THEORY
OF
HEAVY QUANTA

DOOR

F. J. BELINFANTE

'S-GRAVENHAGE
MARTINUS NIJHOFF
1939

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PROEFSCHRIFT

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TER VERKRIJGING VAN DEN GRAAD VAN DOCTOR IN DE WIS- EN NATUURKUNDE AAN DE RIJKSUNIVERSITEIT TE LEIDEN, OP GEZAG VAN DEN RECTOR MAGNIFICUS DR FRED. MULLER, HOOGLEERAAR IN DE FACULTEIT DER LETTEREN EN WIJSBEGEERTE, VOOR DE FACULTEIT DER WIS- EN NATUURKUNDE TE VERDEDIGEN OP DINSDAG 10 OCTOBER 1939, DES NAMIDDAGS TE 4 UUR

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VOORWOORD

Bij deze bijzondere gelegenheid is het mij een behoefte, een woord van dank te richten tot al degenen, die hebben bijgedragen tot mijn vorming.

Het voortreffelijke onderricht van den heer H. Corver op het *Nederlandsch Lyceum* te 's-Gravenhage deed een sluimerende liefde voor de natuurkunde bij mij ontwaken. Zijn lessen, doch ook een voordracht van wijlen prof. dr. P. Ehrenfest aan deze school en de raad van den toenmaligen rector prof. R. Casimir, bepaalden mijn latere studierichting.

In mijn eerste studiejaren aan de universiteit te Leiden had in het bijzonder de kinetische gastheorie mijn belangstelling, waarin mevr. dr. G. L. de Haas-Lorentz college gaf. Ik was nog een jaar lang in de gelegenheid, de levendige colleges van prof. Ehrenfest te volgen, die een diepen indruk op mij maakten.

Toen prof. Ehrenfest ons in 1933 zoo onverwacht ontviel, waart gij het, mijn waarde prof. dr. H. B. G. Casimir, die, toen nog assistent, een jaar lang als „invaller" het college golfmechanica hebt gegeven. Hiervoor, maar nog meer voor de vriendelijke welwillendheid, waarmee gij mij gedurende mijn geheele studie te Leiden in wetenschappelijke moeilijkheden hebt willen helpen, ben ik U van harte dankbaar.

U, hooggeachte Promotor, hooggeleerde Kramers, ben ik veel dank verschuldigd voor de belangstelling, waarmee U mijn studie hebt gevolgd en gestimuleerd. Veel heb ik van U geleerd: op Uw college, uit Uw college-dictaat, uit Uw boek over golfmechanica, op het door U ingestelde „seminarium" voor theoretische natuurkunde, maar het meest nog tijdens mijn assistentschap bij U uit gesprekken en uit de studie, waartoe zulke gesprekken steeds weer aanleiding gaven.

Een woord van dank past mij aan mijn ouders, die mij in de gelegenheid stelden, een wetenschappelijke opleiding te genieten en te voltooien. Het is ondoenlijk, hier verder al degenen te noemen, die aan mijn vorming en opvoeding hebben meegewerkt. Mogen echter zij allen overtuigd zijn van mijn erkentelijkheid.

In 1938 werd ik door stipendia van de *Nederlandsch-Amerikaansche Fundatie* en van het *Lorentz-fonds* in staat gesteld, den zomercursus aan de *University of Michigan* te Ann Arbor (Mich., U.S.A.) te volgen. Door colleges van prof. dr. H. A. Bethe en prof. dr. G. Breit maakte ik hier kennis met het onderwerp, dat in dit proefschrift wordt besproken. Prof. dr. H. A. Kramers stelde mij tot probleem, de vergelijkingen voor het zware quantum neer te schrijven in een vorm, analoog aan de z.g. „photon"-vergelijkingen uit De Broglie's „neutrino-theorie" van het licht. De onderzoekingen, die hierop volgden, leidden niet alleen tot het ontstaan van dit proefschrift, maar ook tot het publiceeren in het tijdschrift „Physica" van een artikel over het spin-impulsmoment van golfvelden, en tot verdere studiën, die ik in de komende tijden hoop te mogen voltooien.

In het eerste hoofdstuk van dit proefschrift wordt het formalisme behandeld van de *„undor-rekening"*; het derde hoofdstuk bestudeert de algemeene *theorie der zware quanta*. Het tweede hoofdstuk vormt hiertusschen de schakel. Ik ben den uitgever en de Redactie van „Physica", in het bijzonder prof. dr. A. D. Fokker, zeer erkentelijk voor de door hen geschapen mogelijkheid, de eerste twee hoofdstukken te laten verschijnen als artikels in dit tijdschrift.

F. J. B.

UNDOR CALCULUS AND CHARGE-CONJUGATION

Zusammenfassung

Grössen, welche sich transformieren wie Produkte 4-komponentiger D i r a c scher Wellenfunktionen, werden „Undoren" genannt. Zu jedem Undor ψ kann durch Linearkombination der Komponenten seines komplex Konjugierten ψ^* ein neuer „ladungskonjugierter" Undor $\psi^{\mathfrak{C}}$ gebildet werden. Wenn $\psi = \psi^{\mathfrak{C}}$, wird die Wellenfunktion ψ ein „Neutrettor" genannt. Undoren zweiter Stufe entsprechen gewisse Tensoren, Neutrettoren zweiter Stufe entsprechen *reelle* Tensoren. Mit Hilfe eines „metrischen" Undors zweiter Stufe werden „kontravariante" Undoren definiert, welche sich kontragredient zu den gewöhnlichen („kovarianten") Undoren transformieren. Endlich wird der „Gradient-Neutrettor" eingeführt.

Resumo

Kvantojn, kiuj transformiĝas kiel produtoj de kvar-komponantaj D i r a c'aj ondofunkcioj, ni nomas „undoroj". El ĉiu undoro ψ per unuagrada kombinado de la komponantoj de ĝia komplekse konjugito ψ^* nova „ŝarge konjugita" undoro $\psi^{\mathfrak{C}}$ povas esti konstruata. Se $\psi = \psi^{\mathfrak{C}}$, ni nomas la ondofunkcion ψ „neŭtretoro". Duaŝtufaj undoroj reprezentas certajn tensorojn, duaŝtufaj neŭtretoroj reprezentas realajn tensorojn. Per „metrika" undoro duaŝtufa „kontraŭvariantaj" undoroj estas difinataj, kiuj transformiĝas kontraŭpaŝe al la ordinaraj („kunvariantaj") undoroj. Fine la „gradiento-neŭtretoro" estas prezentata.

§ 1. *Introduction.* It is well known that — with respect to the *restricted* L o r e n t z group, excluding spatial reflections through the origin — tensors can be expressed in terms of spinors [1] [2]. As soon as spatial reflections are taken into account, however, it is necessary to consider *pairs* of spinors transforming one into the other by a reflection through the origin. An example of such a pair of spinors is the wave function of the D i r a c electron.

In the following we shall investigate the properties of quantities transforming like products of such D i r a c wave-functions [3]. Such quantities we shall call "undors" *). They form a generalization of Dirac wave-functions in the same sense as tensors form a generalization of vectors. Just as the representations of the Lorentz group by the transformations of most tensors are not *ausreduziert*, so the representations by most undors are likewise reducible.

In particular we shall discuss, in the following, the relation between undors of the second rank and tensors, and the analogon in undor-calculus to *real* tensors: "neutrettors" [4]). Finally we shall deduce the metrical undor and define the gradient undor. The whole set of mathematical relations will be built up in such a form, that we shall be able to apply it later on to the theory of mesons and neutrettos [5]) [6]) [7]) [8]) [9]).

§ 2. *The Dirac wave-function* (*undor of the first rank*). The Dirac-equation of a positive particle (a positon or a proton according to the value of $\varkappa = mc/\hbar$) can be written in the following form:

$$\{i\varkappa + (\vec{\gamma}\,.\,\vec{D}) + \boldsymbol{\beta} D_0\}\,\psi = 0, \tag{1}$$

if we put

$$\vec{\gamma} = \boldsymbol{\beta}\vec{\alpha}, \tag{2}$$

$$\vec{D} = \vec{\nabla} + \frac{e}{i\hbar c}\vec{\mathfrak{A}}, \quad D_0 = \frac{1}{c}\frac{\partial}{\partial t} - \frac{e}{i\hbar c}\mathfrak{V} \tag{3}$$

Here e is the elementary charge ($e > 0$) and $\vec{\mathfrak{A}}$, $\mathfrak{V}$ is the potential four-vector of the electromagnetic field. As in (1) the interaction with *heavy* quanta is neglected, this equation does not account for the anomalous magnetic moment of the proton [10]) [5]) [6]) [11]).

We shall call quantities transforming like the four-component Dirac wave-function ψ, *four-spinors* or *undors of the first rank*. In the following we shall often use a representation of them, which is *ausreduziert* with respect to the group of restricted Lorentz transformations, and in which the first two components of a four-spinor transform like the two-component quantity called a covariant conjugated spinor by Van der Waerden [1]), Laporte and Uhlenbeck [2]), and called a regular spinor by Kramers [12]), whereas the last two components transform like the spin-conjugated †)

*) Derived from *unda* = wave.

†) See H. A. Kramers, loc. cit. [12]), page 263.

of such a quantity, called a contravariant regular spinor by Van der Waerden[1]) and others[2]). Explicitly:

$$\left.\begin{aligned} x' &= x \\ y' &= y\cos\vartheta + z\sin\vartheta \\ z' &= -y\sin\vartheta + z\cos\vartheta \\ t' &= t \end{aligned}\right\} \rightleftarrows \left\{\begin{aligned} \psi_1' &= \psi_1\cos\frac{\vartheta}{2} + \psi_2\, i\sin\frac{\vartheta}{2} \\ \psi_2' &= \psi_1\, i\sin\frac{\vartheta}{2} + \psi_2\cos\frac{\vartheta}{2} \\ \psi_3' &= \psi_3\cos\frac{\vartheta}{2} + \psi_4\, i\sin\frac{\vartheta}{2} \\ \psi_4' &= \psi_3\, i\sin\frac{\vartheta}{2} + \psi_4\cos\frac{\vartheta}{2} \end{aligned}\right. \tag{4a}$$

$$\left.\begin{aligned} x' &= x\cos\varphi + y\sin\varphi \\ y' &= -x\sin\varphi + y\cos\varphi \\ z' &= z \\ t' &= t \end{aligned}\right\} \rightleftarrows \left\{\begin{aligned} \psi_1' &= \psi_1\, e^{i\phi/2} \\ \psi_2' &= \psi_2\, e^{-i\phi/2} \\ \psi_3' &= \psi_3\, e^{i\phi/2} \\ \psi_4' &= \psi_4\, e^{-i\phi/2} \end{aligned}\right. \tag{4b}$$

$$\left.\begin{aligned} x' &= x \\ y' &= y \\ z'\sqrt{1-(v/c)^2} &= z - vt \\ t'\sqrt{1-(v/c)^2} &= t - zv/c^2 \end{aligned}\right\} \rightleftarrows \left\{\begin{aligned} \psi_1' &= \psi_1/\tau \\ \psi_2' &= \tau\psi_2 \\ \psi_3' &= \tau\psi_3 \\ \psi_4' &= \psi_4/\tau, \end{aligned}\right. \tag{4c}$$

where

$$\tau = +\sqrt[4]{\frac{c+v}{c-v}}.$$

If we require that (1) is Lorentz-invariant, we find that the Dirac matrices are given in the representation (4) by

$$\vec{\alpha} = \rho_z\,\vec{\sigma} \tag{5}$$

and

$$\beta = (A + B\rho_z)\rho_x, \tag{6}$$

if we denote by

$$\rho_x,\ \rho_y,\ \rho_z \quad \text{and} \quad \vec{\sigma}\,\{\sigma_x,\ \sigma_y,\ \sigma_z\}$$

the Pauli matrices [13]) operating on the discrete arguments r and s of the four-spinor

$$\psi_{s,r} = \begin{vmatrix} \psi_{\frac{1}{2},\frac{1}{2}} \\ \psi_{-\frac{1}{2},\frac{1}{2}} \\ \psi_{\frac{1}{2},-\frac{1}{2}} \\ \psi_{-\frac{1}{2},-\frac{1}{2}} \end{vmatrix} = \begin{vmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{vmatrix} = \psi_k . \tag{7}$$

From $\boldsymbol{\beta}^2 = 1$ we deduce $A^2 - B^2 = 1$, therefore,

$$\boldsymbol{\beta} = \begin{vmatrix} 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \\ 1/\lambda & 0 & 0 & 0 \\ 0 & 1/\lambda & 0 & 0 \end{vmatrix}, \tag{8}$$

with $\lambda = A + B = 1/(A - B)$.

Apart from a numerical factor the transformation-matrix of ψ for a spatial reflection through the origin [14]) must be equal to $\boldsymbol{\beta}$:

$$\psi'(-x, -y, -z, t) = j\,\boldsymbol{\beta}\,\psi(x, y, z, t). \tag{9}$$

As a double reflection should not change the geometrical meaning of the four-spinor, j must be a square root of ± 1:

$$j^2 = \pm 1. \tag{10}$$

By (4) a representation of the *complete* Lorentz group by transformations of ψ is not yet uniquely given. If we complete (4) by making a definite choice for the representation of reflections, the matrix $\boldsymbol{\beta}$ will be fixed by (9), (10) apart from a square root of ± 1. On the other hand we may choose $\lambda = 1$, that is,

$$\boldsymbol{\beta} = \boldsymbol{\rho}_{\mathbf{x}}; \tag{11}$$

the representation by ψ of the Lorentz group including reflections is then given by (9), (10) together with (4), apart from the same factor j in the reflection.

In the following we shall denote the conjugate complex of a matrix by an asterisk $*$; by a cross $\dagger$ the adjoint ("Hermitian conjugate") of a matrix. For instance

$$\psi = \begin{vmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{vmatrix} \rightarrow \psi^* = \begin{vmatrix} \psi_1^* \\ \psi_2^* \\ \psi_3^* \\ \psi_4^* \end{vmatrix} \text{ and } \psi^\dagger = |\psi_1^* \psi_2^* \psi_3^* \psi_4^*|.$$

Further, we put

$$\Omega^{\dagger *} = \Omega^{\infty}, \quad (\Omega^{*} = \Omega^{\dagger\infty}); \tag{12}$$

for instance:

$$\psi^{\infty} \equiv |\psi_1 \psi_2 \psi_3 \psi_4|.$$

We introduce the *normalization-* or *density*-matrix $\boldsymbol{\vartheta}$, which is defined apart from a real numerical factor by

$$\boldsymbol{\beta}^{\dagger}\boldsymbol{\vartheta} = \boldsymbol{\vartheta}\boldsymbol{\beta}, \quad \vec{\boldsymbol{\alpha}}^{\dagger}\boldsymbol{\vartheta} = \boldsymbol{\vartheta}\vec{\boldsymbol{\alpha}}, \quad \boldsymbol{\vartheta}^{\dagger} = \boldsymbol{\vartheta}. \tag{13}$$

Further, we shall postulate that under a linear transformation by a non-singular matrix S to another representation:

$$\psi' = S\psi, \quad \psi^{\dagger\prime} = \psi^{\dagger}S^{\dagger}, \tag{14}$$

$$\boldsymbol{\beta}' = S\boldsymbol{\beta}S^{-1}, \quad \vec{\boldsymbol{\alpha}}' = S\vec{\boldsymbol{\alpha}}S^{-1}, \tag{14a}$$

the real expression

$$\psi^{\dagger}\boldsymbol{\vartheta}\boldsymbol{\beta}\psi$$

shall be invariant, so that

$$\boldsymbol{\vartheta}' = S^{\dagger -1}\boldsymbol{\vartheta}S^{-1}. \tag{14b}$$

The definitions (13) are indeed invariant under these transformations (14*a*, *b*).

In our particular representation (4), in which

$$\vec{\boldsymbol{\alpha}}^{\dagger} = \vec{\boldsymbol{\alpha}}, \tag{15}$$

we have on account of (13), apart from a numerical factor,

$$\boldsymbol{\vartheta} = \begin{vmatrix} |1/\lambda| & 0 & 0 & 0 \\ 0 & |1/\lambda| & 0 & 0 \\ 0 & 0 & |\lambda| & 0 \\ 0 & 0 & 0 & |\lambda| \end{vmatrix}; \tag{16}$$

after the choice (11) of $\boldsymbol{\beta}$ we have, therefore,

$$\boldsymbol{\vartheta} = 1. \tag{17}$$

The density operator will remain unity so long as, starting from the representation (4), (5), (11), (17), we admit only *unitary transformations*, for which

$$SS^{\dagger} = 1. \tag{18}$$

In the particular representation (5), (11) it is easily verified that, with regard to the complete L o r e n t z group,

$$\psi^\dagger \boldsymbol{\vartheta}\boldsymbol{\beta}\psi \tag{19}$$

s a scal ar, and that

$$\vec{j}/c = \psi^\dagger \boldsymbol{\vartheta} \vec{\boldsymbol{\alpha}} \psi \text{ with } \rho = \psi^\dagger \boldsymbol{\vartheta} \psi \tag{20}$$

together form a four-vector (like $\vec{r}$ with ct), which satisfies the continuity equation

$$\operatorname{div} \vec{j} + \frac{\partial \rho}{\partial t} = 0 \tag{21}$$

as a consequence of (13) and the D i r a c equation (1):

$$\operatorname{div}(\psi^\dagger \boldsymbol{\vartheta} \vec{\boldsymbol{\alpha}} \psi) + \frac{1}{c}\frac{\partial}{\partial t}(\psi^\dagger \boldsymbol{\vartheta} \psi) =$$

$$= \left(\vec{\nabla}\psi^\dagger \,.\, \vec{\boldsymbol{\alpha}}^\dagger + \frac{1}{c}\frac{\partial \psi^\dagger}{\partial t}\right) \boldsymbol{\vartheta}\psi + \psi^\dagger \boldsymbol{\vartheta} \left(\vec{\boldsymbol{\alpha}} \,.\, \vec{\nabla} + \frac{1}{c}\frac{\partial}{\partial t}\right)\psi =$$

$$= i\varkappa(\psi^\dagger \boldsymbol{\beta}^\dagger \boldsymbol{\vartheta} \psi - \psi^\dagger \boldsymbol{\vartheta}\boldsymbol{\beta}\psi) + \frac{e}{i\hbar c}\psi^\dagger\{(\vec{\mathfrak{A}} \,.\, \vec{\boldsymbol{\alpha}}^\dagger - \mathfrak{V})\boldsymbol{\vartheta} - \boldsymbol{\vartheta}(\vec{\mathfrak{A}} \,.\, \vec{\boldsymbol{\alpha}} - \mathfrak{V})\}\psi = 0.$$

It is, therefore, possible to regard $\vec{j}$ and ρ as the probability current-density and we shall normalize c-number solutions of (1) in one-particle wave mechanics by

$$\iiint dx\, dy\, dz\, (\psi^\dagger \boldsymbol{\vartheta} \psi) = 1. \tag{22}$$

In the following we shall confine ourselves mainly to those representations, in which (17) is fulfilled, and we shall therefore drop $\boldsymbol{\vartheta}$ practically throughout. As regards the representation (4), this restriction (17) means taking $A = A^*$, $B = -B^*$ in (6) and $|\lambda| = 1$ in (8), so that

$$\boldsymbol{\beta}^\dagger = \boldsymbol{\beta} \tag{17a}$$

becomes a H e r m i t i a n matrix. As regards transformations by (14) to other representations, it means restriction to unitary transformations (18).

§ 3. *Charge-conjugation of four-spinors.* A matrix £ is defined apart from a complex unity factor e^{iC} by *)

$$\vec{\gamma}\pounds = -\pounds\vec{\gamma}^*, \quad \beta\pounds = -\pounds\beta^*, \quad \pounds\pounds^* = 1; \tag{23}$$

so

$$\vec{\alpha}\pounds = \pounds\vec{\alpha}^*, \quad \vec{\varsigma}\pounds = -\pounds\vec{\varsigma}^*, \tag{23a}$$

where

$$\varsigma_x = -i\alpha_y\alpha_z, \text{ cycl.}, \tag{24}$$

so that in the representation (5), (6)

$$\vec{\varsigma} = \vec{\sigma}. \tag{25}$$

In this representation (5), (8) this matrix £ is equal to

$$\pounds = e^{iC}\begin{vmatrix} 0 & 0 & 0 & -|\lambda| \\ 0 & 0 & |\lambda| & 0 \\ 0 & |1/\lambda| & 0 & 0 \\ -|1/\lambda| & 0 & 0 & 0 \end{vmatrix}, \tag{26}$$

therefore the restriction (17) makes it equal to

$$\pounds = e^{iC}\,.\,\rho_y\sigma_y \tag{27}$$

in the representation (4), (5), (8).

In the following the particular representation (4), (5), (11), (17), (27) with $e^{iC} = 1$ will be called the K r a m e r s [15]) representation:

$$\vec{\alpha} = \rho_z\vec{\sigma}, \quad \beta = \rho_x, \quad \pounds = \rho_y\sigma_y, \quad \vartheta = 1; \quad \vec{\varsigma} = \vec{\sigma}. \tag{28}$$

By means of the matrix £ we construct from the conjugate complex (ψ^*) of the four-spinor wave-function (ψ) of a positive particle (1), another four-component quantity

$$\psi^{\pounds} = \pounds\psi^*. \tag{29}$$

From (1), (3) and (23) we can easily deduce that $\psi^{\pounds}$ satisfies the equation [15])

$$\{i\varkappa + (\vec{\gamma}\,.\,\vec{D}^*) + \beta D^*_{0}\}\psi^{\pounds} = 0, \tag{30}$$

that is, the wave equation for a negative particle (a negaton or a hystaton †) according to the value of $\varkappa$). For this reason $\psi^{\pounds}$ is called

*) This matrix £ is identical with the matrix C^* introduced by P a u l i, loc. cit. [14]).

