Answers to the Exam Quantum Theory, 12 March 2020

each item gives 2 points for a fully correct answer, grade = total $\times 9/24 + 1$

1. *a)* $\hat{p}\Phi(p) = p\Phi(p)$, $\hat{x}\Phi(p) = i\hbar d\Phi/dp$.
   b) $[x, -i\hbar d/x] = -i\hbar(xd/x - d/dxx) = i\hbar$, $[i\hbar d/dp, p] = i\hbar(d/dpp - pd/dp) = i\hbar$.
   c) $(\hat{x}\hat{p})^\dagger = \hat{p}\hat{x} = \hat{x}\hat{p} - i\hbar$, so $\hat{O}_1$ is not Hermitian; $(\hat{x}^2\hat{p} - i\hbar\hat{x})^\dagger = \hat{p}\hat{x}^2 + i\hbar\hat{x} = \hat{x}^2\hat{p} - 2i\hbar\hat{x} + i\hbar\hat{x} = \hat{x}^2\hat{p} - i\hbar\hat{x}$, so $\hat{O}_2$ is Hermitian.

2. *a)* $U = \sum_{n,m} c_n^* c_m \langle\psi_n|\hat{H}|\psi_m\rangle = \sum_{n,m} c_n^* c_m E_n \delta_{nm} = \sum_n |c_n|^2 E_n$. Since $E_n \geq E_0$ for all n, we have $U \geq E_0 \sum_n |c_n|^2 = E_0$, because $\sum_n |c_n|^2 = \langle\phi|\phi\rangle = 1$.
   b) $\sum_n |c_n|^2 (E_n - U)^2 = \sum_n |c_n|^2 (E_n^2 - 2UE_n + U^2) = V - 2U^2 + U^2 = V - U^2$.
   c) Assuming that $|E_p - U| \leq |E_n - U|$ for all n, we have $V - U^2 \geq (E_p - U)^2 \sum_n |c_n|^2 = (E_p - U)^2$. Taking the square root gives $|E_p - U| \leq \sqrt{V - U^2} \Rightarrow -\sqrt{V - U^2} \leq E_p - U \leq \sqrt{V - U^2}$, which is equation (1). The ground state E_0 may be smaller than $U - \sqrt{V - U^2}$ if there is a higher energy level in the range of equation (1), so this is not a lower bound on the ground state. (We only have the upper bound $E_0 \leq U$.)

3. *a)* $ae^{\beta a^\dagger}|0\rangle = \sum_{n=0}^{\infty} \frac{1}{n!}\beta^n a(a^\dagger)^n|0\rangle = \sum_{n=0}^{\infty} \frac{1}{n!}\beta^n n(a^\dagger)^{n-1}|0\rangle = \beta e^{\beta a^\dagger}|0\rangle$
   b) $\langle\alpha|\beta\rangle = e^{-|\alpha|^2/2}\langle 0|e^{\alpha^* a}|\beta\rangle = e^{-|\alpha|^2/2}e^{\alpha^*\beta}\langle 0|\beta\rangle = e^{-|\alpha|^2/2 - |\beta|^2/2}e^{\alpha^*\beta}$, hence $|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha|^2 - |\beta|^2}e^{\alpha^*\beta + \alpha\beta^*} = e^{-|\alpha - \beta|^2}$.
   c) $\bar{n} = \langle\beta|a^\dagger a|\beta\rangle = \beta^*\beta$, $\overline{n^2} = \langle\beta|(a^\dagger a)^2|\beta\rangle = \langle\beta|(a^\dagger)^2 a^2|\beta\rangle + \langle\beta|a^\dagger a|\beta\rangle = (\beta^*\beta)^2 + \beta^*\beta$, so $\operatorname{var} n = \beta^*\beta = \bar{n}$. For Poisson statistics the average and the variance of n should be the same, which is indeed the case.

4. *a)* Insert a resolution of the identity, $\mathbb{1} = \sum_n |n\rangle\langle n|$, to arrive at $\int dx\, G(x, x; t) = \int dx\, \sum_n \langle x|e^{-iHt/\hbar}|n\rangle\langle n|x\rangle = \sum_n e^{-iE_n t/\hbar} \int dx\, |\langle x|n\rangle|^2 = \sum_n e^{-iE_n t/\hbar}$.
   b) $S[x(t')] = \int_0^t L[x(t'), \dot{x}(t')]dt'$, with $L(x, \dot{x}) = p\dot{x} - H = \dot{x}^2/2m - V(x)$ the Lagrangian.
   c) The Fourier transform $F(t)$ of $\rho(E)$ contains only paths which return to their starting point, and in the semiclassical limit $\hbar \to 0$ only classical paths contribute. In an x–t diagram this is a straight line from $(0, 0)$ to (W, T) and then from (W, T) back to $(0, 2T)$, where $T = W(2E/m)^{-1/2}$.