4 Quantum electrodynamics

4.1 Gauge transformation, Aharonov-Bohm effect & Byers-Yang theorem

Consider the Hamiltonian of an electron in a magnetic field $\boldsymbol{B}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}(\boldsymbol{r})$,

$$H = \frac{1}{2m}[\boldsymbol{p} - e\boldsymbol{A}(\boldsymbol{r})]^2.$$

a) Show that the gauge transformation $\boldsymbol{A}'(\boldsymbol{r}) = \boldsymbol{A}(\boldsymbol{r}) + \nabla\chi(\boldsymbol{r})$ of the vector potential is equivalent to a unitary transformation $H = UHU^{-1}$ of the Hamiltonian, so it leaves all physical properties invariant.

A ring enclosing a line of magnetic flux Φ at the origin has vector potential $\boldsymbol{A}(r,\phi) = (\Phi/2\pi r)\hat{\phi}$ in polar coordinates. Because $\boldsymbol{B} = 0$ for all $r \neq 0$, we can perform a gauge transformation with $\chi(r,\phi) = (\Phi/2\pi)\phi$ that removes the vector potential from the ring, $\boldsymbol{A}' = \boldsymbol{A} + \nabla\chi = 0$ for $r \neq 0$.

b) Explain why this does not invalidate the existence of the Aharonov-Bohm effect.

c) Derive the *Byers-Yang theorem* that all physical properties are periodic in Φ with period h/e.

4.2 Persistent currents

Consider a ring (radius R) enclosing a magnetic flux Φ. For simplicity, we assume that the ring is one-dimensional and take the coordinate x along the ring, $0 < x < L$ ($L = 2\pi R$). The single-electron Hamiltonian is

$$\hat{H} = \frac{1}{2m}(\hat{p} - eA)^2 + V(\hat{x}),$$

with vector potential $A = \Phi/L$ and electrical potential $V(\hat{x})$. The first term in the Hamiltonian is the kinetic energy $\frac{1}{2}m\hat{v}^2$, with velocity operator $\hat{v} = (\hat{p} - eA)/m$. The corresponding electrical[1] current operator is $\hat{I} = e\hat{v}/L$.

a) Use the Hellmann-Feynman theorem to prove that the expectation value $I_0 = \langle\hat{I}\rangle_0$ of the electrical current operator in the ground state equals the derivative of the ground state energy E_0 with respect to the flux,

$$I_0 = -\frac{dE_0}{d\Phi}.$$

[1]To avoid confusion with minus signs, we take the electron charge as e.

This current will not decay, because the ground state is time-independent, so it is a *persistent* current even if the electron is scattered as it moves along the ring (an unexpected discovery in a non-superconducting system by Büttiker, Imry, and Landauer).

b) Show that the persistent current $I_0(\Phi)$ is periodic in Φ with period h/e, as required by the Byers-Yang theorem.
Hint: Examine the effect of the unitary transformation $\hat{H} \mapsto \hat{U}\hat{H}\hat{U}^{-1}$ with $\hat{U} = \exp(2\pi i\hat{x}/L)$.

c) Calculate the magnitude of the persistent current in the simplest case $V \equiv 0$ of a free particle. At what value of Φ is it largest?
Hint: Take notice of the periodic boundary condition $\psi(x) = \psi(x+L)$ when searching for a plane-wave eigenstate $\psi(x) = L^{-1/2}e^{ikx}$.

4.3 Casimir effect

Two parallel metallic plates in the x–y plane are separated by a distance d. The plates are uncharged and the electromagnetic field is in the vacuum state, so one would not expect the energy of the system to depend on d. In fact it does, as discovered by Casimir (1948), because of the d-dependence of the vacuum energy.

The zero-point energy of a photon of wave vector $\boldsymbol{k} = (k_x, k_y, k_z)$ and frequency $\omega = c|\boldsymbol{k}|$ is $E(\boldsymbol{k}) = \frac{1}{2}\hbar\omega = \frac{1}{2}\hbar c|\boldsymbol{k}|$. In free space the energy density $\mathscr{E}_0$ (energy per unit volume) is given by

$$\mathscr{E}_0 = 2 \times \tfrac{1}{2}\hbar c \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{\infty} \frac{dk_y}{2\pi} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} (k_x^2 + k_y^2 + k_z^2)^{1/2}.$$

The factor 2 in front accounts for the two photon polarizations. The integrals diverge without a high-frequency cutoff, which we will introduce later on.

a) Use periodic boundary conditions to explain the factors of 2π in the denominator.

Because the wave function must vanish on the metallic plates at $z = 0$ and $z = d$, the wave vector component k_z perpendicular to the plates must be an integer multiple of π/d, so the expression for the energy density between the plates reads

$$\mathscr{E}_{\text{plates}} = 2 \times \tfrac{1}{2}\hbar c \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{\infty} \frac{dk_y}{2\pi} \frac{1}{2d} \sum_{n=-\infty}^{\infty} (k_x^2 + k_y^2 + n^2\pi^2/d^2)^{1/2}.$$

b) Check that in the limit $d \to \infty$ the expression for $\mathscr{E}_0$ is recovered.

c) Transform to polar coordinates in the k_x–k_y and derive the following expression for the difference $\delta\mathscr{E} = \mathscr{E}_{\text{plates}} - \mathscr{E}_0$,

$$\begin{aligned}\delta\mathscr{E} &= \frac{\hbar c\pi^2}{8d^4}\int_0^\infty du\left(\sum_{n=-\infty}^{\infty} - \int_{-\infty}^{\infty} dn\right)(u+n^2)^{1/2}\\ &= \frac{\hbar c\pi^2}{4d^4}\left(\sum_{n=-\infty}^{\infty} - \int_{-\infty}^{\infty} dn\right)\int_{|n|}^{\infty}\omega^2\,d\omega.\end{aligned}$$

We now introduce the high-frequency cutoff, in the form of a function $F(\omega)$ that equals 1 for $\omega \lesssim \omega_c$, while it vanishes smoothly for $\omega \gg \omega_c$. The cutoff frequency ω_c is the plasma frequency, above which metals become transparent for electromagnetic radiation. We rewrite $\delta\mathscr{E}$ as

$$\delta\mathscr{E} = \frac{\hbar c\pi^2}{2d^4}\left(\tfrac{1}{2}\mathscr{F}(0) + \sum_{n=1}^{\infty}\mathscr{F}(n) - \int_0^\infty \mathscr{F}(n)\,dn\right),\quad \mathscr{F}(n) = \int_n^\infty F(\omega)\omega^2\,d\omega.$$

d) Evaluate the difference between sum and integral using the Euler-MacLaurin formula

$$\sum_{n=1}^{\infty}\mathscr{F}(n) = \int_0^\infty \mathscr{F}(n)\,dn + \tfrac{1}{2}[\mathscr{F}(\infty) - \mathscr{F}(0)] + \sum_{p=1}^{\infty}\frac{B_{2p}}{(2p)!}[\mathscr{F}^{(2p-1)}(\infty) - \mathscr{F}^{(2p-1)}(0)],$$

where $B_2, B_4, B_6, \ldots = \frac{1}{6}, -\frac{1}{30}, \frac{1}{42}, \ldots$ is a Bernoulli number and $\mathscr{F}^{(p)}$ is the p-th derivative.

e) Explain why the result $\delta\mathscr{E} = -\frac{\hbar c\pi^2}{720d^4}$ implies an attractive force between the plates, given per unit area by

$$F_{\text{Casimir}} = \frac{\hbar c\pi^2}{240d^4}.$$