†) Hystaton (antiproton) is derived from ὕστατος = last; proton from πρωτος = first.

the *charge-conjugated* [15]) of ψ. From (30) we can conclude [14]) that — with respect to *restricted* Lorentz transformations — the charge-conjugated of an undor of the first rank is again an undor [15]).

As regards *reflection*: from (30) we can conclude only that the transformation of $\psi^{\pounds}$ must be again of the form (9). But the representations of the complete Lorentz group by the transformations of ψ and by those of $\psi^{\pounds}$ might be different with respect to the sign of j. In order to examine this, we compare the charge-conjugated four-spinor in the reflected system of co-ordinates, which is defined by

$$\psi^{\pounds'} \equiv \psi'^{\pounds} = \pounds\psi'^* = j^* \pounds \beta^* \psi^*, \tag{31}$$

with the charge-conjugated four-spinor $\psi^{\pounds}$ in the original system of co-ordinates. This last, (29), will be transformed into (31) by (9) with perhaps a different value of j, — say $j^{(\pounds)}$:

$$\psi^{\pounds'} = j^{(\pounds)} \beta \psi^{\pounds} = j^{(\pounds)} \beta \pounds \psi^*. \tag{32}$$

From (31), (32) and (23) we find

$$j^{(\pounds)} = -j^*. \tag{33}$$

The charge-conjugated of an undor of the first rank is therefore itself an undor with respect to the *complete* Lorentz group ($j^{(\pounds)} = j$), if

$$j = -j^* \text{ (so that } j = \pm i \text{ on account of (10))}. \tag{34}$$

We can also prove directly that $\psi^{\pounds}$ is an undor with the choice (34) of j, without making use of the equation (30). Let Λ be the linear operator of some Lorentz transformation of the undor ψ. Then $\psi^{\pounds}$ transforms like an undor if

$$\psi'^{\pounds} = \Lambda \psi^{\pounds}, \tag{35}$$

or

$$\pounds \Lambda^* \psi^* = \Lambda \pounds \psi^*, \tag{35a}$$

that is, if

$$\Lambda \pounds = \pounds \Lambda^*. \tag{36}$$

Indeed this condition is satisfied for the restricted Lorentz transformations on account of (23*a*). For the reflection (9) it is satisfied, if

$$j \beta \pounds = \pounds j^* \beta^*, \tag{37}$$

or, on account of (23), if

$$j = -j^*. \tag{34}$$

This result of Majorana[4]) and Racah[16]) means that it serves a useful purpose to define the reflection of a Dirac wave-function in such a way that a double spatial reflection through the origin inverts its sign. In the following we shall see that this same definition enables us to describe the Proca field (that is, a field consisting of a four-vector $\vec{\mathbf{A}}$, $\mathbf{V}$ and an antisymmetrical tensor of the second rank $\vec{\mathbf{E}}$, $\vec{\mathbf{H}}$) by means of a *symmetrical* undor.

Since according to (34) $\psi^{\pounds}$ can now be regarded as a regular undor, it is natural to postulate that by a transformation (14) to another representation of undors, the undor $\psi^{\pounds}$ shall be transformed in the same way as all other undors ψ. This means that

$$\psi^{\pounds'} \equiv \psi'^{\pounds} = \pounds' \psi'^* = \pounds' S^* \psi^* \tag{38a}$$

must be obtained from $\psi^{\pounds}$ (29) by a transformation (14):

$$\psi^{\pounds'} = S\psi^{\pounds} = S\pounds\psi^* \tag{38b}$$

This holds independently of the choice of ψ, if $\pounds' S^* = S\pounds$, or

$$\pounds' = S\pounds S^{*-1}. \tag{14c}$$

Under the transformation (14*a*, *c*) the definitions (23) are indeed invariant, that is to say, if $\pounds$ is defined in one representation in accordance with (23) and is then transformed to another representation according to (14*c*), the relations (23) will hold again between the transformed charge-conjugator $\pounds'$ and the transformed Dirac matrices $\boldsymbol{\beta}'$, $\vec{\boldsymbol{\alpha}}'$, etc. In the same way the relation

$$\vartheta\pounds = \pounds^{\backsim}\vartheta^* = (\vartheta\pounds)^{\backsim}, \tag{23b}$$

which is valid in Kramers' representation, is invariant by a transformation (14*b*, *c*) and in consequence holds in every representation of undors.

It follows from (27) that

$$\pounds = \pounds^{\backsim} \tag{17b}$$

holds in the representation (4), (17); and from (14*c*) we deduce that *this relation* (17*b*) *holds in all representations* (17), *for which the density matrix is unity*, because (17*b*) is invariant by a *unitary* transformation ($S^\dagger = S^{-1}$, $S^{\backsim} = S^{*-1}$):

$$\pounds'^{\backsim} = S^{\dagger -1}\pounds^{\backsim}S^{\backsim} = S\pounds S^{*-1} = \pounds'$$

If F is an operator, which operates on four-spinors, we can define a *charge-transformed* operator $F^{\mathfrak{L}}$ by

$$(F\psi)^{\mathfrak{L}} = F^{\mathfrak{L}}\psi^{\mathfrak{L}}, \tag{39}$$

so

$$F^{\mathfrak{L}} = \mathfrak{L} F^* \mathfrak{L}^*. \tag{40}$$

On the other hand, if F depends on the electric elementary charge e, we can define the *charge-inverse* F^L of F by

$$\{F(e)\}^L = F(-e). \tag{41}$$

Then we can summarize the connection between (1) or

$$H_{op}\psi = i\hbar \frac{\partial}{\partial t}\psi \equiv \mathcal{E}_{op}\psi \tag{1a}$$

and (30) or

$$H_{op}^L \psi^{\mathfrak{L}} = \mathcal{E}_{op}\psi^{\mathfrak{L}} = -\mathcal{E}_{op}^{\mathfrak{L}}\psi^{\mathfrak{L}} \tag{30a}$$

by stating that

$$H_{op}^L = -H_{op}^{\mathfrak{L}}, \quad \mathcal{E}_{op}^L = -\mathcal{E}_{op}^{\mathfrak{L}}. \tag{42}$$

Kramers[15]) has pointed out that a description of electrons by means of ψ and H_{op}, and a *charge-conjugated description* by means of $\psi^{\mathfrak{L}}$ and H_{op}^L, should be equivalent. It can be shown that the meson theory shows a similar kind of *charge-invariance*.

§ 4. *Neutrettors of the first rank*. We shall call *self-charge-conjugated* four-spinors

$$\psi^{\mathfrak{L}} = \psi \tag{43}$$

neutrettors of the first rank *). These quantities are adequate for the description of the neutral particles of the theory of Majorana [4]). It can easily be shown that Majorana's "real" Dirac wave-functions are exactly what we call *neutrettors*. Proceeding by a transformation (14) with

$$S = S^{\dagger-1} = (1 + i\rho_y\sigma_y)/\sqrt{2i} \tag{44}$$

*) The name "neutr*inors*" (in analogy to sp*inors* proposed by me in a Letter to the Editor in Nature [9])), seems to be less adequate, since it suggests that neutrinos can be described by these quantities. By neutrettors, however, only photons and neutrettos are described.

from the Kramers representation (28) to a Majorana representation

$$\pounds_{MAJ} = 1, \quad \vartheta_{MAJ} = 1, \tag{45}$$

we deduce from (23):

$$\beta^*_{MAJ} = -\beta_{MAJ}, \ \vec{\gamma}^*_{MAJ} = -\vec{\gamma}_{MAJ}, \ \vec{\alpha}^*_{MAJ} = \vec{\alpha}_{MAJ}, \ \vec{\varsigma}^*_{MAJ} = -\vec{\varsigma}_{MAJ}. \tag{45a}$$

This is indeed the characteristic of the representation discussed by Majorana *). Four-spinors, which are *self*-charge-conjugated, are on account of (29) and (45) *real in a Majorana representation*. According to Majorana [4]) *neutrettors* can describe *neutral* particles †).

For the purpose of building up a canonical theory of Majorana particles the Kramers representation is very convenient. Denoting in this representation (28) the first two components of ψ as a Kramers spinor by

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = u = \begin{pmatrix}u_{\dot{1}}\\ u_{\dot{2}}\end{pmatrix} \tag{46a}$$

and the last two components as the *spin-conjugated* $v^{\$}$ of another Kramers spinor v by

$$\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix} = v^{\$} = \begin{pmatrix}v^{\$1}\\ v^{\$2}\end{pmatrix} = \begin{pmatrix}v_{\dot{2}}\\ -v_{\dot{1}}\end{pmatrix}^* = \begin{pmatrix}v_{\dot{1}}\\ v_{\dot{2}}\end{pmatrix}^{\$}, \tag{46b}$$

we can write (29) in the following form:

$$\psi^{\pounds} = \begin{vmatrix}u\\ v^{\$}\end{vmatrix}^{\pounds} = \begin{vmatrix}v\\ u^{\$}\end{vmatrix}, \tag{47}$$

so

$$\psi^{\pounds\pounds} = \begin{vmatrix}v\\ u^{\$}\end{vmatrix}^{\pounds} = \begin{vmatrix}u\\ v^{\$}\end{vmatrix} = \psi. \tag{48}$$

This four-spinor is a neutrettor, if

$$\begin{vmatrix}u\\ v^{\$}\end{vmatrix} = \psi = \psi^{\pounds} = \begin{vmatrix}v\\ u^{\$}\end{vmatrix}, \text{ that means, if } u \equiv v. \tag{49}$$

*) The representation actually used by Majorana is obtained from (28) not by (44), but by taking $S = \frac{1}{2}\sqrt{i}\,.\{\rho_x(1 - \sigma_y) + \rho_z(1 + \sigma_y)\}$ in (14).

†) *Note added in proofs.* The Majorana theory of neutral particles and the transformation properties of $\pounds$ have also been investigated by W. H. Furry, Phys. Rev. **54**, 56, 1938.

In other words, *a neutrettor of the first rank consists of a two-spinor and its spin-conjugated* and can be written as

$$\psi = \psi^{\mathfrak{C}} = \begin{vmatrix} u \\ u^{\mathfrak{s}} \end{vmatrix} \qquad (50)$$

in a Kramers representation.

In a canonical theory of neutral particles only the two-spinor u should be regarded as a canonical variable like ψ in Dirac's theory; u^* takes a place comparable with that of ψ^* in Dirac's theory and $u^{\mathfrak{s}}$ can be expressed in terms of u^*.

§ 5. *Undors of the second rank.* We shall call a sixteen-component quantity

$$\Psi_{k_1k_2} = \Psi_{s_1r_1,s_2r_2} \quad (k_1, k_2 = 1, 2, 3, 4;\ s_1, r_1, s_2, r_2 = +\tfrac{1}{2}, -\tfrac{1}{2}) \qquad (51)$$

transforming like the product $\psi_{k_1}\psi'_{k_2} = \psi_{s_1,r_1}\psi'_{s_2,r_2}$ of two four-spinors an *undor of the second rank.* With respect to the undor-indices we regard it as a *matrix with one column and* 16 *rows.* Still, we shall *write* it as a square matrix with 4 × 4 elements:

$$\Psi_{k_1k_1} = \begin{vmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ \Psi_{21} & \Psi_{22} & \Psi_{23} & \Psi_{24} \\ \Psi_{31} & \Psi_{32} & \Psi_{33} & \Psi_{34} \\ \Psi_{41} & \Psi_{42} & \Psi_{43} & \Psi_{44} \end{vmatrix} . \qquad (52)$$

Matrices like $\rho_x^{(n)}$, $\rho_y^{(n)}$, $\rho_z^{(n)}$, $\vec{\sigma}^{(n)}$, $\beta^{(n)}$, $\vec{\gamma}^{(n)}$, $\vec{\alpha}^{(n)}$, $\vec{\varsigma}^{(n)}$, $\vartheta^{(n)}$, $\mathfrak{L}^{(n)}$, $S^{(n)}$, etc. are assumed to operate on the argument k_n of $\Psi_{k_1k_2}$. Taking these operators as unity matrices with respect to the index, on which they do not operate, we can regard them as matrices with 16 rows and 16 colums.

With respect to the *restricted* Lorentz group an undor of the second rank represents one regular spinor, one conjugate complex spinor and two mixed spinors of the second rank; it represents, therefore [2] [12]), two four-vectors $\vec{K}$, K^0 and $\vec{L}$, L^0 (transforming like $\vec{r}$, ct), two scalars F_0 and G_0, one regular complex three-vector (the *Kennzahlen* of an antisymmetrical self-dual tensor [2]))

$$\vec{F} = \vec{H}_1 - i\vec{E}_1 \qquad (53a)$$

and one conjugate complex three-vector

$$\vec{G} = \vec{H}_2 + i\vec{E}_2, \tag{53b}$$

where $\vec{E}_1$, $\vec{H}_1$ and $\vec{E}_2$, $\vec{H}_2$ form two antisymmetrical tensors of the second rank. Generally all these quantities are complex. In the Kramers representation (28) we can write:

$$\begin{vmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} \\ \Psi_{21} & \Psi_{22} & \Psi_{23} & \Psi_{24} \\ \Psi_{31} & \Psi_{32} & \Psi_{33} & \Psi_{34} \\ \Psi_{41} & \Psi_{42} & \Psi_{43} & \Psi_{44} \end{vmatrix} = \frac{1}{2} \begin{vmatrix} F_x - iF_y & F_0 - F_z & K_x - iK_y & -K^0 - K_z \\ -F_0 - F_z & -F_x - iF_y & K^0 - K_z & -K_x - iK_y \\ L_x - iL_y & L^0 - L_z & -G_x + iG_y & G_0 + G_z \\ -L^0 - L_z & -L_x - iL_y & -G_0 + G_z & G_x + iG_y \end{vmatrix}. \tag{54}$$

In an arbitrary representation of undors obtainable from (28) by a transformation (14) we should replace the left hand member Ψ of (54) by

$$S^{(1)-1} S^{(2)-1} \Psi' = \Psi. \tag{55}$$

By spatial reflection through the origin the undor Ψ will transform according to (9) by

$$\Psi' = j^2 \boldsymbol{\beta}^{(1)} \boldsymbol{\beta}^{(2)} \Psi. \tag{56}$$

In the representation (28) used by us we have $\boldsymbol{\beta}^{(1)}\boldsymbol{\beta}^{(2)} = \boldsymbol{\rho}_{\mathbf{x}}^{(1)}\boldsymbol{\rho}_{\mathbf{x}}^{(2)}$, so that this transformation (56) can be written, on account of (54), as

$$\begin{aligned} \vec{K}' &= j^2 \vec{L}, \quad K^{0\prime} = -\, j^2 L^0, \quad \vec{F}' = -\, j^2 \vec{G}, \quad F_0' = j^2 G_0, \\ \vec{L}' &= j^2 \vec{K}, \quad L^{0\prime} = -\, j^2 K^0, \quad \vec{G}' = -\, j^2 \vec{F}, \quad G_0' = j^2 F_0. \end{aligned} \tag{57}$$

Putting

$$\begin{aligned} \vec{K} &= (\vec{\mathbf{A}} + \vec{\mathbf{B}})j/i, \quad K^0 = (\mathbf{V} + \mathbf{W})j/i, \\ \vec{L} &= (\vec{\mathbf{A}} - \vec{\mathbf{B}})i/j, \quad L^0 = (\mathbf{V} - \mathbf{W})i/j, \\ \vec{F} &= (\vec{\mathbf{H}} - i\vec{\mathbf{E}})j/i, \quad F_0 = (-\,\mathbf{S} - i\mathbf{Y})j/i, \\ \vec{G} &= (\vec{\mathbf{H}} + i\vec{\mathbf{E}})i/j, \quad G_0 = (\mathbf{S} - i\mathbf{Y})i/j, \end{aligned} \tag{58}$$

the new quantities $\vec{\mathbf{A}}$, $\mathbf{V}$; $\vec{\mathbf{B}}$, $\mathbf{W}$; $\vec{\mathbf{E}}$, $\vec{\mathbf{H}}$; $\mathbf{S}$ and $\mathbf{Y}$ are still tensors with respect to restricted Lorentz transformations (two four-vectors, an anti-symmetrical tensor and two scalars), whereas we can now write (57) in the following form:

$$\vec{\mathbf{A}}' = -\vec{\mathbf{A}},\ \mathbf{V}' = +\mathbf{V};\ \ \vec{\mathbf{E}}' = -\vec{\mathbf{E}},\ \vec{\mathbf{H}}' = +\vec{\mathbf{H}};$$
$$\vec{\mathbf{B}}' = +\vec{\mathbf{B}},\ \mathbf{W}' = -\mathbf{W};\ \mathbf{Y}' = -\mathbf{Y};\ \mathbf{S}' = +\mathbf{S}. \tag{59}$$

With respect to the *complete* L o r e n t z group including reflections, an undor of the second rank consists therefore of a regular scalar **S**; a regular four-vector $\vec{\mathbf{A}}$, **V**; an antisymmetrical tensor of the second rank $\vec{\mathbf{E}}$, $\vec{\mathbf{H}}$ (which can be regarded as a *pseudo*-tensor of the second rank $\vec{\mathbf{H}}$, $\vec{\mathbf{E}}$); a *pseudo*-four-vector (that is, an antisymmetrical tensor of the third rank) $\vec{\mathbf{B}}$, **W**, and a *pseudo*-scalar (anti-symmetrical tensor of the fourth rank) **Y**.

It is often assumed [6]) that the meson field can be regarded as a P r o c a field [17]), that is, a field consisting of a four-vector $\vec{\mathbf{A}}$, **V** and a six-vector $\vec{\mathbf{E}}$, $\vec{\mathbf{H}}$ only (case (*b*) of K e m m e r [6])). Such a field can be described by an undor of the second rank satisfying the relations

$$\Psi_{12} = \Psi_{21},\ \Psi_{34} = \Psi_{43};$$
$$\Psi_{13} = -j^2\Psi_{31},\ \Psi_{14} = -j^2\Psi_{41},\ \Psi_{23} = -j^2\Psi_{32},\ \Psi_{24} = -j^2\Psi_{42}. \tag{60}$$

The most symmetrical method to achieve this and at the same time the one and only possibility to achieve (60) in a way which is independent of the representation of undors, that is, invariant under the transformation

$$\Psi' = S^{(1)}S^{(2)}\Psi, \tag{55a}$$

is to postulate that the undor describing the P r o c a field must be *symmetrical* with respect to its two indices, and at the same time that for undors j^2 in (9), (10) must be equal to minus unity:

$$\Psi_{k_1k_2} = \Psi_{k_2k_1}, \tag{61}$$

$$j^2 = -1. \tag{61a}$$

Indeed, in general a *symmetrical* undor of the second rank represents:

(*b*). a regular 4-vector $\vec{\mathbf{A}}$, **V** and a 6-vector $\vec{\mathbf{E}}$, $\vec{\mathbf{H}}$, if $j^2 = -1$;
(*c*). a pseudo-4-vector $\vec{\mathbf{B}}$, **W** and a 6-vector $\vec{\mathbf{E}}$, $\vec{\mathbf{H}}$, if $j^2 = +1$; (62)

whereas an *antisymmetrical* undor of the second rank represents:

$$\begin{aligned}&(d).\ \text{a pseudo-4-vector } \vec{\mathbf{B}}, \mathbf{W}, \text{a scalar } \mathbf{S} \text{ and a pseudo-scalar } \mathbf{Y}, \quad \text{if } j^2 = -1;\\ &(a).\ \text{a regular 4-vector } \vec{\mathbf{A}}, \mathbf{V}, \text{a scalar } \mathbf{S} \text{ and a pseudo-scalar } \mathbf{Y}, \quad \text{if } j^2 = +1.\end{aligned} \tag{62a}$$

Here (*a*), (*b*), (*c*), (*d*) refer to the tensors composing the field in the four cases considered by Kemmer [6]).

