d\lambda(k) \hat{\underline{\phi}}_k \underline{c}_k \,, \qquad \tilde{\underline{\underline{\Psi}}} = \int_C d\lambda(k) \hat{\underline{\psi}}_k \underline{c}_k \,, \tag{A.10}$$

in which $\hat{\underline{\phi}}_k$ and $\hat{\underline{\psi}}_k$ are the solutions of (4.2) and (4.3) with

$$\rho_k \to \hat{\rho}_k = \frac{p-k}{p^*-k}\, \rho_k \,. \tag{A.11}$$

Appendix B

Eq. (2.32) is most easily derived using two relations of the linear problem in subsection 2.6. From the integrated version of (2.48) we have

$$i\partial_x u^-_{1,1} = -u^-_{1,0} + \tfrac{1}{2} i u^-_{1,0} u^+_{0,1} \,, \tag{B.1}$$

$$i\partial_x u^-_{0,1} = -u^-_{0,0} + \tfrac{1}{2} i u^-_{0,0} u^+_{0,1} \,, \tag{B.2}$$

from which

$$u^-_{0,0} = \frac{\partial_x (u^-_{1,0})^2}{2\partial_x u^-_{1,1}} \,. \tag{B.3}$$

From the integrated version of (2.41), using also (B.1) it follows

$$2pu^-_{1,1} = \tfrac{1}{2}i(u^+_{1,0}+2i)^2 - \tfrac{1}{2}iu^{-2}_{1,0} + 2i = \frac{2i(\partial_x u^-_{1,1})^2}{(u^-_{1,0})^2} - \tfrac{1}{2}iu^{-2}_{1,0} + 2i \,, \tag{B.4}$$

from which one can solve

$$(u^-_{1,0})^2 = 2 + 2ipu^-_{1,1} \pm 2[(1+ipu^-_{1,1})^2 + (\partial_x u^-_{1,1})^2]^{\frac{1}{2}} \,. \tag{B.5}$$

Substituting (B.3) and (B.5) in the (1,1) element of (2.30), i.e.

$$(\partial_t - \partial_x^3)u^-_{1,1} = (3ipu^-_{1,0} - \tfrac{3}{2}u^-_{1,0}u^-_{0,0})\partial_x u^-_{0,1} \,, \tag{B.6}$$

one obtains

$$(\partial_t - \partial_x^3)u^-_{1,1} = -\tfrac{3}{2}(\partial_x u^-_{1,1})^{-1}\Big[\partial_x[(1+ipu^-_{1,1})^2 + (\partial_x u^-_{1,1})^2]^{\frac{1}{2}}\Big]^2 - \tfrac{3}{2}p^2\partial_x u^-_{1,1} \,. \tag{B.7}$$

Note that eq. (B.7) can also be derived from the Bäcklund transformation for eq. (2.15)

$$(1-2\partial_x\tilde{u}_{1,1})(1-2\partial_x u_{1,1}) = [1+ip(\tilde{u}_{1,1}-u_{1,1})]^2 \,, \tag{B.8}$$

which can be inferred from the (1,1) element of (2.29), using also (2.17).

It is now easy to show that the quantity

$$b \equiv 2\partial_x \ln\left(u^-_{1,1} - \frac{i}{p}\right) \tag{B.9}$$

obeys the following PDE

$$(\partial_t - \partial_x^3)b = \frac{3}{2}\,\partial_x\left[\frac{b(\partial_x b)^2}{4p^2-b^2} - \frac{1}{12}b^3\right] \,, \tag{B.10}$$

and eq. (2.32) is obtained substituting $b = 2ip \sinh z$.

Appendix C

In this appendix we give the derivation of the Bäcklund transformation (3.41) for the PDE (3.19). As a first step, eq. (3.39) is rewritten

$$
\begin{aligned}
(r-k)\tilde{\underline{u}}_k^{+\alpha} = r\underline{u}_k^{+\alpha} - \underline{\underline{J}}^T\cdot\underline{u}_k^{+\alpha} + \tfrac14 i\Big[\,&\tilde{\underline{\underline{U}}}^{++}\cdot\underline{\underline{Q}}\cdot\underline{u}_k^{+\alpha} + \tilde{\underline{\underline{U}}}^{+-}\cdot\underline{\underline{Q}}\cdot\underline{u}_k^{+\bar{\alpha}} \\
&+ \tilde{\underline{\underline{U}}}^{-+}\cdot\underline{\underline{Q}}\cdot\underline{u}_k^{-\alpha} + \tilde{\underline{\underline{U}}}^{--}\cdot\underline{\underline{Q}}\cdot\underline{u}_k^{-\bar{\alpha}}\Big] \ ,
\end{aligned}
\tag{C.1}
$$

$$
\begin{aligned}
(r-k)\tilde{\underline{u}}_k^{-\alpha} = r\underline{u}_k^{-\alpha} - \underline{\underline{J}}^T\cdot\underline{u}_k^{-\alpha} + \tfrac14 i\Big[\,&\tilde{\underline{\underline{U}}}^{++}\cdot\underline{\underline{Q}}\cdot\underline{u}_k^{-\alpha} + \tilde{\underline{\underline{U}}}^{+-}\cdot\underline{\underline{Q}}\cdot\underline{u}_k^{-\bar{\alpha}} \\
&+ \tilde{\underline{\underline{U}}}^{-+}\cdot\underline{\underline{Q}}\cdot\underline{u}_k^{+\alpha} + \tilde{\underline{\underline{U}}}^{--}\cdot\underline{\underline{Q}}\cdot\underline{u}_k^{+\bar{\alpha}}\Big] \ .
\end{aligned}
\tag{C.2}
$$

Integrating (C.1) over $d\lambda_2(k)$, and using the relation $(r-k)d\lambda_2(k) = (r+k)d\tilde{\lambda}_2(k)$ to evaluate the left-hand side, we have

$$
\begin{aligned}
r(\tilde{\underline{\underline{U}}}^{++} - \underline{\underline{U}}^{++}) = {}& -\underline{\underline{J}}^T\cdot\underline{\underline{U}}^{++} - \tilde{\underline{\underline{U}}}^{++}\cdot\underline{\underline{J}} \\
&+ \tfrac14 i\Big[\tilde{\underline{\underline{U}}}^{++}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{++} + \tilde{\underline{\underline{U}}}^{+-}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{+-} + \tilde{\underline{\underline{U}}}^{-+}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{-+} + \tilde{\underline{\underline{U}}}^{--}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{--}\Big] \ .
\end{aligned}
\tag{C.3}
$$

Using (C.3), its inverse with $r \leftrightarrow -r$, $\underline{\underline{U}}^{\lambda\mu} \leftrightarrow \tilde{\underline{\underline{U}}}^{\lambda\mu}$, and the relation

$$
\begin{aligned}
\underline{\underline{J}}^T\cdot\underline{\underline{U}}^{++} + \underline{\underline{U}}^{++}\cdot\underline{\underline{J}} = -2p\underline{\underline{U}}^{-+} + \tfrac14 i\Big[\,&\underline{\underline{U}}^{++}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{++} + \underline{\underline{U}}^{+-}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{+-} \\
&- \underline{\underline{U}}^{-+}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{-+} - \underline{\underline{U}}^{--}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{--}\Big] \ ,
\end{aligned}
\tag{C.4}
$$

cf. eqs. (3.8) and (3.14), we find

$$
\begin{aligned}
2r(\tilde{\underline{\underline{U}}}^{++} - \underline{\underline{U}}^{++}) = {}& 2p(\tilde{\underline{\underline{U}}}^{-+} + \underline{\underline{U}}^{-+}) - \tfrac14 i(\tilde{\underline{\underline{U}}}^{++} - \underline{\underline{U}}^{++})\cdot\underline{\underline{Q}}\cdot(\tilde{\underline{\underline{U}}}^{++} - \underline{\underline{U}}^{++}) \\
&- \tfrac14 i(\tilde{\underline{\underline{U}}}^{+-} - \underline{\underline{U}}^{+-})\cdot\underline{\underline{Q}}\cdot(\tilde{\underline{\underline{U}}}^{+-} - \underline{\underline{U}}^{+-}) + \tfrac14 i(\tilde{\underline{\underline{U}}}^{-+} + \underline{\underline{U}}^{-+})\cdot\underline{\underline{Q}}\cdot(\tilde{\underline{\underline{U}}}^{-+} + \underline{\underline{U}}^{-+}) \\
&+ \tfrac14 i(\tilde{\underline{\underline{U}}}^{--} + \underline{\underline{U}}^{--})\cdot\underline{\underline{Q}}\cdot(\tilde{\underline{\underline{U}}}^{--} + \underline{\underline{U}}^{--}) \ .
\end{aligned}
\tag{C.5}
$$

From (C.2) we obtain after integration over $d\lambda_2(k)$

$$
\begin{aligned}
r(\tilde{\underline{\underline{U}}}^{-+} - \underline{\underline{U}}^{-+}) = {}& -\underline{\underline{J}}^T\cdot\underline{\underline{U}}^{-+} - \tilde{\underline{\underline{U}}}^{-+}\cdot\underline{\underline{J}} \\
&+ \tfrac14 i\Big[\tilde{\underline{\underline{U}}}^{++}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{-+} + \tilde{\underline{\underline{U}}}^{+-}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{--} + \tilde{\underline{\underline{U}}}^{-+}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{++} + \tilde{\underline{\underline{U}}}^{--}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{+-}\Big] \ ,
\end{aligned}
\tag{C.6}
$$

and from (C.6), the inverse relation with $r \leftrightarrow -r$, $\underline{\underline{U}}^{\lambda\mu} \leftrightarrow \tilde{\underline{\underline{U}}}^{\lambda\mu}$, and the relation

$$\underline{\underline{J}}^T\cdot\underline{\underline{U}}^{-+} + \underline{\underline{U}}^{-+}\cdot\underline{\underline{J}} = -2i\partial_x\underline{\underline{U}}^{-+}$$
$$+ \tfrac{1}{4}i\left[\underline{\underline{U}}^{-+}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{++} + \underline{\underline{U}}^{--}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{+-} + \underline{\underline{U}}^{++}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{-+} + \underline{\underline{U}}^{+-}\cdot\underline{\underline{Q}}\cdot\underline{\underline{U}}^{--}\right], \quad (C.7)$$

cf. (3.7) and (3.14), it can be shown that

$$2i\partial_x(\tilde{\underline{\underline{U}}}^{-+} + \underline{\underline{U}}^{-+}) = 2r(\tilde{\underline{\underline{U}}}^{-+} - \underline{\underline{U}}^{-+}) + \tfrac{1}{2}i\left[(\tilde{\underline{\underline{U}}}^{-+} - \underline{\underline{U}}^{-+})\cdot\underline{\underline{Q}}\cdot(\tilde{\underline{\underline{U}}}^{++} - \underline{\underline{U}}^{++})\right]^s$$
$$+ \tfrac{1}{2}i\left[(\tilde{\underline{\underline{U}}}^{--} - \underline{\underline{U}}^{--})\cdot\underline{\underline{Q}}\cdot(\tilde{\underline{\underline{U}}}^{+-} - \underline{\underline{U}}^{+-})\right]^s . \quad (C.8)$$

In a similar way one can derive two relations for $\tilde{\underline{\underline{U}}}^{+-} - \underline{\underline{U}}^{+-}$ and $\tilde{\underline{\underline{U}}}^{--} - \underline{\underline{U}}^{--}$, but we shall not give them here.

From the (0,0) element of (C.8), using (3.16) and (3.35) for $i\partial_x u^{-+}_{0,0}$ and $u^{+-}_{0,0}$ respectively, one has

$$\tfrac{1}{2}i(\tilde{u}^{++}_{0,0} - u^{++}_{0,0}) + 2r = \pm\frac{(\tilde{u}^{--}_{0,0} + u^{--}_{0,0})}{2(\tilde{u}^{-+}_{0,0} - u^{-+}_{0,0})} \times$$
$$\times\left\{\left[4(q^2-p^2) - (\tilde{u}^{-+}_{0,0} - 2pi)^2\right]^{\frac{1}{2}} + \left[4(q^2-p^2) - (u^{-+}_{0,0} - 2pi)^2\right]^{\frac{1}{2}}\right\}, \quad (C.9)$$

and eq. (3.41) can be derived inserting (C.9) and (3.35) in (C.5).

Appendix D

In this appendix we derive eq. (4.65). From (4.62) and (4.63) we obtain the relations

$$(q-k)\tilde{\underline{\phi}}^{\pm}_k = q\underline{\phi}^{\pm}_k - \underline{\underline{J}}^T\cdot\underline{\phi}^{\pm}_k + \tfrac{1}{2}(\tilde{\underline{\underline{\Psi}}}^{+*}\cdot\underline{\underline{Q}}\cdot\underline{\phi}^{\pm}_k + \tilde{\underline{\underline{\Psi}}}^{-*}\cdot\underline{\underline{Q}}\cdot\underline{\phi}^{\mp}_k + \tilde{\underline{\underline{\Phi}}}^{+}\cdot\underline{\underline{Q}}\cdot\underline{\psi}^{\pm}_k + \tilde{\underline{\underline{\Phi}}}^{-}\cdot\underline{\underline{Q}}\cdot\underline{\psi}^{\mp}_k) , \quad (D.1)$$

$$(q-k)\tilde{\underline{\psi}}^{\pm}_k = q^*\underline{\psi}^{\pm}_k - \underline{\underline{J}}^T\cdot\underline{\psi}^{\pm}_k + \tfrac{1}{2}(\tilde{\underline{\underline{\Psi}}}^{+}\cdot\underline{\underline{Q}}\cdot\underline{\psi}^{\pm}_k + \tilde{\underline{\underline{\Psi}}}^{-}\cdot\underline{\underline{Q}}\cdot\underline{\psi}^{\mp}_k - \tilde{\underline{\underline{\Phi}}}^{+*}\cdot\underline{\underline{Q}}\cdot\underline{\phi}^{\pm}_k - \tilde{\underline{\underline{\Phi}}}^{-*}\cdot\underline{\underline{Q}}\cdot\underline{\phi}^{\mp}_k) . \quad (D.2)$$

Integrating (D.1) over $d\lambda_1(k)$ and using $(q-k)d\lambda_1(k) = (q^*-k)d\tilde{\lambda}_1(k)$ to evaluate the left-hand side, we find

$$q^*\tilde{\underline{\underline{\Phi}}}^{-} - q\underline{\underline{\Phi}}^{-} = -(\underline{\underline{J}}^T\cdot\underline{\underline{\Phi}}^{-} - \tilde{\underline{\underline{\Phi}}}^{-}\cdot\underline{\underline{J}}) + \tfrac{1}{2}(\tilde{\underline{\underline{\Psi}}}^{-*}\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Phi}}^{+} + \tilde{\underline{\underline{\Psi}}}^{+*}\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Phi}}^{-} + \tilde{\underline{\underline{\Phi}}}^{-}\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Psi}}^{+} + \tilde{\underline{\underline{\Phi}}}^{+}\cdot\underline{\underline{Q}}\cdot\underline{\underline{\Psi}}^{-}). \quad (D.3)$$

