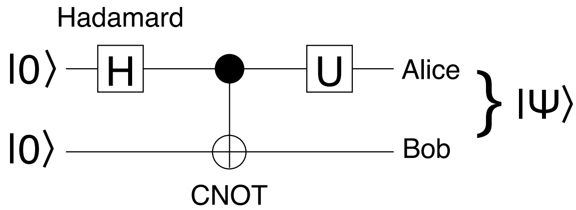
Exam Quantum Information, 18 November 2022, 9–12 hours.

1. The z-component S_z of the angular momentum operator of a spin-1 particle has three eigenvalues, $+1$, 0, -1, with eigenstates $|+1\rangle$, $|0\rangle$, $|-1\rangle$. In this basis the operator S_z and the density matrix ρ of the particle are given by

$$S_z = \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \rho = \frac{1}{4}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

- *a)* Is the particle in a pure state or in a mixed state? Motivate your answer.
- *b)* Calculate the expectation value of S_z.
- *c)* A measurement of S_z gives the value 0. What is the density matrix of the particle after the measurement? Is the state pure or mixed?

2. Consider the circuit shown in the figure, consisting of a Hadamard gate, a controlled NOT gate, and an arbitrary single-qubit gate U.



- *a)* Are the two qubits entangled? Does it depend on U? Motivate your answer by calculating the concurrence of $|\Psi\rangle$.

  Alice and Bob know the circuit and they both know that U is either the identity $U_0 \equiv I$ or one of the three Pauli gates $U_1 \equiv X$, $U_2 \equiv Y$, $U_3 \equiv Z$. Only Alice knows which of these possible four gates was applied, Bob does not know. Alice sends her qubit to Bob.

- *b)* Because Bob does not know U, he has received one of four possible two-qubit states. Show that these are mutually orthogonal.
- *c)* How can Bob determine which of the gates U_0, U_1, U_2, U_3 was applied to Alice's qubit?

  hint: the circuit contains again a Hadamard gate and a CNOT gate.

  comment: This protocol is called "dense coding", because Alice has transmitted two classical bits of information to Bob via one single qubit.

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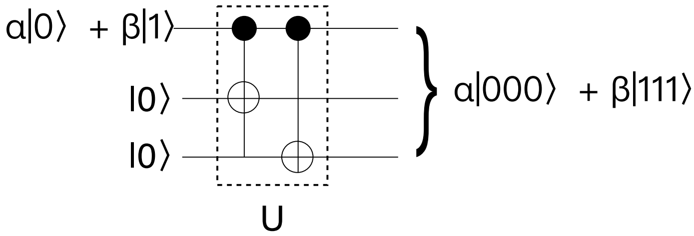
3. The “no-cloning theorem” says that it is impossible to copy an unknown quantum state.

- *a)* Formulate this theorem in mathematical terms.
- *b)* Give a proof of the theorem.
- *c)* Explain what is meant by the statement: “The quantum internet protects the privacy of communications by means of the no-cloning theorem".

4. A qubit in the initial state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ interacts with the enviroment for certain time t; after that time it has suffered a bit-flip error with probability p.

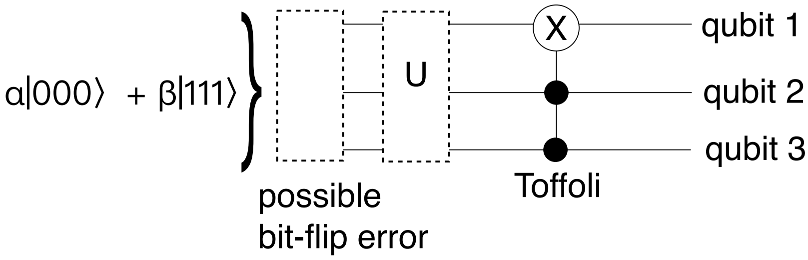
- *a)* What is the density matrix ρ_t of the qubit at time t?

  To protect the qubit from the error, the initial single-qubit state is encoded into three qubits by means of two CNOT gates, see the figure.



- *b)* The no-cloning theorem forbids you from making copies of an unknown state. Explain how the encoded state $\alpha|000\rangle + \beta|111\rangle$ differs from three copies of the initial state $\alpha|0\rangle + \beta|1\rangle$.

  For $p \ll 1$ you may neglect the possibility that more than a single qubit was flipped, so you may assume that at most one of the three qubits in the encoded state is flipped. The final state is first passed once more through the operator U, and then a Toffoli gate* is applied, with the second and third qubits as the controls (see figure).



- *c)* Explain why after this procedure the first qubit has returned to the initial state $\alpha|0\rangle + \beta|1\rangle$. Is the first qubit entangled with the other two qubits?

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*The Toffoli gate is a controlled-controlled-not operation: the first qubit is flipped if and only if the second and third qubits are both equal to 1.