Answers to the Exam Quantum Theory, 4 January 2021

each item gives 2 points for a fully correct answer, grade = total $\times\, 9/24 + 1$

1. *a)* Expand $e^{i\beta\tau}$ in a Taylor series,
$$[e^{i\beta\tau}, H] = \sum_{p=0}^{\infty}[\frac{1}{p!}(i\beta\tau)^p, H] = -\hbar\beta\sum_{p=1}^{\infty}\frac{1}{(p-1)!}(i\beta\tau)^{p-1} = -\hbar\beta e^{i\beta\tau}.$$
b) $HU|E\rangle = He^{i\beta\tau}|E\rangle = e^{i\beta\tau}H|E\rangle + \hbar\beta e^{i\beta\tau}|E\rangle = (E+\hbar\beta)|E\rangle$. So eigenstate at eigenvalue $E+\hbar\beta$. Since U is unitary (for Hermitian τ), the norm of $U|E\rangle$ is conserved, equal to unity, so this state cannot be equal to zero.
c) Since β can take on any real value, the spectrum of H must be the entire real axis: continuous, unbounded.

2. *a)* $[a, a^\dagger] = i[p, q] = 1$; $q = (a + a^\dagger)/\sqrt{2}$, $p = -i(a - a^\dagger)/\sqrt{2}$; $q^2 + p^2 = aa^\dagger + a^\dagger a = 2a^\dagger a + 1$; $\Rightarrow R(\theta) = e^{i\theta/2}e^{i\theta a^\dagger a}$.
b) $db/d\theta = R^\dagger(\theta)[a, ia^\dagger a]R(\theta) = ib(\theta) \Rightarrow b(\theta) = e^{i\theta}b(0) = e^{i\theta}a$.
c) $R^\dagger(\theta)qR(\theta) = 2^{-1/2}e^{i\theta}a + 2^{-1/2}e^{-i\theta}a^\dagger = q\cos\theta - p\sin\theta$,
hence $R^\dagger(-\pi/2)qR(-\pi/2) = p$, so the eigenstates transform as
$R^\dagger(-\pi/2)|s\rangle_q = |s\rangle_p$ and $R^\dagger(-\pi/2)|s\rangle_q = R(\pi/2)|s\rangle_q$.

3. *a)* The velocity, which is a physical observable, cannot depend on the choice of gauge, so $\phi \mapsto \phi + (2e/\hbar)\chi$ to cancel the change in $\vec{A}$.
b) $F = \oint_{\delta S}(\vec{A} + m\vec{v}/e)\cdot d\vec{l} = (1/\hbar)\oint_{\delta S}\nabla\phi\cdot d\vec{l} = (\hbar/2e)\Delta\phi$, where $\Delta\phi$ is the change in ϕ on going once around the perimeter δS. Because the wave function must be single-valued, $\Delta\phi$ must be an integer multiple of 2π, hence F must be an integer multiple of $2\pi\hbar/2e = h/2e$.
c) A change in flux by $h/2e$ is represented by a change in the vector potential in the superconducting disc by $\delta\vec{A} = (\hbar/2e)\hat{\theta}/r$, in polar coordinates r, θ. I can remove this change by a gauge transformation with $\chi = -(\hbar/2e)\theta$, which changes the phase by $\delta\phi = -\theta$, leaving the velocity unaffected. The phase remains single-valued, so this is allowed.
The Byers-Yang theorem says that all physical properties are periodic with period h/e, a periodicity $h/2e$ still satisfies that requirement.

4. *a)*
$$E(L) = \frac{\pi\hbar c}{2L}\sum_{n=1}^{\infty}ne^{-\alpha n}, \text{ with } \alpha = \frac{\pi}{Lk_c},$$
and then substitute $\sum_{n=1}^{\infty}ne^{-\alpha n} = 1/\alpha^2 - 1/12 + \text{order}(\alpha^2)$.
b) $E_{\text{tot}}(a, b) = E(a) + E(b) = \frac{1}{2}\pi\hbar c\big((a+b)k_c^2/\pi^2 - 1/12a - 1/12b\big)$
c) Vary a at fixed $L = a + b$, so $b = L - a$; $F = -dE_{\text{tot}}(a, L-a)/da = -\frac{1}{2}\pi\hbar c\big(1/12a^2 - 1/12(L-a)^2\big) \to -\pi\hbar c/24a^2$.
The force points to the right in the figure.

For more on this calculation of the Casimir effect, see

https://en.wikiversity.org/wiki/Quantum_mechanics/Casimir_effect_in_one_dimension