From (D.3) and the relation, cf. (4.31) and (4.50),

$$\underline{\underline{J}}^{T}\cdot\underline{\underline{\Phi}}^{-} + \underline{\underline{\Phi}}^{-}\cdot\underline{\underline{J}} = -2i\partial_x\underline{\underline{\Phi}}^{-} - \tfrac{1}{2}(\underline{\underline{\Phi}}^{-}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Psi}}^{+} + \underline{\underline{\Phi}}^{+}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Psi}}^{-}) + \tfrac{1}{2}(\underline{\underline{\Psi}}^{-*}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Phi}}^{+} + \underline{\underline{\Psi}}^{+*}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Phi}}^{-}). \quad (D.4)$$

and the inverse relations with $q \leftrightarrow q^{*}$, $\underline{\underline{\Phi}}^{\pm} \leftrightarrow \underline{\underline{\tilde{\Phi}}}^{\pm}$, $\underline{\underline{\Psi}}^{\pm} \leftrightarrow \underline{\underline{\tilde{\Psi}}}^{\pm}$ it can be shown that

$$2(q^{*}\underline{\underline{\tilde{\Phi}}}^{-} - q\underline{\underline{\Phi}}^{-}) = -2i\partial_x(\underline{\underline{\tilde{\Phi}}}^{-} - \underline{\underline{\Phi}}^{-}) - \tfrac{1}{2}(\underline{\underline{\tilde{\Phi}}}^{-} + \underline{\underline{\Phi}}^{-})\cdot\underline{\underline{O}}\cdot(\underline{\underline{\tilde{\Psi}}}^{+} - \underline{\underline{\Psi}}^{+}) - \tfrac{1}{2}(\underline{\underline{\tilde{\Phi}}}^{+} + \underline{\underline{\Phi}}^{+})\cdot\underline{\underline{O}}\cdot(\underline{\underline{\tilde{\Psi}}}^{-} - \underline{\underline{\Psi}}^{-})$$

$$+ \tfrac{1}{2}(\underline{\underline{\tilde{\Psi}}}^{-*} - \underline{\underline{\Psi}}^{-*})\cdot\underline{\underline{O}}\cdot(\underline{\underline{\tilde{\Phi}}}^{+} + \underline{\underline{\Phi}}^{+}) + \tfrac{1}{2}(\underline{\underline{\tilde{\Psi}}}^{+*} - \underline{\underline{\Psi}}^{+*})\cdot\underline{\underline{O}}\cdot(\underline{\underline{\tilde{\Phi}}}^{-} + \underline{\underline{\Phi}}^{-}) \quad . \qquad (D.5)$$

Integrating (D.2) over $d\lambda_1(k)$ we have

$$q^{*}(\underline{\underline{\tilde{\Psi}}}^{+} - \underline{\underline{\Psi}}^{+}) = -(\underline{\underline{J}}^{T}\cdot\underline{\underline{\Psi}}^{+} - \underline{\underline{\tilde{\Psi}}}^{+}\cdot\underline{\underline{J}}) + \tfrac{1}{2}(\underline{\underline{\tilde{\Psi}}}^{-}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Psi}}^{-} + \underline{\underline{\tilde{\Psi}}}^{+}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Psi}}^{+}) - \tfrac{1}{2}(\underline{\underline{\tilde{\Phi}}}^{-*}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Phi}}^{-} + \underline{\underline{\tilde{\Phi}}}^{+*}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Phi}}^{+}) \; . \qquad (D.6)$$

From (D.5) and the relation, cf. (4.32) and (4.50),

$$\underline{\underline{\Psi}}^{+}\cdot\underline{\underline{J}} - \underline{\underline{J}}^{T}\cdot\underline{\underline{\Psi}}^{+} = -\tfrac{1}{2}(\underline{\underline{\Psi}}^{-}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Psi}}^{-} + \underline{\underline{\Psi}}^{+}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Psi}}^{+}) + \tfrac{1}{2}(\underline{\underline{\Phi}}^{-*}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Phi}}^{-} + \underline{\underline{\Phi}}^{+*}\cdot\underline{\underline{O}}\cdot\underline{\underline{\Phi}}^{+}) \quad , \qquad (D.7)$$

in combination with the inverse relations, it can be shown that

$$(q^{*}-q)(\underline{\underline{\tilde{\Psi}}}^{+} - \underline{\underline{\Psi}}^{+}) = -\tfrac{1}{2}(\underline{\underline{\tilde{\Psi}}}^{-} - \underline{\underline{\Psi}}^{-})\cdot\underline{\underline{O}}\cdot(\underline{\underline{\tilde{\Psi}}}^{-} - \underline{\underline{\Psi}}^{-}) - \tfrac{1}{2}(\underline{\underline{\tilde{\Psi}}}^{+} - \underline{\underline{\Psi}}^{+})\cdot\underline{\underline{O}}\cdot(\underline{\underline{\tilde{\Psi}}}^{+} - \underline{\underline{\Psi}}^{+})$$

$$+ \tfrac{1}{2}(\underline{\underline{\tilde{\Phi}}}^{-*} - \underline{\underline{\Phi}}^{-*})\cdot\underline{\underline{O}}\cdot(\underline{\underline{\tilde{\Phi}}}^{-} - \underline{\underline{\Phi}}^{-}) + \tfrac{1}{2}(\underline{\underline{\tilde{\Phi}}}^{+*} - \underline{\underline{\Phi}}^{+*})\cdot\underline{\underline{O}}\cdot(\underline{\underline{\tilde{\Phi}}}^{+} - \underline{\underline{\Phi}}^{+}) \; . \qquad (D.8)$$

Eq. (4.65) can now be derived solving the imaginary quantity $\tilde{\psi}^{+}_{0,0} - \psi^{+}_{0,0} + q^{*} - q$ from (D.5) and inserting the result, as well as (4.35) and (4.39) in the (0,0) element of (D.8).

Appendix E

In this appendix we discuss the relation between the integral equation (2.1) in the case $\omega(k) = k^{-1}$, and the sine-Gordon equation and a shallow water wave equation.

Consider the Miura transformation which maps the function $v_{0,0}$ defined by (5.7) and (5.8), on the function $u_{0,0}$ defined by (2.1) and (2.6), i.e.

$$\partial_x u_{0,0} = \partial_x v_{0,0} + v_{0,0}^{2} , \qquad (E.1)$$

cf. eq. (2.29) with $u^-_{0,0} = -2v_{0,0}$ and $p=0$.

Using eqs. (7.5), (3.26b) and (3.14b) for $p=-1$, and (6.16), with the minus sign, of ref. 18, it can be shown that

$$\partial_t u_{0,0} = \partial_t v_{0,0} - \tfrac{1}{2} + \tfrac{1}{2}[1 + 4(\partial_t v_{0,0})^2]^{\frac{1}{2}} . \tag{E.2}$$

From (E.2), together with (5.9), we have

$$v_{0,0} = \frac{\partial_x \partial_t u_{0,0}}{1 + 2\partial_t u_{0,0}} . \tag{E.3}$$

Solving $\partial_t v_{0,0}$ from (E.2) and using (E.3) one can derive

$$\partial_t \left(\frac{\partial_x \partial_t u_{0,0}}{1 + 2\partial_t u_{0,0}} \right) = \frac{(\partial_t u_{0,0})^2 + \partial_t u_{0,0}}{1 + 2\partial_t u_{0,0}} . \tag{E.4}$$

Eq. (E.4) can be expressed as

$$2\partial_x \partial_t \ln[1 + 2\partial_t u_{0,0}] = [1 + 2\partial_t u_{0,0}] - \frac{1}{[1 + 2\partial_t u_{0,0}]} , \tag{E.5}$$

which is equivalent to the sine-Gordon equation (SG).

From (E.5) it can also be shown that the matrix element $u_{1,0}$, defined by (2.6), satisfies a PDE which is equivalent to the sine-Gordon equation. In fact, integrating (2.4) with $\underline{\underline{R}} = \underline{\underline{J}}^T \cdot \underline{\underline{O}} \cdot \underline{\underline{J}}$, and taking the (0,0) element, we immediately obtain the relation

$$2i\partial_t u_{0,0} = 2u_{1,0} - iu_{1,0}^2 . \tag{E.6}$$

The same fact can also be seen in the following way. Using eqs. (7.5) and (6.15) of ref. 18, we obtain the relation

$$u_{1,0} = v_{1,0} - i + i[1 - v_{1,0}^2]^{\frac{1}{2}} , \tag{E.7}$$

showing that the PDE's for $u_{1,0}$ and $v_{1,0}$ are equivalent, independent of the dispersion $\omega(k)$. For the special case $\omega(k) = k^{-1}$, $v_{1,0}$ satisfies an equation equivalent to the sine-Gordon equation, as follows from eq. (6.16) of ref. 18.

On the other hand, from (E.1) together with (E.3) one has

$$\partial_x\left(\frac{\partial_x\partial_t u_{0,0}}{1+2\partial_t u_{0,0}}\right)=\partial_x u_{0,0}-\left(\frac{\partial_x\partial_t u_{0,0}}{1+2\partial_t u_{0,0}}\right)^2 . \tag{E.8}$$

Substituting $s\equiv u_{0,0}+\frac{1}{2}t$, eq. (E.8) can be rewritten as

$$\tfrac{1}{2}(\partial_x^2\partial_t s)(\partial_t s)=\tfrac{1}{4}(\partial_x\partial_t s)^2+(\partial_t s)^2(\partial_x s) , \tag{E.9}$$

which after differentiation with respect to x gives the shallow water wave equation [36,37]

$$\tfrac{1}{2}\partial_x^3\partial_t s=(\partial_x^2 s)(\partial_t s)+2(\partial_x\partial_t s)(\partial_x s) . \tag{E.10}$$

Therefore, from the linear integral equation (2.1) with $\omega(k)=k^{-1}$ one can obtain solutions of the SG (E.5), as well as of the shallow water wave equation (E.10). This is not surprising, since on general grounds it can be shown that any solution of the SG leads to a solution of the shallow water wave equation. For that purpose we consider the equation

$$\frac{\partial_x\partial_t w}{w}-\frac{(\partial_t w)(\partial_x w)}{w^2}=w+\frac{g(t)-\frac{1}{4}}{w} , \tag{E.11}$$

where we have included an arbitrary function $g(t)$. For $g(t)=0$, eq. (E.11) reduces to the SG eq. (E.5) with $w=\frac{1}{2}+\partial_t u_{0,0}$, and in the case $g(t)=\frac{1}{4}$, eq. (E.11) is equivalent to the Liouville equation $\partial_x\partial_t p=e^p$.

It is now easy to show that any solution of (E.11) for arbitrary $g(t)$ leads to a solution of (E.9). In fact, from (E.11) we have

$$\partial_x^2\partial_t w-\frac{(\partial_t w)(\partial_x^2 w)}{w}-\frac{(\partial_x\partial_t w)\partial_x w}{w}+\frac{(\partial_t w)(\partial_x w)^2}{w^2}=2w\partial_x w , \tag{E.12}$$

and (E.12) is the compatibility condition for a variable $s(x,t)$ defined by

$$\partial_t s=w , \tag{E.13}$$

$$\partial_x s=\frac{1}{2}\frac{\partial_x^2 w}{w}-\frac{1}{4}\frac{(\partial_x w)^2}{w^2} . \tag{E.14}$$

Inserting (E.13) in the right-hand side of (E.14), one immediately obtains (E.9).

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CHAPTER IV

LINEAR INTEGRAL EQUATIONS AND DIFFERENCE-DIFFERENCE EQUATIONS

1. Introduction

In the last decade a lot of insight has been gained in the integrability of nonlinear partial differential equations (PDE's) [1-3]. One of the most successful methods has been the inverse-scattering transform formalism, which provides an exact linearization in the sense that the initial value problem of the nonlinear PDE is reduced to the solution of only linear equations. In fact, for suitable boundary conditions at infinity, the time evolution of scattering data is governed by linear relations, so that the solutions of the PDE can be obtained with the help of a Gel'fand-Levitan equation.

More recently, Fokas and Ablowitz [4] have proposed a direct method of exact linearization of the Korteweg-de Vries (KdV) equation, based on a singular linear integral equation involving an arbitrary contour and measure. Since then, other singular linear integral equations have been studied for various other PDE's as well, including the nonlinear Schrödinger equation, the (complex and real) sine-Gordon equation, the Boussinesq equation, the equation of motion for the Heisenberg spin chain etc. [5-7], see also ref. 8 for a treatment of the Kadomtsev-Petviashvili equation and the Benjamin-Ono equation. It has also been shown that Bäcklund transformations (BT's) for PDE's can be generated by an appropriate singular transformation of the measure or by an equivalent transformation of the plane-wave factor occurring in the integral equation [9,10].

In this chapter we study the problem of discretizing nonlinear PDE's to obtain difference-difference equations, while retaining their integrability. This problem has of late been addressed by several authors using different starting points. Ablowitz and Ladik [11] started from a discretized linear problem, Hirota [12] started from a discretized bilinear differential equation, and Date et al. used a discretized bilinear identity [13], cf. also ref. 14. The treatment in the present paper is based on the singular linear integral equations associated with the various PDE's mentioned above, and it will be shown that these same integral equations, after no more than a simple and straightforward discretization of the plane-wave factor, also yield a direct linearization of nonlinear difference-difference equations which may be regarded as the double-discrete analogues of the PDE's

As corollaries the following two results will be given. First of all it is shown that the integrable nonlinear difference-difference equations we obtain are equivalent to Bianchi identities expressing the commutativity of Bäcklund transformations, cf. refs. 3, 15-21. Secondly we will see that after applying a suitable continuum limit to the difference-difference equations as well as to the associated wave factors in the singular integral equations, we obtain integrable differential-difference equations together with their direct linearizations. (A relationship between BT's and differential-difference equations was presented in refs. 22, 23.)

The outline of this chapter is as follows. Sections 2-5 give a treatment of the difference-difference equation of the KdV type. In particular, section 2 summarizes the main results, and some details of the derivation are presented in section 3. The continuum limit, as well as the differential-difference equation that is obtained in this limit, are treated in section 4, along with some interesting special cases, and in section 5 the relation with Bäcklund transformations and Bianchi identities is discussed. The last two sections are devoted to a treatment of the difference-difference versions of the nonlinear Schrödinger equation, the sine-Gordon equation and the equation of motion for the Heisenberg spin chain, with a summary of the main results in section 6, and a discussion of the continuum limit and a discussion of the connection with Bäcklund transformations in section 7. (A preliminary account of the considerations in this chapter was given in ref. 24.)

2. The KdV class; results

In this section we present two linear integral equations involving an arbitrary contour and measure, which linearize a certain class of nonlinear difference-difference equations and differential-difference equations of the Korteweg-de Vries type.

Proposition

Let $u_k(n,m)$ be a solution of the linear integral equation

$$u_k(n,m) + i\rho_k(n,m) \int_C d\lambda(\ell)\, \frac{u_\ell(n,m)}{k+\ell} = \frac{\rho_k(n,m)}{k+\alpha} \quad , \qquad n,m \in \mathbb{Z}, \quad k,\alpha \in \mathbb{C} \,, \tag{2.1}$$

where C and $d\lambda(k)$ are an arbitrary contour and measure in the complex k-plane, and

$$\rho_k(n,m) = \left(\frac{p+k}{p-k}\right)^n \left(\frac{q+k}{q-k}\right)^m \rho_k(0,0) \,, \qquad p,q \in \mathbb{C} \,. \tag{2.2}$$

Let the contour C and measure $d\lambda(k)$ be such that the homogeneous integral equation corresponding to (2.1) has only the zero solution. Then the function

$$u(n,m) \equiv i \int_C \frac{u_k(n,m)}{k+\beta}\, d\lambda(k) \,, \qquad n,m \in \mathbb{Z}, \quad \beta \in \mathbb{C} \,, \tag{2.3}$$

obeys the following nonlinear difference-difference equation

$$\begin{aligned} &[(p-\alpha)u(n,m) - (p+\beta)u(n+1,m) + 1][(p-\beta)u(n,m+1) - (p+\alpha)u(n+1,m+1) + 1] \\ &= [(q-\alpha)u(n,m) - (q+\beta)u(n,m+1) + 1][(q-\beta)u(n+1,m) - (q+\alpha)u(n+1,m+1) + 1] \,, \end{aligned} \tag{2.4}$$

for fixed α, β, p and q.

