# Answers to the Exam Quantum Information, 19 November 2021

each item gives 2 points for a fully correct answer, grade = total $\times 9/24 + 1$

1. *a)* The matrix $\rho$ is Hermitian, it has trace 1, and it has non-negative eigenvalues ($N^{-1}(1-\eta)$ and $N^{-1}(1-\eta)+\eta$).
   *b)* $P = N^{-1}(1-\eta)^2 + \eta^2 + 2N^{-1}\eta(1-\eta) = N^{-1}[1+\eta^2(N-1)]$; $P = 1$ for $\eta = 1$.
   *c)* $\rho(t) = e^{-(i/\hbar)Ht}\rho(0)e^{(i/\hbar)\hbar Ht} \Rightarrow \operatorname{tr}\rho(t)^2 = \operatorname{tr}\rho(0)$. Since $\eta$ is uniquely determined by $P$, also $\eta$ must be time independent.

2. *a)* That is possible, Alice would need two additional qubits, each of which forms a Bell pair with a qubit of Bob. She would then perform the entanglement protocol on each of the two qubits in the state $|\psi\rangle$, to teleport that state to Bob.
   *b)* That is possible, Alice would execute the usual teleportation protocol, and would communicate her measurement result to Bob. No classical communication with Charlie is needed for either Alice or Bob.
   *c*) Bob has no way to find out if his qubit has collapsed, so no information is transfered. If Bob could clone his qubit in many copies, then he could determine its state and so he would have a way of finding out whether or not his qubit has collapsed.

3. *a)* $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, $|\phi\rangle = \alpha'|0\rangle + \beta'|1\rangle$, then $S|\psi\rangle|\phi\rangle = \alpha\alpha'|0\rangle|0\rangle + \beta\beta'|1\rangle|1\rangle + \alpha\beta'|1\rangle|0\rangle + \alpha'\beta|0\rangle|1\rangle = |\phi\rangle|\psi\rangle$.
   *b)* in the basis $|i\rangle|j\rangle$, $i, j \in \{0,1\}$, the operator $S$ has only nonzero matrix elements $S_{ij,kl} = 1$ if $i = l$, $j = k$. Hence $S_{ij,kl} = S^*_{kl,ij}$ and $S$ is Hermitian ($S = S^\dagger$). Moreover, since $S^2 = 1$ also $SS^\dagger = 1$, hence $S$ is unitary.
   *c)* for general states $|\psi\rangle$ and $|\phi\rangle$ of the second and third qubit, the result of the circuit is $\frac{1}{2}|0\rangle(|\psi\phi\rangle + |\phi\psi\rangle) + \frac{1}{2}|1\rangle(|\psi\phi\rangle - |\phi\psi\rangle)$; taking $|\psi\rangle = |0\rangle$, $|\phi\rangle = |1\rangle$ gives $\frac{1}{2}|0\rangle(|01\rangle + |10\rangle) + \frac{1}{2}|1\rangle(|01\rangle - |10\rangle)$, so the measurement of the first qubit gives 1 with probability 1/2; if $|\psi\rangle = \phi\rangle$, instead, the result is $|0\rangle|\psi\psi\rangle$, and the probability for 1 is zero.

4. *a)* Bob tells Alice the result of his coin tosses, one by one; Alice compares these results with her own coin tosses, and she tells Bob which results match; those measurements form the shared secret code. The measurement outcomes are discarded for those qubits for which the coin tosses of Alice and Bob do not match.
   *b)* The random bit string can be added bitwise by Alice to the string that encodes the information. The resulting bit string can then be communicated to Bob in a nonsecure way, who will subtract the shared code to recover the information.
   *c)* Alice and Bob can disclose a part of their shared code and check if the bits are indeed the same. If the qubits were measured by an adversary before reaching Bob, there will be errors with high probability.