Exam Quantum Theory, 8 January 2023, 9–12 hours.

1. For any complex number β, consider the state $|\beta\rangle$, defined as an infinite sum of eigenstates $|n\rangle = (n!)^{-1/2}(a^\dagger)^n|0\rangle$ of the harmonic oscillator,

$$|\beta\rangle = C\sum_{n=0}^{\infty}\frac{\beta^n}{\sqrt{n!}}|n\rangle. \tag{1}$$

The coefficient C is a normalization constant.

- *a)* Show that $|\beta\rangle$ is an eigenstate with eigenvalue β of the bosonic annihilation operator a. (Such a state is called a *coherent* state.)
- *b)* Calculate the value of C such that $|\beta\rangle$ is normalized to unity.
- *c)* Is it possible to construct a state of the form $|\Psi\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$, with coefficients c_n such that $|\Psi\rangle$ is an eigenstate of the *creation* operator $a^\dagger$? Motivate your answer.

2. A quantum particle is in the ground state of a one-dimensional potential well between two infinite barriers. One barrier is fixed at $x = 0$, the other barrier is at a time-dependent position $x = L(t)$. We assume that $L(t)$ first increases from L_1 to L_2 and then decreases back to L_1. You may assume that the adiabatic approximation holds, so the particle remains in the ground state. The final wave function differs from the initial wave function by a phase factor $e^{i\phi}$. The phase ϕ is the sum of the dynamical phase and the Berry phase.

- *a)* How slow should the motion of the barrier be for the adiabatic approximation to hold?
- *b)* Calculate the dynamical phase for a given time dependence $L(t)$. (You may leave the final answer in the form of an integral over time.)
- *c)* Show that the Berry phase equals zero for any $L(t)$.

Reminder: the Berry phase γ_B for a state that depends on a time-dependent parameter α is given by $\gamma_B = i\oint\langle\alpha|\frac{\partial}{\partial\alpha}|\alpha\rangle\, d\alpha$.

3. The Hamiltonian of a two-dimensional Dirac fermion is given by

$$H = vp_x\sigma_x + vp_y\sigma_y + \mu\sigma_z, \tag{2}$$

in terms of Pauli matrices σ_α, velocity v, and momentum operator $\boldsymbol{p} = (p_x, p_y)$. In graphene the coefficient $\mu = 0$, but here we consider the more general case $\mu > 0$.

- *a)* Does this Hamiltonian satisfy the conditions for the Kramers degeneracy theorem to apply? Motivate you answer.
- *b)* Calculate the energy spectrum and plot it as a function of momentum. *Hint: try squaring H.*
- *c)* The ratio μ/v^2 is called the *mass* of the Dirac fermion. Can you explain why?

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4. A particle of mass m moves freely along the x-axis, with Hamiltonian $H = \frac{1}{2}p^2/m$.

- *a)* Calculate the quantum mechanical propagator

$$G(x_2, t_2; x_1, t_1) = \langle x_2 | e^{-(i/\hbar)(t_2 - t_1)H} | x_1 \rangle.$$

Hint: insert a resolution of the identity in terms of momentum eigenstates. You may use the integral $\int_{-\infty}^{\infty} e^{ias - ibs^2} ds = \sqrt{\frac{\pi}{ib}} \exp\left(\frac{ia^2}{4b}\right)$.

- *b)* Calculate the classical action $S_{\text{class}} = \int_{t_1}^{t_2} L dt$ from the Lagrangian $L = \frac{1}{2}m\dot{x}^2$, for the classical path from point x_1 at time t_1 to point x_2 at time t_2.
- *c)* Discuss the relation between G and S_{class} in the context of Feynman's path integral formula.