Corollary

Let $u_k(n,t)$ be a solution of the linear integral equation

$$u_k(n,t) + i\rho_k(n,t) \int_C d\lambda(\ell)\, \frac{u_\ell(n,t)}{k+\ell} = \frac{\rho_k(n,t)}{k+\alpha} \quad , \qquad n \in \mathbb{Z}, \; t,k,\alpha \in \mathbb{C} \,, \tag{2.5}$$

where C and $d\lambda(k)$ are an arbitrary contour and measure in the complex k-plane, and

$$\rho_k(n,t) = \left(\frac{p+k}{p-k}\right)^n \exp\left(\frac{-2kt}{p^2-k^2}\right) \rho_k(0,0) \quad , \qquad p \in \mathbb{C} \,. \tag{2.6}$$

Let the contour C and measure $d\lambda(k)$ be such that the homogeneous integral equation corresponding to (2.5) has only the zero solution. Then the function

$$u(n,t) \equiv i \int_C \frac{u_k(n,t)}{k+\beta}\, d\lambda(k) \quad , \quad n \in \mathbb{Z} \,, \quad t,\beta \in \mathbb{C} \,, \tag{2.7}$$

obeys the following nonlinear differential-difference equation

$$\partial_t u(n,t) = -\left[2p-(p+\alpha)(p+\beta)u(n+1,t) + (p-\alpha)(p-\beta)u(n-1,t)\right]^{-1} \times$$

$$\left[u(n+1,t) - u(n-1,t) + 2pu(n+1,t)u(n-1,t) + 2pu^2(n,t)\right.$$

$$\left. - (2p+\alpha+\beta)u(n,t)u(n+1,t) - (2p-\alpha-\beta)u(n,t)u(n-1,t)\right] , \qquad (2.8)$$

for fixed α, β and p.

Remarks

(i) The integral equations (2.1) and (2.5) may be regarded as the double-discrete and the single-discrete analogues of the integral equation

$$u_k(x,t) + i\rho_k(x,t) \int_C d\lambda(\ell)\, \frac{u_\ell(x,t)}{k+\ell} = \rho_k(x,t), \qquad k,x,t \in \mathbb{C} , \qquad (2.9)$$

where

$$\rho_k(x,t) = e^{i(kx-k^3t)}\, \rho_k(0,0) , \qquad (2.10)$$

which was proposed by Fokas and Ablowitz [4] for the linearization of the Korteweg-de Vries equation, cf. also refs. 5 and 25.

ii) From the proposition and its corollary it is clear that the difference-difference equation (2.4) as well as the differential-difference equation (2.8) are both also completely integrable, since solutions can be obtained from the linear integral equations (2.1) and (2.5) respectively.

(iii) In the continuum limit with $r \equiv n - \frac{t}{p} \to \infty$, $p\to\infty$, $\frac{2r}{p} \to ix$, and using the time scaling $\frac{4}{3}\frac{t}{p^4} \to it$, we obtain from (2.6) the result

$$\rho_k(n,t) = \exp\left[(r+\frac{t}{p})\ \ln\frac{p+k}{p-k} - \frac{2kt}{p^2-k^2}\right]\rho_k(0,0)$$

$$\to \rho_k(x,t) = e^{i(kx-k^3t)}\rho_k(0,0) , \qquad (2.11)$$

and from (2.5) we obtain an integral equation generalizing (2.9).

3. *The KdV class; derivation*

In this section we give the derivation of the proposition presented in section 2. For that purpose, consider the linear integral equation

$$u_k^i(n,m;\alpha) + i\rho_k(n,m)\int_C d\lambda(\ell)\,\frac{u_\ell^i(n,m;\alpha)}{k+\ell} = \frac{\rho_k(n,m)k^i}{k+\alpha}, \quad n,m,i\in\mathbb{Z},\ \ k,\alpha\in\mathbb{C}, \tag{3.1}$$

with $\rho_k(n,m)$ given by (2.2). Define

$$u^{i,j}(n,m;\alpha,\beta) \equiv i\int_C d\lambda(k)\,\frac{u_k^j(n,m;\alpha)k^i}{k+\beta}, \qquad n,m,i,j\in\mathbb{Z},\ \ \alpha,\beta\in\mathbb{C}. \tag{3.2}$$

Then from (3.1) and (3.2), using the relation

$$\rho_k(n+1,m) = \left(\frac{p+k}{p-k}\right)\rho_k(n,m), \tag{3.3}$$

we can derive

$$(p-k)u_k^i(n+1,m;\alpha) + i\rho_k(n,m)\int_C d\lambda(\ell)\,\frac{(p-\ell)u_\ell^i(n+1,m;\alpha)}{k+\ell}$$

$$= (p+k)\,\frac{\rho_k(n,m)k^i}{k+\alpha} - \rho_k(n,m)u^{1,i}(n+1,m;\alpha,0). \tag{3.4}$$

Taking into account that the homogeneous integral equation corresponding to (3.1) has only the zero solution, we have

$$(p-k)u_k^i(n+1,m;\alpha) = pu_k^i(n,m;\alpha) + u_k^{i+1}(n,m;\alpha) - u^{1,i}(n+1,m;\alpha,0)u_k^1(n,m;0). \tag{3.5}$$

Using the relation

$$u_k^{i+1}(n,m;0) = u_k^{i+1}(n,m;\alpha) + \alpha u_k^i(n,m;\alpha), \tag{3.6}$$

we have, taking $i=0$,

$$(p-k)u_k^0(n+1,m;\alpha) = (p-\alpha)u_k^0(n,m;\alpha) + [1 - u^{1,0}(n+1,m;\alpha,0)]u_k^1(n,m;0), \tag{3.7}$$

and dividing by $(k+\beta)$ and integrating over C we obtain, using (3.2),

$$(p-\alpha)u^{0,0}(n,m;\alpha,\beta) - (p+\beta)u^{0,0}(n+1,m;\alpha,\beta) + 1$$

$$= [1 - u^{1,0}(n+1,m;\alpha,0)][1 - u^{1,0}(n,m;\beta,0)] \quad , \tag{3.8}$$

where we have also used the symmetry property

$$u^{i,j}(n,m;\alpha,\beta) = u^{j,i}(n,m;\beta,\alpha) \ . \tag{3.9}$$

Because eq. (3.3) is invariant under $p \to -p$, $n \to n+1$, $n+1 \to n$, we also have a second relation which can be obtained from (3.8), replacing p by $-p$ and interchanging n and $n+1$, i.e.

$$(-p-\alpha)u^{0,0}(n+1,m;\alpha,\beta) - (-p+\beta)u^{0,0}(n,m;\alpha,\beta) + 1$$

$$= [1 - u^{1,0}(n,m;\alpha,0)][1 - u^{1,0}(n+1,m;\beta,0)] \quad . \tag{3.10}$$

Finally, in view of (2.2), we also have two relations which can be found from (3.8) and (3.10), replacing p by q, and $(n+1,m)$ and (n,m) by $(n,m+1)$ and (n,m) respectively, i.e.

$$(q-\alpha)u^{0,0}(n,m;\alpha,\beta) - (q+\beta)u^{0,0}(n,m+1;\alpha,\beta) + 1$$

$$= [1 - u^{1,0}(n,m+1;\alpha,0)][1 - u^{1,0}(n,m;\beta,0)] \quad , \tag{3.11}$$

$$(-q-\alpha)u^{0,0}(n,m+1;\alpha,\beta) - (-q+\beta)u^{0,0}(n,m;\alpha,\beta) + 1$$

$$= [1 - u^{1,0}(n,m;\alpha,0)][1 - u^{1,0}(n,m+1;\beta,0)] \quad . \tag{3.12}$$

Eq. (2.4) can now be derived directly by eliminating $u^{1,0}$ from (3.8) and (3.10)-(3.12). In fact, dividing eq. (3.8) by (3.11), and eq. (3.12) with $n \to n+1$ by eq. (3.10) with $m \to m+1$, we obtain equation (2.4).

4. The KdV class; continuum limit

Eq. (2.4) is a difference-difference equation which may be regarded as a discrete analogue of a differential-difference equation. To obtain a corresponding differential-difference equation we consider a limit

$$m \to \infty, \quad b \to 0, \quad mb = t \ , \tag{4.1}$$

in which b is a suitably chosen lattice parameter characterizing the distance between two successive time-points, i.e. two successive sites m and $m+1$, and in which t can be identified with the continuous time variable.

Rewriting the difference-difference equation (2.4) as follows

$$\begin{aligned}
&[p - q - (p+\alpha)(p+\beta)u(n+1,m) + (p-\alpha)(p-\beta)u(n,m+1)][u(n+1,m+1) - u(n,m)] \\
&\quad + (p+q)[u(n+1,m) - u(n,m+1) + (p-q)u(n,m+1)u(n+1,m) \\
&\qquad + (p-q)u(n,m)u(n+1,m+1) - (p-q+\alpha+\beta)u(n,m)u(n+1,m) \\
&\qquad - (p-q-\alpha-\beta)u(n,m+1)u(n+1,m+1)] = 0 ,
\end{aligned} \tag{4.2}$$

it is clear that we can take a continuum limit with $b = p+q$ as lattice parameter, provided that

$$u(n+1,m+1) - u(n,m) = \mathcal{O}(p+q) . \tag{4.3}$$

Eq. (4.3) can be satisfied by relabeling the sites (n,m) of the two-dimensional lattice as (n',m), $n' \equiv n-m$, and by using the relation

$$a(n',m+1) = a(n',t) + (p+q)\partial_t a(n',t) + \mathcal{O}([p+q]^2) , \tag{4.4}$$

for an arbitrary function $a(n',m)$, in the continuum limit. Up to the order $(p+q)$, we then have the relations

$$\begin{aligned}
&u(n,m) \to u(n',t) , \\
&u(n+1,m) \to u(n'+1,t) , \\
&u(n,m+1) \to u(n'-1,t) + (p+q)\partial_t u(n'-1,t) , \\
&u(n+1,m+1) \to u(n',t) + (p+q)\partial_t u(n',t) .
\end{aligned} \tag{4.5}$$

Eq. (2.8) can now be obtained inserting (4.5) in (4.2) and taking only the terms $\mathcal{O}(p+q)$, in the limit $(p+q) \to 0$.

To obtain eq. (2.5) we consider the continuum limit of the factor (2.2), which we rewrite as follows

$$\rho_k(n,m) = \left(\frac{p+k}{p-k}\right)^{n'} \left(1 + \frac{2k(p+q)}{(p-k)(q-k)}\right)^m \rho_k(0,0), \qquad n' \equiv n-m, \tag{4.6}$$

and in the limit (4.1) with $b = p+q$, we immediately obtain

$$\rho_k(n,m) \rightarrow \rho_k(n',t) = \left(\frac{p+k}{p-k}\right)^{n'} \exp\left(\frac{-2kt}{p^2-k^2}\right) \rho_k(0,0) \ . \tag{4.7}$$

If we now drop the primes, it is clear from the proposition given in section 2 that the function $u(n,t)$ defined by (2.7) and (2.5) satisfies the differential-difference equation (2.8), and thus the corollary of the proposition has been proved. (It is also possible to prove this corollary directly from (2.5)-(2.7) without using the proposition itself.)

Note that the integral equation (2.5) can also be formulated in terms of the variable $z \equiv (p+k)(p-k)^{-1}$, with the corresponding factor $\rho_z(n,t) = z^n \exp[\frac{-1}{2p}(z-z^{-1})t]$, cf. ref. 26.

We shall now consider some special cases of eq. (2.8). For that purpose we first rewrite (2.8) using the substitutions

$$\begin{aligned} &u(n,t) \rightarrow \frac{2pu(n,t)}{(p+\alpha)(p+\beta)} \ , \qquad \frac{t}{2p} \rightarrow t \ , \\ &a \equiv \frac{p-\alpha}{p+\alpha} \ , \quad b \equiv \frac{p-\beta}{p+\beta} \ . \end{aligned} \tag{4.8}$$

Eq. (2.8) then becomes

$$\begin{aligned} \partial_t u(n,t) = {} & [1-u(n+1,t)+abu(n-1,t)]^{-1}[u(n-1,t)-u(n+1,t) \\ & + (2ab+a+b)u(n,t)u(n-1,t) + (2+a+b)u(n,t)u(n+1,t) \\ & - (1+a)(1+b)u(n-1,t)u(n+1,t) - (1+a)(1+b)u^2(n,t)] \ . \end{aligned} \tag{4.9}$$

Some special cases of eq. (4.9) are

(i) $a=0$, $b=0$, $A(n,t) \equiv -\ln[1-u(n,t)]$:

$$\partial_t A(n,t) = 2 - e^{A(n,t)-A(n-1,t)} - e^{A(n+1,t)-A(n,t)} \ , \tag{4.10}$$

and $B(n,t) \equiv A(n-2,t) - A(n,t)$ obeys the equation of motion for the Toda lattice [27,28], i.e.

$$\partial_t^2 B(n,t) = 2e^{-B(n,t)} - e^{-B(n+2,t)} - e^{-B(n-2,t)} \ . \tag{4.11}$$

Note that in passing from (4.10) to (4.11) the first-order differential equation (4.10) involving all lattices sites,is decomposed into two identical second-order differential equations (4.11) for the even and odd sites respectively.

(ii) $au(n,t) \to u(n,t)$, $a\to\infty$, $b=0$:

$$\partial_t u(n,t) = [1 - u(n,t) + u(n-1,t)][1 - u(n+1,t) + u(n,t)] - 1 . \tag{4.12}$$

Under the substitution $C(n,t) \equiv u(n,t) - u(n-1,t)$, eq. (4.12) reduces to the discrete Korteweg-de Vries equation, cf. refs. 29 and 30,

$$\partial_t C(n,t) = [1 - C(n,t)][C(n-1,t) - C(n+1,t)] . \tag{4.13}$$

Eq. (4.13) can also be derived from (4.10) using the substitution

$$A(n,t) - A(n-1,t) = \ln[1 - C(n,t)] .$$

(iii) $a=-1$, $b=1$, $D(n,t) = \frac{1}{2}\ln[2u(n,t) - 1]$:

$$\partial_t D(n,t) = \tanh[D(n-1,t) - D(n+1,t)] . \tag{4.14}$$

(iv) $au(n,t) \to -u(n,t)$, $a\to\infty$, $b=1$,

$$E(n,t) \equiv [u(n,t) - u(n-1,t)][u(n-1,t) - 1]^{-1} :$$

$$\partial_t E(n,t) = [1 - E^2(n,t)][E(n-1,t) - E(n+1,t)] , \tag{4.15}$$

the discrete modified Korteweg-de Vries equation [31].

From the considerations given above it is clear that all differential-difference equations (4.10)-(4.15) are integrable, since their solutions can be obtained from the linear integral equation (2.5).

5. The KdV class; connection with Bäcklund transformations

From appendix A of ref. 10 it is clear that the transformation

$$\rho_k \to \tilde{\rho}_k = \frac{q+k}{q-k}\,\rho_k \tag{5.1}$$

induces a Bäcklund transformation of the singular integral equation

$$u_k + i\rho_k \int_C d\lambda(\ell)\, \frac{u_\ell}{k+\ell} = \frac{\rho_k}{k+\alpha} \quad , \tag{5.2}$$

with a corresponding transformation

$$u = \int_C d\lambda(k)\, \frac{u_k}{k+\beta} \rightarrow \tilde{u} = \int_C d\lambda(k)\, \frac{\tilde{u}_k}{k+\beta} \quad , \tag{5.3}$$

where $\tilde{u}_k$ is the solution of (5.2) with $\tilde{\rho}_k$ instead of ρ_k. (Note that it is also possible to obtain the function $\tilde{u}$, defined in (5.3),by a singular transformation of the measure $d\lambda(k) \rightarrow d\tilde{\lambda}(k) = (q+k)(q-k)^{-1}d\lambda(k)$ as $\tilde{u} = \int_C d\tilde{\lambda}(k)\tilde{u}_k(k+\beta)^{-1}$, where now $\tilde{u}_k$ is the solution of (5.2) with the measure $d\tilde{\lambda}(k)$ instead of $d\lambda(k)$, see ref. 10.)

