Answers to the Exam Quantum Information, 18 November 2022

each item gives 2 points for a fully correct answer, grade = total $\times 9/24 + 1$

1. *(a)* $\rho^2 \neq \rho$, so the particle is in a mixed state.
   (b) $\langle S_z \rangle = \mathrm{Tr}\, S_z \rho = 1/4$
   (c) $\rho = |0\rangle\langle 0| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\rho^2 = \rho$, so the particle is in a pure state.

2. *a)* A local unitary operation cannot change the degree of entanglement, so might as well take the identity for U. Then the state is $(|00\rangle + |11\rangle)/\sqrt{2}$, with concurrence 1 (maximally entangled).
   b) Depending on whether $U = U_0 \equiv I$, $U = U_1 \equiv X$, $U = U_2 \equiv Y$, $U = U_3 \equiv Z$, the state received by Bob is $|\Psi_0\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, $|\Psi_1\rangle = (|10\rangle + |01\rangle)/\sqrt{2}$, $|\Psi_2\rangle = (|10\rangle - |01\rangle)/\sqrt{2}$, $|\Psi_3\rangle = (|00\rangle - |11\rangle)/\sqrt{2}$. These are orthogonal.
   c) Bob inverts the circuit, by first applying a CNOT gate (with Alice's qubit as the control) and then a Hadamard gate on Alice's qubit. He then measures both qubits. The answer is 00 for $|\Psi_0\rangle$, 01 for $|\Psi_1\rangle$, 11 for $|\Psi_2\rangle$, 10 for $|\Psi_3\rangle$.

3. *a)* there exists no unitary operator U such that for any pure state $|\phi\rangle$

$$U|\phi\rangle_A|0\rangle_B = e^{i\alpha(\phi)}|\phi\rangle_A|\phi\rangle_B$$

   b) Proof:

$$\begin{aligned}
\langle\phi|\psi\rangle &= \langle 0|_B\langle\phi|_A|\psi\rangle_A|0\rangle_B = \langle 0|_B\langle\phi|_A U^\dagger U|\psi\rangle_A|0\rangle_B \\
&= e^{i(\alpha(\psi)-\alpha(\phi))}\langle\phi|_B\langle\phi|_A|\psi\rangle_A|\psi\rangle_B \\
&= e^{i(\alpha(\psi)-\alpha(\phi))}\langle\phi|\psi\rangle^2 \\
&\Rightarrow |\langle\phi|\psi\rangle| = |\langle\phi|\psi\rangle|^2 \Rightarrow |\langle\phi|\psi\rangle| = 0 \text{ or } 1.
\end{aligned}$$

   This can not be the case for two arbitrary states.
   c) An encryption key cannot be intercepted and copied without the recipient noticing.

4. *a)* $\rho_t = (1-p)|\psi\rangle\langle\psi| + p|\psi'\rangle\langle\psi'|$, with $|\psi'\rangle = \sigma_x|\psi\rangle = \alpha|1\rangle + \beta|0\rangle$.

   b) the copied state would be $(\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)$, it has cross-terms that do not appear in the encoded state $\alpha|000\rangle + \beta|111\rangle$.
   c) the final state when no error has occurred is $(\alpha|0\rangle + \beta|1\rangle)|0\rangle|0\rangle$; if an error occurred on the first qubit it is $(\alpha|0\rangle + \beta|1\rangle)|1\rangle|1\rangle$; if an error occurred on the second qubit it is $(\alpha|0\rangle + \beta|1\rangle)|1\rangle|0\rangle$; if an error occurred on the third qubit it is $(\alpha|0\rangle + \beta|1\rangle)|0\rangle|1\rangle$; so the state of the first qubit factors out from the state of the second and third qubits, it is not entangled with them.