

Quantum Theory

lecture 3: fermions & bosons

quantum statistics

- creation/annihilation operators S-2.3
- fermionic/bosonic Fock space S-7.2
- field operators S-7.5
- coherent states
- Bogoliubov & Majorana quasiparticles

S = Sakurai (2nd edition)

creation/annihilation operators

the harmonic oscillator revisited

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2, \quad a = x\sqrt{\frac{m\omega}{2\hbar}} + ip\sqrt{\frac{1}{2\hbar m\omega}}$$
$$\Rightarrow H = \hbar\omega a^\dagger a + \frac{1}{2}\hbar\omega, \quad [x, p] = i\hbar \Rightarrow [a, a^\dagger] = 1.$$

if Ψ_E is an eigenstate at energy E , then $a\Psi_E$ is an eigenst. at $E - \hbar\omega$
the ground state has $a|0\rangle = 0$, $E_0 = \frac{1}{2}\hbar\omega$.

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle, \quad E_n = (n + 1/2)\hbar\omega.$$

the operator a^\dagger creates an excitation ("phonon")
the operator a destroys (annihilates) it.

Fermions versus bosons

statistics of identical particles

$$|\xi_1, \xi_2, \dots, \xi_N\rangle = \frac{1}{\sqrt{N!}} \sum_P |\xi_{P_1}\rangle |\xi_{P_2}\rangle \cdots |\xi_{P_N}\rangle \times \begin{cases} 1 & \text{bosons} \\ \sigma_P & \text{fermions} \end{cases}$$

sum over all permutations P , with parity $\sigma_P = \pm 1$

bosons: symmetric under exchange (permanent)

fermions: antisymmetric under exchange

(determinant — Slater determinant)

cumbersome notation, limited to a fixed number of particles

use creation/annihilation operators for more flexibility

Fock space

specify a many-particle state in terms of occupation numbers

one harmonic oscillator: $|n\rangle = C(a^\dagger)^n|0\rangle$

several harmonic oscillators:

$$|n_1, n_2, n_3 \dots\rangle \propto (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} (a_3^\dagger)^{n_3} \dots |\Omega\rangle$$

with vacuum state $|\Omega\rangle$ defined by $a_i|\Omega\rangle = 0$ for all i

number operator: $N = \sum_i a_i^\dagger a_i$

$$\langle x_1, x_2 | \hat{a}_1^\dagger \hat{a}_2^\dagger | \Omega \rangle \propto \phi_1(x_1) \phi_2(x_2) \pm \phi_2(x_1) \phi_1(x_2)$$

bosons: $[a_i, a_j] = 0, [a_i^\dagger, a_j^\dagger] = 0, [a_i, a_j^\dagger] = \delta_{ij}$

fermions: $\{a_i, a_j\} = 0, \{a_i^\dagger, a_j^\dagger\} = 0, \{a_i, a_j^\dagger\} = \delta_{ij}$

$[a, b] = ab - ba$ (commutator); $\{a, b\} = ab + ba$ (anticommutator)

field operators

(historically known as “second quantization”)

$$\hat{\psi}(\mathbf{x}) = \sum_i \phi_i(\mathbf{x}) \hat{a}_i, \quad \hat{\psi}^\dagger(\mathbf{x}) = \sum_i \phi_i^*(\mathbf{x}) \hat{a}_i^\dagger$$

(anti)-commutator preserved: $[\hat{\psi}(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}')$

special case: when $\phi_n(\mathbf{x}) \propto e^{i\mathbf{p}\mathbf{x}/\hbar}$ is a momentum eigenstate

$$\hat{\psi}(\mathbf{x}) = \int \frac{d\mathbf{p}}{2\pi\hbar} e^{i\mathbf{p}\mathbf{x}/\hbar} \hat{a}(\mathbf{p})$$

$$[\hat{a}(\mathbf{p}), \hat{a}^\dagger(\mathbf{p}')] = 2\pi\hbar\delta(\mathbf{p} - \mathbf{p}') \Rightarrow [\hat{\psi}(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}')$$

particle density operator $\hat{n}(\mathbf{x}) = \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x})$

$$\hat{N} = \int \hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x}) d\mathbf{x} = (2\pi\hbar)^{-1} \int \hat{a}^\dagger(\mathbf{p})\hat{a}(\mathbf{p}) d\mathbf{p}$$

coherent states

see exercise 3.2

a laser produces radiation in an eigenstate of the annihilation operator,

$$a|\beta\rangle = \beta|\beta\rangle, |\beta\rangle = e^{-|\beta|^2/2}e^{\beta a^\dagger}|0\rangle; \text{ check: } \langle\beta|\beta\rangle = e^{-|\beta|^2/2}\langle 0|e^{\beta^* a}|\beta\rangle = e^{|\beta|^2/2}\langle 0|\beta\rangle = 1$$

$$\text{normal ordering: } \langle\beta|f(a^\dagger)g(a)|\beta\rangle = f(\beta^*)g(\beta)$$

a coherent state has a Poisson distribution of the particle number

$$P(n) = |\langle n|\beta\rangle|^2, \text{ with } |n\rangle = (n!)^{-1/2}(a^\dagger)^n|0\rangle \Rightarrow P(n) = \frac{1}{n!}|\beta|^{2n}|\langle 0|\beta\rangle|^2 = \frac{1}{n!}|\beta|^{2n}e^{-|\beta|^2}$$

Majorana fermions

see exercise 3.4

a non-Hermitian fermionic operator a (*Dirac fermion*) can be decomposed into two Hermitian

$$\text{operators } \gamma_1, \gamma_2 \text{ (Majorana fermion): } a = \frac{1}{2}(\gamma_1 + i\gamma_2) \Leftrightarrow \gamma_1 = a + a^\dagger, \gamma_2 = -i(a - a^\dagger)$$

$$\{a, a\} = 0, \{a, a^\dagger\} = 1 \Leftrightarrow \{\gamma_n, \gamma_m\} = 2\delta_{nm}$$

fermion parity operator $\mathcal{P}_{12} = \frac{1}{2}i\gamma_1\gamma_2$ has eigenvalues ± 1 (*Majorana qubit*)