Comparing eq. (5.1) with the relation

$$\rho_k(n,m+1) = \rho_k(n,m)\, \frac{q+k}{q-k} \quad , \tag{5.4}$$

which follows from (2.2), it is clear that eq. (2.4) with $u(n,m) \rightarrow u(n',t)$, $u(n+1,m) \rightarrow u(n'+1,t)$, $u(n,m+1) \rightarrow \tilde{u}(n'-1,t)$, $u(n+1,m+1) \rightarrow \tilde{u}(n',t)$, i.e.

$$\begin{aligned} &\left[(p+\beta)u(n'+1,t) - (p-\alpha)u(n',t) - 1\right]\left[(p-\beta)\tilde{u}(n'-1,t) - (p+\alpha)\tilde{u}(n',t) + 1\right] \\ &= \left[(q+\beta)\tilde{u}(n'-1,t) - (q-\alpha)u(n',t) - 1\right]\left[(q-\beta)u(n'+1,t) - (q+\alpha)\tilde{u}(n',t) + 1\right] , \end{aligned} \tag{5.5}$$

provides a Bäcklund transformation of the differential-difference equation (2.8).

Furthermore introducing a second Bäcklund transformation

$$\rho_k \rightarrow \hat{\rho}_k = \frac{p+k}{p-k}\, \rho_k \quad , \tag{5.6}$$

leading to

$$u \rightarrow \hat{u} = \int_C d\lambda(k)\, \frac{\hat{u}_k}{k+\beta} \quad , \tag{5.7}$$

where $\hat{u}_k$ is the solution of (5.2) with $\hat{\rho}_k$ instead of ρ_k, it is clear that eq. (2.4) with $u(n,m) \rightarrow u$, $u(n+1,m) \rightarrow \hat{u}$, $u(n,m+1) \rightarrow \tilde{u}$, $u(n+1,m+1) \rightarrow \hat{\tilde{u}}$, i.e.

$$[(p+\beta)\hat{u} - (p-\alpha)u - 1][(p-\beta)\tilde{u} - (p+\alpha)\hat{\tilde{u}} + 1]$$

$$=[(q+\beta)\tilde{u} - (q-\alpha)u - 1][(q-\beta)\hat{u} - (q+\alpha)\hat{\tilde{u}} + 1] \quad , \tag{5.8}$$

is a Bianchi-identity (see refs. 15, 17 and 18) expressing the commutativity of the BT's (5.1) and (5.6). This holds independently of the specific dependence of the factor ρ_k on variables or lattice sites. In fact, (5.5) is a Bianchi-identity for any partial differential equation which can be derived from the integral equation (5.2) with factor $\rho_k = \rho_k(x,t)$, for any differential-difference equation which follows from (5.2) with factor $\rho_k(n,t) = (s+k)^n(s-k)^{-n}\rho_k(0,t)$, as well as for any difference-difference equation which can be derived from (5.2) with arbitrary $\rho_k(n,m)$.

On the other hand, as we have shown in the previous sections, a Bianchi-identity involving u, $\hat{u}$, $\tilde{u}$, $\hat{\tilde{u}}$, following from two BT's, as given by (5.1) and (5.6) of the integral equation (5.2) with the replacements $u \to u(n,m)$, $\hat{u} \to u(n+1,m)$, $\tilde{u} \to u(n,m+1)$ and $\hat{\tilde{u}} \to u(n+1,m+1)$ leads in a natural way to an integrable difference-difference equation (2.4) associated with the integral equation (2.1), (2.2).

The above procedure can be applied to other linear integral equations as well (see e.g. refs. 5 and 6). As a first step one derives the Bianchi-identity expressing the commutativity of Bäcklund transformations of the factor ρ_k in the integral equation. Secondly the Bianchi-identity is interpreted as an (integrable) difference-difference equation which can be derived from the integral equation with a factor ρ_k such as specified in (2.2). Furthermore choosing a small parameter for which one may take a continuum limit of the type (4.1), one can derive an (integrable) differential-difference equation, the solutions of which can be obtained from the integral equation with a factor $\rho_k(n,t) = (p+k)^n(p-k)^{-n}\rho_k(0,t)$ and the Bianchi-identity immediately leads to a BT for the differential-difference equation. In the following sections we shall work out the procedure mentioned above for the integral equation of the NLS type.

6. The NLS class; results

In this section we present two linear integral equations involving an arbitrary contour and measure, which linearize certain nonlinear difference-difference equations and differential-difference equations of the nonlinear

Schrödinger type.

Proposition

Let $\phi_k(n,m;\alpha)$ be a solution of the linear integral equation

$$\phi_k(n,m;\alpha) + \int_C d\lambda(\ell) \int_{C^*} d\lambda^*(\ell') \, \frac{\rho_k(n,m)\rho^*_{\ell'}(n,m)}{(k-\ell')(\ell'-\ell)} \, \phi_\ell(n,m;\alpha) = \frac{\rho_k(n,m)}{k+\alpha} \, ,$$
$$n,m \in \mathbb{Z}, k,\alpha \in \mathbb{C} \, , \qquad (6.1)$$

where C and $d\lambda(k)$ are an arbitrary contour and measure in the complex k-plane, and

$$\rho_k(n,m) = \left(\frac{p-k}{p+k}\right)^n \left(\theta' \frac{q-k}{q^*-k}\right)^m \rho_k(0,0) \, , \qquad p,q,\theta' \in \mathbb{C} \, , \quad p=-p^* , \quad |\theta'| = 1 \, . \qquad (6.2)$$

Let the contour C and the measure $d\lambda(k)$ be such that the homogeneous integral equation corresponding to (6.1) has only the zero solution, and define

$$\phi(n,m;\alpha,\beta) \equiv \int_C \frac{\phi_k(n,m;\alpha)}{k+\beta} \, d\lambda(k) \, , \qquad n,m \in \mathbb{Z}, \quad \beta \in \mathbb{C} \, . \qquad (6.3)$$

Then the following results hold, for special choices of p, q, α, β and θ'.

(1) The function

$$\phi(n,m) \equiv 2p\phi(n,m;-p,p) \qquad (6.4)$$

obeys the double-discrete nonlinear Schrödinger equation (ddNLS), i.e.

$$2|p|^2 + 2|q|^2 + \theta'(q+p)(q-p)\phi(n,m)\phi^*(n,m+1)$$
$$+ \theta'^*(q^*+p^*)(q^*-p^*)\phi^*(n,m)\phi(n,m+1) =$$
$$|q+p|^2\left(1+|\phi(n,m+1)|^2\right) \frac{(q^*-p^*)\phi(n+1,m+1) - \theta'(q-p)\phi(n,m)}{\theta'(q+p)\phi(n+1,m) - (q^*+p^*)\phi(n,m+1)}$$
$$+ |q-p|^2\left(1+|\phi(n,m)|^2\right) \frac{\theta'(q+p)\phi(n+1,m) - (q^*+p^*)\phi(n,m+1)}{(q^*-p^*)\phi(n+1,m+1) - \theta'(q-p)\phi(n,m)} \, . \qquad (6.5)$$

(2) If $\lambda \equiv (q+q^*)|p||q|^{-2}$, $\theta' = q^*q^{-1}$, $|q-q^*|^2|p|^2 = 4|q|^4$, then the vector function

$$\vec{S}(n,m) = \left(\frac{S^+(n,m) + S^{+*}(n,m)}{2} , \frac{S^+(n,m) - S^{+*}(n,m)}{2i} , [1 - |S^+(n,m)|^2]^{\frac{1}{2}} \right) , \tag{6.6}$$

where

$$S^+(n,m) = p\phi(n+1,m;0,0) - p\phi(n,m;0,0) , \tag{6.7}$$

obeys the double-discrete isotropic Heisenberg spin chain (ddIHSC), i.e.

$$\begin{aligned} &\vec{S}(n,m+1) - \vec{S}(n,m) + \lambda \frac{\vec{S}(n,m) \times \vec{S}(n,m+1)}{1 + \vec{S}(n,m)\cdot\vec{S}(n,m+1)} \\ &\quad - [\vec{S}(n,m) + \vec{S}(n,m+1)] \left[1 - \tfrac{1}{2}\lambda^2 \frac{\{1 - \vec{S}(n,m)\cdot\vec{S}(n,m+1)\}}{\{1 + \vec{S}(n,m)\cdot\vec{S}(n,m+1)\}^2} \right]^{\frac{1}{2}} \\ &\quad = \vec{S}(n+1,m) - \vec{S}(n+1,m+1) + \lambda \frac{\vec{S}(n+1,m) \times \vec{S}(n+1,m+1)}{1 + \vec{S}(n+1,m)\cdot\vec{S}(n+1,m+1)} \\ &\quad - \left[\vec{S}(n+1,m) + \vec{S}(n+1,m+1)\right] \left[1 - \tfrac{1}{2}\lambda^2 \frac{\{1 - \vec{S}(n+1,m)\cdot\vec{S}(n+1,m+1)\}}{\{1 + \vec{S}(n+1,m)\cdot\vec{S}(n+1,m+1)\}^2} \right]^{\frac{1}{2}} , \\ &\qquad \vec{S}(n,m)\cdot\vec{S}(n,m) = 1. \end{aligned} \tag{6.8}$$

(3) If $\lambda \equiv qp^{-1}$, $\theta' = 1$, $q^* = -q$, then the function

$$s(n,m) = \tfrac{1}{2}p\phi(n,m;0,0) \tag{6.9}$$

obeys the double-discrete complex sine-Gordon equation (ddCSG) , i.e.

$$\begin{aligned} &\lambda[s(n,m) + s(n,m+1)][1 - 4|s(n,m+1) - s(n+1,m+1)|^2]^{\frac{1}{2}} \\ &\quad + \lambda[s(n+1,m) + s(n+1,m+1)][1 - 4|s(n,m+1) - s(n+1,m+1)|^2]^{\frac{1}{2}} \\ &\quad + [s(n,m+1) - s(n+1,m+1)][1 - 4\lambda^2|s(n,m) + s(n+1,m+1)|^2]^{\frac{1}{2}} \\ &\quad - [s(n,m) - s(n+1,m)][1 - 4\lambda^2|s(n+1,m) + s(n+1,m+1)|^2]^{\frac{1}{2}} = 0 . \end{aligned} \tag{6.10}$$

Remarks

(i) Various double-discrete versions of the NLS and the IHSC have been obtained in the literature with corresponding Gel'fand-Levitan equation [11] or bi-linearization [13].

(ii) In appendix B eqs. (6.5), (6.8) and (6.10) are obtained as special reductions of the two coupled equations

$$\begin{aligned}
&\theta[1 - (p+\beta)\hat{\tilde{\psi}}^* + (p+\alpha)\tilde{\psi}^*][\theta'(q+\beta)\phi - (q^*+\alpha)\tilde{\phi}] \\
&\quad + [1 + (q^*+\alpha)\tilde{\psi}^* - (q^*+\beta)\psi^*][\theta(p+\alpha)\tilde{\phi} - (p^*+\beta)\hat{\tilde{\phi}}] \\
&\quad = \theta'[1 - (q+\beta)\hat{\tilde{\psi}}^* + (q+\alpha)\hat{\psi}^*][\theta(p+\beta)\phi - (p^*+\alpha)\hat{\phi}] \\
&\quad + [1 + (p^*+\alpha)\hat{\psi}^* - (p^*+\beta)\psi^*][\theta'(q+\alpha)\hat{\phi} - (q^*+\beta)\hat{\tilde{\phi}}] \ , \qquad (6.11)
\end{aligned}$$

$$\begin{aligned}
&[\theta(p+\beta)\hat{\tilde{\phi}}^* - (p^*+\alpha)\tilde{\phi}^*][\theta'(q+\beta)\phi - (q^*+\alpha)\tilde{\phi}] \\
&\quad + [1 - (p^*+\beta)\hat{\tilde{\psi}} + (p^*+\alpha)\tilde{\psi}][1 + (q^*+\alpha)\tilde{\psi}^* - (q^*+\beta)\psi^*] \\
&\quad = [\theta'(q+\beta)\hat{\tilde{\phi}}^* - (q^*+\alpha)\hat{\phi}^*][\theta(p+\beta)\phi - (p^*+\alpha)\hat{\phi}] \\
&\quad + [1 - (q^*+\beta)\hat{\tilde{\psi}} + (q^*+\alpha)\hat{\psi}][1 + (p^*+\alpha)\hat{\psi}^* - (p^*+\beta)\psi^*] \ , \qquad (6.12)
\end{aligned}$$

where we have used the notations

$$F = F(n,m;\alpha,\beta) \ , \quad \hat{F} = F(n+1,m;\alpha,\beta) \ ,$$

$$\tilde{F} = F(n,m+1;\alpha,\beta) \ , \quad \hat{\tilde{F}} = F(n+1,m+1;\alpha,\beta) \ ,$$

$$F^* \equiv [F(n,m;\alpha^*,\beta^*)]^* , \ \text{etc., for } F = \phi,\psi,$$

$$\psi(n,m;\alpha,\beta) \equiv \int_C d\lambda(k) \int_{C^*} d\lambda^*(\ell') \, \frac{\rho_k \phi^*_{\ell'}(n,m;\alpha)}{(k+\beta)(k-\ell')} \ , \qquad (6.13)$$

in combination with the relations

$$1 = [1 - (p+\beta)\hat{\psi}^* + (p+\alpha)\psi^*][1 - (p^*+\beta)\hat{\psi} + (p^*+\alpha)\psi]$$
$$+ [\theta(p+\alpha)\phi - (p^*+\beta)\hat{\phi}][\theta^*(p^*+\alpha)\phi^* - (p+\beta)\hat{\phi}^*] \quad , \qquad (6.14)$$

$$1 = [1 - (q+\beta)\tilde{\psi}^* + (q+\alpha)\psi^*][1 - (q^*+\beta)\tilde{\psi} + (q^*+\alpha)\psi]$$
$$+ [\theta'(q+\alpha)\phi - (q^*+\beta)\tilde{\phi}][\theta'^*(q^*+\alpha)\phi^* - (q+\beta)\tilde{\phi}^*] \quad , \qquad (6.15)$$

and

$$[1 + (\alpha-\beta)\psi^*][1 + (\alpha-\beta)\psi] + (\alpha-\beta)^2\phi\phi^* = 1 \quad . \qquad (6.16)$$

In appendix A, eqs. (6.11), (6.12), (6.14)-(6.16) are derived from the integral equation (6.1) with

$$\rho_k(n,m) = \left(\theta \frac{p-k}{p^*-k}\right)^n \left(\theta' \frac{q-k}{q^*-k}\right)^m \rho_k(0,0),$$
$$p,q,\theta,\theta' \in \mathbb{C}, \quad |\theta| = |\theta'| = 1. \qquad (6.17)$$

Equations (6.11) and (6.12) may be regarded as Bianchi-identities expressing the commutativity of the two Bäcklund transformations induced by

$$\rho_k \to \hat{\rho}_k = \theta\left(\frac{p-k}{p^*-k}\right) \rho_k \quad , \qquad |\theta| = 1 \quad , \; (6.18)$$

$$\rho_k \to \tilde{\rho}_k = \theta'\left(\frac{q-k}{q^*-k}\right) \rho_k \quad , \qquad |\theta'| = 1 \; , \; (6.19)$$

cf. (6.17) and (6.13) and (6.3).

