Quantum Theory
lecture 4: quantum electrodynamics

the magic vector potential

- gauge transformations S-2.7
- Byers-Yang theorem & Aharonov-Bohm effect S-5.6
- persistent current
- Casimir effect S-7.6

S = Sakurai (2nd edition)

gauge transformations

the two vector potentials A and $A + \nabla\chi$ (related by a gauge transformation) correspond to the same magnetic field $B = \nabla \times A$; in Maxwell's equation only B appears, but the Schrödinger equation depends explicitly on A:

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2\psi.$$

Q. what is the effect of a gauge transformation on ψ?
A. $\psi \mapsto U\psi$ with $U = \exp(ie\chi/\hbar)$ is equivalent to $A \mapsto A + \nabla\chi$

gauge transformation = unitary transformation
$H \mapsto UHU^\dagger$ of the Hamiltonian
all physical properties unaffected . . .

. . . except if $U\psi$ is no longer single-valued

Byers-Yang theorem & Aharonov-Bohm effect

in a multiply connected geometry (a disc with a hole) we can have a vector potential that is singular at the origin:
$\mathbf{A} = (\Phi/2\pi r)\hat{\phi}$ — a flux Φ line at the origin ($\oint \mathbf{A} dl = \Phi$)
$\mathbf{A} = \nabla\chi$ with $\chi = (\Phi/2\pi)\phi$, so the magnetic field in the disc is zero; in classical mechanics the flux line should have no effect on the disc, but in quantum mechanics it may (*Aharonov-Bohm effect*)

the gauge transformation $\psi \mapsto e^{ie\chi/\hbar}\psi$ is not allowed when $e\Phi/\hbar$ is not a multiple of 2π, because then $\psi(\phi + 2\pi) \neq \psi(\phi)$

Byers-Yang theorem: all physical properties of the disc are periodic in Φ with period h/e

Persistent current

see exercise 4.2

$H = \frac{1}{2}mv^2$ with $mv = p - eA \Rightarrow H = \frac{1}{2m}(p - eA)^2$

(mechanical momentum = canonical momentum – electromagnetic momentum)

in a ring (circumference L) enclosing a flux Φ one has $A = \Phi/L$

current operator $I = ev/L$ (in a time L/v a charge e goes around the ring)

Hellmann-Feynman: $\langle I \rangle = (e/L)\langle \partial H/\partial p \rangle = -(1/L)\langle \partial H/\partial A \rangle = -dE/d\Phi$.

when E is the ground state: non-decaying (persistent) current: Büttiker, Imry, Landauer (1980’s)

Casimir effect

see exercise 4.3

the exercise follows the derivation in Sakurai (§7.6)

here we take an alternative approach, see exam question 4 from January 2020