ANSWERS TO THE EXAM QUANTUM THEORY, 24 JANUARY 2022

each item gives 2 points for a fully correct answer, grade = total $\times 9/24+1$

1. *a)* $U^{-1}\mathcal{T}\Psi = \mathcal{T}U\Psi = \mathcal{T}\lambda\Psi = \lambda^*\mathcal{T}\Psi$, so Ψ' is an eigenstate of U^{-1} with eigenvalue λ^*, and hence an eigenstate of U with eigenvalue $1/\lambda^*$. Since the eigenvalues of the unitary operator U are of the form $e^{i\phi}$ with real ϕ, we have $1/\lambda^* = \lambda$.
   b) Suppose $\Psi' = c\Psi$, so $\mathcal{T}\Psi = c\Psi$; apply $\mathcal{T}$ to both sides, $-\Psi = \mathcal{T}^2\Psi = c^*\mathcal{T}\Psi = |c|^2\Psi$, which is a contradiction since $\Psi \neq 0$.

2. *a)* Alice is right: $|\Psi\rangle^*$ is an eigenstate with eigenvalue λ^* of a^* not of $a^\dagger$.
   b) $\langle n+1|\beta\rangle = \beta^{-1}\langle n+1|a^\dagger|\beta\rangle = \beta^{-1}\sqrt{n+1}\langle n|\beta\rangle = 0$
   c) $\langle 0|\beta\rangle = \beta^{-1}\langle 0|a^\dagger|\beta\rangle = 0$; this is the first step of the induction process, hence $\langle n|\beta\rangle = 0$ for all $n \geq 0$, which means that the state $|\beta\rangle$ is identically zero, so it cannot be an eigenstate. The conclusion is that eigenstates of the annihilation operator do not exist.
   Since $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, no superposition of number states can be annihilated by $a^\dagger$, hence the eigenvalue $\beta = 0$ is not possible.

3. *a)* The operator $A^\dagger = \frac{1}{2}\nu xp - \frac{1}{4}i\hbar\nu = \frac{1}{2}\nu px + \frac{1}{4}i\hbar\nu = A$, so it is Hermitian. Hence $U^\dagger = e^{-itA^\dagger/\hbar} = e^{-itA/\hbar} = U^{-1}$, so it is unitary.
   b) $dP/dt = (i/\hbar)U[A,p]U^{-1} = -\frac{1}{2}\nu UpU^{-1} = -\frac{1}{2}\nu P$;
   $dX/dt = (i/\hbar)U[A,X]U^{-1} = \frac{1}{2}\nu UxU^{-1} = \frac{1}{2}\nu X$. The commutators follow from $[A,p] = \frac{1}{2}\nu(pxp - p^2x) = \frac{1}{2}i\hbar\nu p$, $[A,x] = \frac{1}{2}\nu(px^2 - xpx) = -\frac{1}{2}i\hbar\nu x$.
   c) $P(t) = e^{-\nu t/2}P(0) = e^{-\nu t/2}p$, $X(t) = e^{\nu t/2}X(0) = e^{\nu t/2}x$.
   $U(t)H(0)U^{-1}(t) = P(t)^2/2m_0 + m_0\omega^2X(t)^2/2 = e^{-\nu t}p^2/2m_0 + m_0\omega^2e^{\nu t}x^2/2 = H(t)$.
   d) $i\hbar d\psi/dt = H(t)\psi = U(t)H(0)U^{-1}(t)\psi$, define $\tilde{\psi} = U^{-1}\psi$, then $H_0\tilde{\psi} = i\hbar U^{-1}d\psi/dt = i\hbar d\tilde{\psi}/dt - i\hbar(dU^{-1}/dt)U\tilde{\psi} = i\hbar d\tilde{\psi}/dt + A\tilde{\psi}$. Hence $i\hbar d\tilde{\psi}/dt = (H(0)-A)\tilde{\psi} \Rightarrow \tilde{\psi}(t) = \exp[-(i/\hbar)(H(0)-A)]\tilde{\psi}(0) \Rightarrow \psi(t) = U(t)\tilde{\psi}(t) = U(t)\exp[-(i/\hbar)(H(0)-A)]\psi(0)$.
   So Charlie has overlooked the term A in the exponent. Without that term, the harmonic oscillator remains an eigenstate of $H(t)$ if it starts out as an eigenstate, with the same eigenvalue, there are no transitions to other states — this is the adiabatic approximation.

4. *(a)* free motion at constant velocity $\dot{x} = v = (x_2 - x_1)/(t_2 - t_1)$, so the action is $S_{\text{class}} = \frac{1}{2}mv^2(t_2 - t_1) = \frac{1}{2}m(x_2 - x_1)^2/(t_2 - t_1)$.
   (b) insert a resolution of the identity $\int dp|p\rangle\langle p|$ to write

$$G = (2\pi\hbar)^{-1}\int_{-\infty}^{\infty} dp \exp\left(\frac{i}{\hbar}(x_2 - x_1)p\right)\exp\left(-\frac{i}{\hbar}(t_2 - t_1)\frac{p^2}{2m}\right)$$
$$= \exp\left(\frac{im(x_2 - x_1)^2}{2\hbar(t_2 - t_1)}\right)\sqrt{\frac{m}{2\pi i\hbar(t_2 - t_1)}}.$$

   (c) Feynman's path integral formula: $G(x_2,t_2;x_1,t_1) \propto \sum_{\text{paths}}\exp\left(\frac{i}{\hbar}S_{\text{path}}\right)$. In the semiclassical limit only classical paths contribute; in this case there is a

single classical path, so $G \propto e^{iS_{\text{class}}/\hbar}$; the proportionality constant contains fluctuations around the classical path.