Exam Quantum Information, 16 December 2019, 10.15–13.15 hours.

1. Consider the following two-qubit state

$$|\psi\rangle = \sqrt{\tfrac{1}{3}}|00\rangle - \sqrt{\tfrac{1}{4}}|10\rangle + i\sqrt{\tfrac{1}{6}}|01\rangle - i\sqrt{\tfrac{1}{4}}|11\rangle.$$

The first qubit is measured and the outcome is 0.

- *a)* What is the probability for this outcome to happen?
- *b)* Which density matrix describes the system after this measurement? Is it a pure state or a mixed state?
- *c)* Are the two qubits in the original state $|\psi\rangle$ entangled or not? Motivate your answer.

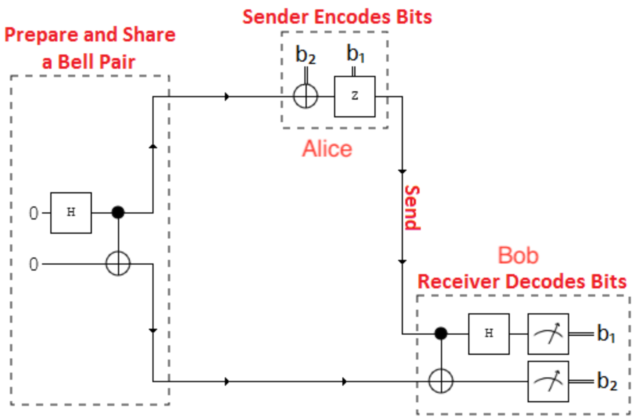
2. Peter has two qubits. The first qubit is in the unknown state $|\psi_1\rangle = \alpha|0\rangle + \beta|1\rangle$, the second qubit is in the known state $|\psi_2\rangle = |0\rangle$.

- *a)* Can you construct a quantum operation that transforms $|\psi_1\rangle|\psi_2\rangle$ into $\alpha|0\rangle|0\rangle + \beta|1\rangle|1\rangle$?
- *b)* Peter says such an operation cannot be a unitary operation because it would violate the "no-cloning theorem". What is your answer to this claim?
- *c)* Give a proof of the no-cloning theorem.

3. In the teleportation protocol Alice and Bob use an entangled qubit pair that they share to transmit an unknown state $\alpha|0\rangle + \beta|1\rangle$ from Alice to Bob.

- *a)* Explain what is meant by the statement that "Alice sends two classical bits of information to Bob in order to transmit the state of one qubit."

The socalled "superdense coding" protocol can be thought of as the inverse of teleportation: Alice sends one qubit to Bob in order to transmit two classical bits of information. This diagram (from Wikipedia) illustrates it:



The dashed box at the left prepares the two-qubit state $2^{-1/2}|0\rangle|0\rangle + 2^{-1/2}|1\rangle|1\rangle$, shared by Alice and Bob. Alice then acts on her qubit with the operation $(\sigma_z)^{b_1}(\sigma_x)^{b_2}$, dependent on the two classical bits b_1, b_2 that she wants to transmit to Bob. When Bob receives her qubit he performs a CNOT *and a Hadamard operation on the two qubits, and finally measures them both.*

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- *b)* First assume that $b_2 = 0$ and write down the state of Bob's two qubits just before and just after he has carried out the CNOT and Hadamard operations. Then do the same for the case $b_2 = 1$.
- *c)* Explain how Bob is able to learn the values of the two classical bits b_1 and b_2 from the single qubit that he has received from Alice.

4. We wish to protect the state of one qubit against the occurrence of an error of the phase-shift type (a σ_z error). For that purpose we encode our qubit into a three-qubit state, according to the rule

$$|0\rangle \to |\psi_0\rangle \equiv 2^{-3/2}(|0\rangle + |1\rangle)^3, \quad |1\rangle \to |\psi_1\rangle \equiv 2^{-3/2}(|0\rangle - |1\rangle)^3.$$

- *a)* Construct a circuit that encodes the state $\alpha|0\rangle + \beta|1\rangle$ as $\alpha|\psi_0\rangle + \beta|\psi_1\rangle$.
- *b)* Show that this encoded state is an eigenstate of each of the two operators $S_1 = \sigma_x \otimes \sigma_x \otimes 1$ and $S_2 = 1 \otimes \sigma_x \otimes \sigma_x$.
- *c)* Suppose we know that at most one of the three qubits in the encoded state has suffered a σ_z-error. Explain how a measurement of S_1 and S_2 allows you to determine on which of the qubits the error has occured — without disturbing the encoded state. A σ_z operation on the erroneous qubit then allows you to recover the original state.