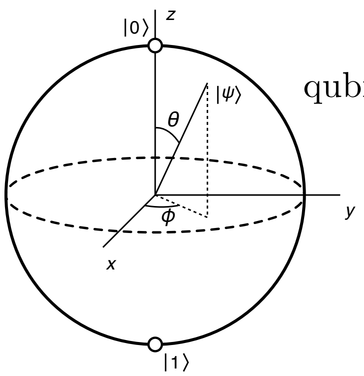

qubit $\leftrightarrow$ Bloch sphere

1. General qubit state (spinor form)

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1.$$

Up to a global phase,

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle, \ \ \theta \in [0, \pi], \ \ \phi \in [0, 2\pi)$$

2. Bloch vector representation

A pure qubit state $\longleftrightarrow$ unit vector on the Bloch sphere.

$$\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z), \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\text{Bloch vector: } \boldsymbol{n} = (n_x, n_y, n_z) = (\sin\theta\cos\phi, \ \sin\theta\sin\phi, \ \cos\theta),$$

$$(\boldsymbol{n}\cdot\boldsymbol{\sigma})|\psi\rangle = |\psi\rangle, \ \ n_\alpha = \langle\psi|\sigma_\alpha|\psi\rangle, \ \ |\langle\psi_1|\psi_2\rangle|^2 = \tfrac{1}{2}(1 + \boldsymbol{n}_1\cdot\boldsymbol{n}_2).$$

3. Unitary action

SU(2) unitaries act as SO(3) rotations

$$|\psi\rangle \mapsto U|\psi\rangle, \quad \boldsymbol{n} \mapsto R\,\boldsymbol{n}, \quad U \in \mathrm{SU}(2), \quad R \in \mathrm{SO}(3).$$

R is a rotation on the Bloch sphere with angle θ about axis $\boldsymbol{n}$

$$U(\boldsymbol{n},\theta) = \exp\left(-i\frac{\theta}{2}\boldsymbol{n}\cdot\boldsymbol{\sigma}\right) = \cos\frac{\theta}{2}\mathbb{I} - i\sin\frac{\theta}{2}\boldsymbol{n}\cdot\boldsymbol{\sigma}.$$

Double cover: $U \mapsto -U$, $\phi \mapsto \phi + 2\pi$, give the same rotation.

4. Density matrix

$$\rho = \tfrac{1}{2}\big(\mathbb{I} + \boldsymbol{a}\cdot\boldsymbol{\sigma}\big), \ \ \mathrm{Tr}\,\rho_1\rho_2 = \tfrac{1}{2}(1 + \boldsymbol{a}_1\cdot\boldsymbol{a}_2)$$

the vector $\boldsymbol{a}$ has norm ≤ 1; for a pure state $\boldsymbol{a} = \boldsymbol{n}$ has norm 1
pure states lie on the Bloch sphere, mixed states lie inside the sphere