(iii) Eq. (6.10) is a double-discrete version of the complex sine-Gordon equation. A double-discrete version of the sine-Gordon equation (cf. refs. 12 and 32) may be obtained from the integral equation (2.1) in the special case that

$$\rho_k(n,m) = \left(\frac{p+k}{p-k}\right)^n \left(\frac{q+k^{-1}}{q-k^{-1}}\right)^m \rho_k(0,0) \; , \qquad n,m \in \mathbb{Z} \; ,p,q,k \in \mathbb{C} \, . \; (6.20)$$

In fact, in ref. 24 it has been shown that the function

$$w(n,m) = \frac{1}{2i} \ln\left[-1 + i\int_C d\lambda(k) u_k(n,m)\right] \quad , \tag{6.21}$$

where $u_k(n,m)$ is the solution of (2.1), with (6.20), and with $\alpha=0$, satisfies

$$\begin{aligned} &\sin[w(n,m) + w(n+1,m) + w(n,m+1) + w(n+1,m+1)] \\ &\quad - pq \sin[w(n,m) + w(n+1,m+1) - w(n+1,m) - w(n,m+1)] = 0 \ , \end{aligned} \tag{6.22}$$

which can be regarded as the double-discrete sine-Gordon equation.

Corollary

Let $\phi_k(n,t;\alpha)$ be a solution of the linear integral equation

$$\phi_k(n,t;\alpha) + \int_C d\lambda(\ell) \int_{C^*} d\lambda^*(\ell') \frac{\rho_k(n,t)\rho^*_{\ell'}(n,t)}{(k-\ell')(\ell'-\ell)} \phi_\ell(n,t;\alpha) = \frac{\rho_k(n,t)}{k+\alpha} \ ,$$
$$n \in \mathbb{Z}, \quad t \in \mathbb{R}, \quad k,\alpha \in \mathbb{C}, \tag{6.23}$$

where C and $d\lambda(k)$ are an arbitrary contour and measure in the complex k-plane. Let the contour C and the measure $d\lambda(k)$ be such that the homogeneous integral equation corresponding to (6.23) has only the zero solution, and define

$$\phi(n,t;\alpha,\beta) \equiv \int_C \frac{\phi_k(n,t;\alpha)}{k+\beta} d\lambda(k) \ , \qquad n \in \mathbb{Z}, \quad t \in \mathbb{R}, \quad \alpha,\beta \in \mathbb{C}. \tag{6.24}$$

Then the following results hold for special choices of $\rho_k(n,t)$, p, α and β.

(1) If

$$\rho_k(n,t) = \left(\frac{p-k}{p+k}\right)^n \exp\left[it \frac{[2pk(f^*-f) + 2k^2(f+f^*)]}{p^2-k^2}\right] \rho_k(0,0) \ ,$$
$$p,f \in \mathbb{C}, \quad p = -p^* \ , \quad |f| = 1 \ , \tag{6.25}$$

then the function

$$\phi(n,t) \equiv 2p\phi(n,t;-p,p) \tag{6.26}$$

obeys the discrete nonlinear Schrödinger equation (dNLS), i.e.

$$i\partial_t\phi(n,t) = (f+f^*)\phi(n,t) - [1 + |\phi(n,t)|^2][f\phi(n+1,t) + f^*\phi(n-1,t)] \ . \quad (6.27)$$

(2) If $\rho_k(n,t)$ is given by (6.25), then the vector function

$$\vec{S}(n,t) = \left(\frac{S^+(n,t) + S^{+*}(n,t)}{2}, \frac{S^+(n,t) - S^{+*}(n,t)}{2i}, [1 - |S^+(n,t)|^2]^{\frac{1}{2}}\right), \quad (6.28)$$

where

$$S^+(n,t) = p\phi(n+1,t;0,0) - p\phi(n,t;0,0) \ , \quad (6.29)$$

obeys the discrete isotropic Heisenberg spin chain (dIHSC), i.e.

$$\partial_t\vec{S}(n,t) = (f+f^*)\left[\frac{\vec{S}(n,t) \times \vec{S}(n+1,t)}{1 + \vec{S}(n,t)\cdot\vec{S}(n+1,t)} - \frac{\vec{S}(n-1,t) \times \vec{S}(n,t)}{1 + \vec{S}(n-1,t)\cdot\vec{S}(n,t)}\right],$$

$$\vec{S}(n,t)\cdot\vec{S}(n,t) = 1. \quad (6.30)$$

(3) If

$$\rho_k(n,t) = \left(\frac{p-k}{p+k}\right)^n \exp\left(-\frac{pt}{k}\right)\rho_k(0,0) \ , \qquad p\in\mathbb{C},\ p=-p^*, \quad (6.31)$$

then the function

$$s(n,t) \equiv \tfrac{1}{2}p\phi(n,t;0,0) \quad (6.32)$$

obeys the discrete complex sine-Gordon equation (dCSG)

$$\partial_t[s(n,t) - s(n-1,t)] = [s(n,t) + s(n-1,t)][1 - 4|s(n,t) - s(n-1,t)|^2]^{\frac{1}{2}} \ . \quad (6.33)$$

Remarks

(i) The integral equations (6.1) and (6.22) may be regarded as double-discrete and single-discrete analogues of the integral equation

$$\phi_k(x,t) + \int_C d\lambda(\ell) \int_{C^*} d\lambda^*(\ell') \frac{\rho_k(x,t)\rho^*_{\ell'}(x,t)}{(k-\ell')(\ell'-\ell)} \phi_\ell(x,t) = \frac{\rho_k(x,t)}{k+\alpha} ,$$

$$\rho_k(x,t) = e^{i[kx-\omega(k)t]}\rho_k(0,0), \qquad x,t \in \mathbb{R}, \quad k \in \mathbb{C} , \tag{6.34}$$

which gives a direct linearization of e.g. the nonlinear Schrödinger equation, the equation of motion for the isotropic Heisenberg spin chain, and the complex sine-Gordon equation, see chapter II of this thesis.

(ii) From the proposition given in this section and from its corollary it is clear that the difference-difference equations (6.5), (6.8) and (6.10), and the differential-difference equations (6.27), (6.30) and (6.33) are also completely integrable, since solutions can be obtained from the linear integral equations (6.1) and (6.23).

7. The NLS class; continuum limit

In section 6 we have given some difference-difference equations of the NLS class, i.e. the ddNLS (6.5), the ddIHSC (6.8) and the ddCSG (6.10). To obtain the corresponding differential-difference equations we consider the limit (4.1), in which b is a suitably chosen lattice parameter.

From eq. (6.5) it is clear that we can take

$$q \to -p , \tag{7.1}$$

$$\theta'(q-p) - (q^*-p^*) \to 0 , \tag{7.2}$$

$$\phi(n+1,m+1;-p,p) \to \phi(n,m;-p,p) . \tag{7.3}$$

Eq. (7.1) suggests that we can choose $p+q$ as a small parameter and eq. (7.2) with $p^*=-p$ is automatically satisfied for $\theta' = q^*/q$.

In eq. (6.8) we can take a limit

$$\lambda \to 0 , \quad |q| \to |p| , \tag{7.4}$$

$$\vec{S}(n+1,m+1) \to \vec{S}(n,m) . \tag{7.5}$$

Therefore, in both cases, $p+q$ can be chosen as a small parameter, provided that (7.3) and (7.5) respectively are satisfied. Accordingly we take

$$p+q \to -2ipf\,\frac{t}{m}\ , \tag{7.6}$$

where f is a phase factor expressing the phase of $p+q$.
Up to the order $(p+q)$ we then have the following relations

$$\begin{aligned}
&\phi(n,m) \to \phi(n',t)\ , \qquad n' \equiv n-m\ ,\\
&\phi(n+1,m) \to \phi(n'+1,t)\ ,\\
&\phi(n,m+1) \to \phi(n'-1,t) + \tfrac{1}{2}i\,\frac{p+q}{p}\,f^*\partial_t\phi(n'-1,t)\ ,\\
&\phi(n+1,m+1) \to \phi(n',t) + \tfrac{1}{2}i\,\frac{p+q}{p}\,f^*\partial_t\phi(n',t)\ ,
\end{aligned} \tag{7.7}$$

and also

$$\begin{aligned}
&\vec{S}(n,m) \to \vec{S}(n',t)\ ,\\
&\vec{S}(n+1,m) \to \vec{S}(n'+1,t)\ ,\\
&\vec{S}(n,m+1) \to \vec{S}(n'-1,t) + \tfrac{1}{2}i\,\frac{p+q}{p}\,f^*\partial_t\vec{S}(n'-1,t)\ ,\\
&\vec{S}(n+1,m+1) \to \vec{S}(n',t) + \tfrac{1}{2}i\,\frac{p+q}{p}\,f^*\partial_t\vec{S}(n',t)\ .
\end{aligned} \tag{7.8}$$

Taking (6.5) with $p^* = -p$, $\theta' = q^*/q$ and (7.1) and neglecting all terms $\mathcal{O}(p+q)$, we have

$$1 = [1 + |\phi(n,m)|^2]\,\frac{(p+q)\phi(n+1,m) + \frac{f^*}{f}\,(p+q)\phi(n,m+1)}{(p+q)(1+\frac{f^*}{f})\phi(n,m) - 2p[\phi(n+1,m+1) - \phi(n,m)]}\ , \tag{7.9}$$

and (6.27) follows immediately inserting (7.7) in (7.9). Analogously eq. (6.30) follows from (6.8) inserting (7.8) and considering only terms $\mathcal{O}(\lambda)$.

The integral equation (6.23), together with eq. (6.25), can be derived from (6.1) taking the continuum limit of the factor $\rho_k(n,m)$ given in (6.2). We first rewrite $\rho_k(n,m)$ as follows,

$$\rho_k(n,m) = \left(\frac{p-k}{p+k}\right)^{n'}\left[1 - \frac{(\theta'pq+p^*q^*) - k(p^*+q^*+p\theta'+q\theta') + k^2(1+\theta')}{(p^*-k)(q^*-k)}\right]^m\ ,$$

$$n' = n-m, \tag{7.10}$$

and from $p^* = -p$, $\theta' = q^*/q$ and eq. (7.5) we have

$$\frac{(\theta' pq+p^*q^*) - k(p^*+q^*+p\theta'+q\theta') + k^2(1+\theta')}{(p^*-k)(q^*-k)} \to \frac{2pki(f^*-f)\frac{t}{m} + 2k^2i(f^*+f)\frac{t}{m}}{|p|^2 + k^2} , \tag{7.11}$$

and from (7.10) we immediately obtain

$$\rho_k(n,m) \to \rho_k(n',t) , \tag{7.12}$$

in which $\rho_k(n',t)$ is given by (6.25).

Finally we can take a limit of eq. (6.10) with $\lambda \to 0$, provided that

$$s(n,m+1) - s(n+1,m+1) - s(n,m) + s(n+1,m) \to 0 . \tag{7.13}$$

Therefore we take

$$\begin{aligned}
&s(n,m) \to s(n,t) , \\
&s(n+1,m) \to s(n+1,t) , \\
&s(n,m+1) \to s(n,t) + 2\lambda\partial_t s(n,t) , \\
&s(n+1,m+1) \to s(n+1,t) + 2\lambda\partial_t s(n+1,t) ,
\end{aligned} \tag{7.14}$$

and keeping only the terms of order λ, eq. (6.10) reduces to eq. (6.33). The integral equation (6.23), together with eq. (6.31), follows by considering the continuum limit of (6.2) with $q = \lambda p = \frac{1}{2}p\frac{t}{m}$, $\theta' = 1$. We have

$$\frac{\rho_k(n,t)}{\rho_k(0,0)} = \left(\frac{p-k}{p+k}\right)^n \left(\frac{\frac{1}{2}p\frac{t}{m} - k}{-\frac{1}{2}p\frac{t}{m} - k}\right)^m \to \left(\frac{p-k}{p+k}\right)^n \exp\left(\frac{-pt}{k}\right) , \tag{7.15}$$

in agreement with (6.31).

Remarks

(i) In this section the corollary of the proposition given in section 6 has been proved by taking a suitably chosen continuum limit of the integral equation, as well as of the difference-difference equation given in the proposition. Of course eqs. (6.27), (6.30) and (6.33) can also be derived directly from (6.23) with (6.25) and (6.31). In fact in ref. 33 the dNLS, dIHSC and dCSG have been obtained from an (equivalent) integral equation in terms of the complex variable $z = (p-k)/(p+k)$ and $\rho_k(n,m) \to z^{n'} e^{i\omega(z)t}$, where the dispersion $\omega(z)$ is given by $\omega(z) = fz + f^*z^{-1} - (f+f^*)$ in the case of the dNLS and the dIHSC and by $\omega(z) = -i(z+1)(z-1)^{-1}$ in the case of the dCSG.

(ii) In the dNLS (6.27) we can take $f = f^* = 1$ without loss of generality, as can be seen introducing a new function $\Phi(n,t) = f^n\phi(n,t)\exp[i(f+f^*-2)t]$. Taking a continuum limit, however, i.e. $\phi(n\pm1,t) = \phi(x,t) \pm a\partial_x\phi(x,t) + \frac{1}{2}a^2\partial_x^2\phi(x,t) \pm \frac{1}{6}a^3\partial_x^3\phi(x,t)$, eq. (6.27) after some obvious transformations changes into the NLS $i\partial_t\phi = -\partial_x^2\phi - 2|\phi|^2\phi$ for $f=1$, and into the CMKdV $\partial_t\phi = \partial_x^3\phi + 6|\phi|^2\partial_x\phi$ for $f=-i$.

(iii) The dIHSC, which was given in refs. 34 and 35 for $f=1$, reduces in the continuum limit to $\partial_t\vec{S} = \vec{S}\times\partial_x^2\vec{S}$, which is the well-known IHSC. For $f=i$, eq. (6.30) is trivial and a higher-order expansion in powers of λ is necessary to obtain a meaningful differential-difference equation.

(iv) In section 5 we discussed the relation between the difference-difference equations for the KdV class on the one hand, and the BT for the differential-difference equations, as well as the Bianchi-identities expressing the commutativity of BT's on the other hand. Such relations exist also in connection with the difference-difference equations of the NLS class.

In fact eq. (B.5) of appendix B, with $p^*=-p$, $\theta=-1$, $2p\phi(-p,p) \to \phi(n',t)$, $2p\hat{\phi}(-p,p) \to \phi(n'+1,t)$, $2p\tilde{\phi}(-p,p) \to \tilde{\phi}(n'-1,t)$ and $2p\hat{\tilde{\phi}}(-p,p) \to \tilde{\phi}(n',t)$ is the BT for the dNLS (6.27) associated with the transformation

$$\rho_k(n,t) \to \tilde{\rho}_k(n,t)\theta'\left(\frac{q-k}{q^*-k}\right) , \tag{7.16}$$

whereas eq. (B.5) with parameters $r=-r^*$ and s, instead of $p=-p^*$ and q, expresses the commutativity of the BT's

$$\rho_k \to \tilde{\rho}_k = \rho_k\theta'\,\frac{s-k}{s^*-k} , \quad \rho_k \to \hat{\rho}_k = -\rho_k\theta\,\frac{r-k}{r+k} , \tag{7.17}$$

for the NLS, the dNLS and ddNLS, independent of the specific dependence of ρ_k on variables or lattice sites.

Furthermore eq. (B.17) with $\vec{S} \to \vec{S}(n',t)$, $\hat{\vec{S}} \to \vec{S}(n'+1,t)$, $\tilde{\vec{S}} \to \tilde{\vec{S}}(n'-1,t)$, $\hat{\tilde{\vec{S}}} \to \tilde{\vec{S}}(n',t)$ provides a BT for the dIHSC (6.30) under the transformation (7.16) with $\theta' = q^*/q$ and eq. (B.19) with $\frac{1}{2}p\phi \to s(n,t)$, $\frac{1}{2}p\hat{\phi} \to s(n+1,t)$, $\frac{1}{2}p\tilde{\phi} \to \tilde{s}(n,t)$, $\frac{1}{2}p\hat{\tilde{\phi}} \to \tilde{s}(n+1,t)$ is the BT for the dCSG (6.33) corresponding to (7.16) with $\theta'=1$. Eqs. (B.17) and (B.19) can also be regarded as expressing the commutativity of Bäcklund transformations.