Now we define the *charge-conjugated* of an undor of the second rank by

$$\Psi^{\mathfrak{C}}_{k_1k_2} = \mathfrak{C}^{(1)}\,\mathfrak{C}^{(2)}\,(\Psi_{k_1k_2})^* \tag{63}$$

and its *charge-adjoint* by

$$\Psi^{\mathfrak{Q}}_{k_1k_2} = \Psi^{\mathfrak{C}}_{k_2k_1} = \mathfrak{C}^{(1)}\,\mathfrak{C}^{(2)}\,(\Psi_{k_2k_1})^*. \tag{63a}$$

Then

$$\psi^{\mathfrak{C}}_k \psi_l = (\psi^{\mathfrak{C}}_k \psi_l)^{\mathfrak{Q}} \tag{63b}$$

is a *self*-charge-adjoint undor of the second rank. Now we can express the tensors represented by $\Psi^{\mathfrak{Q}}$ in terms of those represented according to (54), (58) by Ψ. In this way we find from (28), (54) and (58):

$$\begin{aligned}&\vec{\mathbf{A}}^{\mathfrak{Q}} = \vec{\mathbf{A}}^*,\ \mathbf{V}^{\mathfrak{Q}} = \mathbf{V}^*;\ \vec{\mathbf{E}}^{\mathfrak{Q}} = \mathbf{E}^*,\ \vec{\mathbf{H}}^{\mathfrak{Q}} = \vec{\mathbf{H}}^*;\\ &\vec{\mathbf{B}}^{\mathfrak{Q}} = \vec{\mathbf{B}}^*,\ \mathbf{W}^{\mathfrak{Q}} = \mathbf{W}^*;\ \mathbf{Y}^{\mathfrak{Q}} = \mathbf{Y}^*;\ \mathbf{S}^{\mathfrak{Q}} = \mathbf{S}^*.\end{aligned} \tag{64}$$

We observe that by the choice of constants made in (58) we have achieved that the tensors represented according to (54), (58) by the charge adjoint $\Psi^{\mathfrak{Q}}$ of an undor of the second rank Ψ, are the conjugate complex of the tensors represented by the undor Ψ itself *).

If now a *neutrettor of the second rank* is defined as a *self-charge-adjoint* undor of the second rank, it represents by (54) and (58) according to (64) a set of *real tensors*. Such neutrettors are therefore adequate for the description of the Maxwellian field [2]) and of Kemmer's neutretto field [7]). A specimen of a neutrettor of the second rank is given by $\psi^{\mathfrak{C}}_k \psi_l$ (63*b*).

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*) The constants in the definition by (54), (58) of the tensors $\mathbf{S}$; $\vec{\mathbf{A}}$, $\mathbf{V}$; $\vec{\mathbf{E}}$, $\vec{\mathbf{H}}$; $\vec{\mathbf{B}}$, $\mathbf{W}$ and $\mathbf{Y}$ in terms of the components of the undor Ψ are uniquely determined by the conditions (59) and (64) apart from arbitrary *real* numerical factors to these tensors, which are all chosen equal to unity in (58).

Taking

$$j = i \tag{34a}$$

in the following, in accordance with (34) and (61*a*), all factors j/i and i/j vanish from (58).

§ 6. *Covariant undor calculus.* The fact that the linear combinations (29) (and, apart from linear combinations differing from (29) by a numerical factor, *only* these) of the components of a conjugate complex undor $\psi^*_{\dot{k}}$ form again the components of a regular undor ψ_k, enables us to define a *metrical undor* $\mathfrak{g}_{lk}$, that is, an undor of the second rank, which connects *contravariant* and *covariant* undors with each other.

Since, on account of (19), (23) and (13), the expression

$$\psi^\dagger \boldsymbol{\vartheta\beta}\psi = \psi^\dagger \boldsymbol{\beta}^\dagger \boldsymbol{\vartheta}\psi = \sum_{\dot{k},k} \psi^*_{\dot{k}} (\boldsymbol{\beta}^\dagger \boldsymbol{\vartheta})^{\dot{k}k} \psi_k = \sum_{\dot{k},k} (\boldsymbol{\vartheta}^{\sim\!\sim} \boldsymbol{\beta}^{\dagger\sim\!\sim})^{k\dot{k}} \psi^*_{\dot{k}} \psi_k =$$

$$= \sum_{\dot{k},k} (\boldsymbol{\vartheta}^* \boldsymbol{\beta}^*)^{k\dot{k}} \psi^*_{\dot{k}} \psi_k \equiv \Sigma\, \chi^k \psi_k \tag{65}$$

is a scalar, we can regard

$$\chi^k = \sum_{\dot{k}} (\boldsymbol{\vartheta}^* \boldsymbol{\beta}^*)^{k\dot{k}} \psi^*_{\dot{k}} \tag{66}$$

as a regular *contravariant* undor. We shall connect with this χ^k an ordinary *covariant* undor χ_l by

$$\chi_l = \sum_k \mathfrak{g}_{lk} \chi^k = \sum_{k,\dot{k}} \mathfrak{g}_{lk} (\boldsymbol{\vartheta}^* \boldsymbol{\beta}^*)^{k\dot{k}} \psi^*_{\dot{k}}\,. \tag{67}$$

Since χ_l shall be a regular undor and its components are linear combinations of those of $\psi^*_{\dot{k}}$, the undor χ_l must be equal to $(\psi^{\pounds})_l$ apart from some numerical factor, for which we shall choose unity:

$$\chi = \psi^{\pounds}. \tag{68}$$

As this result should be independent of ψ, we find

$$\sum_k \mathfrak{g}_{lk} (\boldsymbol{\vartheta}^* \boldsymbol{\beta}^*)^{k\dot{k}} = \pounds_l^{\ \dot{k}}, \text{ or } \mathfrak{g}\boldsymbol{\vartheta}^* \boldsymbol{\beta}^* = \pounds. \tag{69}$$

We conclude:

$$\mathfrak{g} = \pounds \boldsymbol{\beta}^* \boldsymbol{\vartheta}^{*-1}. \tag{70}$$

In Kramers' representation ($\boldsymbol{\vartheta\beta} = \boldsymbol{\rho}_x$, $\pounds = \boldsymbol{\rho}_y \boldsymbol{\sigma}_y$) the metrical undor $\mathfrak{g}_{lk}$ takes the following form:

$$\mathfrak{g} = -i\boldsymbol{\rho}_z\boldsymbol{\sigma}_y = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix}, \tag{71}$$

that is,

$$\psi_1 = -\psi^2,\ \psi_2 = \psi^1,\ \psi_3 = \psi^4,\ \psi_4 = -\psi^3. \tag{71a}$$

We observe that in this representation the metrical undor is *antisymmetrical* just as the metrical spinor is in spinor calculus and unlike the metrical tensor in tensor calculus (which is symmetrical):

$$\mathfrak{g}^{\sim} = -\mathfrak{g} \quad (\mathfrak{g}_{kl} = -\mathfrak{g}_{lk}). \tag{72}$$

This property of the metrical undor is invariant under transformations (14) to other representations. This follows from (70), (14*a, b, c*):

$$\mathfrak{g}'_{l'k'} = \pounds' \boldsymbol{\beta}'^{*} \boldsymbol{\vartheta}'^{*-1} = S \pounds \boldsymbol{\beta}^{*} \boldsymbol{\vartheta}^{*-1} S^{\sim} = S \mathfrak{g} S^{\sim} = \sum_{l,k} S_{l'}{}^{l} \mathfrak{g}_{lk} S_{k'}{}^{k}. \tag{14d}$$

A consequence of this antisymmetry of $\mathfrak{g}$ is

$$\chi^k \psi_k = -\chi_k \psi^k. \tag{73}$$

Conjugate complex contravariant and covariant undors are connected by the *conjugate complex metrical undor*

$$\mathfrak{g}^{*}_{\dot{l}\dot{k}} = (\mathfrak{g}_{lk})^{*} = \sum_{k} \pounds^{*}{}_{\dot{l}}{}^{k} (\boldsymbol{\beta\vartheta}^{-1})_{k\dot{k}} \tag{74}$$

This undor is in Kramers' representation given by

$$\mathfrak{g}^{*}_{\dot{l}\dot{k}} = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix} \tag{74a}$$

and transforms to another representation by

$$\mathfrak{g}^{*\prime} = S^{*} \mathfrak{g}^{*} S^{\dagger}. \tag{14e}$$

The *contravariant metrical undor* $\mathfrak{g}^{lm}$ is determined by

$$\sum_{k} \mathfrak{g}_{lk} \mathfrak{g}^{km} = \delta_l^m \tag{75}$$

or

$$\mathfrak{g}^{km} \equiv \mathfrak{g}^{-1}; \quad \mathfrak{g}^{*\dot{k}\dot{m}} \equiv \mathfrak{g}^{*-1}. \tag{75a}$$

In K r a m e r s' representation we find from (71) and (74a)

$$\mathfrak{g}^{km} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \quad \mathfrak{g}^{*\dot{k}\dot{m}} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}. \tag{76}$$

Transformation to another representation changes the contravariant metrical undors $\mathfrak{g}^{km}$ and $\mathfrak{g}^{*\dot{k}\dot{m}}$ according to (14d, e) and (75a).

Co- and contravariant undors are now connected by

$$\psi_l = \sum_k \mathfrak{g}_{lk}\, \psi^k, \qquad \psi^*_{\dot{l}} = \sum_{\dot{k}} \mathfrak{g}^*_{\dot{l}\dot{k}}\, \psi^{*\dot{k}},$$
$$\psi^k = \sum_m \mathfrak{g}^{km}\, \psi_m, \qquad \psi^{*\dot{k}} = \sum_{\dot{m}} \mathfrak{g}^{*\dot{k}\dot{m}}\, \psi^*_{\dot{m}}. \tag{77}$$

Here the summation must always be carried out with respect to the last index of the metrical undor.

From (65), (68), (73) it follows that

$$\psi^\dagger\, \vartheta\, \boldsymbol{\beta}\, \psi = \sum_k \psi^{\mathfrak{C}k}\, \psi_k = -\sum_k \psi^{\mathfrak{C}}_k\, \psi^k. \tag{19a}$$

In the same way we derive

$$\psi^\dagger\, \vartheta\, \boldsymbol{\beta}\, \psi = \sum \psi^*_{\dot{k}}\, \psi^{\mathfrak{C}*\dot{k}} = -\sum \psi^{*\dot{k}}\, \psi^{\mathfrak{C}*}_{\dot{k}}. \tag{19b}$$

For the current-density four-vector (20) we can derive similar expressions, for instance *)

$$\vec{j}/c = \sum \psi^{\mathfrak{C}k}\, \vec{\gamma}_k{}^l\, \psi_l, \quad \rho = \sum \psi^{\mathfrak{C}k}\, \boldsymbol{\beta}_k{}^l\, \psi_l \tag{20a}$$

Inserting (46a, b) into (71a) we find in K r a m e r s' representation

$$\psi^1 = u_{\dot{2}}, \ \psi^2 = -u_{\dot{1}}, \ \psi^3 = -v^{\$2}, \ \psi^4 = v^{\$1}, \tag{78}$$

therefore, following V a n d e r W a e r d e n's notation [1]) and putting $v^\$ \equiv w$:

$$\begin{vmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{vmatrix} \equiv \begin{vmatrix} \begin{pmatrix} u_{\dot{1}} \\ u_{\dot{2}} \end{pmatrix} \\ \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \end{vmatrix} \equiv \begin{vmatrix} u \\ v^\$ \end{vmatrix} \rightleftarrows \begin{vmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{vmatrix} \equiv \begin{vmatrix} \begin{pmatrix} u^{\dot{1}} \\ u^{\dot{2}} \end{pmatrix} \\ \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \end{vmatrix} \equiv \begin{vmatrix} u^{\$*} \\ v^* \end{vmatrix}. \tag{78a}$$

*) Compare P. A. M. D i r a c, loc. cit. [18]), and W. G o r d o n, loc. cit. [19]).

The difference between our undor calculus and Van der Waerden's spinor calculus is, that we have taken care from the beginning that the transformations of "contravariant" undors should indeed be contragredient to those of "covariant" undors by *all* transformations of the complete Lorentz group *including spatial reflections.*

We might have derived (14*e*) and (76) in a simpler way. Making use of our knowledge of Van der Waerden's metrical spinor, we conclude that the contravariant metrical undor $\mathfrak{g}^{km}$ must have the form

$$\begin{vmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & -b \\ 0 & 0 & b & 0 \end{vmatrix} \tag{76a}$$

in Kramers' representation, in which the first two components of a four-spinor behave like the components of a conjugate complex covariant Van der Waerden two-spinor and the last two components like a regular contravariant Van der Waerden spinor. The ratio (a/b) is now determined by the condition that, if φ and ψ are two arbitrary regular covariant undors, the expression

$$\varphi_k\, \mathfrak{g}^{kl}\, \psi_l = \varphi^{\infty}\, \mathfrak{g}^{-1}\, \psi \tag{79}$$

shall be a scalar with respect to the spatial reflection (9), (10) (as well as it was, on account of (76*a*), a scalar by *restricted* Lorentz transformations). In Kramers' representation we find from (76*a*), (9) and $\boldsymbol{\beta} = \boldsymbol{\rho}_x$ (28):

$$\begin{aligned} a(\varphi_1\, \psi_2 - \varphi_2\, \psi_1) - b(\varphi_3\, \psi_4 - \varphi_4\, \psi_3) = \\ = a(\varphi_1'\, \psi_2' - \varphi_2'\, \psi_1') - b(\varphi_3'\, \psi_4' - \varphi_4'\, \psi_3') = \\ = a\, j^2(\varphi_3\, \psi_4 - \varphi_4\, \psi_3) - b\, j^2(\varphi_1\, \psi_2 - \varphi_2\, \psi_1), \end{aligned}$$

therefore,

$$a/b = -j^2 = -1/j^2. \tag{76b}$$

Choosing j according to (34) we find $a = b$ and (76*a*) becomes identical to (76) for Kramers' representation (28).

If, now, we *postulate* that the scalar (79) shall be an invariant under transformation (14) to another representation, we find

$$(\mathfrak{g}^{-1})' = S^{\sim -1}\, \mathfrak{g}^{-1}\, S^{-1}, \tag{14f}$$

in accordance with (14*d*).

A still shorter way of deriving (71) is by making use of (54) and (58). Since the metrical undor is supposed not to change its form by Lorentz transformations and spatial reflections, it must represent a scalar. Therefore it must have, according to (54), (58), the following form in Kramers' representation:

$$\mathfrak{g}_{kl} = \tfrac{1}{2} \cdot \begin{vmatrix} 0 & -S & 0 & 0 \\ S & 0 & 0 & 0 \\ 0 & 0 & 0 & S \\ 0 & 0 & -S & 0 \end{vmatrix}.$$

Taking $S = 2$ we find (71).

We remark that $\mathfrak{g}$ is a *neutrettor* of the second rank on account of the special definition (63*a*) given for the charge-adjoint of an undor of the second rank.

In the literature use is often made of an abbreviated notation for the contravariant charge-conjugated of a regular undor ψ_k or $\Psi_{k_1k_2}$:

$$\begin{aligned} \psi^{\mathfrak{C}k} &= \psi^\dagger\, \vartheta\beta \equiv \psi^+, \\ \Psi^{\mathfrak{C}k_1k_2} &= \Psi^\dagger\, \vartheta^{(1)}\vartheta^{(2)}\, \beta^{(1)}\beta^{(2)} \equiv \Psi^+. \end{aligned} \tag{80}$$

Further, we shall denote by α^μ and γ^μ ($\mu = 0, 1, 2, 3$) the matrices

$$\begin{aligned} \{\alpha_1, \alpha_2, \alpha_3\} &= \{\alpha^1, \alpha^2, \alpha^3\} \equiv \vec{\alpha}, \quad \alpha^0 = -\alpha_0 \equiv 1, \\ \{\gamma_1, \gamma_2, \gamma_3\} &= \{\gamma^1, \gamma^2, \gamma^3\} \equiv \vec{\gamma} = \beta\vec{\alpha}, \quad \gamma^0 = -\gamma_0 \equiv \beta. \end{aligned} \tag{2a}$$

The probability density and current of a Dirac electron are then given by

$$\{j^1, j^2, j^3\} \equiv \vec{j}/c, \quad j^0 = -j_0 \equiv \rho; \quad j^\mu = \psi^+\gamma^\mu\psi. \tag{20b}$$

The relations (13), (23) and (23*a*) can now be written as

$$\begin{aligned} \beta^\dagger\vartheta &= \vartheta\beta, & \alpha^{\mu\dagger}\vartheta &= \vartheta\alpha^\mu, \\ \gamma^\mu\mathfrak{C} &= -\mathfrak{C}\gamma^{\mu *}, & \alpha^\mu\mathfrak{C} &= \mathfrak{C}\alpha^{\mu *}. \end{aligned} \tag{81}$$

In undor calculus the gradient four-vector ∇^μ is, according to (54), (58), (60), (64), represented by a symmetrical neutrettor. In Kram e r s' representation this gradient neutrettor has the following form:

$$\nabla_{kl} = \begin{vmatrix} 0 & 0 & \nabla_x - i\nabla_y & \frac{1}{c}\frac{\partial}{\partial t} - \nabla_z \\ 0 & 0 & -\frac{1}{c}\frac{\partial}{\partial t} - \nabla_z & -\nabla_x - i\nabla_y \\ \nabla_x - i\nabla_y & -\frac{1}{c}\frac{\partial}{\partial t} - \nabla_z & 0 & 0 \\ \frac{1}{c}\frac{\partial}{\partial t} - \nabla_z & -\nabla_x - i\nabla_y & 0 & 0 \end{vmatrix} = \nabla_{lk} \quad (82)$$

The D i r a c equation of a free electron can now be written in covariant undor-notation:

$$\{i\varkappa + \gamma^\mu \nabla_\mu\}\psi = 0 \longrightarrow i\varkappa\psi_k + \nabla_{kl}\psi^l = 0. \quad (1b)$$

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THE UNDOR EQUATION OF THE MESON FIELD

Zusammenfassung

Die Meson-Gleichung von Proca, Kemmer und Bhabha wird mittels Undoren zweiter Stufe dargestellt. Eine Verallgemeinerung der Gleichung führt zu einer neuen Meson-Gleichung, welche im wesentlichen aus einer Kombination der Fälle (*b*) und (*d*) von Kemmer besteht. Die Neutretto-Gleichung wird in ähnlicher Weise erweitert. — Das magnetische Moment der Mesonen wird abgeleitet.

Die ladungskonjugierten Wellenfunktionen genügen einer Gleichung, in welcher die Vorzeichen aller Ladungen umgekehrt sind. Wenn man postuliert, dass sich bei Beschreibung des physikalischen Geschehens mittels der ladungskonjugierten Grössen für alle physikalisch sinnvollen Grössen dieselben Werte ergeben sollen, lässt sich folgern, dass Teilchen mit ganzzahligem Spin der Einstein-Bose-Statistik, und Teilchen mit halbzahligem Spin dem Ausschliessungsprinzip genügen müssen.

Resumo.

La mezona ekvacio de Proca, Kemmer kaj Bhabha estas prezentata per duaŝtufaj undoroj. Pliĝeneraligado de la ekvacio donas novan mezonan ekvacion, kiu ĉefe konsistas el kombinaĵo de la kazoj (*b*) kaj (*d*) de Kemmer. La neŭtreta ekvacio estas plivastigata en la sama maniero. La magneta momanto de la mezonoj estas kalkulata.

La ŝarge konjugitaj ondofunkcioj kontentigas ekvacion, en kiu la antaŭsignoj de ĉiuj ŝargoj estas inversaj. Se oni postulas ke ĉe priskribo de la fizikaj okazaĵoj per la ŝarge konjugitaj grandoj por ĉiuj observebloj devas rezulti la samaj valoroj, oni povas konkludi ke korpuskloj kun entjera spino devas obei la statistikon de Einstein kaj Bose, kaj ke korpuskloj kun entjerplusduona spino devas obei la statistikon de Fermi kaj Dirac.

§ 1. *The Proca-Kemmer meson equation in undor notation.* The usual meson equations of Kemmer[1]), Bhabha[2]) and Yukawa[3]) can be written in the following form:

$$\varkappa(\zeta_{\mu\nu} - f_b\, u_{\mu\nu}) = D_{[\mu}\, \varphi_{\nu]} \equiv D_\mu\, \varphi_\nu - D_\nu\, \varphi_\mu, \qquad (1)$$
$$\varkappa(\varphi_\nu + g_b\, v_\nu) = D^\mu\, \zeta_{\mu\nu}; \quad (\mu, \nu = 0, 1, 2, 3).$$

Here $\varkappa$ can be expressed in terms of the mass $\mathfrak{m}$ of the meson by $\varkappa = \mathfrak{m}c/\hbar$; $u_{\mu\nu}$ is an antisymmetrical tensor and v_ν is a four-vector given by *):

$$u_{\mu\nu} = (i/2) \,.\, \psi_N^+ \gamma_{[\mu} \gamma_{\nu]} \psi_P; \qquad v_\nu = \psi_N^+ \gamma_\nu \psi_P;$$
$$(\psi^+ \equiv \psi^\dagger \beta; \quad \gamma^\mu = \beta\alpha^\mu); \quad (\mu, \nu = 0, 1, 2, 3). \tag{2}$$

By ψ_N and ψ_P we denote the wave-functions of the fields of neutrons and protons, so that after superquantization the expressions (2) represent operators which possess non-vanishing matrix elements for transitions of a proton into a neutron. The operators D_μ in (1) are defined by [U.C. (3)]:

$$D_\mu = \nabla_\mu + \frac{e}{i\hbar c}\mathfrak{A}_\mu; \; \nabla_0 = -\nabla^0 = \frac{1}{c}\frac{\partial}{\partial t}; \; \mathfrak{A}_0 = -\mathfrak{A}^0 = -\mathfrak{B}. \tag{3}$$

We shall now write the Proca-Kemmer equations (1) in vector notation. For this purpose we put

$$\zeta_{a0} = \zeta^{0a} = \mathbf{E}_a; \; \zeta_{bc} = -\zeta_{cb} = \mathbf{H}_a; \; \varphi_a = \mathbf{A}_a; \; -\varphi_0 = \varphi^0 = \mathbf{V}; \tag{4}$$

(a, b, c being a cyclic permutation of 1, 2, 3).

