Exam Quantum Theory, 25 January 2021, 13.30–17.00 hours.

1. The parity operator P can be defined by its action on a wave function $\psi(x)$: $P\psi(x) = \psi(-x)$.

- *a)* Recall the definition of a Hermitian operator and prove that P is Hermitian.
- *b)* Show that P is also unitary and give its eigenvalues.
- *c)* The Hamiltonian $H = p^2/2m + V(x)$ commutes with P if the potential $V(x)$ is an even function of x. Assume that this is the case and prove that the wave function of any nondegenerate energy level must be either an even or an odd function of x. (In your proof, indicate explicitly where you use the nondegeneracy of the energy level.)

2. The *squeezed vacuum* for photons is the state $|s\rangle \equiv S(s)|0\rangle$ obtained by acting on the vacuum state $|0\rangle$ with the squeeze operator

$$S(s) = \exp\left(\tfrac{1}{2}s(aa - a^\dagger a^\dagger)\right).$$

Here s is a real number and $a, a^\dagger$ are bosonic annihilation and creation operators (commutator $[a, a^\dagger] = 1$).

- *a)* Is $S(s)$ unitary? Is it Hermitian?

In what follows you may use the identity

$$S^\dagger(s) a S(s) = a\cosh s - a^\dagger \sinh s.$$

- *b)* The position operator is $\hat{x} = 2^{-1/2}(a + a^\dagger)$ (in dimensionless units). Calculate the variance $\Delta x^2 = \langle s|\hat{x}^2|s\rangle - \langle s|\hat{x}|s\rangle^2$ of the position in the squeezed vacuum state.
- *c)* For $s \to \infty$ the variance of the position goes to zero. Does this contradict the uncertainty principle? Please explain.

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3. We consider a spin-1/2 particle at rest in a time-dependent magnetic field $\vec{B}$ which rotates in the x–y plane, so $\vec{B}(t) = B_0(\cos\omega t, \sin\omega t, 0)$ (with B_0 the field strength and ω the rotation frequency). The Hamiltonian is

$$H[\vec{B}(t)] = -\frac{\mu}{2}\vec{\sigma}\cdot\vec{B}(t) = -\frac{\mu}{2}\begin{pmatrix} 0 & B_x(t) - iB_y(t) \\ B_x(t) + iB_y(t) & 0 \end{pmatrix}.$$

We wish to solve the Schrödinger equation

$$i\hbar\frac{d}{dt}\psi(t) = H[\vec{B}(t)]\psi(t),$$

to obtain the two-component wave function $\psi(t) = (u(t), v(t))$ with initial condition $u(0) = 1$, $v(0) = 0$.

- *a)* Show that H becomes time independent if we make a unitary transformation with the matrix $U = \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix}$.
- *b)* Derive the following evolution equation for $\tilde{\psi}(t) = U\psi(t)$,

$$i\hbar\frac{d}{dt}\tilde{\psi}(t) = \tilde{H}\tilde{\psi}(t), \quad \text{with } \tilde{H} = \frac{1}{2}\begin{pmatrix} -\hbar\omega & -\mu B_0 \\ -\mu B_0 & \hbar\omega \end{pmatrix}.$$

- *c)* Calculate the time dependence of $u(t)$ and $v(t)$.*

4. A particle of mass m moves freely along the x-axis, with Hamiltonian $H(x,p) = \frac{1}{2}p^2/m$ and Lagrangian $L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2$.

- *a)* Calculate the classical action $S_{\text{class}} = \int_{t_1}^{t_2} L\,dt$ for the classical path from point x_1 at time t_1 to point x_2 at time t_2.
- *b)* Calculate the quantum mechanical propagator†

$$G(x_2, t_2; x_1, t_1) = \langle x_2|e^{-(i/\hbar)(t_2-t_1)H}|x_1\rangle.$$

- *c)* Discuss the relation between G and S_{class} in the context of Feynman's path integral formula.

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*You may use the following matrix identity (with $r = \sqrt{a^2+b^2}$):

$$\exp\left[\begin{pmatrix} ia & ib \\ ib & -ia \end{pmatrix}\right] = \begin{pmatrix} \cos r + i(a/r)\sin r & i(b/r)\sin r \\ i(b/r)\sin r & \cos r - i(a/r)\sin r \end{pmatrix}.$$

†You may use the integral $\int_{-\infty}^{\infty} e^{ias-ibs^2}ds = \sqrt{\frac{\pi}{ib}}\exp\left(\frac{ia^2}{4b}\right)$.