# Quantum Theory
# lecture 2: Symmetry

the force of symmetry

- conservation laws — S-4.1
- unitary & anti-unitary symmetries — S-4.4
- parity — S-4.2
- time-reversal & Kramers degeneracy — S-4.4
- Galilean invariance

S = Sakurai (2nd edition)

# conservation laws

Hamiltonian invariant under a unitary transformation

$$\hat{U}\hat{H}\hat{U}^\dagger = \hat{H} \Rightarrow [\hat{U}, \hat{H}] = 0$$

$$\hat{U} = e^{i\hat{A}} \Rightarrow i\hbar\frac{d\hat{A}}{dt} = [\hat{A}, \hat{H}] = 0$$

observable corresponding to $\hat{A}$ is conserved

(the Hermitian operator $\hat{A}$ is called the generator of the unitary symmetry $\hat{U}$)

translational symmetry: $\hat{U} = e^{i\hat{p}a/\hbar} \Rightarrow$ momentum conserved

see exercise 1.2, or consider an infinitesimal translation over $a$:

$\delta\hat{U} = 1 + \delta a\, \partial/\partial x$ with $\delta a = a/N$

then compose N such translations and take the limit $N \to \infty$

$$\hat{U} = \left(1 + \frac{a}{N}\frac{\partial}{\partial x}\right)^N = e^{a\partial/\partial x} = e^{ia\hat{p}/\hbar}$$

# unitary & anti-unitary symmetries

Wigner's theorem: every symmetry $S$ is either a unitary operator or an anti-unitary operator.

Symmetry: $|\langle S\psi|S\chi\rangle|^2 = |\langle\psi|\chi\rangle|^2$ for all states $\psi, \chi$.
Unitary case: $S = U$, $UU^\dagger = U^\dagger U = 1$,
Anti-unitary case: $S = UK$, with K = complex conjugation
for example, time-reversal: $\psi^*(x,t) = \psi(x,-t)$

Sketch of a proof for a 2D Hilbert space (spin-$1/2$):
$|\psi\rangle$ is associated with a vector $\vec{n}$ on the Bloch sphere

$$|\langle\psi|\chi\rangle|^2 = \tfrac{1}{2}(1 + \vec{n}_1 \cdot \vec{n}_2).$$

symmetry is angle-preserving mapping of the unit sphere on itself: only rotations ($S = U$) or rotations + reflection ($S = UK$).

# parity (= spatial inversion)

see exercise 2.1

$$\mathcal{P}\hat{q}\mathcal{P}^{-1} = -\hat{q} \;\text{ and }\; \mathcal{P}\hat{p}\mathcal{P}^{-1} = -\hat{p}, \;\; \mathcal{P}^2 = 1$$

unitary operator, eigenvalues $\pm 1$ (odd or even parity)

# time-reversal

see exercise 2.2

$$\mathcal{T}\hat{q}\mathcal{T}^{-1} = \hat{q} \;\text{ and }\; \mathcal{T}\hat{p}\mathcal{T}^{-1} = -\hat{p}, \;\; \mathcal{T}^2 = \pm 1.$$

anti-unitary operator, examples: $\mathcal{T}\Psi = \Psi^*$ or $\mathcal{T}\Psi = \sigma_y \Psi^*$

# Kramers degeneracy

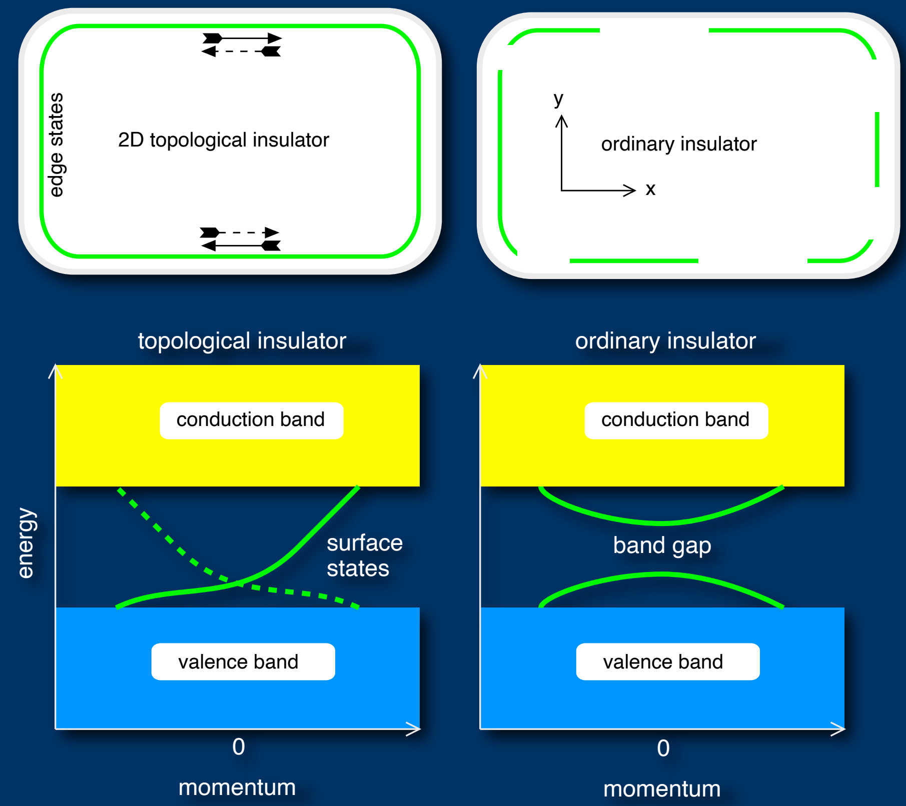
if $\mathcal{H}$ commutes with $\mathcal{T}$ and $\mathcal{T}^2 = -1$, then for every eigenvalue there are at least two independent eigenvectors

if $\mathcal{H}|\psi\rangle = E|\psi\rangle$ then also $\mathcal{H}|\psi'\rangle = E|\psi'\rangle$ with $|\psi'\rangle = \mathcal{T}|\psi\rangle$;

use $\mathcal{T}^2 = -1$ to prove by contradiction that there cannot be a $\lambda \neq 0$ such that $|\psi\rangle = \lambda|\psi'\rangle$

# quantum spin Hall effect

Charles Kane & Gene Mele (2004) — first observation by Laurens Molenkamp (2007)




Kramers degeneracy protects the crossing at zero momentum; mixing of left-movers and right-movers is forbidden

Hellmann-Feynman: $\langle v \rangle = \langle \psi | \partial H / \partial p | \psi \rangle = dE/dp$