# Exam Quantum Theory, 4 January 2021, 13.30–17.00 hours.

1. Pauli's theorem (1926) determines necessary conditions for the existence of a Hermitian "time operator" $\tau$ such that $[\tau, H] = i\hbar$ (with $H$ the Hamiltonian). Let us find these conditions.

- *a)* Define for any real $\beta$ the operator $U = e^{i\beta\tau}$. Derive the commutator

$$[U, H] = -\hbar\beta U.$$

- *b)* Let $|E\rangle$ be an eigenstate of $H$ (normalized to unity), with eigenvalue $E$. Prove that $U|E\rangle$ is also an eigenstate of $H$. Check that it is nonzero!
- *c)* Explain why the existence of a time operator is forbidden if $H$ has a discrete spectrum and also if $H$ has a ground state.

2. For ease of notation, in this exercise we set $\hbar$ to 1, so that the commutator of position and momentum operators is $[q, p] = i$. The state $|s\rangle_q$ is an eigenstate of position with eigenvalue $s$, and the state $|s\rangle_p$ is an eigenstate of momentum with eigenvalue $s$.
Consider the operator

$$R(\theta) = e^{i\theta(q^2+p^2)/2},$$

depending on the real parameter $\theta$. We wish to show that $R(\pi/2)$ transforms eigenstates of position into eigenstates of momentum, and more generally, that $R(\theta)$ "rotates" a state in phase space.

- *a)* The annihilation operator $a$ is defined by $a = (q + ip)/\sqrt{2}$. Calculate the commutator $[a, a^\dagger]$. Derive that

$$R(\theta) = e^{i\theta/2}e^{i\theta a^\dagger a}.$$

- *b)* Define $b(\theta) = R^\dagger(\theta)aR(\theta)$ and calculate the derivative $db/d\theta$. Use this result to demonstrate that

$$R^\dagger(\theta)aR(\theta) = e^{i\theta}a.$$

- *c)* Express $R^\dagger(\theta)qR(\theta)$ in terms of the operators $q$ and $p$ and explain why $|s\rangle_p = R(\pi/2)|s\rangle_q$.

3. The ground state of a superconductor is described by a wave function for paired electrons, known as Cooper pairs. The phase $\phi(\vec{r})$ of that wave function determines the ground-state velocity $\vec{v}$ of the Cooper pairs via the equation

$$2m\vec{v} = \hbar\nabla\phi(\vec{r}) - 2e\vec{A}(\vec{r}),$$

where $\vec{A}$ is the vector potential. (The electron has charge $+e$ and mass $m$.)

- *a)* A gauge transformation changes $\vec{A} \mapsto \vec{A} + \nabla\chi$, where $\chi(\vec{r})$ is an arbitrary scalar field. How does the phase $\phi(\vec{r})$ change under this gauge transformation?

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- *b)* A surface $S$ inside the superconductor has perimeter $\delta S$. The magnetic flux through $S$ equals $\Phi$. The socalled "fluxoid" $F$ is defined by

$$F = \Phi + \frac{m}{e} \oint_{\delta S} \vec{v} \cdot d\vec{l}.$$

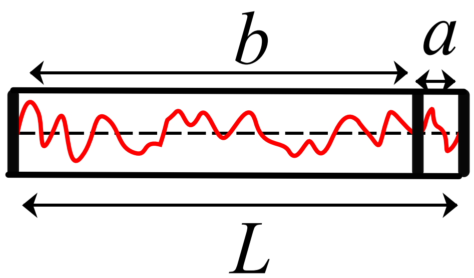
Derive the law of fluxoid quantization: $F$ equals an integer multiple of $h/2e$.
- *c)* Consider a superconducting disc with a hole. A magnetic field is nonzero only inside the hole. The flux through the hole is $\Phi$. Explain why the ground-state velocity $\vec{v}$ must be a periodic function of $\Phi$ with period $h/2e$.
Does this invalidate the Byers-Yang theorem? (Explain.)

4. We consider the vacuum electromagnetic energy $E$ inside a single-mode wave guide, of length $L$, closed at the two ends by metal boundaries. The wave vector $k$ has only components along the wave guide, equal to $k = \pi n/L$, with $n = 1, 2, 3, \ldots$. The vacuum energy contribution from each wave vector (speed of light $c$) is $\frac{1}{2}\hbar c k e^{-k/k_c}$. The exponential factor enters because waves of wave number $k \gtrsim k_c$ are suppressed by the resistivity of the metal boundaries.

- *a)* Show that for large $k_c$ the vacuum energy has the Taylor expansion*

$$E(L) = \tfrac{1}{2}\pi\hbar c \left( \frac{Lk_c^2}{\pi^2} - \frac{1}{12L} + \text{order}(1/k_c^2) \right).$$

- *b)* We insert a metal plate in the wave guide, as shown in the figure, at a distance $a$ from one end and at a distance $b$ from the other end. What is now the vacuum energy of the entire system for large $k_c$?
- *c)* Calculate the force on the metal plate when $b \gg a$. In which direction does it point?




*You may use that $\sum_{n=1}^{\infty} n e^{-\alpha n} = 1/\alpha^2 - 1/12 + \text{order}(\alpha^2)$.