Further, we put

$$u_{0a} = u^{a0} = \mathbf{e}_a, \; -u_{bc} = u_{cb} = \mathbf{h}_a; \; v_a = \mathbf{a}_a, \; v^0 = \mathbf{v}, \tag{5}$$

so that

$$\vec{\mathbf{e}} = -\psi^\dagger i\beta\vec{\alpha}\psi, \quad \vec{\mathbf{h}} = -\psi^\dagger\beta\vec{\sigma}\psi; \quad \mathbf{a} = \psi^\dagger\vec{\alpha}\psi, \quad \mathbf{v} = \psi^\dagger\psi. \tag{6}$$

Then, the equations (1) read †)

$$\begin{aligned}
\varkappa(\vec{\mathbf{E}} + f_b\,\vec{\mathbf{e}}) &= -\vec{\nabla}\mathbf{V} - (\partial/c\partial t)\,\vec{\mathbf{A}} - (e/i\hbar c)\,(\vec{\mathfrak{A}}\mathbf{V} - \mathfrak{B}\vec{\mathbf{A}}),\\
\varkappa(\vec{\mathbf{H}} + f_b\,\vec{\mathbf{h}}) &= \operatorname{rot}\vec{\mathbf{A}} + (e/i\hbar c)\,[\vec{\mathfrak{A}}, \vec{\mathbf{A}}],\\
\varkappa(\vec{\mathbf{A}} + g_b\,\vec{\mathbf{a}}) &= -\operatorname{rot}\vec{\mathbf{H}} + (\partial/c\partial t)\,\vec{\mathbf{E}} - (e/i\hbar c)\,([\vec{\mathfrak{A}}, \vec{\mathbf{H}}] + \mathfrak{B}\vec{\mathbf{E}}),\\
\varkappa(\mathbf{V} + g_b\,\mathbf{v}) &= -\operatorname{div}\vec{\mathbf{E}} - (e/i\hbar c)\,(\vec{\mathfrak{A}}\,.\,\vec{\mathbf{E}}).
\end{aligned} \tag{7}$$

In order to write these equations in undor notation we can make use of the Kramers representation of undors [U.C. (28)], in

*) For the notation used in the present paper we must refer to the preceding paper of the author on undor calculus [4]). References to formulae from that paper will be indicated by [U.C.]. In (2), (6) and (12) we have put $\vartheta = 1$ [U.C. (17)].

†) rot ≡ curl.

which the Dirac matrices $\vec{\boldsymbol{\alpha}}$, $\boldsymbol{\beta}$ occurring in (6) are given by

$$\alpha_x = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{vmatrix}, \quad \alpha_y = \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{vmatrix}, \quad \alpha_z = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix},$$

$$\boldsymbol{\beta} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \tag{8}$$

and the charge-conjugated $\psi^{\mathfrak{C}}$ of a Dirac wave-function ψ is given by

$$\psi^{\mathfrak{C}} = \mathfrak{C}\psi^*, \quad \mathfrak{C} = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}. \tag{9}$$

In this representation the components of the tensors represented by an undor of the second rank are related to the components of this undor by [U.C. (54), (58), (34*a*)]:

$$2\Psi_{k_1k_2} = \begin{vmatrix} F_x - iF_y & F_0 - F_z & K_x - iK_y & -K^0 - K_z \\ -F_0 - F_z & -F_x - iF_y & K^0 - K_z & -K_x - iK_y \\ L_x - iL_y & L^0 - L_z & -G_x + iG_y & G_0 + G_z \\ -L^0 - L_z & -L_x - iL_y & -G_0 + G_z & G_x + iG_y \end{vmatrix}; \tag{10}$$

$$\vec{K} = \vec{\mathbf{A}} + \vec{\mathbf{B}},\ K^0 = \mathbf{V} + \mathbf{W},\ \vec{F} = \vec{\mathbf{H}} - i\vec{\mathbf{E}},\ F_0 = -\mathbf{S} - i\mathbf{Y},$$
$$\vec{L} = \vec{\mathbf{A}} - \vec{\mathbf{B}},\ L^0 = \mathbf{V} - \mathbf{W},\ \vec{G} = \vec{\mathbf{H}} + i\vec{\mathbf{E}},\ G_0 = \mathbf{S} - i\mathbf{Y}.$$

Here **S** is a scalar, $\vec{\mathbf{A}}$, **V** a four-vector, $\vec{\mathbf{E}}$, $\vec{\mathbf{H}}$ a six-vector, $\vec{\mathbf{B}}$, **W** a pseudo-four-vector and **Y** a pseudo-scalar. In particular the undor of the second rank

$$X_{k_1k_2} = 2\varphi^{\mathfrak{C}}_{k_1}\psi_{k_2} \tag{11}$$

represents, according to (10), the following tensors

$$\vec{\mathbf{a}} = \varphi^\dagger \vec{\boldsymbol{\alpha}}\psi, \quad \mathbf{v} = \varphi^\dagger\psi; \quad \vec{\mathbf{e}} = -\varphi^\dagger i\boldsymbol{\beta}\vec{\boldsymbol{\alpha}}\psi, \ \vec{\mathbf{h}} = -\varphi^\dagger\boldsymbol{\beta}\vec{\boldsymbol{\sigma}}\psi;$$
$$\vec{\mathbf{b}} = -\varphi^\dagger\vec{\boldsymbol{\sigma}}\psi, \ \mathbf{w} = -\varphi^\dagger\gamma_5\psi; \ \mathbf{y} = \varphi^\dagger i\gamma_5\boldsymbol{\beta}\psi; \quad \mathbf{s} = \varphi^\dagger\boldsymbol{\beta}\psi. \tag{12}$$

Here we have put

$$\sigma_x = - i\alpha_y\alpha_z, \text{ cycl.}; \ \gamma_5 = - i\alpha_x\alpha_y\alpha_z. \qquad (13)$$

From (10) we see that the Proca field $\vec{A}$, V; $\vec{E}$, $\vec{H}$ can be represented by a symmetrical undor

$$\overset{sym}{\Psi}_{k_1k_2} = \tfrac{1}{2}\{\Psi_{k_1k_2} + \Psi_{k_2k_1}\} \equiv \tfrac{1}{2}\Psi_{\{k_1k_2\}}. \qquad (14)$$

Writing out the meson equations (7) with the help of (10)—(12) in terms of the undor components of $\overset{sym}{\Psi}_{k_1k_2}$ and of

$$\overset{sym}{X}_{k_1k_2} = \psi^{\mathfrak{C}}_{N_{k_1}} \psi_{P_{k_2}} + \psi^{\mathfrak{C}}_{N_{k_2}} \psi_{P_{k_1}} \equiv \psi_{\{N_{k_1}}^{\mathfrak{C}} \psi_{P_{k_2}\}}, \qquad (14a)$$

and collecting the ten equations into one undor equation, we find, on account of (8) and (13):

$$2i\varkappa(\overset{sym}{\Psi}_{k_1k_2} + f_{op}\overset{sym}{X}_{k_1k_2}) + \{\beta^{(1)}(\vec{\alpha}^{(1)}.\vec{D} + D_0) +$$
$$+ \beta^{(2)}(\vec{\alpha}^{(2)}.\vec{D} + D_0)\}\overset{sym}{\Psi}_{k_1k_2} = 0, \qquad (15)$$

where

$$f_{op} = \tfrac{1}{2}\{(f_b + g_b) + (f_b - g_b)\,\gamma_5^{(1)}\gamma_5^{(2)}\} \qquad (16)$$

is a scalar operator which, operating on an undor of the second rank (10), multiplies in this undor $\vec{E}$, $\vec{H}$, **Y** and **S** by f_b and $\vec{A}$, **V** and $\vec{B}$, **W** by g_b. By an index in brackets we have here distinguished Dirac matrices operating on each of the indices k_1 and k_2 of the undor.

We remark that (15) and (16) are invariant by transformation to another representation of undors [4]).

Putting

$$D_{kl} = \begin{vmatrix} 0 & 0 & D_x - iD_y & D_0 - D_z \\ 0 & 0 & -D_0 - D_z & -D_x - iD_y \\ D_x - iD_y & -D_0 - D_z & 0 & 0 \\ D_0 - D_z & -D_x - iD_y & 0 & 0 \end{vmatrix} = D_{lk} \qquad (17)$$

we can write (15) at once in covariant undor notation [4]) (compare [U.C. (82), (1b)]):

$$2i\varkappa(\overset{sym}{\Psi}_{k_1k_2} + f_{k_1k_2}{}^{l_1l_2}\overset{sym}{X}_{l_1l_2}) + D_{k_1l_1}\overset{sym}{\Psi}{}^{l_1}{}_{k_2} + D_{k_2l_2}\overset{sym}{\Psi}_{k_1}{}^{l_2} = 0, \qquad (18)$$

where $f_{k_1k_2}{}^{l_1l_2}$ are the matrix elements of the scalar interaction operator f_{op} (16).

This interaction operator would take a particularly simple form, if the constants f_b and g_b should happen to be equal:

$$f_b = g_b \,(?) \qquad (19)$$

A preliminary interpretation [1] [5]) of the binding energy of the deuteron in the (triplet) ground state and the attraction potential in the singlet state, on which experimental data are available, seemed to indicate that $|f_b/g_b|$ did not differ much from unity indeed, but later investigations [6]), which accounted for the charge-independence of the nuclear forces between heavy particles [7]), gave a different result. The simplification (19) seems, therefore, not to be allowed for the moment and we shall not make use of it in the following.

§ 2. *The generalized meson equation and the neutretto equation.* In the preceding section we have discussed the equation for the Proca-Kemmer meson field (case (*b*) of Kemmer [1])). This field was described by a symmetrical undor $\overset{sym}{\Psi}_{k_1k_2}$ and it interacted with the symmetrical part $\overset{sym}{X}_{k_1k_2}$ of the undor $X_{k_1k_2} = 2\psi^{\mathfrak{C}}_{N k_1}\psi_{P k_2}$. It is plausible now to consider the generalization of the equations (15) and (18) by replacing these symmetrical undors $\overset{sym}{\Psi}_{k_1k_2}$ and $\overset{sym}{X}_{k_1k_2}$ by the unsymmetrized undors $\Psi_{k_1k_2}$ and $X_{k_1k_2}$. The operator f_{op} can then be replaced by a more complicated scalar operator which, operating on an undor of the second rank (10), multiplies $\mathbf{S}$ by f_o; $\vec{\mathbf{A}}$, $\mathbf{V}$ by g_b; $\vec{\mathbf{E}}$, $\vec{\mathbf{H}}$ by f_b, $\vec{\mathbf{B}}$, $\mathbf{W}$ by g_d, and $\mathbf{Y}$ by f_d. The generalized meson equation

$$2i\varkappa\{\Psi_{k_1k_2} + f_{op}\,(2\psi^{\mathfrak{C}}_{N k_1}\psi_{P k_2})\} + \{(\overline{\boldsymbol{\gamma}^{(1)} + \boldsymbol{\gamma}^{(2)}} \,.\, \vec{D}) + (\boldsymbol{\beta}^{(1)} + \boldsymbol{\beta}^{(2)})\, D_0\}\,\Psi_{k_1k_2} = 0 \qquad (20)$$

can then be written, according to (8)—(13), in vector notation:

$$\varkappa(\mathbf{S} + f_o\,\mathbf{s}) = 0; \qquad (21o)$$

$$\varkappa(\vec{\mathbf{A}} + g_b\,\vec{\mathbf{a}}) = D_0\,\vec{\mathbf{E}} - [\vec{D}, \vec{\mathbf{H}}], \quad \varkappa(\mathbf{V} + g_b\,\mathbf{v}) = -(\vec{D}\,.\,\vec{\mathbf{E}});$$
$$\varkappa(\vec{\mathbf{E}} + f_b\,\vec{\mathbf{e}}) = -D_0\,\vec{\mathbf{A}} - \vec{D}\,\mathbf{V}, \quad \varkappa(\vec{\mathbf{H}} + f_b\,\vec{\mathbf{h}}) = [\vec{D}, \vec{\mathbf{A}}]; \qquad (21b)$$

$$\varkappa(\mathbf{W} + g_d\,\mathbf{w}) = -D_0\,\mathbf{Y}, \quad \varkappa(\vec{\mathbf{B}} + g_d\,\vec{\mathbf{b}}) = \vec{D}\,\mathbf{Y};$$
$$\varkappa(\mathbf{Y} + f_d\,\mathbf{y}) = D_0\,\mathbf{W} + (\vec{D}\,.\,\vec{\mathbf{B}}). \qquad (21d)$$

Apart from the first equation (21*o*), which defines the field component **S** in terms of the components of the field of heavy D i r a c particles, these equations represent exactly the cases (*b*) and (*d*) of K e m m e r, that is, the meson field suggested by M ø l l e r and R o s e n f e l d [8]). The P r o c a field (*b*) describes mesons with a spin angular momentum $\hbar$, whereas the "pseudo-scalar field" (*d*) describes spinless mesons [9]).

According to K e m m e r [6]) the neutretto equation is obtained from the meson equation by changing $\psi_N^* \psi_P$ into $\frac{1}{2}(\psi_P^* \psi_P - \psi_N^* \psi_N)$ and D_μ into ∇_μ; and by postulating that the tensors representing the neutretto field $\Phi_{k_1k_2}$ shall be real. Thus (20) changes into

$$2i\varkappa\{\Phi_{k_1k_2} + f_{op}\,(\psi^{\mathfrak{C}}_{P_{k_1}}\,\psi_{P_{k_2}} - \psi^{\mathfrak{C}}_{N_{k_1}}\,\psi_{N_{k_2}})\} + (\gamma_\mu^{(1)} + \gamma_\mu^{(2)})\,\nabla^\mu\,\Phi_{k_1k_2} = 0 \qquad (22)$$

with the additional condition (see [U.C. (64)]) that

$$\Phi_{k_1k_2} = \mathfrak{C}^{(1)}\,\mathfrak{C}^{(2)}(\Phi_{k_2k_1})^* \equiv \Phi^{\mathfrak{C}}_{k_2k_1} \equiv \Phi^{\mathfrak{N}}_{k_1k_2} \qquad (23)$$

is a neutrettor of the second rank.

The compatibility of the condition (23) with the equation (22) must be shown. For this purpose we multiply the conjugate complex of (22) by $-\,\mathfrak{C}^{(1)}\,\mathfrak{C}^{(2)}$ and find, on account of $-\,\mathfrak{C}^{(n)}\,\gamma_\mu^{(n)*} = \gamma_\mu^{(n)}\,\mathfrak{C}^{(n)}$ [U.C. (81)] and $\mathfrak{C}^{(1)}\,\mathfrak{C}^{(2)}\,(\Phi_{k_1k_2})^* = \Phi^{\mathfrak{N}}_{k_2k_1}$ [U.C. (63*a*)]:

$$2i\varkappa\{\Phi^{\mathfrak{N}}_{k_2k_1} + \mathfrak{C}^{(1)}\mathfrak{C}^{(2)}\,f^*_{op}(\psi^{\mathfrak{C}}_{P_{k_1}}\psi_{P_{k_2}} - \psi^{\mathfrak{C}}_{N_{k_1}}\psi_{N_{k_2}})^*\} + (\gamma_\mu^{(1)} + \gamma_\mu^{(2)})\,\nabla^\mu\Phi^{\mathfrak{N}}_{k_2k_1} = 0,$$

or, interchanging k_1 and k_2:

$$2i\varkappa\{\Phi^{\mathfrak{N}}_{k_1k_2} + f^{\mathfrak{C}}_{op}\,(\psi^{\mathfrak{C}}_{P_{k_1}}\,\psi_{P_{k_2}} - \psi^{\mathfrak{C}}_{N_{k_1}}\,\psi_{N_{k_2}})^{\mathfrak{N}}\} + (\gamma_\mu^{(1)} + \gamma_\mu^{(2)})\,\nabla^\mu\,\Phi^{\mathfrak{N}}_{k_1k_2} = 0, \qquad (24)$$

where we have put (compare U.C. (40)):

$$f^{\mathfrak{C}}_{op} = \mathfrak{C}^{(1)}\,\mathfrak{C}^{(2)}\,f^*_{op}\,\mathfrak{C}^{(2)*}\,\mathfrak{C}^{(1)*}, \qquad (25)$$

so that

$$(f_{op}\,\Psi_{k_1k_2})^{\mathfrak{C}} = f^{\mathfrak{C}}_{op}\,\Psi^{\mathfrak{C}}_{k_1k_2}. \qquad (2\epsilon)$$

Now f_{op} only multiplies the tensors represented by $\Psi_{k_1k_2}$ by the constant factors f_o, g_b, f_b, g_d and f_d, whereas, according to [U.C. (63), (63*a*), (64)], charge-conjugation of an undor of the second rank changes the tensors **S**; $\vec{\mathbf{A}}$, **V**; $\vec{\mathbf{E}}$, $\vec{\mathbf{H}}$; $\vec{\mathbf{B}}$, **W** and **Y**, represented by it according to (10), into $-\mathbf{S}^*$; $\vec{\mathbf{A}}^*$, $\mathbf{V}^*$; $\vec{\mathbf{E}}^*$, $\vec{\mathbf{H}}^*$; $-\,\vec{\mathbf{B}}^*$, $-\,\mathbf{W}^*$ and $-\,\mathbf{Y}^*$. We conclude therefore from (26) that $f^{\mathfrak{C}}_{op}$ is the scalar operator multiplying the tensors by the conjugate complex of the factors f_o, g_b, f_b, g_d, f_d. Now, assuming that these constants are real

$$f_o = f_o^*,\ g_b = g_b^*,\ f_b = f_b^*,\ g_d = g_d^*,\ f_d = f_d^*, \tag{27}$$

we find

$$f_{op}^{\mathfrak{C}} = f_{op}. \tag{27a}$$

If we make use of the fact that $\psi_{k_1}^{\mathfrak{C}}\ \psi_{k_2}$ is a neutrettor of the second rank [U.C. (63*b*)], the equation (24) turns into

$$2i\varkappa\{\Phi_{k_1k_2}^{\mathfrak{Q}} + f_{op}\ (\psi_{P_{k_1}}^{\mathfrak{C}}\ \psi_{P_{k_2}} - \psi_{N_{k_1}}^{\mathfrak{C}}\ \psi_{N_{k_2}})\} + (\gamma_\mu^{(1)} + \gamma_\mu^{(2)})\ \nabla^\mu\ \Phi_{k_1k_2}^{\mathfrak{Q}} = 0. \tag{28}$$

This equation for $\Phi_{k_1k_2}^{\mathfrak{Q}}$ is identical with the original equation (22) for $\Phi_{k_1k_2}$, so that the condition $\Phi_{k_1k_2} = \Phi_{k_1k_2}^{\mathfrak{Q}}$ (23) is indeed compatible with (22), if the interaction constants (27) are real *).

§ 3. *The charge-conjugated meson equation.* Kramers[10]) has shown that if ψ is a solution of the Dirac equation for positive particles, then $\psi^{\mathfrak{C}}$ is a solution of the equation for negative particles, that is, of the equation following from the Dirac equation by changing e into $(-e)$. In the present section we shall show that the meson and the neutretto equations possess similar properties and that, if Ψ is a solution of the equation (20) for positive mesons (*"theticons"* †)), then $\Psi^{\mathfrak{C}}$ [U.C. (63)] is a solution of the equation for negative mesons (*"arneticons"* †)).

For this purpose we proceed in a similar way as in the preceding section; only this time we shall not interchange the indices k_1 and k_2. In this way we derive from equation (20):

$$2i\varkappa\{\Psi_{k_1k_2}^{\mathfrak{C}} + 2f_{op}^{\mathfrak{C}}\ (\psi_{P_{k_2}}^{\mathfrak{C}}\ \psi_{N_{k_1}})\} + (\gamma_\mu^{(1)} + \gamma_\mu^{(2)})\ D^{*\mu}\ \Psi_{k_1k_2}^{\mathfrak{C}} = 0, \tag{29}$$

where we have made use of

$$\mathfrak{L}^{(1)}\ \mathfrak{L}^{(2)}\ (\varphi_{k_1}^{\mathfrak{C}}\ \psi_{k_2})^* = \mathfrak{L}^{(2)}\ \psi_{k_2}^* \cdot \mathfrak{L}^{(1)}\ \varphi_{k_1}^{\mathfrak{C}*} = \psi_{k_2}^{\mathfrak{C}}\ \varphi_{k_1}. \tag{30}$$

Comparing (29) with (20) we observe that $\Psi_{k_1k_2}^{\mathfrak{C}}$ satisfies an "arneticonic" equation (29) differing from the "theticonic" equation (20) for $\Psi_{k_1k_2}$ by the inversion of the sign of e (as D is replaced by D^*) and by the change of

$$f_{op}\ (\psi_{N_{k_1}}^{\mathfrak{C}}\ \psi_{P_{k_2}})\ \text{into}\ f_{op}^{\mathfrak{C}}\ (\psi_{P_{k_2}}^{\mathfrak{C}}\ \psi_{N_{k_1}}). \tag{31}$$

*) If neutrettors of the second rank were defined by $\Phi = \Phi^{\mathfrak{C}}$, instead of by $\Phi = \Phi^{\mathfrak{Q}}$, it would have been necessary to take g_b and f_b real, but f_o, g_d and f_d purely imaginary.

†) These names are derived from θετικος = positive and ἀρνητικος = negative.

The electromagnetic potentials, being real, are not changed.

