**EXAM QUANTUM INFORMATION, 16 NOVEMBER 2020, 13.30–17.00 HOURS.**

1. • *a)* Construct* a unitary operation $\Omega$ on a two-qubit state, such that $\Omega|0\rangle|0\rangle = |0\rangle|0\rangle$ and $\Omega|1\rangle|0\rangle = |1\rangle|1\rangle$. Why can this operation not be used to copy an arbitrary state $\alpha|0\rangle + \beta|1\rangle$ of a single qubit onto a second qubit?

• *b)* Two qubits are initialized in the state $|0\rangle|0\rangle$. Construct* a unitary operation $U$ such that

$$U|0\rangle|0\rangle = \tfrac{1}{2}|0\rangle|0\rangle + \tfrac{1}{2}|0\rangle|1\rangle + \tfrac{1}{2}|1\rangle|0\rangle + \tfrac{1}{2}|1\rangle|1\rangle.$$

How would this same operation act on the state $|1\rangle|1\rangle$ ?

• *c)* Find out whether or not the two-qubit state $U|0\rangle|0\rangle$ is entangled by calculating the concurrence. What is the state of the second qubit after the first qubit is measured? Does that state depend on the measurement outcome?

2. The states $|0\rangle$, $|1\rangle$, $|2\rangle$ denote orthonormal states of a quantum system, with density matrix $\hat{\rho}$.

• *a)* List three conditions that *any* valid density matrix should satisfy.

• *b)* Explain, for each of the matrices $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{\rho}_3$, if it is a valid density matrix or not:

$$\hat{\rho}_1 = \tfrac{1}{4}|0\rangle\langle 0| - \tfrac{1}{4}|0\rangle\langle 1| - \tfrac{1}{4}|1\rangle\langle 0| + \tfrac{1}{4}|1\rangle\langle 1| + \tfrac{1}{2}|2\rangle\langle 2|,$$

$$\hat{\rho}_2 = \tfrac{1}{2}|0\rangle\langle 0| - \tfrac{1}{2}|1\rangle\langle 1| + |2\rangle\langle 2|, \quad \hat{\rho}_3 = |0\rangle\langle 0| + |1\rangle\langle 1|,$$

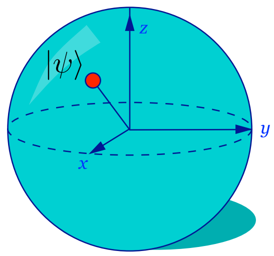
• *c)* Explain, for each of the density matrices $\hat{\rho}_4$, $\hat{\rho}_5$, $\hat{\rho}_6$, if it represents a pure state or not:

$$\hat{\rho}_4 = \tfrac{1}{3}|0\rangle\langle 0| + \tfrac{1}{3}|1\rangle\langle 1| + \tfrac{1}{3}|2\rangle\langle 2|, \quad \hat{\rho}_5 = \tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|0\rangle\langle 1| + \tfrac{1}{2}|1\rangle\langle 0| + \tfrac{1}{2}|1\rangle\langle 1|,$$

$$\hat{\rho}_6 = \tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|1\rangle\langle 1|.$$

3. A bit-flip operation is a single-qubit gate that switches 0 and 1. The unitary matrix for the bit-flip operation is the $\sigma_x$ Pauli matrix.

• *a)* The state of a qubit can be represented by a point on the unit sphere, the Bloch sphere. The figure shows one such point in red, corresponding to a state $|\psi\rangle$. Copy the figure and insert the point that corresponds to the state $\sigma_x|\psi\rangle$. (Please mark the angles, such that the position of the new point can be uniquely identified.) Is the state $\sigma_x|\psi\rangle$ orthogonal to $|\psi\rangle$ or not? Explain your answer.

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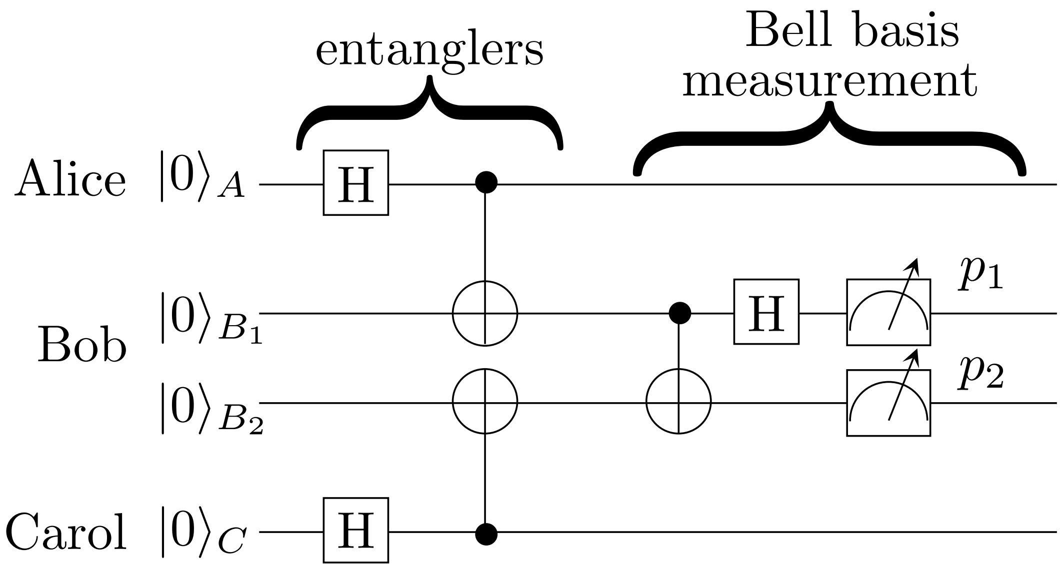
*For the construction you can draw the circuit that represents the operation or you can give the corresponding unitary matrix.*

- *b)* In a classical computer it makes no sense to talk of the "square-root" of a bit flip, but in a quantum computer such a gate exists. What are the elements of a unitary matrix that squares to a bit flip?
- *c)* The SWAP gate is a two-qubit gate that exchanges the states of the first and second qubit: $|01\rangle$ is transformed into $|10\rangle$ and vice versa, while $|00\rangle$ and $|11\rangle$ are unchanged. Can you construct a SWAP gate by combining three CNOT gates? Please show the circuit diagram.




4. Entanglement swapping is a procedure that entangles two distant qubits, qubit $A$ with Alice and qubit $C$ with Carol, without any direct interaction between Alice and Carol. The procedure is carried out by an intermediary, Bob, who has two qubits, $B_1$ and $B_2$. In the first step of the procedure Bob interacts with Alice to entangle qubit $B_1$ with qubit $A$ and he interacts with Carol to entangle qubit $B_2$ with qubit $C$.

- *a)* Explain how the Hadamard and CNOT gates in the figure entangle the qubits of Bob with those of Alice and Carol. Give the four-qubit state after the entanglement step.

  In the second step of the procedure Bob measures both his qubits in the Bell basis, and he communicates the measurement outcomes $p_1$, $p_2$ to Alice and Carol.

- *b)* Show that the qubits of Alice and Carol are entangled for each of the four measurement outcomes $(p_1, p_2) = (0,0), (0,1), (1,0), (1,1)$.
- *c)* If Bob would not tell Alice and Carol what he had measured, the qubits $A$ and $C$ would be in a mixed state, without any entanglement. Calculate the density matrix of this mixed state.