Appendix A

In this appendix we derive the coupled equations for the NLS class (6.11), (6.12) and (6.14)-(6.16) starting from the following two integral equations

$$\phi_k^i(\alpha) + \int_{C^*} d\lambda^*(\ell') \frac{\rho_k}{k-\ell'} \psi_{\ell'}^{i*}(\alpha) = \rho_k \frac{k^i}{k+\alpha} \quad , \tag{A.1}$$

$$\psi_k^i(\alpha) - \int_{C^*} d\lambda^*(\ell') \frac{\rho_k}{k-\ell'} \phi_{\ell'}^{i*}(\alpha) = 0 \quad , \tag{A.2}$$

or equivalently

$$\phi_k^i(\alpha) + \int_C d\lambda(\ell) \int_{C^*} d\lambda^*(\ell') \frac{\rho_k \rho_{\ell'}^*}{(k-\ell')(\ell'-\ell)} \phi_\ell^i(\alpha) = \rho_k \frac{k^i}{k+\alpha} \quad , \tag{A.3}$$

which for $i=0$ with $\rho_k = \rho_k(n,m)$ reduces to (6.1), and

$$\psi_k^i(\alpha) + \int_C d\lambda(\ell) \int_{C^*} d\lambda^*(\ell') \frac{\rho_k \rho_{\ell'}^*}{(k-\ell')(\ell'-\ell)} \psi_\ell^i(\alpha) = \int_{C^*} d\lambda^*(\ell') \frac{\rho_k \rho_{\ell'}^*}{k-\ell'} \frac{\ell'^i}{\ell'+\alpha} . \tag{A.4}$$

Define

$$\phi^{i,j}(\alpha,\beta) \equiv \int_C d\lambda(k) \frac{\phi_k^j(\alpha) k^i}{k+\beta} \quad , \tag{A.5}$$

$$\psi^{i,j}(\alpha,\beta) \equiv \int_C d\lambda(k) \frac{\psi_k^j(\alpha) k^i}{k+\beta} \quad . \tag{A.6}$$

It is easy to show that

$$\phi^{i,j}(\alpha,\beta) = \phi^{j,i}(\beta,\alpha) \quad , \tag{A.7}$$

$$\psi^{i,j}(\alpha,\beta) = -\psi^{j,i*}(\beta,\alpha) \quad , \tag{A.8}$$

$$\phi_k^{i+1}(0) = \phi_k^{i+1}(\alpha) + \alpha\phi_k^i(\alpha) \; , \tag{A.9}$$

$$\psi_k^{i+1}(0) = \psi_k^{i+1}(\alpha) + \alpha\psi_k^i(\alpha) \; . \tag{A.10}$$

Using (A.1) and (A.2) we have

$$k\phi_k^i(\alpha) + \int_{C^*} d\lambda^*(\ell') \frac{\rho_k}{k-\ell'} \ell'\psi_{\ell'}^{i^*}(\alpha) = \rho_k \frac{k^{i+1}}{k+\alpha} - \psi^{1,i^*}(\alpha,0)\rho_k , \tag{A.11}$$

$$k\psi_k^i(\alpha) - \int_{C^*} d\lambda^*(\ell') \frac{\rho_k}{k-\ell'} \ell'\phi_{\ell'}^{i^*}(\alpha) = \phi^{1,i^*}(\alpha,0)\rho_k , \tag{A.12}$$

leading to

$$k\phi_k^i(\alpha) + \int_C d\lambda(\ell) \int_{C^*} d\lambda^*(\ell') \frac{\rho_k\rho_{\ell'}^*}{(k-\ell')(\ell'-\ell)} \ell\phi_\ell^i(\alpha)$$

$$= \rho_k \frac{k^{i+1}}{k+\alpha} - \psi^{1,i^*}(\alpha,0)\rho_k - \phi^{1,i}(\alpha,0) \int_{C^*} d\lambda^*(\ell') \frac{\rho_k\rho_{\ell'}^*}{k-\ell'} , \tag{A.13}$$

and

$$k\psi_k^i(\alpha) + \int_C d\lambda(\ell) \int_{C^*} d\lambda^*(\ell') \frac{\rho_k\rho_{\ell'}^*}{(k-\ell')(\ell'-\ell)} \ell\psi_\ell^i(\alpha)$$

$$= \phi^{1,i^*}(\alpha,0)\rho_k + \int_{C^*} d\lambda^*(\ell') \frac{\rho_k\rho_{\ell'}^*}{k-\ell'} \frac{\ell'^{i+1}}{\ell'+\alpha} - \psi^{1,i}(\alpha,0) \int_{C^*} d\lambda^*(\ell') \frac{\rho_k\rho_{\ell'}^*}{k-\ell'} . \tag{A.14}$$

Taking into account that the solutions of the integral equations (A.3) and (A.4) are unique and using also (A.9) and (A.10), we have

$$(k+\alpha)\phi_k^i(\alpha) = \phi_k^{i+1}(0) - \psi^{1,i^*}(\alpha,0)\phi_k^1(0) - \phi^{1,i}(\alpha,0)\psi_k^1(0) , \tag{A.15}$$

$$(k+\alpha)\psi_k^i(\alpha) = \psi_k^{i+1}(0) - \psi^{1,i}(\alpha,0)\psi_k^1(0) + \phi^{1,i^*}(\alpha,0)\phi_k^1(0) . \tag{A.16}$$

Dividing (A.15) and (A.16) for $i=0$ by $k+\beta$, integrating over the contour C, and using (A.7) and (A.8), we have

$$(\alpha-\beta)\phi^{0,0}(\alpha,\beta) = \left(1-\psi^{1,0^*}(\alpha,0)\right)\phi^{1,0}(\beta,0) - \left(1-\psi^{1,0^*}(\beta,0)\right)\phi^{1,0}(\alpha,0), \tag{A.17}$$

$$1 + (\alpha-\beta)\psi^{0,0}(\alpha,\beta) = \left(1-\psi^{1,0}(\alpha,0)\right)\left(1-\psi^{1,0^*}(\beta,0)\right) + \phi^{1,0^*}(\alpha,0)\phi^{1,0}(\beta,0) , \tag{A.18}$$

which for $\beta=\alpha$ reduces to the identity

$$\left(1-\psi^{1,0}(\alpha,0)\right)\left(1-\psi^{1,0^*}(\alpha,0)\right) + \phi^{1,0^*}(\alpha,0)\phi^{1,0}(\alpha,0) = 1 . \tag{A.19}$$

Let us now consider the Bäcklund transformation (6.18) of the integral equation (A.3). It is then straightforward to show that the solutions $\hat{\phi}^i_k(\alpha)$ and $\hat{\psi}^i_k(\alpha)$ of (A.1) and (A.2) with ρ_k replaced by $\hat{\rho}_k$ satisfy the integral relations

$$(p^*-k)\hat{\phi}^i_k(\alpha) + \int_C d\lambda(\ell) \int_{C^*} d\lambda^*(\ell') \frac{\rho_k\rho^*_{\ell'}}{(k-\ell')(\ell'-\ell)} (p^*-\ell)\hat{\phi}^i_\ell(\alpha)$$

$$= \theta(p-k)\rho_k \frac{k^i}{k+\alpha} + \theta\hat{\psi}^{1,i^*}(\alpha,0)\rho_k + \hat{\phi}^{1,i}(\alpha,0) \int_{C^*} d\lambda^*(\ell') \frac{\rho_k\rho^*_{\ell'}}{k-\ell'} , \tag{A.20}$$

and

$$(p^*-k)\hat{\psi}^i_k(\alpha) + \int_C d\lambda(\ell) \int_{C^*} d\lambda^*(\ell') \frac{\rho_k\rho^*_{\ell'}}{(k-\ell')(\ell'-\ell)} (p^*-\ell)\hat{\psi}^i_\ell(\alpha)$$

$$= -\theta\hat{\phi}^{1,i^*}(\alpha,0)\rho_k + \int_{C^*} d\lambda^*(\ell') \frac{\rho_k\rho^*_{\ell'}}{k-\ell'} \frac{(p^*-\ell')\ell'^i}{\ell'+\alpha} + \hat{\psi}^{1,i}(\alpha,0) \int_{C^*} d\lambda^*(\ell') \frac{\rho_k\rho^*_{\ell'}}{k-\ell'}, \tag{A.21}$$

where $\hat{\psi}^{i,j}(\alpha,\beta)$ and $\hat{\phi}^{i,j}(\alpha,\beta)$ are defined by (A.5) and (A.6) with $\hat{\psi}^j_k(\alpha)$ and $\hat{\phi}^j_k(\alpha)$ instead of $\psi^j_k(\alpha)$ and $\phi^j_k(\alpha)$.

Comparing with the integral equations (A.3) and (A.4) and using also (A.9) and (A.10) we immediately have

$$(p^*-k)\hat{\phi}^i_k(\alpha) = \theta(p+\alpha)\phi^i_k(\alpha) - \theta\phi^{i+1}_k(0) + \theta\hat{\psi}^{1,i^*}(\alpha,0)\phi^1_k(0) + \hat{\phi}^{1,i}(\alpha,0)\psi^1_k(0) , \tag{A.22}$$

$$(p^*-k)\hat{\psi}^i_k(\alpha) = (p^*+\alpha)\psi^i_k(\alpha) - \psi^{i+1}_k(0) + \hat{\psi}^{1,i}(\alpha,0)\psi^1_k(0) - \theta\hat{\phi}^{1,i^*}(\alpha,0)\phi^1_k(0) . \tag{A.23}$$

Hence, after dividing (A.22) and (A.23) for $i=0$ by $k+\beta$, and integrating over the contour C, we have

$$(p^*+\beta)\hat{\phi}(\alpha,\beta) - \theta(p+\alpha)\phi(\alpha,\beta) = \left(1-\psi^{1,0^*}(\beta,0)\right)\hat{\phi}^{1,0}(\alpha,0)$$

$$- \theta\left(1-\hat{\psi}^{1,0^*}(\alpha,0)\right)\phi^{1,0}(\beta,0) , \tag{A.24}$$

$$-1+(p^*+\beta)\hat{\psi}(\alpha,\beta) - (p^*+\alpha)\psi(\alpha,\beta) = -\left(1-\hat{\psi}^{1,0}(\alpha,0)\right)\left(1-\psi^{1,0^*}(\beta,0)\right) - \theta\hat{\phi}^{1,0^*}(\alpha,0)\phi^{1,0}(\beta,0)\ , \qquad (A.25)$$

where we have omitted the superscripts 0,0 on the left-hand side.

Eqs. (6.11), (6.12) and (6.14)-(6.16) are most easily derived introducing the matrix

$$\underline{\underline{g}}(\alpha) \equiv \begin{pmatrix} 1-\psi^{1,0}(\alpha,0) & \phi^{1,0}(\alpha,0) \\ -\phi^{1,0^*}(\alpha,0) & 1-\psi^{1,0^*}(\alpha,0) \end{pmatrix} . \qquad (A.26)$$

Then (A.24) and (A.25) can be cast in matrix notation

$$\begin{pmatrix} \theta\left(1-(p+\beta)\hat{\psi}^*(\alpha,\beta)+(p+\alpha)\psi^*(\alpha,\beta)\right) & -(p^*+\beta)\hat{\phi}(\alpha,\beta)+\theta(p+\alpha)\phi(\alpha,\beta) \\ \theta(p+\beta)\hat{\phi}^*(\alpha,\beta)-(p^*+\alpha)\phi^*(\alpha,\beta) & 1-(p^*+\beta)\hat{\psi}(\alpha,\beta)+(p^*+\alpha)\psi(\alpha,\beta) \end{pmatrix} =$$

$$= \hat{\underline{\underline{g}}}^{-1}(\alpha)\cdot\underline{\underline{\Theta}}\cdot\underline{\underline{g}}(\beta)\ , \qquad \underline{\underline{\Theta}} \equiv \begin{pmatrix}\theta & 0\\ 0 & 1\end{pmatrix}\ , \qquad (A.27)$$

where $\hat{\underline{\underline{g}}}(\alpha)$ can be obtained from $\underline{\underline{g}}(\alpha)$ replacing $\phi^{1,0}$ and $\psi^{1,0}$ by $\hat{\phi}^{1,0}$ and $\hat{\psi}^{1,0}$.

Considering the Bäcklund transformation (6.19), we have in a similar way, omitting the superscripts 0,0 on the left-hand side

$$\begin{pmatrix} \theta'\left(1-(q+\beta)\tilde{\psi}^*(\alpha,\beta)+(q+\alpha)\psi^*(\alpha,\beta)\right) & -(q^*+\beta)\tilde{\phi}(\alpha,\beta)+\theta'(q+\alpha)\phi(\alpha,\beta) \\ \theta'(q+\beta)\tilde{\phi}^*(\alpha,\beta)-(q^*+\alpha)\phi^*(\alpha,\beta) & 1-(q^*+\beta)\tilde{\psi}(\alpha,\beta)+(q^*+\alpha)\psi(\alpha,\beta) \end{pmatrix}$$

$$= \tilde{\underline{\underline{g}}}^{-1}(\alpha)\cdot\underline{\underline{\Theta}}'\cdot\underline{\underline{g}}(\beta), \qquad \underline{\underline{\Theta}}' \equiv \begin{pmatrix}\theta' & 0\\ 0 & 1\end{pmatrix}\ , \qquad (A.28)$$

where $\tilde{\psi}(\alpha,\beta) = \tilde{\psi}^{0,0}(\alpha,\beta)$ and $\tilde{\phi}(\alpha,\beta) = \tilde{\phi}^{0,0}(\alpha,\beta)$ are defined by (A.5) and (A.6) with $\psi_k^j(\alpha)$ and $\phi_k^j(\alpha)$ replaced by the solutions $\tilde{\psi}_k^j(\alpha)$ and $\tilde{\phi}_k^j(\alpha)$ of (A.1) and (A.2) with $\tilde{\rho}_k$ instead of ρ_k, and $\tilde{\underline{\underline{g}}}(\alpha)$ can be obtained from $\underline{\underline{g}}(\alpha)$ replacing $\phi^{1,0}$ and $\psi^{1,0}$ by $\tilde{\phi}^{1,0}$ and $\tilde{\psi}^{1,0}$.

Furthermore eq. (A.19) implies that

$$\underline{\underline{g}}(\alpha)\cdot\underline{\underline{g}}^{\dagger}(\alpha) = \underline{\underline{1}}\ , \qquad (A.29)$$

in which the 2×2 matrix $\underline{\underline{g}}^{\dagger}(\alpha)$ is defined by

$$g^{\dagger}_{pq}(\alpha) = g^{*}_{qp}(\alpha) = \left(g_{qp}(\alpha^{*})\right)^{*} . \tag{A.30}$$

For the matrices $\underline{\underline{g}}(\alpha)$, $\hat{\underline{\underline{g}}}(\alpha)$, $\tilde{\underline{\underline{g}}}(\alpha)$ and the matrix $\hat{\tilde{\underline{\underline{g}}}}(\alpha)$, which can be obtained from (A.26) replacing $\phi^{1,0}$ and $\psi^{1,0}$ by $\hat{\tilde{\phi}}^{1,0}$ and $\hat{\tilde{\psi}}^{1,0}$ respectively, we have the obvious relations

$$\hat{\tilde{\underline{\underline{g}}}}^{-1}(\alpha)\cdot\underline{\underline{\theta}}\cdot\tilde{\underline{\underline{g}}}(\beta)\cdot\tilde{\underline{\underline{g}}}^{-1}(\beta)\cdot\underline{\underline{\theta}}'\cdot\underline{\underline{g}}(\alpha) = \hat{\tilde{\underline{\underline{g}}}}^{-1}(\alpha)\cdot\underline{\underline{\theta}}'\cdot\hat{\underline{\underline{g}}}(\beta)\cdot\hat{\underline{\underline{g}}}^{-1}(\beta)\cdot\underline{\underline{\theta}}\cdot\underline{\underline{g}}(\alpha) , \tag{A.31}$$

$$\underline{\underline{1}} = \hat{\underline{\underline{g}}}^{-1}(\alpha)\cdot\underline{\underline{\theta}}\cdot\underline{\underline{g}}(\beta)\cdot\left(\hat{\underline{\underline{g}}}^{-1}(\alpha)\cdot\underline{\underline{\theta}}\cdot\underline{\underline{g}}(\beta)\right)^{\dagger} = \tilde{\underline{\underline{g}}}^{-1}(\alpha)\cdot\underline{\underline{\theta}}'\cdot\underline{\underline{g}}(\beta)\cdot\left(\tilde{\underline{\underline{g}}}^{-1}(\alpha)\cdot\underline{\underline{\theta}}'\cdot\underline{\underline{g}}(\beta)\right)^{\dagger}, \tag{A.32}$$

$$\underline{\underline{1}} = \underline{\underline{g}}^{-1}(\alpha)\cdot\underline{\underline{g}}(\beta)\cdot\underline{\underline{g}}^{-1}(\beta)\cdot\underline{\underline{g}}(\alpha) . \tag{A.33}$$

Inserting the expressions (A.26)-(A.28) we obtain eqs. (6.11) and (6.12) from eq. (A.31), eqs. (6.14) and (6.15) from eq. (A.32) and eq. (6.16) from eq. (A.33). Eq. (6.16) is an algebraic identity relating $\phi(\alpha,\beta)$ and $\psi(\alpha,\beta)$, eqs. (6.14) and (6.15) are relations between $\phi(\alpha,\beta)$ and $\psi(\alpha,\beta)$ and their Bäcklund transforms under the transformations (6.18) and (6.19), and (6.11) and (6.12) are Bianchi-identities expressing the commutativity of both BT's.