We must now remember that the interaction of mesons with heavy particles, as described by the equations, should consist in the possibility of the absorption of an arneticon or the emission of a theticon by a proton which changes into a neutron, and *vice versa*. That is to say, the wave-functions in the equations (20) and (29) should be superquantized *).

If the wave-functions of anti-protons and neutrons ($\psi_P^{\mathfrak{C}}$ and ψ_N) are assumed to be *anticommutative* with each other (an assumption which simplifies the discussion of the canonical theory of quantized wave-fields and which enables us to introduce the formalism of the *isotopic spin* in a natural way), we can express (31) by stating that, in order to change the equation for Ψ into that for $\Psi^{\mathfrak{C}}$, not only should Ψ be replaced by $\Psi^{\mathfrak{C}}$ and the *electric* charge e by

$$e^L = -e^*, \tag{32}$$

but at the same time the "*mesic*" charges f_o, g_b, etc. should be replaced by

$$f_o^L = -f_o^*,\ g_b^L = -g_b^*,\ f_b^L = -f_b^*,\ \text{etc.}, \tag{32a}$$

and $\psi_N^{\mathfrak{C}}$ by ψ_N and ψ_P by $\psi_P^{\mathfrak{C}}$. There is no need to change the potentials of the Maxwellian field occurring in the meson equation (20). This field can be described by a *symmetrical* neutrettor of the second rank, so that it is equal not only to its own charge-adjoint, but also to its charge-conjugated.

In the same way the neutretto equation for $\Phi_{k_1k_2}$ is changed into the equivalent equation for $\Phi_{k_1k_2}^{\mathfrak{C}}$. This is seen at once by interchanging in (28) again k_1 and k_2 and by making use of the anticommutativity of $\psi_P^{\mathfrak{C}}$ with ψ_P and of $\psi_N^{\mathfrak{C}}$ with ψ_N. Formally the infinities of the δ-functions from the commutation rules of protons and of neutrons cancel each other.

§ 4. *The charge current-density and the magnetic moment of mesons.* Proca [11]) and Bhabha [2]) have derived the electric charge density and current of mesons with a spin $\hbar$ from a Lagrangian, which was chosen in such a way that the Proca equations and the equations for the Maxwellian field could both be deduced from it.

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*) The question of the possibility of quantization of the fields in such a way that the relativistic invariance of the theory is maintained, is not discussed in the present paper.

In a similar way one can proceed for the generalized meson field [9]). If the field is normalized according to Kemmer [1]) in such a way that $\vec{\mathbf{A}}^*$, $-\vec{\mathbf{E}}^*$, $\mathbf{Y}^*$ and $-\mathbf{W}^*$ are the canonical conjugates of $\vec{\mathbf{E}}$, $\vec{\mathbf{A}}$, $\mathbf{W}$ and $\mathbf{Y}$ respectively, the expressions for the electric charge density and current take the following from [9]) [12]):

$$e\rho = (e/i\hbar)\,\{(\vec{\mathbf{A}}^*\,.\,\vec{\mathbf{E}}) - (\vec{\mathbf{E}}^*\,.\,\vec{\mathbf{A}}) + \mathbf{Y}^*\,\mathbf{W} - \mathbf{W}^*\,\mathbf{Y}\} = e\Psi^\dagger\rho_{op}\Psi,$$

$$e\vec{j}/c = (e/i\hbar)\,\{[\vec{\mathbf{A}}^*, \vec{\mathbf{H}}] + [\vec{\mathbf{H}}^*, \vec{\mathbf{A}}] + \tag{33}$$

$$+ \mathbf{V}^*\,\vec{\mathbf{E}} - \vec{\mathbf{E}}^*\,\mathbf{V} + \mathbf{Y}^*\,\vec{\mathbf{B}} - \vec{\mathbf{B}}^*\,\mathbf{Y}\} = (e/c)\,.\,\Psi^\dagger\,\vec{j}_{op}\,\Psi.$$

Here ρ_{op} and $\vec{j}_{op}/c$ are determined by *)

$$\rho_{op} = \rho_{op}^\dagger = \boldsymbol{\vartheta}^{(1)}\,\boldsymbol{\vartheta}^{(2)}\,\frac{\boldsymbol{\beta}^{(1)} + \boldsymbol{\beta}^{(2)}}{2\hbar}\,,$$

$$\vec{j}_{op}/c = \vec{j}_{op}^{\,\dagger}/c = \boldsymbol{\vartheta}^{(1)}\,\boldsymbol{\vartheta}^{(2)}\,\frac{\boldsymbol{\beta}^{(1)}\,\vec{\boldsymbol{\alpha}}^{(2)} + \boldsymbol{\beta}^{(2)}\,\vec{\boldsymbol{\alpha}}^{(1)}}{2\hbar}\,, \tag{34}$$

or, in tensor notation,

$$j^\mu_{op} = j^{\mu\dagger}_{op} = \boldsymbol{\vartheta}^{(1)}\,\boldsymbol{\vartheta}^{(2)}\,\boldsymbol{\beta}^{(1)}\,\boldsymbol{\beta}^{(2)}\,\frac{\boldsymbol{\gamma}^{(1)\mu} + \boldsymbol{\gamma}^{(2)\mu}}{2\hbar}\,. \tag{34a}$$

These expressions are invariant by a transformation [U.C. (14), (14*a*, *b*)] from the Kramers representation to any other representation of undors. The matrices ρ_{op} and $\vec{j}_{op}/c$ take the place here of the matrices $\boldsymbol{\vartheta}$ and $\boldsymbol{\vartheta}\vec{\boldsymbol{\alpha}}$ (that is, $\boldsymbol{\vartheta\beta\gamma}^\mu$) in the case of the Dirac electron. In analogy to [U.C. (20*b*)] it is convenient to write here (see [U.C. (80)])

$$j^\mu_{mes} = \Psi^+\,\frac{\boldsymbol{\gamma}^{(1)\mu} + \boldsymbol{\gamma}^{(2)\mu}}{2\hbar}\,\Psi. \tag{33a}$$

The main difference between the density matrices of electron and meson is that ρ_{op}, being a singular matrix, cannot be made unity by transformation to any representation. A consequence of this singularity of ρ_{op} is that the meson equation (20) contains so called *identities* between the field components (differential equations not containing

*) Compare L. de Broglie, loc. cit. [13]), page 22. The factor $(1/2\hbar)$ is a consequence of Kemmer's [1]) way of normalizing the meson wave-function and of our choice of the constants in (10). For instance, if the factor 2 in (10) is removed, the factor $(1/2\hbar)$ changes into $(1/8\hbar)$ [12]).

derivatives with respect to the time), viz. the equations in the right hand column of (21).

We can split up (20) into the proper equations of motion and the so-called identities by operating on it by $(1 \pm \beta^{(1)}\beta^{(2)})/2$. Abbreviating again by $X_{k_1k_2} = 2\psi^{\mathfrak{C}}_{N_{k_1}}\psi_{P_{k_2}}$ and splitting up $\Psi_{k_1k_2}$ and $X_{k_1k_2}$ according to

$$\Psi_{k_1k_2} = \Psi^{I}_{k_1k_2} + \Psi^{II}_{k_1k_2},$$
$$\Psi^{I} = \tfrac{1}{2}(1 + \beta^{(1)}\beta^{(2)})\,\Psi, \quad \Psi^{II} = \tfrac{1}{2}(1 - \beta^{(1)}\beta^{(2)})\,\Psi, \tag{35}$$

we find the proper equations of motion of Ψ^{I}:

$$2i\varkappa\,(\Psi^{I} + f_{op}\,X^{I}) + (\overrightarrow{\gamma^{(1)} + \gamma^{(2)}}\,.\,\vec{D})\,\Psi^{II} + (\beta^{(1)} + \beta^{(2)})\,D_0\Psi^{I} = 0, \tag{36}$$

and the so-called identities:

$$2i\varkappa\,(\Psi^{II} + f_{op}\,X^{II}) + (\overrightarrow{\gamma^{(1)} + \gamma^{(2)}}\,.\,\vec{D})\,\Psi^{I} = 0. \tag{37}$$

Comparing (36), (37) with (21) we observe without further calculation that Ψ^{I} represents the field components $\vec{\mathbf{E}}$, $\vec{\mathbf{A}}$, $\mathbf{W}$ and $\mathbf{Y}$, and Ψ^{II} the field components $\mathbf{S}$, $\mathbf{V}$, $\vec{\mathbf{H}}$ and $\vec{\mathbf{B}}$. Only of Ψ^{I} the components can be regarded as canonical variables, whereas the components of Ψ^{II} must be regarded as *derived variables*, defined by the equations (37) (like $\mathfrak{H} = \text{rot}\,\mathfrak{A}$ in quantum-electrodynamics).

If for the present the interaction of mesons with heavy particles is neglected, the meson equation (20) takes the form

$$(2i\varkappa + \Gamma_\mu\,D^\mu)\,\Psi = 0, \quad (\Gamma_\mu = \gamma^{(1)}_\mu + \gamma^{(2)}_\mu). \tag{38}$$

Putting $\gamma^{(1)}_\lambda - \gamma^{(2)}_\lambda \equiv \Gamma^{(-)}_\lambda$ and operating on (38) by $(1/2i\varkappa)\,.\,\Gamma^{(-)}_\lambda D^\lambda$ we find

$$\Gamma^{(-)}_\lambda\,D^\lambda\,\Psi + (1/2i\varkappa)\,\Gamma^{(-)}_\lambda\,\Gamma_\mu\,D^\lambda\,D^\mu\,\Psi = 0. \tag{39}$$

From

$$\gamma^{(n)}_{\{\lambda}\,\gamma^{(n)}_{\mu\}} = \gamma^{(n)}_\lambda\,\gamma^{(n)}_\mu + \gamma^{(n)}_\mu\,\gamma^{(n)}_\lambda = -\,2g_{\lambda\mu} \tag{40}$$

we find

$$\Gamma^{(-)}_{\{\lambda}\,\Gamma_{\mu\}} = 0, \tag{41}$$

so that from (39) follows

$$\Gamma^{(-)}_\lambda\,D^\lambda\,\Psi = (i/8\varkappa)\,.\,(\Gamma^{(-)}_\lambda\,\Gamma_\mu - \Gamma^{(-)}_\mu\,\Gamma_\lambda)\,(D^\lambda\,D^\mu - D^\mu\,D^\lambda)\,\Psi. \tag{42}$$

From the definition (3) of D^μ follows

$$D^\lambda D^\mu - D^\mu D^\lambda \equiv D^{[\lambda} D^{\mu]} = (e/i\hbar c) \,.\, \nabla^{[\lambda} \mathfrak{A}^{\mu]} = (e/i\hbar c) \,.\, \mathfrak{H}^{\lambda\mu}, \quad (43)$$

where $\mathfrak{H}^{\lambda\mu}$ denotes the Maxwellian field. The equation (42) can therefore be written in the following form:

$$\Gamma^{(-)}_\lambda D^\lambda \Psi = (e/8\varkappa\hbar c) \,.\, \mathfrak{H}^{\lambda\mu} \Gamma^{(-)}_{[\lambda} \Gamma_{\mu]} \Psi = (e/4\mathrm{m}c^2) \,.\, \mathfrak{H}^{\lambda\mu} \Gamma^{(-)}_\lambda \Gamma_\mu \Psi. \quad (44)$$

Adding (44) to (38) we find the following "equation of motion" for Ψ, from which $\partial\Psi/\partial t$ can be solved by multiplication by $\boldsymbol{\beta}^{(1)}$:

$$(i\varkappa + \gamma^{(1)}_\lambda D^\lambda) \Psi = (e/8\mathrm{m}c^2) \,.\, \mathfrak{H}^{\lambda\mu} \Gamma^{(-)}_\lambda \Gamma_\mu \Psi. \quad (45)$$

The left hand member has the form of a "Dirac equation" for the first index k_1 of the undor Ψ.

If this equation is iterated like the ordinary Dirac equation, we find (compare (40) and (43)):

$$(e/8\mathrm{m}c^2) \,.\, (-\, i\varkappa + \gamma^{(1)}_\rho D^\rho) (\mathfrak{H}^{\lambda\mu} \Gamma^{(-)}_\lambda \Gamma_\mu \Psi) =$$
$$= \{\varkappa^2 + \tfrac{1}{4} \gamma^{(1)}_{\{\lambda} \gamma^{(1)}_{\mu\}} D^{\{\lambda} D^{\mu\}} + \tfrac{1}{4} \gamma^{(1)}_{[\lambda} \gamma^{(1)}_{\mu]} D^{[\lambda} D^{\mu]}\} \Psi =$$
$$= \{(\mathrm{m}^2c^2/\hbar^2) - D_\lambda D^\lambda - \tfrac{1}{2} i\sigma^{(1)}_{\lambda\mu} \,.\, (e/i\hbar c)\, \mathfrak{H}^{\lambda\mu}\} \Psi, \quad (46)$$

where we have put

$$\gamma_{[\lambda} \gamma_{\mu]} = -\, 2i\sigma_{\lambda\mu}. \quad (47)$$

If only a magnetic field is present ($\vec{\mathfrak{E}} = 0$), we have

$$\tfrac{1}{2} \sigma^{(n)}_{\lambda\mu} \mathfrak{H}^{\lambda\mu} \equiv (\vec{\sigma}^{(n)} \,.\, \vec{\mathfrak{H}}) - i(\vec{\alpha}^{(n)} \,.\, \vec{\mathfrak{E}}) = (\vec{\mathfrak{H}} \,.\, \vec{\sigma}^{(n)}). \quad (48)$$

Adding to (46) the corresponding equation with $\sigma^{(2)}_{\lambda\mu}$ (where in the left hand member $\Gamma^{(-)}_\lambda$ occurs with the opposite sign), we find after multiplication by $\hbar^2/2$

$$\{\mathrm{m}^2c^2 + p_\lambda p^\lambda - (e\hbar/2c) \,.\, (\vec{\mathfrak{H}} \,.\, \overrightarrow{\sigma^{(1)} + \sigma^{(2)}})\} \Psi =$$
$$= (e\hbar^2/16\mathrm{m}c^2) \,.\, \Gamma^{(-)}_\rho D^\rho (\mathfrak{H}^{\lambda\mu} \Gamma^{(-)}_\lambda \Gamma_\mu \Psi), \quad (49)$$

where

$$p_\lambda = (\hbar/i) \,.\, D_\lambda \quad (50)$$

is the operator of the kinetic momentum. In non-relativistic approximation we put

$$cp^0 = \mathrm{m}c^2 + T = -\, cp_0, \quad (51)$$

so that T is the operator of the non-relativistic kinetic energy. If, further, according to Yukawa [3]), the right hand member

$\Gamma_{\rho}^{(-)}\Gamma_{\lambda}^{(-)}\Gamma_{\mu}D^{\rho}(\mathfrak{H}^{\lambda\mu}\Psi).(e\hbar^2/16\mathfrak{m}c^2)$ of (49) is neglected in non-relativistic approximation, this equation can be written as a Schrödinger equation ($T \ll \mathfrak{m}c^2$):

$$(\mathcal{E}_{op}^{non\text{-}rel} - e\mathfrak{B})\,\Psi \equiv T\Psi =$$

$$= \{p_{op}^2/2\mathfrak{m} - (e\hbar/2\,\mathfrak{m}c)\,.\,\tfrac{1}{2}\,(\overrightarrow{\overline{\sigma^{(1)} + \sigma^{(2)}}}\,.\,\overrightarrow{\mathfrak{H}})\}\,\Psi, \quad (52)$$

so that the magnetic moment of the meson

$$\overrightarrow{\mu}_{op} = (e/2\mathfrak{m}c)\,.\,\tfrac{1}{2}\,\hbar\,(\overrightarrow{\overline{\sigma^{(1)} + \sigma^{(2)}}}) \equiv (e/2\mathfrak{m}c)\,.\,\overrightarrow{S}_{op}^{\{1\}} \quad (53)$$

is $(e/2\mathfrak{m}c)$ times its spin angular momentum [9] $\overrightarrow{S}_{op}^{\{1\}}$.

Since the energy is given by [9] $\mathcal{E} = \int \Psi^{\dagger}\,\rho_{op}\,\mathcal{E}_{op}\,\Psi$, the *non-relativistic value* of the magnetic moment is actually given by

$$\overrightarrow{\mu} = \int \Psi^{\dagger}\,\rho_{op}\,\overrightarrow{\mu}_{op}\,\Psi. \quad (54)$$

As the *value* of the spin angular momentum is given in the same way by [9] $\overrightarrow{S} = \int \Psi^{\dagger}\,\rho_{op}\,\overrightarrow{S}_{op}^{\{1\}}\,\Psi$, the statement

$$\overrightarrow{\mu} = (e/2\mathfrak{m}c)\,.\,\overrightarrow{S} \quad (55)$$

holds for the values of these quantities as well as for the operators occurring in (53).

§ 5. *Charge-invariance and statistics.* It is well-known that in the hole theory of electrons (superquantized theory of the Dirac electron) there is an infinite c-number difference between the q-number $e\psi^{\dagger}\psi$ (obtained by superquantization of the wave-function ψ from the expression for the electric charge density $e\psi^{\dagger}\psi$ following in the usual way from the Lagrangian of unquantized wave-mechanics) and the q-number representing the *correct* (observable) electric charge density. If the meson field is quantized, (33) must also be corrected by addition of infinite c-numbers.

We have mentioned that to one description of Dirac particles, mesons, neutrettos and the electromagnetic field by undor wave-functions [U.C. (1)], (20), (22) there is an equivalent *charge-conjugated description*, in which some constants like e, f and g are replaced by e^L, f^L and g^L (32), (32*a*), whereas every quantized undor *) is replaced by its charge-conjugated [U.C. (30)], (29). This suggests a kind of

*) We assume that all fields are described by undors (reflection $\psi' = i\beta\psi$, [U.C.(9), (34)]) and *not by "quasi-undors"* (reflection $\psi' = \beta\psi$).

symmetry between both ways of describing physical situations *). By way of hypothesis one might assume that such a symmetry is a *fundamental property of nature*. We shall call this possible property the "*charge-invariance*" of the physical world (not to be confused, however, with the principle of conservation of electric charge!).

Therefore we shall *postulate* that every physically significant quantity in quantum-mechanics (that is, every q-number *correctly* representing the value of an *observable*) is invariant by transition from one description of the fields of wave-functions to the charge-conjugated description, or, in shorter terms, is *charge-invariant*.

This postulate can serve to distinguish between wave-mechanical expressions, which after quantization cannot have a physical meaning any longer, and other analogous expressions, which may represent observables. For the present we shall leave *this* question out of consideration, but we shall show here that the postulate of charge-invariance implies directly that photons and neutrettos *must be* neutral, that Dirac electrons *must* obey Fermi-Dirac statistics and that mesons *must* obey Einstein-Bose statistics. The interesting fact is that this statistical behaviour of particles and quanta follows much more directly from the postulate of charge-invariance then from postulates concerning the positive character of the total energy of free particles or quanta †).

From the Lagrangian of any kind of particles or quanta we can always deduce expressions for the electric charge density, the electric charge current, the total momentum and total energy of these corpuscles.

The terms of the Lagrangian function depending on the derivatives of the field quantities Ψ have always §) the form of [9])

$$iK\,\Psi^{\dagger}\,B\,\Gamma_{\mu}\,\nabla^{\mu}\,\Psi. \tag{56}$$

If Ψ is an undor **) $\Psi_{k_1k_2\ldots k_N}$ of rank N, then [U.C. (12)]:

$$B = B^{\dagger} = \prod_{n=1}^{N} \beta^{(n)};\ \Gamma_{\mu} = \sum_{n=1}^{N} \varepsilon_n \gamma^{(n)},\ (\varepsilon_n = \pm 1);\ B^{*}\Gamma^{*} = \Gamma^{\infty} B^{\infty}; \tag{57}$$

so that, if we put

$$\Psi^{\mathfrak{C}} = \mathfrak{C}\,\Psi^{*},\quad \mathfrak{C} = \mathfrak{C}^{\infty} = \prod_{n=1}^{N} \mathfrak{C}^{(n)};\ \ \mathfrak{C}^{*}\,\mathfrak{C} = 1, \tag{58}$$

*) Compare H. A. Kramers, loc. cit. [10]).

†) Compare for instance H. A. Kramers [14]) and M. Fierz [15]).

§) This and the following considerations apply *at least* to all particles and quanta discussed by Dirac and Fierz [15]) and by Kemmer [1]).

**) In the following, we confine ourselves to representations, for which $\vartheta = 1$ (compare [U.C. (17), (17a–b)]).

we have

$$\mathrm{B}\,\mathfrak{L} = (-1)^N\,\mathfrak{L}\,\mathrm{B}^*, \qquad \Gamma\,\mathfrak{L} = -\,\mathfrak{L}\,\Gamma^*. \tag{59}$$

From (56) we find that the electric charge density, if it exists, is equal to

$$e\rho = (e\mathrm{K}/\hbar c)\,.\,\Psi^{\cdot\dagger}\,\mathrm{B}\,\Gamma^0\,\Psi^{\cdot},\ (-\text{ infinite } c\text{-number}). \tag{60}$$

In the charge-conjugated description this expression is turned, on account of (32), into

$$e^L\rho^L = (-\,e\mathrm{K}/\hbar c)\,.\,\Psi^{\cdot\mathfrak{L}\dagger}\,\mathrm{B}\,\Gamma^0\,\Psi^{\cdot\mathfrak{L}}\ (-\text{ infinite } c\text{-number}), \tag{61}$$

therefore, on account of (57)—(59):

$$e^L\rho^L = (-1)^N\,.\,(e\mathrm{K}/\hbar c)\,.\,\Psi^{\cdot\sim\sim}\Gamma^{0\sim\sim}\mathrm{B}^{\sim\sim}\Psi^{\cdot *}\ (-\text{ infinite } c\text{-number}). \tag{62}$$

If the expressions (60) and (62) for the electric charge density are postulated to be equal, the components of the wave-functions $\Psi^{\cdot}$ and $\Psi^{\cdot *}$ occurring in (60) and (62) must be *commutative* (apart from an infinite c-number term) if N is *even*, and must be *anti-commutative* if N is *odd*.

