



# Quantum Theory

## lecture 6: time-dependent systems

- adiabatic theorem S-5.6
- Landau-Zener transitions
- Berry phase S-5.6
- Dirac fermions in graphene

S = Sakurai (2nd edition)

# *classical* adiabatic theorem

$J = \oint p dq$  is an adiabatic invariant of the time-dependent Hamiltonian  $H[p, q, \alpha(t)]$ , meaning  $J(T) - J(0) \sim \dot{\alpha}$  and not  $\sim \alpha(T) - \alpha(0)$ .

*condition:* frequency change in one period much smaller than frequency itself, so  $|\dot{\omega}| \ll \omega^2$ .

- harmonic oscillator (Lorentz pendulum)

$$J = 4 \int_0^{\sqrt{2mE}/m\omega} \sqrt{2mE - m^2\omega^2x^2} dx = 2\pi E/\omega.$$

- 2D cyclotron motion in a varying field: enclosed flux constant

$$J = \oint (mv + qA) \cdot dq = mv 2\pi l_c - q\Phi = 2qB\pi l_c^2 - q\Phi = q\Phi$$

- channel of varying width: collimation by a horn  $J = 2Wp \sin \theta$

# *quantum* adiabatic theorem

quantization of adiabatic invariant (Ehrenfest):  $J = \oint p dq = nh$   
no transitions between energy levels if the Hamiltonian changes sufficiently slowly

condition:  $\hbar\omega = E_{n+1} - E_n \equiv \delta \Rightarrow \hbar|\dot{\delta}| \ll \delta^2$

adiabaticity breaks down when levels cross, or almost cross (avoided level crossing,  $\delta_{\min} > 0$ )

**Landau-Zener transition:** jump from one level to the other at the avoided crossing

probability  $P_{\text{jump}} \ll 1$  if  $\hbar|\dot{\delta}| \ll \delta_{\min}^2$

$$P_{\text{jump}} = \exp\left(-\frac{\pi\delta_{\min}^2}{2\hbar|\dot{\delta}|}\right)$$

exact for  $H = \begin{pmatrix} \alpha t & \gamma \\ \gamma^* & \beta t \end{pmatrix}$

$$\delta_{\min} = 2|\gamma|, \dot{\delta} = \alpha - \beta$$

$$E_{\pm} = \frac{1}{2}(\alpha + \beta)t \pm \sqrt{|\gamma|^2 + \frac{1}{4}(\alpha - \beta)^2 t^2}$$

# Berry phase

adiabatic evolution with return to starting point (“cyclic”): final state can only differ from initial state by a phase factor (for a non-degenerate state)

$$\psi_{\text{final}} = e^{i\gamma_B} \exp\left(-\frac{i}{\hbar} \int_0^T E(t) dt\right) \psi_{\text{initial}}$$

dynamical phase factor, depending on the period  $T$  of the cycle

**Berry phase**  $\gamma_B$  independent on  $T$  (“geometric” phase)

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H[\alpha(t)] \psi(t), \text{ eigenstate: } H[\alpha(t)] |\alpha(t)\rangle = E(t) |\alpha(t)\rangle$$

$$\text{substitute: } |\psi(t)\rangle = e^{i\gamma(t)} e^{-(i/\hbar) \int_0^t E(t') dt'} |\alpha(t)\rangle \Rightarrow \frac{d\gamma}{dt} = i \langle \alpha(t) | \frac{d}{dt} | \alpha(t) \rangle = i \frac{d\alpha}{dt} \langle \alpha(t) | \frac{\partial}{\partial \alpha} | \alpha(t) \rangle$$

$$\gamma_B = i \oint \left\langle \alpha \left| \frac{\partial}{\partial \alpha} \right| \alpha \right\rangle d\alpha$$

# example: Berry phase of a spin in a rotating magnetic field

$$H[\vec{B}(t)] = -\frac{\mu}{2}\vec{\sigma} \cdot \vec{B}(t) = -\frac{\mu}{2} \begin{pmatrix} B_z(t) & B_x(t) - iB_y(t) \\ B_x(t) + iB_y(t) & -B_z(t) \end{pmatrix}$$

$\vec{B}(t) = B_0(\sin \theta \cos \phi(t), \sin \theta \sin \phi(t), \cos \theta)$  rotates around the  $z$ -axis; consider the evolution of a spin-up eigenstate:

$$|\phi(t)\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi(t)} \sin(\theta/2) \end{pmatrix} \Rightarrow i\langle\phi|\frac{d}{d\phi}|\phi\rangle = -\sin^2(\theta/2)$$

$$\gamma_B = \pm 2\pi \sin^2(\theta/2)$$

*Berry phase is one-half the solid angle covered by the rotating spin*

# Dirac fermions in graphene

see exercises 6.2 and 6.3

anomalous Landau levels

Berry phase of  $\pi$  cancels phase shift from turning points: no zero-point motion offset!

cyclotron radius  $l_c = p/eB$ , energy  $E = vp$

$$\Phi = nh/e, \quad \Phi = B\pi(p/eB)^2 = B\pi(E/eBv)^2 = nh/e \Rightarrow E = \pm\hbar v\sqrt{2neB/\hbar} = \pm(\hbar v/l_m)\sqrt{2n}$$

magnetic length  $l_m = \sqrt{\hbar/eB}$ .