Exercises on RMT for the 2022 Les Houches school

Note: helpful formulas are given at the end

1. The Gaussian Orthogonal Ensemble (GOE) is the ensemble of $N \times N$ real symmetric matrices H with matrix elements drawn from a Gaussian distribution:

$$P(H) \propto \exp\left(-c \sum_{n=1}^{N} \sum_{m=1}^{N} H_{nm}^2\right). \tag{1}$$

a) Derive that $P(H) = P(HO)$ for any $N \times N$ orthogonal matrix O. (This socalled "orthogonal invariance" is what gives the GOE its name.)
b) Show that, as a consequence of *a)*, the matrix Ω that diagonalizes H is uniformly distributed in the orthogonal group.
c) For $N \gg 1$ the matrix elements Ω_{nm} of Ω are approximately independent, with a Gaussian distribution:

$$P(\Omega_{nm}) \propto \exp\left(-\tfrac{1}{2} a \Omega_{nm}^2\right). \tag{2}$$

Derive that $a = N$.
e) The Porter-Thomas distribution from nuclear physics says that the strength $\gamma > 0$ of a resonance in the scattering cross-section of a heavy nucleus has the distribution

$$P(\gamma) \propto \gamma^{-1/2} \exp(-\gamma/2\bar{\gamma}), \tag{3}$$

with $\bar{\gamma}$ the average strength. Show how this distribution results from your earlier findings.

2. The reflection of light by a disordered medium is described by an $N \times N$ reflection matrix r. We assume that the medium is sufficiently thick that there is no transmission through the medium. We also assume there is no absorption in the medium.

a) What constraint is imposed on the reflection matrix r by the absence of transmission and absorption? And what constraint is imposed by time-reversal symmetry? How is the number N determined?

In the Circular Orthogonal Ensemble (COE) we equate $r = UU^{\mathrm{T}}$ and assume that the $N \times N$ matrix U is uniformly distributed in the unitary group.

b) Calculate the ensemble average of $|r_{nm}|^2$ and show that this average is twice as large for $n = m$ than for $n \neq m$. (This is the "coherent backscattering" effect.)
(c) Describe an optical experiment that might be able to observe this effect. Illustrate your description with a drawing, including a plot of what you think will be measured.

3. The eigenvalues T_n $(n = 1, 2, \ldots N)$ of the product $tt^\dagger$ of the $N \times N$ transmission matrix t and its Hermitian conjugate are called "transmission eigenvalues".

a) Why is T_n a real number? Use the unitarity of the scattering matrix to prove that

$0 \leqslant T_n \leqslant 1$ for each n.

For chaotic scattering (circular ensemble) and for $N \gg 1$ the density $\rho(T)$ of transmission eigenvalues is given by

$$\rho(T) = \frac{N}{\pi} \frac{1}{\sqrt{T}\sqrt{1-T}}. \tag{4}$$

b) Discuss, in qualitative terms, how the shape of $\rho(T)$ can explain experiments that are able to image an object behind an optically opaque medium.

c) Sketch a plot of $\rho(T)$ *in a disordered wire*, showing how this density profile changes as you make the wire longer and longer. Discuss the plot in terms of the transition from metallic to insulating behavior.

4. The reflection matrix r from the interface between a normal metal and a superconductor consists of four $N \times N$ blocks,

$$r = \begin{pmatrix} r_{ee} & r_{eh} \\ r_{he} & r_{hh} \end{pmatrix}, \tag{5}$$

describing reflection from electron-to-electron or from hole-to-hole (r_{ee} and r_{hh}), and describing Andreev reflection (r_{eh} and r_{he}).

a) Explain in words why the conductance G of the interface is given by

$$G = G_0 \operatorname{Tr} r_{he} r_{he}^\dagger. \tag{6}$$

Include in your explanation the formula for G_0.

b) We might alternatively write

$$G' = G_0 \operatorname{Tr} r_{eh} r_{eh}^\dagger. \tag{7}$$

Derive $G = G'$. (*Hint:* use unitarity of r.)

In a spin-triplet superconductor the electron and hole blocks are related by

$$r_{hh} = r_{ee}^*, \ \ r_{he} = r_{eh}^*. \tag{8}$$

c) Use these relations to prove that the determinant $Q \equiv \operatorname{Det} r$ equals $+1$ or -1. (This number Q is called a "topological quantum number".)

Hint: compare $\operatorname{Det} r$ with $\operatorname{Det}(\sigma r \sigma)$ for $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

d) Specialize to the case $N = 1$ of a 2×2 matrix r. Derive for this case the following relation between conductance and topological quantum number:

$$G = G_0 \frac{1-Q}{2}. \tag{9}$$

Explain why for $N > 1$ there is no one-to-one relation between G and Q.

Helpful formulas

Averages of an $N \times N$ matrix U that is uniformly distributed over the unitary group:

$$\langle U_{\alpha a} U^*_{\beta b} \rangle = \frac{1}{N} \delta_{\alpha\beta} \delta_{ab}, \tag{10}$$

$$\begin{aligned} \langle U_{\alpha a} U_{\alpha' a'} U^*_{\beta b} U^*_{\beta' b'} \rangle = {} & \frac{1}{N^2-1} \big(\delta_{\alpha\beta} \delta_{ab} \delta_{\alpha'\beta'} \delta_{a'b'} + \delta_{\alpha\beta'} \delta_{ab'} \delta_{\alpha'\beta} \delta_{a'b} \big) \\ & - \frac{1}{N(N^2-1)} \big(\delta_{\alpha\beta} \delta_{ab'} \delta_{\alpha'\beta'} \delta_{a'b} + \delta_{\alpha\beta'} \delta_{ab} \delta_{\alpha'\beta} \delta_{a'b'} \big). \end{aligned} \tag{11}$$

An integral formula:

$$\int_0^\infty x^\alpha e^{-x} \, dx = \Gamma(1+\alpha) \ \text{ for } \ \alpha > -1, \ \ \Gamma(1) = 1, \ \ \Gamma(3/2) = \tfrac{1}{2}\sqrt{\pi}, \ \ \Gamma(1/2) = \sqrt{\pi}. \tag{12}$$