It is not true, of course, that the commutation rules follow *rigorously* from

$$e\rho = e^L\rho^L, \tag{63}$$

since in (63) the sum is taken over the undor indices, and only the wave-functions $\Psi^{\cdot}$ and $\Psi^{\cdot *}$ in *one and the same* point of space are multiplied with each other. In this case the δ-function appearing in the commutation rules becomes infinite; its value corresponds formally to the sum or the difference of the two infinite c-numbers in (60) and (62). Since the infinite c-number in (62) must be the charge-conjugated analogon of the infinite c-number in (60), this may be of some help in the "evaluating" of such infinite c-numbers.

For photons and neutrettos it follows from (23) and from the symmetry of the operator ρ_{op} with respect to both undor indices, on which it operates, that

$$\rho \equiv \Psi^{\cdot\dagger}\,\rho_{op}\,\Psi^{\cdot} = \Psi^{\cdot\mathfrak{Q}\dagger}\,\rho_{op}\,\Psi^{\cdot\mathfrak{Q}} = \Psi^{\cdot\mathfrak{L}\dagger}\,\rho_{op}\,\Psi^{\cdot\mathfrak{L}} \equiv \rho^L. \tag{64}$$

On the other hand we find from (63) and (32) for any particles or quanta

$$\rho = -\,\rho^L. \tag{65}$$

Comparing (64) with (65) we conclude that the electric charge density of the fields of neutrettos and photons must vanish, if it is a charge-invariant expression. In a similar way we derive that by means of neutrettors of the first rank only neutral patricles [16]) can be described. It does not follow from this, however, that neutral particles should necessarily be described by neutrettors!

For electrons we deduce from (65) that

$$\rho = \psi^\dagger \psi - C \tag{66}$$

must be opposite equal to

$$\rho^L = \psi^{\mathfrak{L}\dagger} \psi^{\mathfrak{L}} - C^L, \tag{66a}$$

where C^L takes the place of the infinite c-number C in the charge-conjugated description. From this we deduce

$$\psi^\dagger \psi + \psi^{\mathfrak{L}\dagger} \psi^{\mathfrak{L}} = \psi^\dagger \psi + \psi^{\infty} \psi^* = \sum_k \{(\psi_k)^* \psi_k + \psi_k (\psi_k)^*\} = \\ = C + C^L = c\text{-number}. \tag{67}$$

Similar relations can be deduced by postulating the charge-invariance of the quantized expressions for electric charge current, total momentum and total energy. For instance, from

$$\int \psi^\dagger \frac{\hbar}{i} \vec{\nabla} \psi - C = \int \psi^{\mathfrak{L}\dagger} \frac{\hbar}{i} \vec{\nabla} \psi^{\mathfrak{L}} - C^L = \int \psi^{\infty} \frac{\hbar}{i} \vec{\nabla} \psi^* - C^L = \\ = - \int \left(\frac{\hbar}{i} \vec{\nabla} \psi^{\infty} \right) \psi^* - C^L$$

follows

$$\int \{\psi^\dagger (\vec{\nabla} \psi) + (\vec{\nabla} \psi)^{\infty} \psi^*\} = \int \sum_k \{(\psi_k)^* (\vec{\nabla} \psi_k) + (\vec{\nabla} \psi_k) (\psi_k)^*\} = \\ = \frac{i}{\hbar} (C - C^L) = c\text{-number}. \tag{68}$$

It is obvious that relations like (67) and (68) are consistent with the anticommutativity relations of F e r m i-D i r a c statistics

$$\psi_k(x)^* \psi_{k'}(x') + \psi_{k'}(x') \psi_k(x)^* = \delta_{kk'} \delta(x - x'), \tag{69}$$

but not with E i n s t e i n-B o s e statistics.

In a similar way we find for mesons from

$$\rho = \Psi^\dagger \frac{\beta^{(1)} + \beta^{(2)}}{2\hbar} \Psi - C \tag{70}$$

and

$$\rho^L = \Psi^{\mathfrak{L}\dagger} \frac{\beta^{(1)} + \beta^{(2)}}{2\hbar} \Psi^{\mathfrak{L}} - C^L \tag{70a}$$

that

$$\Psi^{\dagger}\,\frac{\beta^{(1)}+\beta^{(2)}}{2h}\,\Psi+\Psi^{\mathfrak{L}\dagger}\,\frac{\beta^{(1)}+\beta^{(2)}}{2h}\,\Psi^{\mathfrak{L}}=C+C^{L}. \qquad (71)$$

Applying (57)—(59) we find

$$\Psi^{\dagger}\left(\frac{\beta^{(1)}+\beta^{(2)}}{2h}\,\Psi\right)-\left(\frac{\beta^{(1)}+\beta^{(2)}}{2h}\,\Psi\right)^{\backsim}\Psi^{*}=C+C^{L}, \qquad (72)$$

or, on account of (35),

$$\Psi^{I\dagger}\,(\rho_{op}\,\Psi^{I})-(\rho_{op}\,\Psi^{I})^{\backsim}\,\Psi^{I*}=C+C^{L}=c\text{-number}. \qquad (73)$$

Again, in (73) the sum must be taken with respect to the undor indices, as in (67) and (68). It is obvious that (73) is consistent with Einstein-Bose commutativity relations between the components of Ψ^{I} and Ψ^{I*}:

$$\Psi^{I}_{k_1k_2}(x)^{*}\,\Psi^{I}_{k'_1k'_2}(x')-\Psi^{I}_{k'_1k'_2}(x')\,\Psi^{I}_{k_1k_2}(x)^{*}=c\text{-number}, \qquad (74)$$

and not with Fermi-Dirac anticommutativity relations.

The commutation rules for the components of Ψ^{II} must be derived from those for Ψ^{I} by means of the so-called identities (37), so that it is not very alarming that we do not find any indication of them from (71)—(74).

For neutral particles, indication of the commutation rules can be derived in this way from the expressions for the total momentum and the total energy, which are also obtained directly from the Lagrangian. Generally we can postulate that the *total Lagrangian itself* (integrated over space and time) shall be charge-invariant on account of the commutation rules of the field components. It is therefore *not necessary to investigate the sign of the energy* in order to derive the statistical behaviour of the corpuscles concerned [14]) [15]).

It is true, however, that charge-invariance of the quantized expression for the total energy *implies* that by quantization according to the scheme of Pauli and Weisskopf [17]) the so-called "states of negative energy" of free corpuscles (depending on the time by a factor $e^{+2\pi i\nu t}$) can be interpreted, on account of the commutation relations (which do not need specification here!), as states of positive energy *) of corpuscles with opposite electric charge. We can understand this in the following way. By charge-conjugation of the quantized wave-function these states pass into charge-conjugated states of positive energy. If, now, the expression for the total energy is charge-invariant on account of the (un-

*) For the corpuscles under consideration states with $e^{-2\pi i\nu t}$ are of positive energy.

specified) commutation rules of the q-number amplitudes $\boldsymbol{a}$ (Jordan-Wigner or Jordan-Klein matrices), the terms in this expression arising from the so-called states of negative energy are automatically equal to the terms in the charge-conjugated expression arising there from states of *positive* energy of the *charge-conjugated* corpuscles (which are described with the help of the charge-conjugated q-number amplitudes $\boldsymbol{b} = \boldsymbol{a}^*$). Using the latter (charge-conjugated) expression for the description of these terms in the total energy, the energy is given as a sum of only positive energies with amplitudes $\boldsymbol{a}^*\boldsymbol{a}$ or $\boldsymbol{b}^*\boldsymbol{b}$.

We observe that both the statistical behaviour of corpuscles and the possibility of describing so-called states of negative energy (of free corpuscles) as states of positive energy of charge-conjugated corpuscles follow directly from the postulate of charge-invariance of quantum-mechanical theories. The relation between the positive character of the energy of free corpuscles and the charge-invariance of energy seems to be still closer than that between charge-invariance and statistics.

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THE HEAVY QUANTA THEORY OF NUCLEAR AND COSMIC RAY PHENOMENA

§ 1. *Introduction.* In 1935 Yukawa suggested in a paper in the Proceedings of the Physico-Mathematical Society of Japan [1]) that the exchange forces between heavy particles (protons and neutrons) must be attributed to the action of an intervening field and, in particular, may be regarded as a second order effect due to the consecutive emission and absorption of charged "*heavy quanta*", just as the electromagnetic interaction between two charged particles can be described by the quantized electromagnetic field. In order to explain the range of about 2×10^{-13} cm of the nuclear forces, he assumed that this quantum had a mass about 200 times as large as the electron mass: $\mathfrak{m} \approx 200\, m$. In his original theory this field was tentatively regarded as a scalar field. Then, however, it turned out [2]) that, if the energy of the field of heavy quanta was assumed to be positive, the exchange force between a proton and a neutron became repulsive in a ^3S state, in contradiction to the fact that this is the ground state of the deuteron. Since in the mean time experiments on cosmic ray phenomena [3]) [4]) [5]) [6]) had suggested the existence of a charged particle just having a mass of the order of magnitude 200 m, which might be identified with the quantum of Yukawa's theory [7]), Yukawa expressed his intention to investigate whether this difficulty with the sign of the proton-neutron force could be removed by introducing a non-scalar heavy quantum field [2]).

Then, from 1937 on, the non-scalar theory of heavy quanta was gradually developed. In January 1938 its main ideas and applications were announced by Kemmer [8]) and Bhabha [9]) in Letters to the Editor in Nature. It is this theory, partly in a generalized form [10]) [11]) [12]), which will be subject of the present dissertation.

As a name for the heavy quantum of nuclear physics were suggested "heavy quantum", "U-particle", "yukon", "dynaton", "baryt(e)ron", and for the particle composing the penetrating component of the cosmic rays: "heavy electron", "penetron" and "mesot(r)on". The last name was shortened afterwards to the more correct form "meson" [13] [14]. Though it seems to the author that the correct name for a particle of *intermediate* mass should not be "meson", but "*metrion*", and though from a theoretical point of view the meson is no *intermediate electron*, but only a *heavy* *) *quantum*, (that is, a heavy Einstein-Bose particle), the name "meson" seems to be already generally adopted, so that we shall use it in the following.

§ 2. *The four types of meson fields proposed by Kemmer and the simplified deuteron problem.* In an important paper in the Proceedings of the Royal Society of London [10] Kemmer has discussed four different types (*a*, *b*, *c* and *d*) of a heavy quantum field satisfying the Klein-Gordon equation

$$(\Box - \varkappa^2)\Psi = 0, \; (\Box \equiv \Delta - \partial^2/c^2\partial t^2 \equiv \nabla_\mu \nabla^\mu; \; \varkappa \equiv mc/\hbar), \qquad (1)$$

if all interactions with other fields are neglected. In all four cases the field consists of an antisymmetrical tensor of rank n, the potentials, and another antisymmetric tensor, of rank $(n+1)$, the field strengths. In the absence of other fields interacting with the heavy quantum field, the field strengths are the (generalized) curl of the potentials (like in the Maxwellian theory), and the potentials are the four-dimensional divergences of the field strengths (unlike the theory of the electromagnetic field).

In Kemmer's cases *a*, *b*, *c* and *d* the number n is equal to 0, 1, 2 and 3 respectively. Case (*a*) is identical with the field of quanta discussed by Pauli and Weisskopf [15] and used by Yukawa in his original papers [2]. Case (*b*) is identical with the field discussed by Proca [16] and quantized by Durandin and Erschow [17], Kemmer [10], Bhabha [18] and others [19] following the same procedure of Pauli and Weisskopf [15]. The properties of the field of case (*c*) differ from those of the Proca field (*b*) only with respect to the laws of transformation of the field components by a spatial reflection. As a consequence of these different transformation laws, however, the interaction of the heavy quanta with the heavy particles (the proton-neutron, or "*nuclon*", as we

*) „*Baryteron*" = heavier.

shall call it briefly) must be introduced in a different way. Similarly, case (*d*) differs from (*a*) only with respect to the reflection, as long as no interactions are taken into account.

The interactions of these four types of heavy quanta with heavy particles (nuclons) are then introduced by adding to the Lagrangian function scalar terms, in which no derivatives appear; they contain only inner products of the wave-functions of protons, neutrons and heavy quanta. (We have called this a "*Fermi-Ansatz*" in the following). The coefficients of the terms, in which the *potentials* φ of the heavy-quantum fields occur, are called g_a, g_b, g_c, g_d in the four cases, whereas the coefficients f_a, f_b, f_c and f_d appear in the interaction terms containing the *field strengths* ζ. The field equations then take the following form:

$$\begin{aligned} \varkappa(\zeta_{\lambda_0\lambda_1\ldots\lambda_n} - f u_{\lambda_0\lambda_1\ldots\lambda_n}) &= (1/n!)\,\nabla_{[\lambda_0}\varphi_{\lambda_1\ldots\lambda_n]}, \\ \varkappa(\varphi_{\lambda_1\ldots\lambda_n} + g u_{\lambda_1\ldots\lambda_n}) &= \nabla^{\lambda_0}\zeta_{\lambda_0\lambda_1\ldots\lambda_n}. \end{aligned} \tag{2}$$

Here $[\lambda_0\lambda_1\ldots\lambda_n]$ denotes the sum over all even permutations of $\lambda_0\lambda_1\ldots\lambda_n$ minus the sum over all odd permutations; the $u_{\lambda\ldots}$ are linear combinations of the products $\psi_N^*\psi_P$ of the components of the wave-function ψ_P of protons and the conjugate complex ψ_N^* of the wave-function of neutrons.

From (2) we conclude that in Kemmer's theory φ *and* ζ *play an equivalent part*, contrary to the Maxwellian theory, where the potentials cannot be derived from the field strengths and are not uniquely determined by them (possibility of a gauge transformation). In the original paper of Bhabha [9]) the interaction described by the term with f did not appear.

Kemmer calculated the proton-neutron force in each of the cases (*a*), (*b*), (*c*) and (*d*), and found in non-relativistic approximation the following expression for the effective potential $W(1, 2)$ describing the second order *interaction between two nuclons* 1 and 2 through the medium of the field of (charged) mesons [10]):

$$W(1,2) = \tfrac{1}{2}(\boldsymbol{\tau}_x^{(1)}\boldsymbol{\tau}_x^{(2)} + \boldsymbol{\tau}_y^{(1)}\boldsymbol{\tau}_y^{(2)})\{A + B(\vec{\boldsymbol{\sigma}}^{(1)}\cdot\vec{\boldsymbol{\sigma}}^{(2)}) + \\ + (C/\varkappa^2)(\vec{\boldsymbol{\sigma}}^{(1)}\cdot\vec{\nabla}_1)(\vec{\boldsymbol{\sigma}}^{(2)}\cdot\vec{\nabla}_1)\}\,(e^{-\kappa r_{12}}/r_{12}). \tag{3}$$

Here $\boldsymbol{\tau}_x^{(n)}$, $\boldsymbol{\tau}_y^{(n)}$, $\boldsymbol{\tau}_z^{(n)}$ are the *isotopic spin* operators operating on the wave-function of the n^{th} particle; the meaning of the suffixes x, y, z

is only that these operators have the same form as the Pauli matrices [20]) $\boldsymbol{\sigma}_x$, $\boldsymbol{\sigma}_y$, $\boldsymbol{\sigma}_z$; they have nothing to do with the co-ordinates of space. We assume that ψ_P and ψ_N are eigenfunctions of $\boldsymbol{\tau}_z$ belonging to the eigenvalues $\tau = +1$ and $\tau = -1$ respectively.

The constants A, B and C in (3) can be expressed in terms of the coefficients f and g occurring in the meson equations (2). If all types of meson fields are present, in interaction with the nuclons, these coefficients turn out to be equal to [10])

$$\begin{aligned} A &= (c\varkappa^3/4\pi)\,(-\,|g_a|^2 + |g_b|^2), \\ B &= (c\varkappa^3/4\pi)\,(|f_b|^2 - |f_c|^2), \\ C &= (c\varkappa^3/4\pi)\,(-\,|f_b|^2 + |f_c|^2 - |g_c|^2 + |g_d|^2). \end{aligned} \tag{4}$$

It must be pointed out that in (3) some *interaction-potentials of the form of a δ-function* have been omitted (compare § 7). In the literature it is often tried [18]) [21]) to eliminate these δ-function interactions by adding to the Lagrangian some terms, which give rise to a first order interaction between protons and neutrons. It is of interest to remark that, though by these attempts the δ-functions arising in the calculation from

$$\Delta(1/r) = -\,4\pi\,\delta(\vec{r}) \equiv -\,4\pi\,\delta(x)\,\delta(y)\,\delta(z) \tag{5}$$

are taken into account, those arising from *)

$$\begin{aligned} (\vec{A}\,.\,\vec{\nabla})\,(\vec{B}\,.\,\vec{\nabla})\,(1/r) &= -\,4\pi \sum_{i,j} A_i B_j\, \delta_{ij}^{long}(\vec{r}) = \\ &= -\tfrac{4\pi}{3}(\vec{A}\cdot\vec{B})\,\delta(\vec{r}) + r^{-5}\{3(\vec{A}\,.\,\vec{r})\,(\vec{B}\,.\,\vec{r}) - (\vec{A}\,.\,\vec{B})r^2\} \end{aligned} \tag{5a}$$

have been forgotten [18]) [21]) (compare § 7). For the present problem these δ-function interactions are of little importance [22]). Though similar terms in the Lagrangian (which for the sake of simplicity can be introduced in exactly the same form as the terms yielding a direct (δ-function) interaction between nuclons) are of importance for the *theory of β-disintegration*, only the terms with ordinary δ-functions (5) give rise to a direct (first order) Fermi-interaction between the nuclon field and the field of light particles [23]), whereas the terms of the form of (5*a*) can be neglected there (§ 11).

The terms with A and B in the expression (3) for the effective second order potential between two nuclons are commutative with

*) For the definition of the longitudinal δ-function $\delta_{ij}^{long}(r)$ we refer to the foot-note on page 72. Generally: $\nabla_i \nabla_j (1/r) = -\,4\pi\,\delta_{ij}^{long}(\vec{r}) = r^{-5}(3x_i x_j - r^2 \delta_{ij}) - (4\pi/3)\,\delta_{ij}\,\delta(\vec{r})$

independent rotations of the spatial and the spin co-ordinates, so that the energy levels of the two-particle problem can be characterized by *quantum numbers* l, s, j, m_j. The last term of (3) (with C), however, is commutative only with a *simultaneous* rotation of spatial and spin co-ordinates, so that this term will give rise to a coupling of states with different quantum number l. As (3) is invariant with respect to permutation of the ordinary spin operators, states with different quantum number s are not coupled. Finally (3) is invariant with respect to permutation of the isotopic spin operators, so that the ^{1}S, ^{3}P, ^{1}D, ^{3}F .... states, which are antisymmetrical functions of the spatial and spin co-ordinates and thus symmetrical functions of the isotopic spin co-ordinates, are not coupled with the ^{3}S, ^{1}P, ^{3}D, ^{1}F .... states, which are just antisymmetric in the isotopic spin co-ordinates.

We conclude that the ^{1}S state of the deuteron is not coupled with any other state by the term with C, but that the ^{3}S state is coupled with the $^3D_1$ state. We shall discuss this question afterwards and, to start with, we shall with Fröhlich, Heitler and Kemmer[19]) *neglect this coupling*. Then we can write (3) in the form (compare § 8)

$$W^{(S)}(1,2) = \tfrac{1}{2}(\tau_x^{(1)}\tau_x^{(2)} + \tau_y^{(1)}\tau_y^{(2)})\{A + B'(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)})\}(e^{-\kappa r}/r), \quad B' = B + \tfrac{1}{3}C. \tag{6}$$

The effective potentials for the ^{3}S and the ^{1}S state of the deuteron are now given by

$$^3\text{S}: \quad W(r) = -(A + B')e^{-\kappa r}/r, \qquad ^1\text{S}: \quad W(r) = +(A - 3B')e^{-\kappa r}/r. \tag{7}$$

The Schrödinger equation of the simplified deuteron problem is given by

$$\{\mathcal{E} + (\hbar^2/M)\Delta + \mathcal{K}e^{-\kappa r}/r\}\psi = 0; \quad (\varkappa = mc/\hbar). \tag{8}$$

Here

$$M = 2M_N M_P/(M_N + M_P) = 1{\cdot}672_3 \times 10^{-24}. \tag{9}$$

The eigenvalue problem (8) was numerically solved by Wilson[24]) and by Sachs and Goeppert-Mayer[25]). The former calculated

$$b = (\mathcal{K}M/\hbar^2\varkappa)$$

as a function of

$$a = (-\mathcal{E}M/\hbar^2\varkappa^2);$$

the latter calculated

$$a/b = (-\,\mathcal{E}/\mathcal{K}\varkappa)$$

as a function of b. The results of W i l s o n can be represented to a good approximation by

$$a = 0{\cdot}23_6 \times (b - 1{\cdot}70)^2, \qquad (10.\mathrm{W})$$

those of S a c h s and G o e p p e r t-M a y e r by

$$a = 0{\cdot}19_{05} \times (b - 1{\cdot}683)^2,\ (b \leq 2{\cdot}70), \qquad (10.\mathrm{S\text{-}GM})$$

The actual value of b for the ^{3}S ground state of the deuteron is in the neighbourhood of 2·8. We remark that (10.W) and (10.S-GM) do not fit exactly in this region.

