Chaos in Quantum Billiards

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Quantum billiards is a game played by physicists at a few academic and industrial laboratories in various parts of the world. It's a serious game: we are actually getting paid for it. It's also fun and exciting. I would like to share some of this excitement with you.

1 Mesoscopic physics

We are used to dividing nature into a macroscopic and a microscopic world. The macroscopic world contains the things we can see with our eyes. The microscopic world contains the building blocks of matter, the atoms and molecules. We know they are there, but we can't see them directly. The mesoscopic world is in between the microscopic and the macroscopic world. The boundaries are not sharp, but can be roughly indicated. Mesoscopic and macroscopic objects have in common that they both contain a large number of atoms. A first difference is that the macroscopic object can be well described by the average properties of the material from which it is made. The mesoscopic object, in contrast, is so small that fluctuations around the average become important. A second difference is that the macroscopic object obeys (to a good approximation) the laws of classical mechanics, whereas the mesoscopic object is so small that these laws no longer hold. Mesoscopic and microscopic systems both belong to the wonderful world of quantum mechanics.

Mesoscopic physics addresses fundamental physical problems which occur when a macroscopic object is miniaturized. The field originated some ten years ago, motivated largely by the electronics industry. As you know, that industry makes money out of the miniaturization of transistors, which switch the electrical current on a computer chip. I was fortunate enough to work at the Philips Research Labs. when the field was just starting, and have been deeply involved in it ever since.

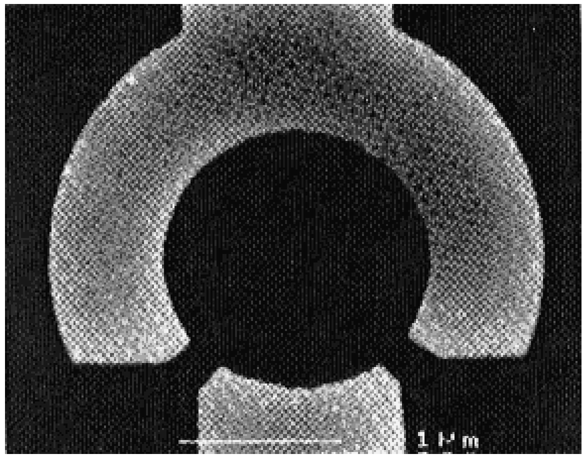


Figure 1: Scanning electron micrograph of a mesoscopic billiard. The white bar at the bottom is 1 micrometer long. The gray area is an electrode deposited on top of a gallium-arsenide semiconductor. A two-dimensional electron gas is formed below the surface (not visible). The electrons can move over the black area but are repelled from the electrode. [From M. J. Berry, J. A. Katine, R. M. Westervelt, and A. C. Gossard, Phys. Rev. B **50**, 17721 (1994).]

2 Quantum billiards

The quantum billiard is a mesoscopic billiard. Imagine a billiard table which is so small, that one hundred of them would fit on the tip of a needle, and fabricated with such an accuracy that it is completely smooth and flat. Billiard balls shoot over this table with velocities of a hundred miles per second, colliding elastically with the walls until they disappear into one of the pockets. An example of such a billiard is shown in figure 1.

The billiard balls I have in mind are the electrons. You should know that the motion of an electron in a typical metal wire does not resemble at all the rectilinear motion of a billiard ball. The impurities in the metal scatter the electrons in all directions, so that the motion becomes completely unpredictable. One speaks of a "random walk". A measure for the influence of impurities on the electron motion is the socalled "mean free path". This is the typical distance an electron can move before colliding with an impurity. What we need for our game, is a mean free path much greater than the size of the billiard.

