Exam Quantum Information, 11 November 2019, 14.15–17.15 hours.

1. The NOT gate Ω transforms a qubit in the state $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ into the orthogonal state $\Omega|\Psi\rangle = \beta^*|0\rangle - \alpha^*|1\rangle$.

- *a)* Show in a drawing the relative positions of the two states $|\Psi\rangle$ and $\Omega|\Psi\rangle$ on the Bloch sphere.
- *b)* What is Ω if α and β are both real numbers?
- *c)* Explain why there is no such thing as a "universal" NOT gate, which would work for arbitrary complex α, β.

2. The density matrix ρ has the general expression

$$\rho = \sum_n p_n |\Psi_n\rangle\langle\Psi_n|.$$

The coefficients p_n are real positive and $\sum_n p_n = 1$. Each state $|\Psi_n\rangle$ is normalized to unity, but pairs of states $|\Psi_n\rangle$ and $|\Psi_m\rangle$ need not be orthogonal.

- *a)* Derive that $\langle\psi|\hat{\rho}|\psi\rangle \geq 0$ for any arbitrary state $|\psi\rangle$.
- *b)* Show, using the Schrödinger equation with Hamiltonian H for $\Psi_n(t)$, that the density matrix evolves in time according to

$$i\hbar\frac{d}{dt}\rho(t) = H\rho(t) - \rho(t)H.$$

- *c)* The density matrix of a pure state satisfies $\rho^2 = \rho$. Show that a state is pure at time $t > 0$ if and only if it is pure at time $t = 0$.

3. Two qubits A and B are in the state

$$|\Psi\rangle = \tfrac{1}{2}|0\rangle_A|0\rangle_B + \tfrac{1}{2}|1\rangle_A|1\rangle_B + \tfrac{1}{2}|1\rangle_A|0\rangle_B + \tfrac{1}{2}|0\rangle_A|1\rangle_B.$$

- *a)* Are the qubits entangled? Explain your answer.
- *b)* Calculate the reduced density matrix ρ_A of qubit A. Does this density matrix represent a pure state or a mixed state?
- *c)* You are given two qubits C and D in the state $|\Phi\rangle = |0\rangle_C|0\rangle_D$. Construct a circuit using a CNOT gate and any desired single-qubit gate that entangles the qubits C and D.

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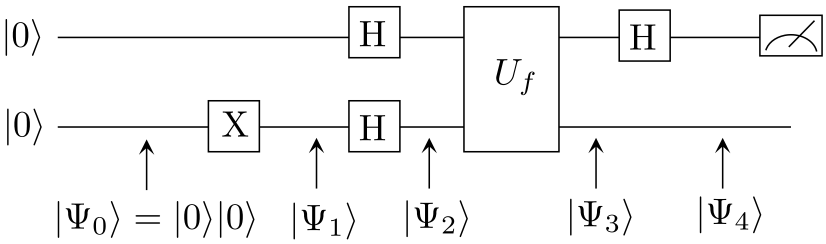
4. The Deutsch algorithm can tell whether a function f from $\{0,1\}$ to $\{0,1\}$ satisfies $f(0) = f(1)$ or $f(0) \neq f(1)$. It does so with a *single* evaluation of f in a two-qubit gate U_f that maps $|x\rangle|y\rangle \mapsto |x\rangle|y \oplus f(x)\rangle$.

- *a)* Why is it in general not possible to represent f by a single-qubit gate?

  The diagram below shows the circuit, containing in addition to the gate U_f three single-qubit Hadamard gates H and a Pauli gate $X = \sigma_x$.



- *b)* Give the expressions for the two-qubit states $|\Psi_n\rangle$ at each stage $n = 1, 2, 3, 4$ of the quantum computation.
- *c)* After the final Hadamard gate that qubit is measured. Explain how the measurement outcome decides whether $f(0) = f(1)$ or $f(0) \neq f(1)$.