Identifying $\phi(\alpha,\beta)$, $\hat{\phi}(\alpha,\beta)$, $\tilde{\phi}(\alpha,\beta)$, $\hat{\tilde{\phi}}(\alpha,\beta)$, $\psi(\alpha,\beta)$, $\hat{\psi}(\alpha,\beta)$, $\tilde{\psi}(\alpha,\beta)$, $\hat{\tilde{\psi}}(\alpha,\beta)$ with $\phi(n,m;\alpha,\beta)$, $\phi(n+1,m;\alpha,\beta)$, $\phi(n,m+1;\alpha,\beta)$, $\phi(n+1,m+1;\alpha,\beta)$, $\psi(n,m;\alpha,\beta)$, $\psi(n+1,m;\alpha,\beta)$, $\psi(n,m+1;\alpha,\beta)$ and $\psi(n+1,m+1;\alpha,\beta)$ respectively, we obtain from (6.11), (6.12) and (6.14)-(6.16) a set of difference-difference equations for the four coupled fields $\phi,\psi,\phi^{*},\psi^{*}$, defined on a two-dimensional lattice. (Note that for general α and β, $\phi^{*}(n,m;\alpha,\beta) = [\phi(n,m;\alpha^{*},\beta^{*})]^{*}$). This set of equations is completely integrable, since the solutions follow from (6.1) with the factor $\rho_k(n,m)$ as given in (6.17). In the general case the set of equations is rather complicated, but in appendix B we shall work out the special cases mentioned in the proposition given in section 6.

Appendix B

In this appendix we derive the ddNLS, the ddIHSC and the ddCSG, as given by eqs. (6.5), (6.8) and (6.10).

(1) For the ddNLS we take $\alpha=-p$, $\beta=p$, $p^{*}=-p$. From (6.14) and (6.16), with $\psi \equiv \psi(-p,p)$; $\phi \equiv \phi(-p,p)$ etc., using also (A.7) and (A.8), one can derive

$$1 = \left(1 + 2p\hat{\psi}(p,-p)\right)\left(1 - 2p\psi(-p,p)\right) \quad , \tag{B.1}$$

and

$$\frac{1 + 2p\psi(p,-p)}{1 + 2p\hat{\psi}(p,-p)} = 1 + 4|p|^2|\phi(-p,p)|^2 \quad , \tag{B.2}$$

and from (6.11) we have

$$\frac{1 + 2p\hat{\tilde{\psi}}(p,-p)}{1 + 2p\hat{\psi}(p,-p)} = \frac{\theta\theta'(q-p)\phi(-p,p) + (q^*-p^*)\hat{\tilde{\phi}}(-p,p)}{\theta(q^*+ p^*)\tilde{\phi}(-p,p) + \theta'(q+p)\hat{\phi}(-p,p)} \quad . \tag{B.3}$$

From (6.15) we obtain

$$\frac{1 - \left(\theta'(q-p)\phi(-p,p) - (q^*-p^*)\tilde{\phi}(-p,p)\right)\left(\theta'^*(q^*+p^*)\phi^*(-p,p) - (q+p)\tilde{\phi}^*(-p,p)\right)}{1 + 4|p|^2|\tilde{\phi}(-p,p)|^2}$$

$$= \left(\frac{q+p}{2p}\,\frac{1 + 2p\tilde{\psi}(p,-p)}{1 + 2p\psi(p,-p)} - \frac{q-p}{2p}\right)\left(\frac{q^*+p}{2p}\,\frac{1 - 2p\tilde{\psi}(-p,p)}{1 - 2p\psi(-p,p)} - \frac{q^*-p}{2p}\right) \quad . \tag{B.4}$$

Using (B.1)-(B.3) to eliminate the ψ's we have

$$1 + \frac{|q|^2 - |p|^2}{2|p|^2} + \theta'(q+p)(q-p)\phi(-p,p)\tilde{\phi}^*(-p,p) + \theta'^*(q^*+p^*)(q^*-p^*) \times$$

$$\times\ \phi^*(-p,p)\tilde{\phi}(-p,p)$$

$$= \frac{|q+p|^2}{4|p|^2}\left(1 + 4|p|^2|\tilde{\phi}(-p,p)|^2\right)\frac{\theta\theta'(q-p)\phi(-p,p) + (q^*-p^*)\hat{\tilde{\phi}}(-p,p)}{\theta(q^*+p^*)\tilde{\phi}(-p,p) + \theta'(q+p)\hat{\phi}(-p,p)}$$

$$+ \frac{|q-p|^2}{4|p|^2}\left(1 + 4|p|^2|\phi(-p,p)|^2\right)\frac{\theta(q^*+p^*)\tilde{\phi}(-p,p) + \theta'(q+p)\hat{\phi}(-p,p)}{\theta\theta'(q-p)\phi(-p,p) + (q^*-p^*)\hat{\tilde{\phi}}(-p,p)} \quad , \tag{B.5}$$

and eq. (6.5) follows from (B.5) using (6.13), (6.4), and $\theta=-1$.

(2) In the case that $\alpha=\beta=\alpha^*=\beta^*$, one can solve $\hat{\psi}-\psi \equiv \hat{\psi}(\alpha,\alpha) - \psi(\alpha,\alpha)$ from (6.14) as a function of $\phi \equiv \phi(\alpha,\alpha)$. We have

$$\hat{\psi}(\alpha,\alpha) - \psi(\alpha,\alpha)$$

$$= \frac{p-p^*}{2|p+\alpha|^2} \pm \frac{i}{2|p+\alpha|^2}\left[|p^*-p|^2 - 4|p+\alpha|^2|\theta(p+\alpha)\phi(\alpha,\alpha) - (p^*+\alpha)\hat{\phi}(\alpha,\alpha)|^2\right]^{\frac{1}{2}}, \tag{B.6}$$

and in a similar way we have from (6.15)

$$\tilde{\psi}(\alpha,\alpha) - \psi(\alpha,\alpha)$$

$$= \frac{q-q^*}{2|q+\alpha|^2} \pm \frac{i}{2|q+\alpha|^2}\left[|q^*-q|^2 - 4|q+\alpha|^2|\theta'(q+\alpha)\phi(\alpha,\alpha) - (q^*+\alpha)\tilde{\phi}(\alpha,\alpha)|^2\right]^{\frac{1}{2}}. \tag{B.7}$$

Inserting (B.6) and (B.7), eqs. (6.11) and (6.12) can be expressed in terms of the ϕ's. We now restrict ourselves to the special case $p^*=-p$, $\theta=-1$, $\alpha=\beta=0$. In that case we obtain from (6.11) and (6.12), with $\phi \equiv \phi(0,0)$, $\psi \equiv \psi(0,0)$, the following equations

$$\pm \frac{i|p|}{p}\left(\theta' q\phi(0,0) - q^*\tilde{\phi}(0,0)\right)[1 - |p|^2|\tilde{\phi}(0,0) - \hat{\tilde{\phi}}(0,0)|^2]^{\frac{1}{2}}$$

$$\pm \frac{i|p|}{p}\left(\theta' q\hat{\phi}(0,0) - q^*\hat{\tilde{\phi}}(0,0)\right)[1 - |p|^2|\phi(0,0) - \hat{\phi}(0,0)|^2]^{\frac{1}{2}}$$

$$- p\left(\tilde{\phi}(0,0) - \hat{\tilde{\phi}}(0,0)\right)\left\{\frac{q+q^*}{2q} \mp \frac{i|q|}{q}\left[\frac{|q^*-q|^2}{4|q|^2} - |\theta' q\phi(0,0) - q^*\tilde{\phi}(0,0)|^2\right]^{\frac{1}{2}}\right\}$$

$$+ \theta' p\left(\phi(0,0) - \hat{\phi}(0,0)\right)\left\{\frac{q+q^*}{2q^*} \pm \frac{i|q|}{q^*}\left[\frac{|q^*-q|^2}{4|q|^2} - |\theta' q\hat{\phi}(0,0) - q^*\hat{\tilde{\phi}}(0,0)|^2\right]^{\frac{1}{2}}\right\} = 0, \tag{B.8}$$

and

$$p\left(\tilde{\phi}^*(0,0) - \hat{\tilde{\phi}}^*(0,0)\right)\left(\theta' q\phi(0,0) - q^*\tilde{\phi}(0,0)\right)$$

$$+ p\left(\phi(0,0) - \hat{\phi}(0,0)\right)\left(\theta' q\hat{\tilde{\phi}}^*(0,0) - q^*\hat{\phi}^*(0,0)\right)$$

$$\mp \frac{i|p|}{p}\left[1 - |p|^2|\tilde{\phi}(0,0) - \hat{\tilde{\phi}}(0,0)|^2\right]^{\frac{1}{2}}$$

$$\left\{ \frac{q+q^*}{2q} \mp \frac{i|q|}{q}\left[\frac{|q^*-q|^2}{4|q|^2} - |\theta' q\phi(0,0) - q^*\tilde{\phi}(0,0)|^2\right]^{\frac{1}{2}} \right\}$$

$$\pm \frac{i|p|}{p}\left[1 - |p|^2|\phi(0,0) - \hat{\phi}(0,0)|^2\right]^{\frac{1}{2}}$$

$$\left\{ \frac{q+q^*}{2q} \mp \frac{i|q|}{q}\left[\frac{|q^*-q|^2}{4|q|^2} - |\theta' q\hat{\phi}(0,0) - q^*\hat{\tilde{\phi}}(0,0)|^2\right]^{\frac{1}{2}} \right\} = 0 \ . \quad \text{(B.9)}$$

For the IHSC we consider the special case $\theta' = q^*/q$, in addition to $p^* = -p$. $\theta = -1$, $\alpha = \beta = 0$. Introducing real vectors $\vec{S}$ and $\vec{U}$ by

$$\vec{S} \equiv \left(\frac{S^+ + S^{+*}}{2}, \frac{S^+ - S^{+*}}{2i}, [1 - |S^+|^2]^{\frac{1}{2}}\right),$$

$$S^+ \equiv p\hat{\phi}(0,0) - p\phi(0,0), \qquad \vec{S}\cdot\vec{S} = 1, \quad \text{(B.10)}$$

and

$$\vec{U} \equiv \left(\frac{U^+ + U^{+*}}{2}, \frac{U^+ - U^{+*}}{2i}, [\mu - |U^+|^2]^{\frac{1}{2}}\right),$$

$$U^+ \equiv p\tilde{\phi}(0,0) - p\phi(0,0), \qquad \vec{U}\cdot\vec{U} = \mu \equiv \tfrac{1}{4}|q-q^*|^2|p|^2/|q|^4, \quad \text{(B.11)}$$

equations (B.8) and (B.9), with the upper signs, can be combined to give the vector equation

$$\tilde{\vec{S}}\times\vec{U} + \vec{S}\times\hat{\vec{U}} + \tfrac{1}{2}\lambda(\tilde{\vec{S}}-\vec{S}) = -i(\vec{U}\cdot\tilde{\vec{S}} - \hat{\vec{U}}\cdot\vec{S})\vec{e}_z = 0, \quad \text{(B.12)}$$

where

$$\lambda \equiv \frac{|p|(q+q^*)}{|q|^2}, \quad \text{(B.13)}$$

and $\vec{e}_z$ is a unit vector in the z direction. (In (B.12) use has been made of the fact that $\vec{S}$ and $\vec{U}$ are real vectors, so that the first and second member vanish independently.)

From the definitions of S^+ and U^+ we have the obvious relation

$$U^+ + \tilde{S}^+ = S^+ + \hat{U}^+ \, . \tag{B.14}$$

From (B.12) we have $\vec{U}\cdot\tilde{\vec{S}} = \hat{\vec{U}}\cdot\vec{S}$. Furthermore, by taking first the outer product with $\vec{S}$ and then the inner product with $\tilde{\vec{S}}$, one can show that $(\tilde{\vec{S}}-\vec{S})\cdot(\hat{\vec{U}}+\vec{U}) = 0$. From (B.12) we then have $(\vec{U}\times\hat{\vec{U}})\cdot(\tilde{\vec{S}}-\vec{S}) = 0$. From these two equations, together with $\tilde{\vec{U}}\cdot\tilde{\vec{U}} = \vec{U}\cdot\vec{U}$, we see that $\tilde{\vec{S}}-\vec{S} = \sigma(\hat{\vec{U}}-\vec{U})$. Comparing with (B.14) we conclude that we must have $\sigma=1$, and hence

$$\vec{U} + \tilde{\vec{S}} = \vec{S} + \hat{\vec{U}} \, . \tag{B.15}$$

Using (B.15) to eliminate $\vec{U}$ from (B.12), we obtain

$$(\vec{S}+\tilde{\vec{S}}) \times \hat{\vec{U}} = \vec{S}\times\tilde{\vec{S}} - \tfrac{1}{2}\lambda(\tilde{\vec{S}}-\vec{S}) \, , \tag{B.16}$$

and from (B.12) and (B.15) with $\vec{S}\to\hat{\vec{S}}$, $\tilde{\vec{S}}\to\hat{\tilde{\vec{S}}}$, $\vec{U}\to\hat{\vec{U}}$, $\hat{\vec{U}}\to\hat{\hat{\vec{U}}}$, eliminating $\hat{\hat{\vec{U}}}$, we also obtain a second relation

$$(\hat{\vec{S}}+\hat{\tilde{\vec{S}}}) \times \hat{\vec{U}} = -\hat{\vec{S}}\times\hat{\tilde{\vec{S}}} - \tfrac{1}{2}\lambda(\hat{\tilde{\vec{S}}}-\hat{\vec{S}}) \, . \tag{B.17}$$

Taking into account that $\vec{U}\cdot\vec{U} = \mu$, one can solve $\hat{\vec{U}}$ from (B.16) as well as from (B.17) to obtain the equation

$$\mp \frac{\vec{S}+\tilde{\vec{S}}}{1+\vec{S}\cdot\tilde{\vec{S}}}\left[2\mu - \tfrac{1}{2}\lambda^2 - 1 + (2\mu+\tfrac{1}{2}\lambda^2)\vec{S}\cdot\tilde{\vec{S}} + (\vec{S}\cdot\tilde{\vec{S}})^2\right]^{\frac{1}{2}} + \tilde{\vec{S}}-\vec{S} + \lambda\,\frac{\vec{S}\times\tilde{\vec{S}}}{1+\vec{S}\cdot\tilde{\vec{S}}}$$

$$= \mp \frac{\hat{\vec{S}}+\hat{\tilde{\vec{S}}}}{1+\hat{\vec{S}}\cdot\hat{\tilde{\vec{S}}}}\left[2\mu - \tfrac{1}{2}\lambda^2 - 1 + (2\mu+\tfrac{1}{2}\lambda^2)\hat{\vec{S}}\cdot\hat{\tilde{\vec{S}}} + (\hat{\vec{S}}\cdot\hat{\tilde{\vec{S}}})^2\right]^{\frac{1}{2}} + \hat{\vec{S}}-\hat{\tilde{\vec{S}}} + \lambda\,\frac{\hat{\vec{S}}\times\hat{\tilde{\vec{S}}}}{1+\hat{\vec{S}}\cdot\hat{\tilde{\vec{S}}}} \, . \tag{B.18}$$

In the special case $\mu=1$, taking the upper signs and using the identifications $\vec{S}\to\vec{S}(n,m)$, $\hat{\vec{S}}\to\vec{S}(n+1,m)$, $\tilde{\vec{S}}\to\vec{S}(n,m+1)$ and $\hat{\tilde{\vec{S}}}\to\vec{S}(n+1,m+1)$, eq. (B.18) immediately reduces to the ddIHSC given in (6.8).