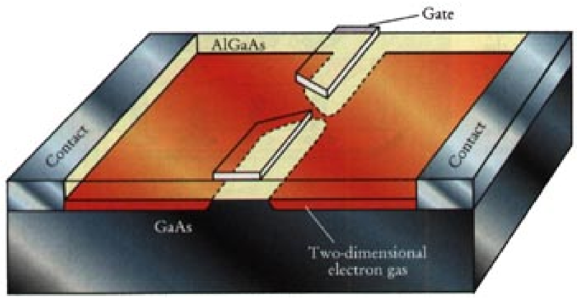


Figure 2: Cross-sectional view of a two-dimensional electron gas (black) at the interface between gallium-arsenide (GaAs) and aluminum-gallium-arsenide (AlGaAs). A gate electrode creates a barrier with a small opening. A current flows through the opening if a voltage is applied between the two contact pads.

The electronic industry has become very skilled at fabricating semiconducting materials (out of which transistors are made) with a large mean free path. The world record is around 100 micrometer. To eliminate the impurities a technique is used called "molecular beam epitaxy". A crystal is grown one atomic layer at the time, until a nearly perfect lattice of atoms is obtained. Because the process is controlled on the atomic scale, it is possible to vary the composition of the individual atomic layers in the crystal. This is used to fabricate a billiard table for electrons which is not only very *smooth* (i.e. with a large mean free path), but also perfectly *flat*.

The way in which this is done is quite ingenious. The crystal grower varies the composition of the atomic layers in such a way that a potential well for electrons is created inside the material gallium-arsenide. This potential well confines the electrons in the direction perpendicular to the layers, but does not hinder their motion parallel to the layers. One boundary of the well is formed by the interface between gallium-arsenide and the alloy aluminum-gallium-arsenide. The other boundary is formed by the attractive force of positively charged silicon donors in the aluminum-gallium-arsenide. The width of the potential well is only a few atomic layers. In such a narrow well the motion of the electrons perpendicular to the layers is completely suppressed, because the width is smaller than the electron wave length. All motion takes place in the two-dimensional plane perpendicular to the atomic layers. One speaks of a two-dimensional electron gas.

It reminds me of the novel *Flatland* by Edwin Abbott. In the last century

Abbott fantasized about a world where only two spatial dimensions exist. He imagined *"a vast sheet of paper, on which straight Lines, Triangles, Squares, Pentagons, Hexagons, and other figures, instead of remaining fixed in their places, move freely about, on or in the surface, but without the power of rising above it, very much like shadows."* One hundred years later, this fantasy has become true — to some extent — in a two-dimensional electron gas.

There is a problem with this billiard table: the two-dimensional electron gas is buried in the interior of the crystal. How to reach it from the outside? Fortunately, a method exists. The materials gallium-arsenide and aluminum-gallium-arsenide are semiconductors. Unlike a metal, a semiconductor can be deeply penetrated by an electric field. A spatially confined electric field can form a barrier in the two-dimensional electron gas, which repels the electrons in a completely elastic way. Such an electric field is generated by a metal (typically gold) electrode on top of the semiconducting crystal. The electrode is called a "gate". An opening in the gate creates a hole in the barrier, through which electrons can enter and leave the billiard (see figure 2). Using the technique of electron beam litography one can fabricate electrodes with openings of less than 0.1 micrometer. In this way one obtains the micron-sized billiard shown in figure 1.

3 Interfering billiard balls

The miniaturisation of the billiard leads to two remarkable changes in the rules of the game. The first new rule is the *uncertainty principle*, formulated in 1927 by the physicist Werner Heisenberg. The uncertainty principle states that it is not possible as a matter of principle to fix with complete certainty both the position and the velocity of the electron. To be specific, if the opening through which the electrons are shot into the billiard is narrower than about 0.05 micrometer, then all control over the direction of motion is lost. The direction in which an electron leaves the opening has become completely random. An accurately aimed shot is therefore impossible in principle in the miniaturized billiard.