We shall put

$$\mathfrak{m} = 100\,\xi m, \quad (\xi \approx 1{\cdot}75 \pm {\cdot}25?) \qquad (11)$$

and calculate $\mathcal{K}/\hbar c$ as a function of ξ. Writing (10.W) and (10.S-GM) in the form

$$a = a_0(b - b_0)^2, \qquad (10)$$

we can express b as a function of ξ by

$$b = b_0 + (1\,/\,100\,\xi m c)\,.\,\sqrt{-\,\mathcal{E}M/a_0}.$$

From the definitions of b and $\varkappa$ we now deduce

$$\mathcal{K}/\hbar c = 100\, b_0 (m/M)\,\xi + \sqrt{-\,\mathcal{E}/Mc^2 a_0}.$$

Putting here $M = 1{\cdot}672_3 \times 10^{-24}$, $m = 0{\cdot}909_4 \times 10^{-27}$, $c = 2{\cdot}9979_6 \times$ $\times\ 10^{10}$, and for the triplet ground state of the deuteron $-\,^3\mathcal{E} =$ $(= 2{\cdot}17_4\ \mathrm{MeV}) = 3{\cdot}45_6 \times 10^{-6}$ (erg), we find for the coefficient $\mathcal{K}$ in the potential of the ^{3}S state:

$$^3\mathcal{K}/\hbar c = 0{\cdot}0543_8\,.\,b_0\xi + 0{\cdot}0479_5/\sqrt{a_0}. \qquad (12)$$

Substituting for a_0 and b_0 the values from (10.W) and (10.S-GM) we find

$$^3\mathcal{K}/\hbar c = 0{\cdot}092_4\xi + 0{\cdot}098_7 = 0{\cdot}092_4(\xi + 1{\cdot}06_8) \qquad (12.\mathrm{W})$$

and

$$^3\mathcal{K}/\hbar c = 0{\cdot}091_5\xi + 0{\cdot}110 = 0{\cdot}091_5(\xi + 1{\cdot}20). \quad (12.\mathrm{S\text{-}GM})$$

So the difference between (10.W) and (10.S-GM) is equivalent with an incertitude of a little more then ten electron masses in the mass $\mathfrak{m}$ of the meson.

From experiments on scattering of neutrons by protons we know that the virtual ^1S level has an energy, which is small ($\approx$ 0·05 MeV) in comparison with the binding energy of the deuteron ($= 2{\cdot}17_4$ MeV). We conclude that a has for the ^1S level only about 1/40 of the value, which it has for the ^3S level, and the opposite sign. Now, the virtual levels of the Schrödinger problem (8) have not yet been determined. If we tentatively assume that for $a < 0$ (virtual levels) an equation of the same kind as (10) is valid:

$$|a| \approx 0{\cdot}2 \times (b - 1{\cdot}68)^2, \qquad (10a)$$

we find for the ^1S level $b \approx 1{\cdot}5_2$. In the literature the value of b for the ^1S level is usually supposed to be equal to $b_0 \approx 1{\cdot}68$, that is, the energy of the ^1S level is *entirely neglected* [24]) [25]). This may be a little dangerous, since for virtual just as for real levels, $(b - b_0)$ may be very sensitive for the exact value of a, if the latter lies in the neighbourhood of zero. This, indeed, would follow from (10*a*). From this formula one finds (putting $^1\mathcal{E} \approx 0{\cdot}05$ MeV):

$$^1\mathcal{K}/\hbar c = 100\, b_0\, (m/M)\, \xi - \sqrt{^1\mathcal{E}/Mc^2 a_0} \approx 0{\cdot}09_2\, \xi - 0{\cdot}01_6. \qquad (13)$$

Neglecting $^1\mathcal{E}$ entirely one finds:

$$^1\mathcal{K}/\hbar c \approx 0{\cdot}09_2\, \xi. \qquad (13a)$$

If $\xi \approx 1{\cdot}75$, the difference between (12.W) and (12.S-GM) is about $3\frac{1}{2}\%$ and that between (13) and (13*a*) is as much as 10%.

Comparing (13) with (12) we find

$$(^3\mathcal{K} - {}^1\mathcal{K})/\hbar c \approx 0{\cdot}11_5, \quad (13.\mathrm{W}); \qquad \approx 0{\cdot}12_6, \quad (13.\mathrm{S\text{-}GM}),$$

if (10*a*) is valid. Here we have put $^1\mathcal{E} \approx 0{\cdot}05$ MeV again.

Putting $\xi \approx 1{\cdot}75$ in (12) and (13), we find from (7):

$$A/\hbar c \approx 0{\cdot}16, \qquad B'/\hbar c \approx 0{\cdot}10.$$

From (4) and (6) we see that among Kemmer's cases (*a*) — (*d*) only (*b*) offers the possibility of making alone both A and B' positive. This was the reason why Kemmer and most other authors have investigated this case in more detail, that is, they consider the meson field as a *pure Proca field*. Of course it is also possible to consider the meson field as a composition of several cases. The advantages of doing so [11]) will be discussed afterwards (§ 8).

We remark that the scalar field, originally discussed by Yukawa [2]) (case (*a*)), gives the wrong sign for A (compare (4)).

§ 3. *The charge-dependence of nuclear forces.* In a theory of charged mesons the proton-proton force and the neutron-neutron force are obtained only in a *fourth approximation*. For a field of Proca-mesons the calculation was performed by Fröhlich, Heitler and Kemmer [19]). The effective potential in the ^1S state turns out to be *repulsive* and very strong for $r < 1/2\varkappa$. The range is smaller than that of the second order forces of the preceding section (§ 2). A similar short-range strong repulsion is found between a neutron and a proton. This indicates that the theory does not allow to determine the exact form of the effective potential between nucleons for small values of r, by taking into account a finite number of successive approximations yielded by the perturbation method.

Experimental data on the scattering of protons by protons [26]) [27]) can be explained very well [28]), if one assumes that the proton-proton ^1S potential, in as far as not of electromagnetic origin, is (within 1%) equal to the proton-neutron ^1S potential [29]). It is obvious that the meson theory as we have presented it until now does not explain this fact. For this reason several authors have assumed the *existence of neutral mesons*. One argument of Bhabha [18]) for the existence of these hypothetical *neutrettos* was, that it would make the theory "more symmetrical": charged and neutral particles would exist with small, intermediate and large masses. We shall see, however, that this argument is hardly tenable. Indeed, the neutrettos actually introduced by the theory cannot be compared with neutrons or neutrinos, since the corresponding *anti-neutrettos do not exist* [30]). There is more reason to draw a parallel between neutrettos and photons. If the arguments of Bhabha were reversed, we should have to expect the existence of "charged photons".

A theory of mesons and neutrettos and their interaction with nucleons was developed in a very elegant way by Kemmer [30]). According to him, neutrettos are just as photons emitted and absorbed by particles jumping from one state into the other without changing their charge. If an antineutretto existed, it would be absorbable by those particles that can emit neutrettos, and *vice versa*. It is obvious that such a particle would behave exactly like a neutretto behaves itself. Therefore it seems to be prudent, not to introduce two kinds of neutral heavy quanta, which cannot be distinguished at all, but to assume, just as in the case of the electromagnetic field, that the neutretto field is to be described by real tensors [30]), or, in an "undor" terminology [31]), by means of "neutrettors" [12]). In a symbolic way we may say that the antineutretto is identical with a neutretto, just as an antiphoton is a photon again. (The Jordan-Klein matrix in a Fourier analysis of the quantized photon field, which should describe the antiphoton, is identical with one of the Jordan-Klein matrices describing the photons; compare § 6).

The quantities $fu_{\lambda\ldots}$ and $gu_{\lambda\ldots}$ in (2) have the form of $\frac{1}{2} g_n \psi^\dagger \boldsymbol{\omega}(\boldsymbol{\tau}_x - i\boldsymbol{\tau}_y)\psi$, where ψ is the wave function of a nuclon and $\boldsymbol{\omega}$ is a combination of Dirac matrices; g_n denotes the constants f and g. The neutretto field $\overline{\zeta}$, $\overline{\varphi}$ should satisfy equations of the same kind as (2), only $\bar{f}\bar{u}_{\lambda\ldots}$ and $\bar{g}\bar{u}_{\lambda\ldots}$ in these equations should now be of the form of $\frac{1}{2}\psi^\dagger \boldsymbol{\omega}(a_n + b_n \boldsymbol{\tau}_z)\psi$. If $\overline{\zeta}$ and $\overline{\varphi}$ are real and if $\boldsymbol{\omega}$ is a *self-adjoint* (Hermitian) matrix ($\boldsymbol{\omega} = \boldsymbol{\omega}^\dagger$), we must assume [30]) (compare [M.F. (27)] *)) that all a_n and b_n are real:

$$a_n^* = a_n, \quad b_n^* = b_n. \tag{14}$$

The effective (second order) potential between two nuclons, due to the interaction through the meson field, turns out (compare § 7) to be a sum of terms, which are proportional to

$$g_m^* g_n \frac{\boldsymbol{\tau}_x^{(1)} + i\boldsymbol{\tau}_y^{(1)}}{2} \frac{\boldsymbol{\tau}_x^{(2)} - i\boldsymbol{\tau}_y^{(2)}}{2} + g_n^* g_m \frac{\boldsymbol{\tau}_x^{(2)} + i\boldsymbol{\tau}_y^{(2)}}{2} \frac{\boldsymbol{\tau}_x^{(1)} - i\boldsymbol{\tau}_y^{(1)}}{2}.$$

In the *non-relativistic approximation*, only terms with $m = n$ occur, so that the terms of this effective potential are simply proportional to

$$\boldsymbol{\tau}_x^{(1)} \boldsymbol{\tau}_x^{(2)} + \boldsymbol{\tau}_y^{(1)} \boldsymbol{\tau}_y^{(2)}. \tag{15}$$

This would still be true in the "relativistic" approximation †), *if* we assumed that it is possible to make *all g_n real at the same time* by a choice of the phase of the meson field:

$$g_n^* = g_n. \tag{16}$$

It is easily seen that the effective (second order) potential, due to

*) By [U.C.] will be referred to formulae from the paper of the author on undor calculus [31]) (first chapter of this thesis); by [M.F.] to formulae from the paper of the author on the undor equation of the meson field [12]) (second chapter of this thesis).

†) In Kemmer's papers [10]) [30]) terms with $m \neq n$ do not appear at all, so that for him the condition (16) is not essential in this connection. Compare § 7.

the interaction through the neutretto field, will be a sum of terms, which are proportional to

$$\frac{a_m^* + b_m^* \tau_z^{(1)}}{2} \frac{a_n + b_n \tau_z^{(2)}}{2} + \frac{a_n^* + b_n^* \tau_z^{(2)}}{2} \frac{a_m + b_m \tau_z^{(1)}}{2}.$$

In *non-relativistic* approximation again there are only terms with $m = n$, so that in this case the terms of the effective potential due to the interaction through the neutretto field are simply proportional to

$$\tfrac{1}{2}\{|a_n|^2 + \tfrac{1}{2}(a_n^* b_n + a_n b_n^*)(\tau_z^{(1)} + \tau_z^{(2)}) + |b_n|^2 \tau_z^{(1)} \tau_z^{(2)}\}.$$

The total non-relativistic interaction is then a sum of terms proportional to

$$\tfrac{1}{2}\{|a_n|^2 + |g_n|^2 (\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)}) + (|b_n|^2 - |g_n|^2)\tau_z^{(1)} \tau_z^{(2)} + \\ + \tfrac{1}{2}(a_n^* b_n + a_n b_n^*)(\tau_z^{(1)} + \tau_z^{(2)})\}. \quad (17)$$

Now, the proton-proton and the proton-neutron interaction in states, which are *antisymmetrical* with respect to the spatial and spin co-ordinates (so that $(\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)}) = 1$), become equal in this approximation, if we assume

$$|b_n|^2 - |g_n|^2 + \tfrac{1}{2}(a_n^* b_n + a_n b_n^*) = 0.$$

From nuclear physics we know that the non-electromagnetic proton-proton forces are also approximately equal to the neutron-neutron forces, so that

$$\tfrac{1}{2}(a_n^* b_n + a_n b_n^*) \ll |a_n|^2 + |b_n|^2.$$

These two conditions are satisfied, if we choose

$$|b_n|^2 = |g_n|^2; \quad a_n^* b_n = \text{purely imaginary}.$$

Now, in order to avoid in our theory the existence of antineutrettos, we have already assumed in (14) that all a_n and b_n are real. Thus we conclude:

$$b_n = |g_n|, \quad a_n b_n = 0, \quad (18)$$

so that there are only two possibilities:

1°. the "*symmetrical theory*", proposed by Kemmer[30]):

$$a_n = 0, \quad b_n = |g_n|; \quad (19)$$

2°. the "*neutral theory*", proposed by Bethe[32]):

$$b_n = g_n = 0. \quad (19a)$$

In the latter theory, the nuclon interaction is entirely due to the (electromagnetic and the) neutretto field, and mesons do not intervene ($g_n = 0$). Since from cosmic ray phenomena seems to follow that the interaction between nuclons and *charged* heavy quanta cannot be neglected, we shall mainly confine ourselves in the following to Kemmer's *symmetrical* theory. Then, the second order effective potential between nuclons is proportional to

$$(\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)}). \quad (20)$$

The heavy quanta forces between nuclons are now independent of the charge of the nuclons.

Comparing (20) with (15) we deduce from (3) the new form of the non-relativistic effective second order potential between two nuclons:

$$W(1,2) = \tfrac{1}{2}(\vec{\tau}^{(1)} \cdot \vec{\tau}^{(2)})\{A + B(\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)}) + $$
$$+ (C/\varkappa^2)(\vec{\sigma}^{(1)} \cdot \vec{\nabla}_1)(\vec{\sigma}^{(2)} \cdot \vec{\nabla}_1)\}(e^{-\kappa r_{12}}/r_{12}) \quad (21)$$

Here A, B and C are still given by (4). The potentials for a *pure* ^{3}S state and for the ^{1}S state of the deuteron are now given (compare (6)—(7)) by

$$\begin{aligned} {}^3\mathrm{S}: \quad W(r) &= -\tfrac{3}{2}(A + B')e^{-\kappa r}/r, \\ {}^1\mathrm{S}: \quad W(r) &= +\tfrac{1}{2}(A - 3B')e^{-\kappa r}/r, \end{aligned} \qquad B' = B + \tfrac{1}{3}C, \quad (22)$$

so that for $\xi = 1{\cdot}5 \sim 2{\cdot}0$ we find (assuming (10*a*) and (13.W) or (13.S-GM)):

m =	150 *m*	175 *m*	200 *m*
$A/\hbar c =$ (*Wilson*)	$0{\cdot}05_7$	$0{\cdot}05_7$	$0{\cdot}05_7$
$A/\hbar c =$ (*Sachs-GM*)	$0{\cdot}06_3$	$0{\cdot}06_3$	$0{\cdot}06_3$
$B'/\hbar c =$	$0{\cdot}10_1$	$0{\cdot}11_7$	$0{\cdot}13_2$

(23)

Comparing (22)—(23) with (4) and putting

$$g_1 = |g_b|\sqrt{c\varkappa^3/4\pi}, \quad g_2 = |f_b|\sqrt{c\varkappa^3/4\pi}, \quad g_3 = |g_d|\sqrt{c\varkappa^3/4\pi} \quad (24)$$

we find for a combination of the cases (*b*) and (*d*) of Kemmer:

$$g_1^2/\hbar c = A/\hbar c, \qquad (2g_2^2 + g_3^2)/3\hbar c = B'/\hbar c,$$

so that $g_1^2/\hbar c$ is of the order of magnitude 1/17 or 1/16, and $g_2^2/\hbar c$ of the order of magnitude $\frac{1}{5} \sim \frac{1}{6}$, if $g_3 = 0$, and of the order of magnitude $\frac{1}{8} \sim \frac{1}{9}$, if $g_3 \approx g_2$. *In the latter case* $(g_1^2 + 2g_2^2)/3\hbar c$ *is of the order of magnitude* 1/10. It must be remembered, however, that we have here entirely neglected the coupling of the ^3S state with the $^3D_1$ state by the term with C (compare § 8).

With Kemmer [30]) we shall now assume that the g_n are real indeed (16). Then $gu_{\lambda...}$ and $\bar{g}\bar{u}_{\lambda...}$ take according to (19) the form of

$$gu = \tfrac{1}{2} g \psi^\dagger \boldsymbol{\omega} (\boldsymbol{\tau}_x - i\boldsymbol{\tau}_y) \psi = g \psi_N^\dagger \boldsymbol{\omega} \psi_P; \qquad (25)$$
$$\bar{g}\bar{u} = \tfrac{1}{2} g \psi^\dagger \boldsymbol{\omega} \boldsymbol{\tau}_z \psi = \tfrac{1}{2} g (\psi_P^\dagger \boldsymbol{\omega} \psi_P - \psi_N^\dagger \boldsymbol{\omega} \psi_N).$$

Putting in a symbolic way [30])

$$\Psi = \boldsymbol{\Phi}_x - i\boldsymbol{\Phi}_y, \quad \Psi^{\mathfrak{L}} = \boldsymbol{\Phi}_x + i\boldsymbol{\Phi}_y, \quad \bar{\Psi} = \boldsymbol{\Phi}_z, \qquad (26)$$

(where Ψ is the meson field, $\Psi^{\mathfrak{L}}$ its charge-adjoint [31]) and $\bar{\Psi}$ the neutretto field), we can write the Lagrangian *)

$$L = -\mathrm{K} \int \{\Psi^\dagger \mathrm{B}(2\varkappa - i\Gamma_\lambda \nabla^\lambda) \Psi + \bar{\Psi}^\dagger \mathrm{B}(2\varkappa - i\Gamma_\lambda \nabla^\lambda) \bar{\Psi} +$$
$$+ 2\varkappa [\Psi^\dagger \mathrm{B} . f_{op} \psi^{\mathfrak{C}} (\boldsymbol{\tau}_x - i\boldsymbol{\tau}_y) \psi + \bar{\Psi}^\dagger \mathrm{B} . f_{op} \psi^{\mathfrak{C}} \boldsymbol{\tau}_z \psi + \text{conj. compl.}] + ...\} \qquad (27)$$

in symbolic vector notation in the form of

$$L = -\mathrm{K} \int \{(\vec{\boldsymbol{\Phi}} . \mathfrak{L}^* \mathrm{B} [2\varkappa - i\Gamma_\lambda \nabla^\lambda] \vec{\boldsymbol{\Phi}}) +$$
$$+ 2\varkappa [(\vec{\boldsymbol{\Phi}} . \mathfrak{L}^* \mathrm{B} f_{op} \psi^{\mathfrak{C}} \vec{\boldsymbol{\tau}} \psi) + \text{conj. compl.}] + \ldots .\}, \qquad (28)$$

where

$$\mathfrak{L} \Phi^*_{k_1 k_2} = \mathfrak{C}^{(1)} \mathfrak{C}^{(2)} \Phi^*_{k_2 k_1} = \Phi^{\mathfrak{C}}_{k_2 k_1} = \Phi^{\mathfrak{L}}_{k_1 k_2}. \qquad (29)$$

Kemmer [30]) has pointed out that, if (16) is assumed, the effective potential between two nuclons in any higher than second order approximation of perturbation calculus can, on account of the "invariant" vector form of (28), be written as

$$W'' + (\vec{\boldsymbol{\tau}}^{(1)} . \vec{\boldsymbol{\tau}}^{(2)}) W', \qquad (20a)$$

where W' and W'' are potentials depending on $\vec{r}_{12}$ and the spin coordinates of the two nuclons only.

§ 4. *Quantization and relativistic invariance of the theory.* In 1929 Heisenberg and Pauli [33]) [34]) have developed a quantum

*) Compare the notation in [M.F.] [12]).

theory for wave fields and have demonstrated its relativistic invariance. In this theory *all* components of a set of quantities $q(x)$ transforming irreducibly among each other by Lorentz transformations are assumed to be canonical co-ordinates, as soon as *one* of these quantities q is a canonical co-ordinate. Thus, if for instance one component of a four-vector is a canonical co-ordinate, all components are so. For this reason Kemmer [10]), Bhabha [9]) [18]) and Yukawa [2]) [21]) built up their theory in such a way that *all components of* φ in (2) were regarded as *canonical co-ordinates* $q(x)$. Apart from some possible normalization factor, the canonical conjugates of $\varphi_{\lambda_1\lambda_2\ldots}$ are then given by $p(\varphi_{\lambda_1\lambda_2\ldots}) = \zeta^*_{0\lambda_1\lambda_2\ldots}$; the canonical conjugates of $\varphi^*_{\lambda_1\lambda_2\ldots}$ by $\zeta_{0\lambda_1\lambda_2\ldots}$. Particularly by Bhabha [18]) the quantization of the meson field was performed in a logical way starting from this point of view.