(3) Finally for the ddCSG we consider the special case $p = \mp i|p|$, $\theta' = -q^*/q$, $\lambda = q/p$, in addition to $\theta = -1$, $\alpha=\beta=0$. From (B.8) we obtain

$$\begin{aligned}
&\lambda^*\big(\phi(0,0) + \tilde{\phi}(0,0)\big)\ \left[1 - |p|^2|\tilde{\phi}(0,0) - \hat{\tilde{\phi}}(0,0)|^2\right]^{\frac{1}{2}} \\
&+ \lambda^*\big(\hat{\phi}(0,0) + \hat{\tilde{\phi}}(0,0)\big)\ \left[1 - |p|^2|\phi(0,0) - \hat{\phi}(0,0)|^2\right]^{\frac{1}{2}} \\
&+ \big(\tilde{\phi}(0,0) - \hat{\tilde{\phi}}(0,0)\big)\left\{\frac{\lambda-\lambda^*}{2\lambda} + \frac{|\lambda|}{\lambda}\left[\frac{(\lambda^*+\lambda)^2}{4|\lambda|^2} - |\lambda|^2|p|^2|\phi(0,0) + \tilde{\phi}(0,0)|^2\right]^{\frac{1}{2}}\right\} \\
&- \big(\phi(0,0) - \hat{\phi}(0,0)\big)\left\{\frac{\lambda-\lambda^*}{2\lambda} - \frac{|\lambda|}{\lambda}\left[\frac{(\lambda^*+\lambda)^2}{4|\lambda|^2} - |\lambda|^2|p|^2|\hat{\phi}(0,0) + \hat{\tilde{\phi}}(0,0)|^2\right]^{\frac{1}{2}}\right\} = 0,
\end{aligned} \tag{B.19}$$

which for λ real, i.e. $q^* = -q$, $\theta' = 1$, together with (6.9) and (6.13) reduces to (6.10).

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SAMENVATTING

Solitonen zijn gelokaliseerde golven die na een onderlinge botsing hun oorspronkelijke vorm en snelheid behouden. Zij treden op in uiteenlopende gebieden van de natuurkunde, zoals de hydrodynamica, optica, plasmafysica, veldentheorie en vaste-stoffysica.

Lang niet alle fysische systemen vertonen soliton-gedrag. Dit treedt met name op in de zogenaamde <u>integreerbare</u> dynamische systemen, waarmee bedoeld wordt <u>niet-lineaire</u> systemen waarvan de oplossing teruggebracht kan worden tot het oplossen van uitsluitend <u>lineaire</u> problemen. De hoop bestaat dat de eigenschappen van niet-integreerbare systemen, die in de meerderheid zijn, benaderd kunnen worden met behulp van integreerbare systemen.

Een bekende methode om solitonsystemen te bestuderen is de methode van de inverse verstrooiïng, ontdekt door Gardner, Greene, Kruskal en Miura. In dit proefschrift worden solitonen echter bestudeerd met de methode van directe linearisatie, ingevoerd door Fokas en Ablowitz. Deze methode gaat uit van een lineaire singuliere integraalvergelijking, met een integraal over een willekeurige contour en maat in het complexe vlak, en leidt hieruit de oplossingen en eigenschappen van het niet-lineaire solitonsysteem af. De methode heeft het voordeel dat fysisch zeer verschillende solitonsystemen op een unificerende manier behandeld worden.

De inhoud van dit proefschrift is in het kort als volgt. Hoofdstuk I behelst een inleiding tot dit proefschrift en een samenvatting van de belangrijkste resultaten. In hoofdstuk II wordt de directe linearisatie van verschillende partiële differentiaalvergelijkingen gegeven, zoals de Korteweg-de Vries vergelijking, de gemodificeerde Korteweg-de Vries vergelijking, de sine-Gordon vergelijking, de niet-lineaire Schrödinger vergelijking en de bewegingsvergelijking voor de klassieke isotrope Heisenberg spinketen; tevens worden verscheidene verbanden tussen deze vergelijkingen uitgewerkt.

In hoofdstuk III worden de Bäcklund transformaties van deze vergelijkingen behandeld op grond van een singuliere transformatie van de maat die in de integraalvergelijking voorkomt en de Bäcklund transformaties worden gebruikt om de directe linearisatie van een keten van zogenaamde gemodificeerde partiële differentiaalvergelijkingen af te leiden. Zo wordt bijvoorbeeld uit de transformatie van de maat in de integraalvergelijking voor de niet-lineaire Schrödinger vergelijking de directe linearisatie van de bewegingsvergelijking voor de klassieke anisotrope Heisenberg spinketen afgeleid.

Tenslotte wordt in hoofdstuk IV aangetoond hoe singuliere lineaire integraalvergelijkingen op een natuurlijke wijze leiden tot de directe linearisatie van verscheidene niet-lineaire differentie-differentievergelijkingen. Deze vergelijkingen voor functies van twee discrete variabelen gaan over in bovengenoemde partiële differentiaalvergelijkingen na twee opeenvolgende continuüm-limieten. Als tussenresultaat wordt de directe linearisatie afgeleid van de differentie-differentiaalvergelijkingen die worden verkregen na een enkele continuüm-limiet, bijvoorbeeld de bewegingsvergelijking voor het Toda rooster, de discrete niet-lineaire Schrödinger vergelijking en de discrete complexe sine-Gordon vergelijking.

CURRICULUM VITAE

De schrijver van dit proefschrift werd geboren op 8 oktober 1953 te Bilthoven. In 1970 behaalde hij het einddiploma Gymnasium β aan het Nieuwe Lyceum te Bilthoven. Hierna behaalde hij aan de Rijksuniversiteit te Utrecht in 1973 het kandidaatsexamen scheikunde, in 1976 het kandidaatsexamen natuurkunde en in 1979 het doctoraalexamen theoretische natuurkunde met bijvakken wiskunde en mathematische fysica met onderwijsbevoegdheid in de wis- en natuurkunde. Zijn doctoraal-scriptie over solitonen in de Heisenberg spinketen schreef hij onder supervisie van Prof.dr. Th.W. Ruijgrok.

In 1979 trad hij in het kader van een beleidsruimteproject toe tot de werkgroep VS-th-L van de Stichting voor Fundamenteel Onderzoek der Materie. Onder leiding van Prof.dr. H.W. Capel verrichtte hij op het Instituut-Lorentz voor Theoretische Natuurkunde te Leiden onderzoek op het gebied van een-dimensionale integreerbare systemen; bovendien vervulde hij enige onderwijstaken. Een aantal resultaten van het bovengenoemde onderzoek zijn in dit proefschrift vastgelegd. Voorts nam hij deel aan de zomerschool "Fundamental Problems in Statistical Mechanics V" in Enschede (1980) en aan de conferenties over niet-lineaire evolutievergelijkingen en dynamische systemen in Trieste (1981), Edinburgh (1982) en Chania (1983). Vanaf 1 oktober 1983 is hij verbonden aan de vakgroep theoretische natuurkunde van de Technische Hogeschool Twente, als wetenschappelijk medewerker van Dr. R.H.G. Helleman.

LIST OF PUBLICATIONS

1. G.R.W. Quispel and H.W. Capel.
Equation of motion for the Heisenberg spin chain.
Phys. Lett. 85A (1981) 248.

2. G.R.W. Quispel and H.W. Capel.
Equation of motion for the Heisenberg spin chain.
Physica 110A (1982) 41.

3. G.R.W. Quispel and H.W. Capel.
The nonlinear Schrödinger equation and the anisotropic Heisenberg spin chain.
Phys. Lett. 88A (1982) 371.

4. F.W. Nijhoff, J. van der Linden, G.R.W. Quispel and H.W. Capel.
Linearization of the nonlinear Schrödinger equation and the isotropic Heisenberg spin chain.
Phys. Lett. 89A (1982) 106.

5. F.W. Nijhoff, J. van der Linden, G.R.W. Quispel, H.W. Capel and J. Velthuizen.
Linearization of the nonlinear Schrödinger equation and the isotropic Heisenberg spin chain.
Physica 116A (1982) 1.

6. G.R.W. Quispel and H.W. Capel.
The anisotropic Heisenberg spin chain and the nonlinear Schrödinger equation.
Physica 117A (1983) 76.

7. G.R.W. Quispel, F.W. Nijhoff and H.W. Capel.
Linearization of the Boussinesq equation and the modified Boussinesq equation.
Phys. Lett. 91A (1982) 143.

8. F.W. Nijhoff, H.W. Capel, G.R.W. Quispel and J. van der Linden.
The derivative nonlinear Schrödinger equation and the massive Thirring model.
Phys. Lett. 93A (1983) 455.

9. F.W. Nijhoff, G.R.W. Quispel, J. van der Linden and H.W. Capel.
On some linear integral equations generating solutions of nonlinear partial differential equations.
Physica 119A (1983) 101.

10. F.W. Nijhoff, G.R.W. Quispel and H.W. Capel.
Linearization of nonlinear differential-difference equations.
Phys. Lett. 95A (1983) 273.

11. G.R.W. Quispel, F.W. Nijhoff, H.W. Capel and J. van der Linden.
Bäcklund transformations and singular integral equations.
Physica A, in press.

12. J.H.H. Perk, H.W Capel, G.R.W. Quispel and F.W. Nijhoff.
Finite-temperature correlations for the Ising chain in a transverse field.
Physica A, in press.

13. F.W. Nijhoff, G.R.W. Quispel and H.W. Capel.
Direct linearization of nonlinear difference-difference equations.
Phys. Lett. 97A (1983) 125.

14. F.W. Nijhoff, H.W. Capel and G.R.W. Quispel.
Integrable lattice version of the massive Thirring model and its linearization.
Phys. Lett. A, in press.

15. G.R.W. Quispel, F.W. Nijhoff, H.W. Capel and J. van der Linden.
Linear integral equations and nonlinear difference-difference equations.
Submitted to Physica A.

Apart from minor modifications chapters II, III and IV of this thesis are contained in the publications 9, 11 and 15, respectively.

ACKNOWLEDGEMENTS

I want to thank Nel for her love and support, and for designing the cover of this thesis.

This investigation is part of the research program of the Stichting voor Fundamenteel Onderzoek der Materie (F.O.M.) which is financially supported by the Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek (Z.W.O.).

STELLINGEN behorend bij het proefschrift "Linear integral equations and soliton systems" te verdedigen door G.R.W. Quispel, 2 november 1983 om 16.15 uur

1. Alternatieve singuliere integraalvergelijkingen, zoals voorgesteld door Fokas en Ablowitz, bieden de mogelijkheid nieuwe Miuratransformaties af te leiden.

   A.S. Fokas en M.J. Ablowitz, in Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems, M. Tabor en Y.M. Treve eds.

2. De door de Vries e.a. gemeten parameters in de uitdrukking voor de isomere verschuiving van Eu^{2+} volgens het Miedemamodel kunnen afgeleid worden uit de eerder gevonden parameters voor Gd^{3+} met behulp van een schalings-argument.

   J.W.C. de Vries, R.C. Thiel en K.H.J. Buschow, Physica 121B (1983) 100.

3. De bij lage temperaturen gemeten afwijking in de soortelijke warmtecurve van α-CuNSal. ten opzichte van de voorspelling voor een homogene lineaire keten kan verklaard worden door aan te nemen dat in de magnetische ketens in deze verbinding random defecten aanwezig zijn.

   L.J. Azevedo, W.G. Clark, D. Hulin en E.O. McLean, Phys.Lett. 58A (1976) 255.
   G. Mennenga, L.J. de Jongh, W.J. Huiskamp en J. Reedijk, wordt gepubliceerd.

4. Het argument dat Muthukumar gebruikt om de effectieve viscositeit van een suspensie te kunnen vergelijken met die van een poreus medium is onjuist.

   M. Muthukumar, J. Chem. Phys. 77 (1982) 959.

5. De berekening door Boiti en Pempinelli van de parametertransformatie behorend bij de Bäcklundtransformatie van de vierde Chazyvergelijking kan worden kortgesloten met behulp van een eenvoudige symmetriebeschouwing.

M. Boiti en F. Pempinelli, Nuovo Cim. *__59B__* *(1980) 40.*
G.R.W. Quispel en H.W. Capel, Physica *__117A__* *(1983) 76.*

6. Het is mogelijk een ijkinvariante beschrijving te geven van "eindstraling" zowel in $q\bar{q}\to Z_0\to e^+e^-(\gamma)$ als in $q\bar{q}'\to W^-\to e^-\bar{\nu}_e(\gamma)$. In deze beschrijving zijn de botsingsdoorsneden voor de eindstraling analoog aan die voor het verval van een vrij W^- of Z_0 deeltje. In het bijzonder geldt dit voor de hierbij optredende infrarooddivergenties.

7. De opmerking van Tanaka dat het resultaat van de theorie van Rosenstock voor het gemiddeld aantal stappen tot vangst in een zelfmijdende stochastische wandeling op een rooster met een random verdeling van valpunten duidelijk fout is, is duidelijk fout.

F. Tanaka, J. Phys. *__A16__* *(1983) L489.*
H.B. Rosenstock, J. Math. Phys. *__21__* *(1980) 1643.*

8. Steeb geeft in zijn studie van de Rikitakedynamo een onvolledige beschrijving van de mogelijke bewegingsconstanten.

W.-H. Steeb, J. Phys. *__A15__* *(1982) L389.*

9. Het bestuderen van Bianchiïdentiteiten die het commuteren van Bäcklundtransformaties van partiële differentiaalvergelijkingen uitdrukken kan op natuurlijke wijze leiden tot een klasse van integreerbare differentie-differentievergelijkingen.

Dit proefschrift, hoofdstuk IV.

10. Evenals de Korteweg-de Vriesvergelijking bezit de bewegingsvergelijking van Fermi, Pasta en Ulam voor een rooster met kubische interactie een oplossing die uitgedrukt kan worden met behulp van Airyfuncties.

E. Fermi, Collected Works, p. 978.

11. Het is op fysische gronden in te zien dat het thermomagnetisch drukverschil in de lage-druklimiet niet met de eerste maar met de tweede macht van de druk evenredig is.

H. Hulsman, G.F. Bulsing, G.E.J. Eggermont, L.J.F. Hermans, J.J.M. Beenakker, Physica <u>*72*</u> *(1974) 287.*

12. Het is in het belang van de wetenschap dat de briefwisseling tussen Pauli en Jung wordt gepubliceerd.

G. Quispel, Bres Symposium (1977) 7.