The second new rule is that of *interfering paths*. In a usual billiard there might be different ways to shoot the ball into one of the pockets. Suppose that the ball has to pass an obstacle (another ball, for example) in order to reach a pocket. The player can try to pass it from the left or from the right. What he or she will do, is to choose the path which looks most promising and ignore the other one. In the miniaturized billiard the situation is very different. First of all it is not possible to determine a priori which path the electron will follow. The uncertainty principle does not allow that. Only the probability of each path is determined. According to the usual rules of probability one would conclude, that the total probability for a hit is the sum of the two probabilities for a hit via the left and right paths. The rule for adding probabilities in the miniaturized billiard is different. Under certain circumstances the total probability for reaching the pocket can be zero, even though the individual left and right probabilities are non-zero. One speaks of *destructive interference*. Can you imagine, seeing two paths into the pocket, knowing that the electron has to follow one of these two, and yet finding the pocket empty no matter how often you repeat your shot.

Destructive interference occurs if the two paths differ in length. How much they should differ depends on a property of the electron called its wavelength. In a two-dimensional electron gas the wavelength is about 0.05 micrometer. The condition for destructive interference is that the path length should be an odd multiple of half the wavelength. It is one of the remarkable predictions of quantum mechanics that particles such as an electron sometimes behave as a wave. The miniaturized billiard is called a "quantum billiard" if interference effects govern the motion of the electrons. This is the case if the wavelength of the electrons is not much smaller than the size of the openings. A real billiard ball has a wavelength too, but it is so extremely small that even professional billiard players do not need to worry about interference effects.

4 Quantum point contacts

Now that we have discussed how a quantum billiard is fabricated, let us examine its properties in some detail. We begin with the little openings in the boundary through which the electrons enter and leave the billiard. The openings are called quantum point contacts. A special name for a hole, because it has a special property. The current of electrons through the hole is induced by application of a voltage and can be varied by adjusting the size of the opening. The bigger the opening, the bigger the current. The special property of the quantum point contact is that upon widening the opening the current does not increase gradually but *stepwise*. The ratio of current and voltage is the conductance. The stepwise increase of the current at a

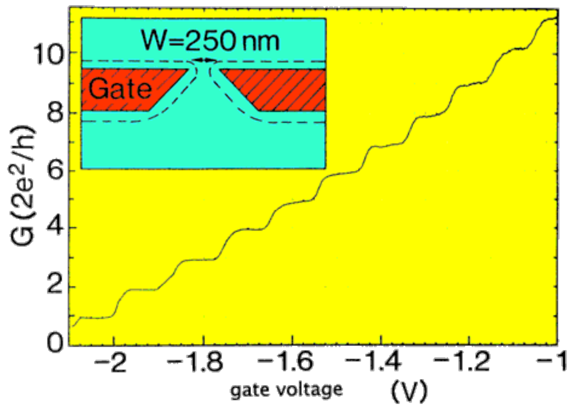


Figure 3: Quantization of the conductance of a quantum point contact. Upon increasing the width of the opening (by varying the voltage on the gate electrode shown in the inset), the conductance increases stepwise. The stepheight of $2e^2/h$ depends only on fundamental constants of nature. [From B. J. van Wees et al., Phys. Rev. Lett. **60**, 848 (1988); similar results were published by D. A. Wharam et al., J. Phys. C **21**, L209 (1988).]

given voltage implies that the conductance can take on only discrete values. This is called the *quantization* of the conductance.

The term "quantization" is used in physics to indicate that some quantities can not be varied continuously, but only occur as discrete multiples of some elementary value, which is called a "quantum". We know only a few examples of quantized quantities in nature. The oldest example is the quantization of charge in multiples of the charge e of a single electron, discovered by Robert Millikan. The quantization of the magnetic flux enclosed by a superconducting ring was predicted in 1950 by Fritz London. The quantum in this case is $h/2e$, where h is Planck's constant. Around the same time Lars Onsager and Richard Feynman predicted the quantization of vortices in superfluid helium. A more recent quantization effect is the quantum Hall effect, discovered in 1980 by Klaus von Klitzing. The quantized conductance of a quantum point contact (shown in figure 3) was discovered in 1988 by a Dutch group (from Philips and the University of Delft) and almost simultaneously by a group in England (from the University of Cambridge).

