5 Semiclassics

5.1 Triangular potential well: variational and semiclassical approximations

A particle of mass m bounces vertically on a perfectly reflecting, rigid floor, under the action of the gravitational potential $V(z) = mgz$ for $z > 0$. (We may take $V(z) = \infty$ for $z < 0$.)

a) Use the variational principle to calculate the ground state energy E_0.
Hint: Use trial wave function $\psi(z) = ze^{-az}$. Why must it vanish at $z = 0$?

The exact answer in this case involves the first zero of the Airy function, $E_0 = 2.33811\,(mg^2\hbar^2/2)^{1/3}$. How accurate is the variational estimate?

Let us next calculate the entire spectrum in the semiclassical approximation, using the Bohr-Sommerfeld quantization rule,

$$\frac{1}{\hbar}\oint p_z dz + \gamma = 2\pi n, \quad n = 0, 1, 2, \ldots$$

b) What is the appropriate value of the phase shift γ ?
c) Calculate the energy levels E_n in this triangular potential well.
d) Compare the semiclassical result for the ground state E_0 with that obtained in *a)* using the variational principle. Which estimate is more accurate?

5.2 Landau levels

We consider the motion of an electron in the x-y plane with a perpendicular magnetic field $\boldsymbol{B} = B\hat{z}$. The classical equations of motion produce a circular orbit, with radius $l_c = mv/eB$ at energy $E = \frac{1}{2}mv^2$. (The radius l_c is called the cyclotron length and $\omega_c = eB/m$ is called the cyclotron frequency.)

To apply the Bohr-Sommerfeld quantization rule we need the total (canonical) momentum $\boldsymbol{p} = m\boldsymbol{v} + e\boldsymbol{A}$, consisting of a mechanical contribution $\boldsymbol{p}_{\rm mech} = m\boldsymbol{v}$ and an electromagnetic contribution $\boldsymbol{p}_{\rm magn} = e\boldsymbol{A}$. There is a certain freedom to choose the vector potential $\boldsymbol{A}(\boldsymbol{r})$, the socalled Landau gauge is a convenient choice for which $\boldsymbol{A}$ points in the y-direction. Then one simply has $p_x = mv_x$.

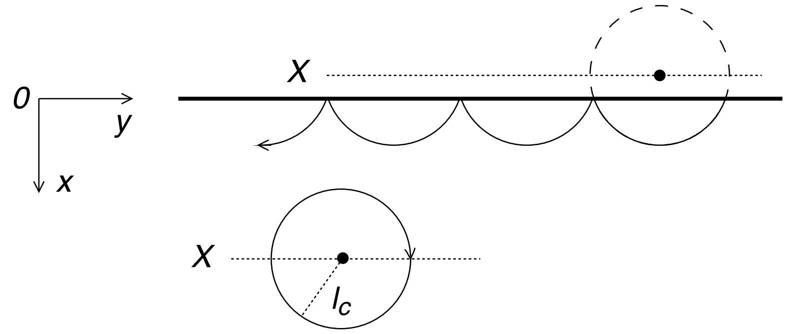
a) Find the vector potential in the Landau gauge.

b) Calculate $\int p_x dx$ over one period of the motion and show that the energy levels (Landau levels) are given by

$$E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega_c, \quad n = 0, 1, 2, \ldots$$

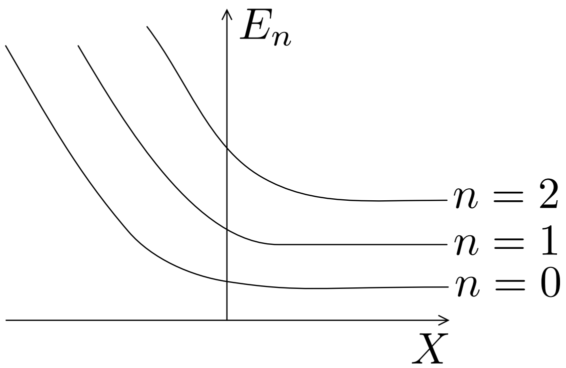
This semiclassical result turns out to be exact.

c) Show that the quantization rule can also be expressed in terms of the flux Φ enclosed by the cyclotron orbit: $\Phi_n = (n+1/2)h/e,\ \ n = 0,1,2,\dots$



Now consider what happens if we introduce an impenetrable wall along the line $x = 0$. The classical motion near the wall is the "skipping orbit" shown in the figure. Since the motion in the x-direction remains periodic, we can still use the Bohr-Sommerfeld quantization rule.

d) Through which area is the flux now quantized? Use the flux quantization rule $\Phi_n = (n+3/4)h/e$ (why +3/4 instead of +1/2?) to explain the plot of the energy of the n-th level as a function of the orbit center X. Explain why the wall pushes the Landau levels up in energy.



e) An eigenstate has the form $\psi(x,y) = \psi(x)e^{iyp_y/\hbar}$. Show that the canonical momentum p_y along the wall is related to the center X of the skipping orbit by $p_y = eBX$. What can you say about the expectation value of the velocity?

g) Imagine that a potential $V(x,y)$ forms a local obstacle for motion along the wall. Show (no calculation needed) that an electron incident on this obstacle is transmitted to the other side with probability one..

This absence of reflection of edge states is at the origin of the socalled "quantum Hall effect".

5.3 Resonant tunneling

An electron at energy E (momentum $\hbar k$) is incident on two tunnel barriers in series, at $x = \pm L/2$. Each barrier has transmission probability Γ. We will use the WKB approximation to calculate the probability T that the electron is transmitted through both barriers. To construct the transmitted wave function we need to sum the amplitudes of all possible classical trajectories that start at the left of the first barrier and end up to the right of the second barrier. Here is one such amplitude:

$$A_n = \Gamma e^{ikL}[(1-\Gamma)e^{2ikL}]^n.$$

a) What classical trajectory corresponds to this wave amplitude?

b) Sum the geometric series to obtain the transmission probability,

$$T = \frac{\Gamma^2}{1 + (1-\Gamma)^2 - 2(1-\Gamma)\cos(2kL)}.$$

Notice that $T = 1$ when kL is an integer multiple of π, no matter how small Γ. This phenomenon of unit transmission through almost impenetrable barriers is known as *resonant tunneling.*

c) What is the interpretation of the resonance condition $kL = n\pi$?