Quantum Theory
lecture 7: path integrals

- Lagrangian & principle of least action S-2.4
- quantum propagator & Feynman path integral S-2.6
- stationary phase approximation

S = Sakurai (2nd edition)

Lagrangian & principle of least action

classical mechanics

Hamiltonian: $H(q,p) = T + V,\ \dot{q} = \frac{\partial H}{\partial p},\ \dot{p} = -\frac{\partial H}{\partial q}.$

Lagrangian: $L(q,\dot{q}) = T - V,\ p = \frac{\partial L}{\partial \dot{q}},\ \dot{p} = \frac{\partial L}{\partial q}.$ Euler-Lagrange equations

why bother? integral formulation of classical equations of motion:

Action: $S[q(t)] = \int_{t_1}^{t_2} L(q,\dot{q})\,dt \Rightarrow \delta S[q(t)] = 0$

fixed $q(t_1), q(t_2)$ principle of least action — Fermat, Maupertuis, Euler, Hamilton

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{\partial L}{\partial q} \delta q \right) dt = \int_{t_1}^{t_2} \left(-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q} \right) \delta q\, dt = 0$$

- free motion: minimize $\int_{t_1}^{t_2} v^2(t)\,dt$ at fixed $\int_{t_1}^{t_2} v(t)\,dt \to v =$ const.
- gravitational field: minimize $\int_{t_1}^{t_2} (mv^2/2 - mgz)\,dt$
  for $z_2 = z_1 = 0, x_2 = x_1 + L \to$ parabola

quantum propagator & path integrals

quantum mechanics

Thirty-one years ago, Dick Feynman told me about his "sum over histories" version of quantum mechanics. "The electron does anything it likes," he said. "It just goes in any direction at any speed, . . . however it likes, and then you add up the amplitudes and it gives you the wavefunction." I said to him, "You're crazy." But he wasn't. *Freeman Dyson*

probability = |complex amplitude|^2,
Feynman: amplitude = sum over *all paths* of $e^{iS[q(t)]/\hbar}$
$\to$ intuitive explanation of the double-slit experiment

in the $\hbar \to 0$ limit only *classical* paths contribute
(stationary phase approximation $\to$ WKB approximation)

a little bit of philosophy...

Is it true that the particle doesn't just "take the right path" but that it looks at all the other possible trajectories? The miracle of it all is, of course, that it does just that. It isn't that a particle takes the path of least action but that it smells all the paths in the neighborhood and chooses the one that has the least action (Feynman, 1964)

Newton's law: force is a cause, deviation from straight path is the effect, deterministic evolution

Hamilton's principle of least action: past and future are connected, "bird's-eye" view of the entire history

— variational (as opposed to differential) formulation of the laws of physics —
classical mechanics: follow the path of least action!
quantum mechanics: explore all paths!
general relativity: follow the path of maximal aging!

derivation of the path integral formula

propagator (Green function)

$$G(q_1, q_0; T) = \langle q_1 | e^{-iHT/\hbar} | q_0 \rangle = \sqrt{\frac{m}{2\pi i \hbar T}} \int_{q(0)=q_0}^{q(T)=q_1} \mathcal{D}[q]\, e^{iS[q]/\hbar}$$

time slicing ($\delta t = T/N$)

$$G(q_1, q_0; T) = \int dq_1' \int dq_2' \cdots \int dq_N'\, G(q_1, q_N'; \delta t) \cdots G(q_2', q_1'; \delta t) G(q_1', q_0; \delta t)$$

$$G(q + \delta q, q; \delta t) = \langle q + \delta q | 1 - iH\delta t/\hbar + \mathcal{O}(\delta t)^2 | q \rangle = \int dp\, \langle q + \delta q | p \rangle \langle p | 1 - iH\delta t/\hbar | q \rangle$$

$$= \int \frac{dp}{2\pi\hbar} e^{ip\delta q/\hbar} \left(1 - \frac{i\delta t}{\hbar} \left[\frac{p^2}{2m} + V(q) \right] \right) \to \int \frac{dp}{2\pi\hbar} e^{ip\delta q/\hbar} e^{-(i\delta t/\hbar)[p^2/2m + V(q)]}$$

$$= \text{constant} \times \exp \left[i \frac{\delta t}{\hbar} \left(\tfrac{1}{2} m (\delta q/\delta t)^2 - V(q) \right) \right] = e^{iL(q,\dot{q})\delta t/\hbar}$$

$$G(q_1, q_0; T) \simeq \int dq_1' \int dq_2' \cdots \int dq_N' \exp \left[\frac{i\delta t}{\hbar} \left(L(q_1, \dot{q}_1) \cdots + L(q_2', \dot{q}_2') + L(q_1', \dot{q}_1') + L(q_0, \dot{q}_0) \right) \right]$$

(action exponent OK, prefactor requires more work)

stationary phase approximation

see exercise 4

WKB approximation (path from q_0 to q in time T, at energy E):

$$\psi = \frac{1}{\sqrt{v}} \exp\left(\frac{i}{\hbar}\int_{q_0}^{q} p(q')\,dq' - iET/\hbar\right) =$$

$$= \frac{1}{\sqrt{v}} \exp\left(\frac{i}{\hbar}\int_0^T \left[p\dot{q} - E\right]\,dt\right)$$

$$p\dot{q} - E = p\dot{q} - \tfrac{1}{2}p\dot{q} - V = T - V = L \Rightarrow \psi = \frac{1}{\sqrt{v}} \exp\left(\frac{i}{\hbar}\int_0^T L(q,\dot{q})\,dt\right) = \frac{1}{\sqrt{v}} \exp\left(\tfrac{i}{\hbar} S_{\text{class}}\right)$$

only classical path contributes in the WKB approximation

dominant contribution to quantum path integral in limit $\hbar \to 0$ is the path of stationary action $\delta S = 0$, which gives the classical action.

in this way classical mechanics is recovered as the $\hbar \to 0$ limit of quantum mechanics.