Quantum Theory
lecture 3: fermions & bosons

quantum statistics

- creation/annihilation operators S-2.3
- fermionic/bosonic Fock space S-7.2
- field operators S-7.5
- coherent states
- Bogoliubov & Majorana quasiparticles

S = Sakurai (2nd edition)

creation/annihilation operators

the harmonic oscillator revisited

$$H = \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2 x^2, \quad a = x\sqrt{\frac{m\omega}{2\hbar}} + ip\sqrt{\frac{1}{2\hbar m\omega}}$$

$$\Rightarrow H = \hbar\omega a^\dagger a + \tfrac{1}{2}\hbar\omega, \quad [x, p] = i\hbar \Rightarrow [a, a^\dagger] = 1.$$

if Ψ_E is an eigenstate at energy E, then $a\Psi_E$ is an eigenst. at $E - \hbar\omega$

the ground state has $a|0\rangle = 0$, $E_0 = \tfrac{1}{2}\hbar\omega$.

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle, \quad E_n = (n + 1/2)\hbar\omega.$$

the operator $a^\dagger$ creates an excitation (“phonon”)

the operator a destroys (annihilates) it.

Fermions versus bosons

statistics of identical particles

$$|\xi_1, \xi_2, \dots \xi_N\rangle = \frac{1}{\sqrt{N!}} \sum_P |\xi_{P_1}\rangle|\xi_{P_2}\rangle \cdots |\xi_{P_N}\rangle \times \begin{cases} 1 & \text{bosons} \\ \sigma_P & \text{fermions} \end{cases}$$

sum over all permutations P, with parity $\sigma_P = \pm 1$
bosons: symmetric under exchange (permanent)
fermions: antisymmetric under exchange
(determinant — Slater determinant)

cumbersome notation, limited to a fixed number of particles
use creation/annihilation operators for more flexibility

Fock space

specify a many-particle state in terms of occupation numbers

one harmonic oscillator: $|n\rangle = C(a^\dagger)^n|0\rangle$

several harmonic oscillators:

$$|n_1, n_2, n_3 \ldots\rangle \propto (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}(a_3^\dagger)^{n_3} \ldots |\Omega\rangle$$

with vacuum state $|\Omega\rangle$ defined by $a_i|\Omega\rangle = 0$ for all i

number operator: $N = \sum_i a_i^\dagger a_i$

$$\langle x_1, x_2|\hat{a}_1^\dagger \hat{a}_2^\dagger|\Omega\rangle \propto \phi_1(x_1)\phi_2(x_2) \pm \phi_2(x_1)\phi_1(x_2)$$

bosons: $[a_i, a_j] = 0, \quad [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \delta_{ij}$

fermions: $\{a_i, a_j\} = 0, \quad \{a_i^\dagger, a_j^\dagger\} = 0, \quad \{a_i, a_j^\dagger\} = \delta_{ij}$

$[a, b] = ab - ba$ (commutator); $\{a, b\} = ab + ba$ (anticommutator)

field operators

(historically known as "second quantization")

$$\hat{\psi}(x) = \sum_i \phi_i(x)\hat{a}_i, \quad \hat{\psi}^\dagger(x) = \sum_i \phi_i^*(x)\hat{a}_i^\dagger$$

(anti)-commutator preserved: $[\hat{\psi}(x), \hat{\psi}^\dagger(x')] = \delta(x - x')$

special case: when $\phi_n(x) \propto e^{ipx/\hbar}$ is a momentum eigenstate

$$\hat{\psi}(x) = \int \frac{dp}{2\pi\hbar} e^{ipx/\hbar}\,\hat{a}(p)$$

$$[\hat{a}(p), \hat{a}^\dagger(p')] = 2\pi\hbar\delta(p - p') \Rightarrow [\hat{\psi}(x), \hat{\psi}^\dagger(x')] = \delta(x - x')$$

particle density operator $\hat{n}(x) = \hat{\psi}^\dagger(x)\hat{\psi}(x)$

$$\hat{N} = \int \hat{\psi}^\dagger(x)\hat{\psi}(x)\,dx = (2\pi\hbar)^{-1}\int \hat{a}^\dagger(p)\hat{a}(p)\,dp$$

coherent states

see exercise 3.2

a laser produces radiation in an eigenstate of the annihilation operator,

$a|\beta\rangle = \beta|\beta\rangle,\ \ |\beta\rangle = e^{-|\beta|^2/2}e^{\beta a^\dagger}|0\rangle;$ check: $\langle\beta|\beta\rangle = e^{-|\beta|^2/2}\langle 0|e^{\beta^* a}|\beta\rangle = e^{|\beta|^2/2}\langle 0|\beta\rangle = 1$

normal ordering: $\langle\beta|f(a^\dagger)g(a)|\beta\rangle = f(\beta^*)g(\beta)$

a coherent state has a Poisson distribution of the particle number

$P(n) = |\langle n|\beta\rangle|^2$, with $|n\rangle = (n!)^{-1/2}(a^\dagger)^n|0\rangle \Rightarrow P(n) = \frac{1}{n!}|\beta|^{2n}|\langle 0|\beta\rangle|^2 = \frac{1}{n!}|\beta|^{2n}e^{-|\beta|^2}$

Majorana fermions

see exercise 3.4

a non-Hermitian fermionic operator a (*Dirac fermion*) can be decomposed into two Hermitian operators γ_1, γ_2 (*Majorana fermion*): $a = \frac{1}{2}(\gamma_1 + i\gamma_2) \Leftrightarrow \gamma_1 = a + a^\dagger, \gamma_2 = -i(a - a^\dagger)$

$\{a, a\} = 0,\ \ \{a, a^\dagger\} = 1 \Leftrightarrow \{\gamma_n, \gamma_m\} = 2\delta_{nm}$

fermion parity operator $\mathcal{P}_{12} = i\gamma_1\gamma_2$ has eigenvalues ± 1 (*Majorana qubit*)