The quantum of conductance is $2e^2/h$, corresponding to a resistance of 12906 Ohm. The conductance G of a quantum point contact is an integer N

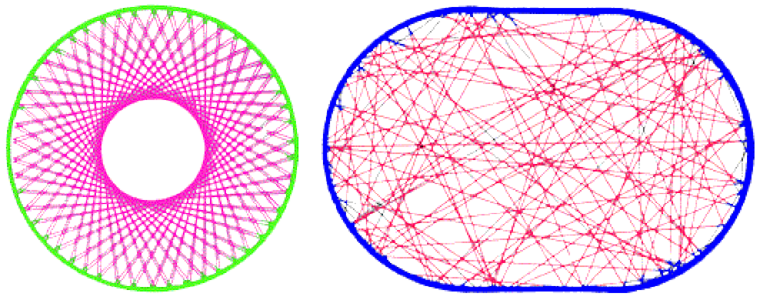
(a)  (b)

Figure 4: Trajectories in a circular billiard (a) and a stadium billiard (b). The motion in the circle is regular or ordered, while in the stadium it is irregular or chaotic.

times the conductance quantum,

$$G = N \times \frac{2e^2}{h}.$$

The number N is approximately equal to the width of the contact divided by half the electron wave length. The electron wave can only pass through the hole in one of a few modes of vibration, for which the interference is constructive rather than destructive. The number N counts the number of modes with constructive interference. The quantization of conductance occurs because of the *equipartitioning* of current among these modes: each mode carries the same current, equal to $2e^2/h$ times the voltage.

5 Order versus chaos

So much for the holes, now let's turn to the billiard itself. Depending on their shape, billiards come in two types: ordered and chaotic. The trajectories in an ordered and chaotic billiard are contrasted in figure 4. An ordered billiard has a special, highly symmetric shape, such as the circle in figure 4a. At each collision with the boundary both the magnitude and the direction of the velocity (relative to the boundary) are the same. The chaotic billiard has no symmetry, so that only the magnitude of the velocity is a constant of the motion. The direction varies in an irregular way from one collision to the next (figure 4b).

In a quantum billiard one can *not* distinguish order from chaos by inspecting the trajectories. This would require the specification of position and velocity of an electron, which violates the uncertainty principle. So, what happens if we miniaturize the two billiards of figure 4? Is it meaningful to distinguish order from chaos in a quantum billiard? Can one, by measuring the conductance, tell whether the billiard has an ordered or a chaotic shape? These are the issues I would like to address at the meeting.

6 Further reading

- Special issue on mesoscopic physics of Physics Today, December 1988.

- Review on *Quantum Chaology* by M. V. Berry, Proceedings Royal Society London A **413**, 183 (1987).

- Two recent theoretical papers:
  Mesoscopic Transport through Chaotic Cavities: A Random S-Matrix Theory Approach, H. U. Baranger and P. A. Mello, Physical Review Letters **73**, 142 (1994).
  Universal Quantum Signatures of Chaos in Ballistic Transport, R. A. Jalabert, J.-L. Pichard, and C. W. J. Beenakker, Europhysics Letters **27**, 255 (1994).

- Two recent experimental papers:
  Weak-Localization in Chaotic versus Non-Chaotic Cavities: A Striking Difference in Line Shape, A. M. Chang, H. U. Baranger, L. N. Pfeiffer, and K. W. West, Physical Review Letters **73**, 2111 (1994).
  Ballistic Conductance Fluctuations in Shape Space, I. H. Chan, R. M. Clarke, C. M. Marcus, K. Campman, and A. C. Gossard, Physical Review Letters **74**, 3876 (1995).