Exam Quantum Theory, 20 January 2016, 12–15 hours.

1. The wave function of a particle in position representation is $\psi(x)$, normalized to unity: $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$. (The particle is spinless and confined to the x-axis.)

- *a)* What is the corresponding wave function $\psi(p)$ in momentum representation? Verify that it is also normalized to unity.
- *b)* The time-reversal operation $\mathcal{T}$ in position representation is $\mathcal{T}\psi(x) = \psi^*(x)$. What is the corresponding time-reversal operation in momentum representation?
- *c)* The Kramers theorem says that, under certain conditions, the eigenstates in the presence of time reversal symmetry are twofold degenerate. Does Kramers theorem apply in this case? Motivate your answer.

2. Consider the creation and annihilation operators $c_\alpha^\dagger, c_\beta^\dagger$ and c_α, c_β of a particle for two *different* states labeled α and β. The particle may be either a boson or a fermion.

- *a)* If we exchange two bosons the wave function remains the same, if we exchange two fermions the wave function acquires a minus sign. Explain how this difference manifests itself in the creation and annihilation operators.
- *b)* Calculate the expectation value $\langle 0|c_\alpha c_\beta c_\alpha^\dagger c_\beta^\dagger|0\rangle$ in the vacuum state $|0\rangle$, both for the bosonic and for the fermionic case.

  Now assume that the particle is a fermion and and that it can occupy one of N states, labeled $i = 1, 2, \ldots N$. Let U be an $N \times N$ unitary matrix.
- *c)* Show that the transformation from c_i to a_i given by

$$a_i = \sum_{j=1}^{N} U_{ij} c_j, \quad i = 1, 2, \ldots N,$$

  has no effect on the commutation relation of the creation and annihilation operators.

3. The Hamiltonian H has eigenvalues E_n, $n = 0, 1, 2, \ldots$, with corresponding eigenfunctions Φ_n. The ground state Φ_0 has the lowest eigenvalue E_0. An arbitrary wave function ψ can be expanded in the basis of eigenfunctions, $\psi = \sum_n \alpha_n \Phi_n$, with complex coefficients α_n.

- *a)* Express the inner product $\langle\psi|\psi\rangle$ in terms of these expansion coefficients.

  The variational theorem says that

$$\frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \geq E_0.$$

- *b)* Prove this theorem.

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The one-dimensional harmonic oscillator has Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx} + \tfrac{1}{2}m\omega^2 x^2.$$

For a given parameter $a > 0$ we approximate the ground state wave function by

$$\Phi_a(x) \approx C \exp\left(-\frac{x^2}{2a}\right),$$

with normalization constant $C = (2\pi a)^{-1/2}$.

- *c)* Explain how you would use the variational theorem to approximate the ground state energy E_0. (You don't have to actually carry out the calculation, just explain which steps you would take.)

4. We consider the Hamiltonian H of a particle of mass m moving along the x-axis in a confining potential $V(x)$. The eigenvalues of H form the discrete spectrum $E_0, E_1, E_2, \ldots$. We define the density of states $\rho(E) = \sum_{n=0}^{\infty} \delta(E - E_n)$ and its Fourier transform

$$F(t) = \int_{-\infty}^{\infty} \rho(E) e^{-iEt/\hbar} dE = \sum_{n=0}^{\infty} e^{-iE_n t/\hbar}.$$

The dynamics from position x_0 to x_1 in a time t is described by the propagator

$$G(x_1, x_0; t) = \langle x_1 | e^{-iHt/\hbar} | x_0 \rangle.$$

- *a)* Derive the following relation between $F(t)$ and the integral of the propagator for equal initial and final position:

$$\int_{-\infty}^{\infty} G(x, x; t) dx = F(t).$$

Feynman showed that the propagator $G(x_1, x_0; t)$ can be written as an integral over all paths $x(t')$ with $x(0) = x_0$ and $x(t) = x_1$,

$$G(x_1, x_0; t) = \sqrt{\frac{m}{2\pi i\hbar t}} \int_{x(0)=x_0}^{x(t)=x_1} \mathcal{D}[x(t')]\, e^{iS[x(t')]/\hbar}.$$

- *b)* Explain how the path-dependent quantity $S[x(t')]$ is related to the Hamiltonian H.

- *c)* Which paths contribute predominantly to the density of states $\rho(E)$ in the limit $\hbar \to 0$?