EXAM QUANTUM INFORMATION, 17 NOVEMBER 2023, 9–12 HOURS.

1.
- *a)* Given the two-qubit state $|\psi\rangle = 2^{-1/2}(|00\rangle + |11\rangle)$, compute the reduced density matrix ρ_A for the first qubit by tracing out the second qubit.
- *b)* Calculate the entanglement entropy $-\mathrm{Tr}\,(\rho_A{}^2 \log \rho_A)$.
- *c)* More generally, the pure state of two qubits A, B has the form

  $$|\Psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle.$$

  By tracing out either the second or the first qubit we obtain the reduced density matrices ρ_A or ρ_B, respectively. Prove that

  $$\mathrm{Tr}\,\rho_A{}^2 \log \rho_A = \mathrm{Tr}\,\rho_B{}^2 \log \rho_B.$$

  (You may use that the matrix products UV and VU have the same eigenvalues, for any pair of square matrices U, V.)

2. The purity P of a density matrix ρ is defined by $\mathrm{Tr}\rho^2$.
- *a)* Use the known properties of the eigenvalues of ρ to prove that $0 < P \leq 1$.
- *b)* The system evolves in time under the action of a Hamiltonian H. Prove that the purity does not change in time.
- *c)* Decoherence of a qubit is the process that a qubit which is initially prepared in a pure state becomes after some time a mixed state with smaller purity. Explain, in words, how that can happen in view of the statement in question 2b?

3. Consider the two-qubit state

  $$|A\rangle|B\rangle = (\alpha|0\rangle + \beta|1\rangle)(\gamma|0\rangle + \delta|1\rangle).$$

  You may assume that the coefficients $\alpha, \beta, \gamma, \delta$ are real, with $\alpha^2 + \beta^2 = \gamma^2 + \delta^2 = 1$.
- *a)* Perform a CNOT operation on this state, with qubit A as the control and qubit B as the target. What is the resulting state?
- *b)* The CNOT operation has entangled qubits A and B. Quantify the degree of entanglement by calculating the concurrence.

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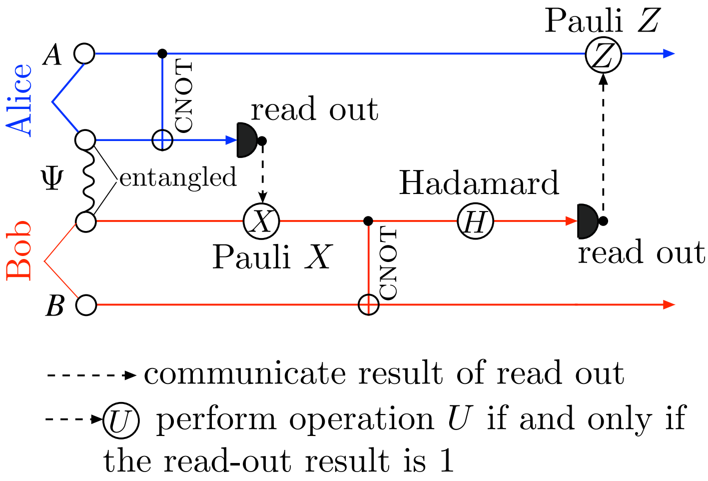
Suppose that Alice holds qubit A, in the state $|A\rangle$ and Bob holds qubit B, in the state $|B\rangle$. Alice and Bob also share an entangled pair of qubits in the state

$$|\Psi\rangle = 2^{-1/2}(|0\rangle|0\rangle + |1\rangle|1\rangle).$$

Alice and Bob live far apart, they can only communicate classically. Teleportation can be used to implement the CNOT operation on the two distant qubits A and B. The circuit for this "gate teleportation" is shown below.



- *c)* Show, by calculating the final state, that this circuit indeed carries out the CNOT operation with qubit A as the control and qubit B as the target. *To simplify this demonstration, you may assume that the first and second read-out both give 1 as outcome.*

4. Consider the 3-qubit error-correcting code where the state $|0\rangle$ is encoded as $|000\rangle$ and the state $|1\rangle$ is encoded as $|111\rangle$.

- *a)* Given an arbitrary state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, write down its encoded state using the 3-qubit code. How does this state change if a bit-flip error occurs on the second qubit?
- *b)* Outline the procedure for detecting and correcting the bit-flip error.
- *c)* Suppose you wish to perform the Hadamard operation on the encoded qubit, so you wish to transform $|000\rangle$ into $2^{-1/2}(|000\rangle + |111\rangle)$ and to transform $|111\rangle$ into $2^{-1/2}(|000\rangle - |111\rangle)$. Explain why this cannot be done using only single-qubit gates.