Answers to the Exam Quantum Theory, 9 January 2017

each item gives 2 points for a fully correct answer, grade = total $\times\, 9/24 + 1$

1. *(a)*

$$\langle n'|\hat{x}(t)|n\rangle = \langle n'|e^{i\hat{H}t/\hbar}\hat{x}e^{-i\hat{H}t/\hbar}|n\rangle = e^{(it/\hbar)(E_{n'}-E_n)}\langle n'|\hat{x}|n\rangle.$$

   b)

$$i\hbar d\hat{x}/dt = [\hat{x},\hat{H}] = i\hbar\omega\hat{p}, \quad i\hbar d\hat{p}/dt = [\hat{p},\hat{H}] = -i\hbar\omega\hat{x}.$$

   $d^2\hat{x}/dt^2 = -\omega^2\hat{x} \Rightarrow \hat{x}(t) = C_1\cos\omega t + C_2\sin\omega t$, initial conditions give $C_1 = \hat{x}(0)$, $\omega C_2 = \hat{x}'(0) = \omega\hat{p}(0)$.
   (c) Equate terms with positive and negative frequency. For $E_{n'} > E_n$ we must have $Q_+ = 0$ and

$$e^{(it/\hbar)(E_{n'}-E_n)}\langle n'|\hat{x}|n\rangle = \tfrac{1}{2}e^{i\omega t}Q_- = e^{i\omega t}\langle n'|\hat{x}|n\rangle.$$

   Alternatively, you can equate cosine and sine terms,

$$\cos\frac{t(E_{n'}-E_n)}{\hbar}\langle n'|\hat{x}|n\rangle + i\sin\frac{t(E_{n'}-E_n)}{\hbar}\langle n'|\hat{x}|n\rangle = \cos\omega t\langle n'|\hat{x}|n\rangle + \sin\omega t\langle n'|\hat{p}|n\rangle,$$

   to conclude that either $\langle n'|\hat{x}|n\rangle = 0$ or $\hbar\omega = |E_{n'} - E_n| = E_{n'} - E_n$, since $E_{n'} > E_n$.

2. *(a)* $\langle 0|\hat{x}^2|0\rangle = (1/2)\langle 0|\hat{a}\hat{a}^\dagger|0\rangle = 1/2$, $\langle 0|\hat{p}^2|0\rangle = (1/2)\langle 0|\hat{a}\hat{a}^\dagger|0\rangle = 1/2$
   (b) define $|\psi\rangle = \hat{a}^\dagger|N\rangle$, then

$$\hat{a}^\dagger\hat{a}|\psi\rangle = \hat{a}^\dagger(\hat{a}^\dagger\hat{a}+1)|N\rangle = (N+1)|\psi\rangle$$

   so $|\psi\rangle = C|N+1\rangle$. The coefficient $C = (N+1)^{1/2}$ follows from $\langle\psi|\psi\rangle = \langle N|\hat{a}^\dagger\hat{a}+1|N\rangle = N+1 = C^2$.
   (c) the expectation values of $\hat{x}$ and $\hat{p}$ vanish because $\langle N|\hat{a}^\dagger|N\rangle \propto \langle N|N+1\rangle = 0$ and $\langle N|\hat{a}|N\rangle \propto \langle N+1|N\rangle = 0$.
   the second moment follows from $\langle N|\hat{x}^2|N\rangle = \frac{1}{2}\langle 0|\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger|0\rangle = \frac{1}{2}(2N+1)$, and similarly for the momentum.

3. *(a)* Denote by ϕ' the phase shift accumulated on going from beam splitter 1 to beam splitter 2; the amplitude of the transmitted wave is

$$\Psi_{\rm out} = \tfrac{1}{2}\Psi_{\rm in}e^{i\phi'}\left[1 + \tfrac{1}{2}e^{i\phi} + (\tfrac{1}{2}e^{i\phi})^2 + (\tfrac{1}{2}e^{i\phi})^3 + \cdots\right] = \frac{\frac{1}{2}\Psi_{\rm in}e^{i\phi'}}{1-\frac{1}{2}e^{i\phi}}.$$

   The absolute value squared of $\Psi_{\rm out}/\Psi_{\rm in}$ then gives $T = \frac{1}{4}(1+\frac{1}{4}-\cos\phi)^{-1}$.
   (b) Constructive interference of the waves at beam splitter 2 for $\phi = 0$ gives resonant transmission with unit probability. Destructive interference for $\phi = \pi$ gives a joint transmission probability 1/9 smaller than the product of the two individual transmission probabilities.
   (c) For an enclosed flux $\Phi = BL^2$ the electron picks up an Aharonov-Bohm phase $2\pi\Phi e/h$ which adds to ϕ, so the transmission probability oscillates between 1/9 and 1 with period $\Delta\Phi = h/e \Rightarrow \Delta B = L^{-2}h/e$.

4. *(a)* free motion at constant velocity $\dot{x} = v = (x_2 - x_1)/(t_2 - t_1)$, so the action is $S_{\text{class}} = \frac{1}{2}mv^2(t_2 - t_1) = \frac{1}{2}m(x_2 - x_1)^2/(t_2 - t_1)$.
(b) insert a resolution of the identity $\int dp|p\rangle\langle p|$ to write

$$
\begin{aligned}
G &= (2\pi\hbar)^{-1}\int_{-\infty}^{\infty} dp\, \exp\left(\frac{i}{\hbar}(x_2 - x_1)p\right)\exp\left(-\frac{i}{\hbar}(t_2 - t_1)\frac{p^2}{2m}\right) \\
&= \exp\left(\frac{im(x_2 - x_1)^2}{2\hbar(t_2 - t_1)}\right)\sqrt{\frac{m}{2\pi i\hbar(t_2 - t_1)}}.
\end{aligned}
$$

(c) Feynman's path integral formula: $G(x_2, t_2; x_1, t_1) \propto \sum_{\text{paths}} \exp\left(\frac{i}{\hbar}S_{\text{path}}\right)$. In the semiclassical limit only classical paths contribute; in this case there is a single classical path, so $G \propto e^{iS_{\text{class}}/\hbar}$; the proportionality constant contains fluctuations around the classical path.