Answers to the Exam Quantum Information, 17 November 2023

each item gives 2 points for a fully correct answer, grade = total $\times 9/24 + 1$

1. *a)* $\rho_A = (1/2)|0\rangle\langle 0| + (1/2)|1\rangle\langle 1| = (1/2)I$.
   b) the two eigenvalues are both 1/2, so we have
   $-(1/2)\,{}^2\log(1/2) - (1/2)\,{}^2\log(1/2) = 1$.
   c) The coefficient matrix of ρ_A is $cc^\dagger$ and for ρ_B it is $c^\top c^* = (c^\dagger c)^\top$ The eigenvalues of $cc^\dagger$ and $c^\dagger c$ are the same, and taking the transpose also does not change the eigenvalues, so ρ_A and ρ_B have the same eigenvalues λ_i and hence the same entanglement entropy $\sum_i \lambda_i\,{}^2\log\lambda_i$.

2. *a)* the eigenvalues λ_i of ρ are positive and sum to unity, so $0 \le \lambda_i^2 \le \lambda_i \le 1$; hence $0 \le P = \sum_i \lambda_i^2 \le \sum_i \lambda_i = 1$.
   b) $d\rho/dt = i[\rho, H]$, $d\rho^2/dt = \rho(d\rho/dt) + (d\rho/dt)\rho = i\rho[\rho,H] + i[\rho,H]\rho = i\rho^2 H - iH\rho^2$, and the trace vanishes
   c) if the qubit interacts with the environment, it will become entangled with external degrees of freedom; the combined state of qubit plus environment is still pure, but if we trace out the degrees of the environment we arrive at a reduced density matrix which is mixed. There is no contradiction with $dP/dt = 0$ for evolution under the action of a Hamiltonian, because the reduction to a partial density matrix is not described by Hamiltonian evolution.

3. *a)* $\text{CNOT}|A\rangle|B\rangle = \alpha|0\rangle(\gamma|0\rangle + \delta|1\rangle) + \beta|1\rangle(\gamma|1\rangle + \delta|0\rangle$.
   b) the coefficient matrix is

   $$c = \begin{pmatrix} \alpha\gamma & \alpha\delta \\ \beta\delta & \beta\gamma \end{pmatrix}.$$

   The concurrence is $C = 2|\det c| = 2|\alpha\beta(\gamma^2 - \delta^2)|$.
   c) initial state after the first CNOT gate is (ignoring factors $1/\sqrt{2}$)

   $$\begin{aligned} &\alpha|0\rangle(|00\rangle + |11\rangle)(\gamma|0\rangle + \delta|1\rangle)+ \\ &+\beta|1\rangle(|10\rangle + |01\rangle)(\gamma|0\rangle + \delta|1\rangle) \end{aligned} \tag{1}$$

   the read out of the second qubit is assumed to give 1, so we apply the Pauli X on the third qubit,

   $$\alpha|0\rangle|1\rangle|0\rangle(\gamma|0\rangle + \delta|1\rangle) + \beta|1\rangle|1\rangle|1\rangle(\gamma|0\rangle + \delta|1\rangle)$$

   then we perform a CNOT on the third and fourth qubit

   $$\alpha|0\rangle|1\rangle|0\rangle(\gamma|0\rangle + \delta|1\rangle) + \beta|1\rangle|1\rangle|1\rangle(\gamma|1\rangle + \delta|0\rangle)$$

   next a Hadamard on the third qubit,

   $$\alpha|0\rangle|1\rangle(|0\rangle + |1\rangle)(\gamma|0\rangle + \delta|1\rangle) + \beta|1\rangle|1\rangle(|0\rangle - |1\rangle)(\gamma|1\rangle + \delta|0\rangle)$$

the read out of the third qubit is also assumed to give 1, so we apply the Pauli Z on the first qubit,

$$\alpha|0\rangle|1\rangle|1\rangle(\gamma|0\rangle + \delta|1\rangle) + \beta|1\rangle|1\rangle|1\rangle(\gamma|1\rangle + \delta|0\rangle)$$

the second and third qubit are discarded, the remaining state of the first qubit (A with Alice) and the fourth qubit (B with Bob) is the desired outcome of the CNOT operation.

4. *a)* The encoded state is $\alpha|000\rangle + \beta|111\rangle$, after the bit flip error it is $\alpha|010\rangle + \beta|101\rangle$
b) carry out a parity-check measurement (by means of two CNOT gates on a target ancilla qubit) with the first two qubits as control and another parity-check measurement (using another ancilla as target) with the last two qubits as control; measurement of the two ancilla's reveals which qubit has been flipped; this can then be corrected with a σ_x operation, without knowledge of the value of the qubit
c) The Hadamard operation on the encoded state entangles the three qubits; no local operation can do that.