Then, however, a difficulty arose. From the antisymmetry of $\zeta_{\lambda_0\lambda_1\lambda_2\ldots}$ it follows that $\varphi_{0\lambda_2\ldots}$ does not possess a canonical conjugate at all. Thus *the scheme of Heisenberg and Pauli cannot be applied*. For in both proofs, given by them [33]) [34]) for the relativistic invariance of each of the commutation relations, an essential use was made of the commutation rules holding for the other co-ordinates $q(x)$ transforming together with the co-ordinate in question. The commutation relations assumed by Heisenberg and Pauli, however, do not hold between $\varphi_{0\lambda_2\ldots}$ and the other canonical variables, if the meson equations are regarded as q-number relations.

For instance, no δ-function is yielded by the commutator of $\varphi_{0\lambda_2\ldots}$ with its canonical conjugate, since the latter, being zero, is (anti)commutative with any quantity. A similar difficulty appeared in quantum-electrodynamics. There it can be removed in a natural way [35]) [36]) by assuming that the canonical conjugate $\mathfrak{S}$ of the electric potential is not identically equal to zero, but is a q-number which, operating on the situation function, multiplies the latter by a constant factor only, for instance by zero. *This q-number must be introduced as a new variable.*

One may try, of course, to proceed in a similar way in the meson theory. There it is possible, indeed, to introduce similar help-quantities U in such a way that finally every canonical co-ordinate $q(x)$ possesses a conjugated momentum. If φ is of rank n, we must introduce a set of tensors U of rank $(n - 1)$, $(n - 2)$, ..., 1, 0 for this purpose *). In this way it is possible to find a consistent set of commutation relations, the relativistic invariance of which follows automatically by an application of the arguments of Heisenberg and Pauli. It must be borne in mind, however, that *these commutation rules are not identical with those, which are in use in the current theory*. Now, for instance, $\varphi_{0\lambda_2...}$ is commutative †) with $\varphi^*_{\lambda_1\lambda_2...}$, and the "identities" expressing the $\varphi_{0\lambda_2...}$ and $\varphi^*_{0\lambda_2...}$ in terms of the ζ and ζ^* are no longer valid as q-number relations. Thus the meson equations are affected and they can only be valid as a condition imposed on the situation function (like in quantum-electrodynamics), if we postulate that U, operating on the situation function, multiplies the latter by a constant factor (for instance zero) only. The advantage of such a procedure would be, that there would exist some possibility of separating a part of the nuclon-nuclon forces by a canonical transformation [11] [37]), like this is done in quantum-electrodynamics, where the static Coulomb force is separated and the longitudinal electromagnetic field is eliminated from the theory [38]) [36]) (compare § 5).

However, *such a procedure is impossible in the theory of heavy quanta*, since from the condition "U = constant" imposed on the situation function would follow that *the situation function vanishes itself*. This is a consequence of the fact that in the meson theory U will not be commutative with its own derivatives with respect to the time (contrary to $\mathfrak{S}$ in quantum-electrodynamics), as $\varphi_{0\lambda_2...}$ appears itself in the left hand member of the field equation $\delta L/\delta\varphi_{0\lambda_2...} =$ $= -\dot{U}$ replacing the "identity" $\delta L/\delta\varphi_{0\lambda_2...} = 0$. *From a non-relativistic point of view*, again, this occurrence of $\varphi_{0\lambda_2...}$ in the corresponding Lagrangian field equation (identity) means that *the introduction of help-quantities U is entirely superfluous*. The identity can be regarded as a *definition* of $\varphi_{0\lambda_2...}$ in terms of the other canonical variables and can be used directly for the derivation of the commutation relations of $\varphi_{0\lambda_2...}$. Thus $\varphi_{0\lambda_2...}$ is no longer treated as an ordinary *canonical* variable, but only as a "*derived variable*", in analogy to $\mathfrak{H}$ in quantum-electrodynamics, and *not to the electric po-*

*) For Kemmer's case (*b*) this reduces to a single scalar (and its conjugate complex). If in this case, instead of φ_λ and φ^*_λ, the $\zeta_{\lambda\mu}$ and $\zeta^*_{\lambda\mu}$ are regarded as canonical co-ordinates, one must introduce a four-vector and a scalar. If φ_λ and $\zeta_{\lambda\mu}$ are regarded as canonical co-ordinates, like it would be natural in an *undor* theory of mesons, one must even introduce two scalars and a four-vector.

†) This is one of the conditions, indeed, on which an application of the scheme of Heisenberg and Pauli is possible. It is *not* realized in the *current* meson theory.

tential $\mathfrak{V}$. In this way we find the commutation relations, which are actually in use in the literature. Indeed this treatment seems to be the more natural one. But *its relativistic invariance has never been proved*, though there seems not to be a particular reason to doubt its existence. Perhaps the proof can be given on the basis of a suitable generalization of the theory of Heisenberg and Pauli.

Since by Kemmer, Bhabha and Yukawa only the components of φ and φ^* are regarded as independent variables in the Lagrangian variational principle, the other quantities then must be "defined" by (2) in terms of these variables. Thus, the Lagrangian function, regarded as a function of the independent variables only, is *of the second degree in the gradient operator* ∇. One may call this an "$L(\nabla\nabla)$-theory". The *field equations* following from such a theory are of the *second order*; the first order equations are arrived at by assuming some of them as definitions.

Although the elegance of such a procedure is questionable even in electrodynamics, it can be defended there, since the field strengths are uniquely determined, if the potentials are known as functions of x, y, z, t. In these "definitions" the variables describing other fields do not occur. The potentials, on the other hand, cannot be expressed in terms of the field strengths.

We have already seen that this is not the case in the meson theory. There, of course, it is possible to introduce with Kemmer [10]) and Bhabha [18]) the quantities $\chi = (\zeta - fu)$, which can be expressed directly in terms of the φ; but the possibility remains of expressing φ or at least $(\varphi + gu) = \eta$ in terms of the ζ. So it is not clear why *in a Lagrangian variational principle* one of the sets of quantities φ and ζ should be treated differently from the other.

For this reason it seems to be more elegant to derive directly the *complete set of first order equations* from a Lagrangian, which is linear in the ∇ operators, like the Lagrangian in the wave-mechanical theory of electrons. We shall call this an "$L(\nabla)$-theory".

A more serious objection against an $L(\nabla\nabla)$-theory of hardly known particles seems to be the following. Since we know that the interaction of Dirac particles with the Maxwellian field can be described by changing in the Lagrangian function ∇_λ into $D_\lambda \equiv \nabla_\lambda + (e/i\hbar c)\,\mathfrak{A}_\lambda$, whereever it operates on the wave-function describing the annihilation of a positively charged particle or the creation of a negative particle, and by $D_\lambda^* \equiv \nabla_\lambda - (e/i\hbar c)\,\mathfrak{A}_\lambda$, whereever it operates on the conjugate wave-function describing the creation of positive and the annihilation of negative particles, we are accustomed to use the same scheme of introducing the interaction with the electromagnetic field, for any hitherto unknown particles. One might desire that this scheme is uniquely determined.

This happens to be the case, if L is linear in the gradient operators, like the Lagrangian used as a rule for the electron wave field ($L(\nabla)$-theory), but in an $L(\nabla\nabla)$-theory the above-mentioned prescription ($\nabla \rightarrow D$) is not sufficient to determine a unique interaction with the Maxwellian field. We shall show this for Dirac particles.

Instead of deriving the Dirac-equation from a Lagrangian

$$L(\nabla) = i\int \psi^\dagger \beta(i\varkappa + \gamma_\mu \nabla^\mu)\,\psi, \quad (\gamma_\mu = \beta\alpha_\mu), \tag{30}$$

we can derive it from the (second order) Klein-Gordon equation [39]) *)

$$(\varkappa^2 - \Box)\,\psi = 0, \qquad (\Box = \nabla_\mu \nabla^\mu), \tag{31}$$

which is derived from a Lagrangian

$$L(\nabla\nabla) = \int \psi^\dagger \beta(\varkappa^2 - \Box)\psi, \tag{32}$$

$$= L(\nabla\nabla) = \int \psi^\dagger \beta\,(-i\varkappa + \gamma_\lambda \nabla^\lambda)\,(i\varkappa + \gamma_\mu \nabla^\mu)\,\psi. \tag{33}$$

On account of $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = -2g_{\mu\nu}$ the integrands appearing in the two expressions (32) and (33) are identical; their difference is equal to

$$(-i/2)\,\psi^\dagger \beta\sigma_{\lambda\mu}\,(\nabla^\lambda\nabla^\mu - \nabla^\mu\nabla^\lambda)\,\psi = 0\;. \quad (2\sigma_{\lambda\mu} = i\gamma_{[\lambda}\gamma_{\mu]}). \tag{34}$$

If only first order derivatives are allowed, we can integrate (32) and (33) by parts:

$$L(\nabla\nabla) = \varkappa^2 \int \psi^\dagger \beta\,\psi + \int (\nabla_\mu \psi)^\dagger \beta\,(\nabla^\mu \psi), \tag{32a}$$

$$= L(\nabla\nabla) = \varkappa^2 \int \psi^\dagger \beta\,\psi - \int (\gamma_\lambda \nabla^\lambda \psi)^\dagger \beta\,(\gamma_\mu \nabla^\mu \psi). \tag{33a}$$

Now changing $\nabla_\lambda \psi$ into $D_\lambda \psi$, the second order equations following from (32*a*) are given by

$$(\varkappa^2 - D^2)\,\psi = 0, \tag{32b}$$

and those following from (33*a*) are given by

$$\{\varkappa^2 - D^2 - (e/i\hbar c)\,(\vec{\mathfrak{H}}\,.\,\vec{\sigma} - i\,\vec{\mathfrak{E}}\,.\,\vec{\alpha})\}\,\psi = 0. \tag{33b}$$

The latter equation describes a Dirac particle with a magnetic moment, as it should be; but the former equation can at most describe a particle, which does not possess a magnetic moment, though it has a spin angular momentum [40]).

Generally the interaction of a particle with the electromagnetic field is not determined by the prescription $\nabla \rightarrow D$, so long as there is a possibility of introducing some terms with $(\nabla_\lambda\nabla_\mu - \nabla_\mu\nabla_\lambda)$ before applying that prescription. The procedure of starting from an $L(\nabla\nabla)$ seems to be especial-

*) Generally from (31) follows $i\varkappa\psi + \gamma_\mu \nabla^\mu \varphi = i\varkappa\varphi + \gamma_\mu \nabla^\mu \psi = 0$, where φ is another undor. If the total field of the quantities satisfying the first order equations consists of one undor only, we can conclude that either $\psi = \varphi$ or $\psi = -\varphi$.

ly dangerous since the (wrong) expression (32) seems to be simpler than the (right) expression (33).

The ambiguity here discussed can be removed by inserting D in the place of ∇ in the first order equations instead of in the Lagrangian itself. This, however, may be dangerous, since some first order equations, such as for instance div $\mathbf{H} = 0$ and div $\mathbf{A} + \dot{\mathbf{V}}/c = 0$ in the case of the meson field in empty space, may possess a character different from that of the regular field equations (compare [M.F. (44)—(45)] !), and this can be verified only by examining whether the equations in question can be derived, without introducing auxiliary variables, from a Lagrangian function that is linear in ∇.

Since no direct experimental data exist on the interaction between mesons and the electromagnetic field, it seems to be necessary — at least for a satisfactory theoretical derivation of an expression for this interaction — to derive the first order meson equations (2) from a Lagrangian, which is linear in the gradient operators, $L(\nabla)$. We shall see that in this way we arrive exactly at the equations given explicitly by Bhabha [18]) and Yukawa [21]). This is not very surprizing, since they have changed ∇ into D in the right first order equations.

So long as the theory of Heisenberg and Pauli has not yet been generalized, there is as little direct proof for the relativistic invariance of the procedure of quantization by starting from $L(\nabla)$, as there was for the $L(\nabla\nabla)$-method used by the cited authors. When a more general relativistic quantum theory of wave fields will be formulated, it should be formulated in such a way that the invariance of the $L(\nabla)$-method is generally warranted. Since this method leads to *the same Hamiltonian* (*quadratic in* ∇) as the $L(\nabla\nabla)$-method, the proof of relativistic invariance of both methods will be substantially identical.

So we shall regard in the following both φ and ζ and their conjugate complex as *independent variables in the Lagrangian function* $L(\nabla)$; the $\varphi_{a_1a_2\ldots}$, $\varphi^*_{a_1a_2\ldots}$, $\zeta_{0a_1a_2\ldots}$ and $\zeta^*_{0a_1a_2\ldots}$ $(a_1, a_2, \ldots = = 1, 2, 3)$ will turn out to be the *canonical* variables q and p, and the variables $\varphi_{0a_2\ldots}$, $\varphi^*_{0a_2\ldots}$, $\zeta_{a_0a_1a_2\ldots}$ and $\zeta^*_{a_0a_1a_2\ldots}$ can be expressed in terms of the canonical variables. The commutation relations for these *"derived variables"* are then derived from those for the *canonical* variables, so that, if the interaction between mesons and nuclons is taken into account, the *derived variables of the meson field will no longer be commutative with the wave-function of the nuclon field*. This is exactly the way, in which Kemmer, Bhabha and Yukawa actually quantize the meson field. The only difference with the treatment of Bhabha[18]) is, that we shall treat $\varphi_{0a_2\ldots}$ *from the beginning* in the same way as the other derived variables $\zeta_{a_0a_1a_2\ldots}$, whereas this is done by Bhabha only after some special arguments, which seem to be superfluous, since they do not prove the relativistic invariance of the theory.

We shall now write down the Lagrangian function $L(\nabla)$ describing the fields of nuclons and light Dirac particles, mesons and neutrettos in interaction with each other and with the Maxwellian field. In the preceding dissertation of the author on the undor equation of the meson field [12]) a formal argument was given for treating the cases (*b*) and (*d*) of Kemmer *together*. We shall return to this question afterwards when we are able to discuss the physical properties of such a generalized meson field (§ 8); *only experimental data* can decide if the spinless [40]) mesons of case (*d*) exist or not. For the present we shall introduce these spinless mesons. Afterwards they can always be eliminated again, if it would turn out that we do not need them, by putting $f_d = g_d = 0$.

We shall make a *"Fermi-Ansatz"* (compare page 41) for the interaction between mesons and light Dirac particles, that is, we shall assume that this interaction is given by adding to fu and gu in (2) similar expressions $f'u'$ and $g'u'$, where the u' are linear combinations of the 4×4 components of $\psi_\nu^* \psi_\pi$. Here ψ_π is the wave-function of a positon π and ψ_ν is the wave-function of the particle ν called a *neutrino* in the original theory of Fermi [41]) and called an *antineutrino* by other authors [42]) [23]). If ψ' is the 8-component wave-function of the light particles, we can write

$$u' = \psi_\nu^\dagger \boldsymbol{\omega} \psi_\pi = \tfrac{1}{2}\, \psi'^\dagger \boldsymbol{\omega}(\boldsymbol{\tau}_x - i\boldsymbol{\tau}_y)\psi'. \tag{35}$$

As for the interaction of light particles with neutrettos, we shall make again the "C.I." (charge-independency) assumption of Kemmer's "symmetrical theory" [30]) (compare § 3: (25)):

$$\bar{u}' = \tfrac{1}{2}(\psi_\pi^\dagger \boldsymbol{\omega}\psi_\pi - \psi_\nu^\dagger \boldsymbol{\omega}\psi_\nu) = \tfrac{1}{2}\psi'^\dagger \boldsymbol{\omega}\boldsymbol{\tau}_z\psi'. \tag{35a}$$

In the literature u' is usually expressed in terms of the wave functions $\psi_\epsilon = \psi_\pi^{\mathfrak{C}}$ of the negaton ε and $\psi_o = \psi_\nu^{\mathfrak{C}}$ of the antiparticle o of the (anti)neutrino ν. It is easily shown, however, that, apart from some signs (which can be added to the constants f' and g'), this does not make any difference, since on account of [U.C. (12), (23), (29)] *) we have

$$\begin{aligned} \psi_\nu^\dagger\psi_\pi &= \psi_o^{\sim}\mathfrak{L}^*\mathfrak{L}\psi_\epsilon^* = \psi_o^{\sim}\psi_\epsilon^* = (-)\,\psi_\epsilon^\dagger\psi_o, \\ \psi_\nu^\dagger\vec{\boldsymbol{\alpha}}\psi_\pi &= \psi_o^{\sim}\mathfrak{L}^*\vec{\boldsymbol{\alpha}}\mathfrak{L}\psi_\epsilon^* = \psi_o^{\sim}\vec{\boldsymbol{\alpha}}^*\psi_\epsilon^* = (-)\,\psi_\epsilon^\dagger\vec{\boldsymbol{\alpha}}\psi_o, \\ \psi_\nu^\dagger\boldsymbol{\beta}\psi_\pi &= \psi_o^{\sim}\mathfrak{L}^*\boldsymbol{\beta}\mathfrak{L}\psi_\epsilon^* = -\,\psi_o^{\sim}\boldsymbol{\beta}^*\psi_\epsilon^* = -\,(-)\,\psi_\epsilon^\dagger\boldsymbol{\beta}\psi_o, \text{ etc.} \end{aligned} \tag{36}$$

The minus signs in brackets result from the anticommutativity of ψ_ϵ^* and ψ_o, the other from [U.C. (23)]. Comparing (36) with [M.F. (11), (12)] and [U.C. (62), (62*a*)] we observe that the tensors $(\psi_\nu^\dagger\boldsymbol{\omega}\psi_\pi)$ arising from the symmetrical part of $\psi_{\nu_{k_1}}^{\epsilon}\,\psi_{\pi_{k_2}}$ are changed into $(-)\,(\psi_\epsilon^\dagger\boldsymbol{\omega}\psi_o)$, whereas those arising from the antisymmetrical part change into $(-)\,(-\,\psi_\epsilon^\dagger\boldsymbol{\omega}\psi_o)$.

In the following we have called o a neutrino and ν an antineutrino, following K o n o p i n s k i and U h l e n b e c k [42]). If a *positon* is regarded as the counterpart of a *proton*, then the counterpart of a *neutron* is called an *antineutrino*, in this terminology. With this convention a neutrino and a *negative* electron are regarded as two states of one and the same particle. Something can be said for returning to the original terminology of F e r m i [41]), but here we have not done so.

A *"K o n o p i n s k i-U h l e n b e c k Ansatz"* for the interaction of mesons with light particles, involving derivatives of the neutrino wave-function, was tentatively tried by Y u k a w a [23]). We shall return to this question in the discussion of the spontaneous meson disintegration (§ 10).

We shall introduce at once the *F e r m i-v a r i a b l e* $\mathfrak{S}$, in order to avoid the difficulties with the commutation rules of a general quantum-electrodynamical theory, in which gauge transformations would be possible [34]) [43]) [44]). The total L a g r a n g i a n function then reads:

$$\begin{aligned} L = {} & i\mathrm{K}(\Psi^\dagger \mathrm{B}\Gamma_\mu\nabla^\mu\Psi + \overline{\Psi}^\dagger\mathrm{B}\Gamma_\mu\nabla^\mu\overline{\Psi}) + i\hbar c \sum_{(P,N,\pi,\nu)} \psi^\dagger\boldsymbol{\beta}\boldsymbol{\gamma}_\mu\nabla^\mu\psi - \\ & - (1/4\pi)\,(\mathfrak{H}_{\mu\nu}\nabla^\mu\mathfrak{A}^\nu + \mathfrak{S}\nabla^\mu\mathfrak{A}_\mu) - 2\varkappa\mathrm{K}(\Psi^\dagger\mathrm{B}\Psi + \Psi^\dagger\mathrm{B}Z + \\ & + Z^\dagger\mathrm{B}\Psi + Z^\dagger\mathrm{B}C_{op}Z + \overline{\Psi}^\dagger\mathrm{B}\overline{\Psi} + \overline{\Psi}^\dagger\mathrm{B}\overline{Z} + \overline{Z}^\dagger\mathrm{B}\overline{\Psi} + \overline{Z}^\dagger\mathrm{B}\overline{C}_{op}\overline{Z}) - \\ & - \sum_{(P,N,\pi,\nu)} mc^2\psi^\dagger\boldsymbol{\beta}\psi + (1/8\pi)\,(\tfrac{1}{2}\mathfrak{H}_{\mu\nu}\mathfrak{H}^{\mu\nu} + \mathfrak{S}^2) + (e\mathrm{K}/\hbar c)\,\Psi^\dagger\,\mathrm{B}\Gamma_\mu\mathfrak{A}^\mu\Psi + \\ & + e\sum_{(P,\pi)} \psi^\dagger\boldsymbol{\beta}\boldsymbol{\gamma}_\mu\mathfrak{A}^\mu\psi. \end{aligned} \tag{37}$$

*) Compare the first foot-note on page 47.

Here the notation is the following: ∇^{μ} denotes $\{\partial/\partial x, \partial/\partial y, \partial/\partial z, -\partial/c\partial t\}$; Ψ is the undor of the second rank describing according to [M.F. (10)] the components $\mathbf{S}$; $\vec{\mathbf{A}}$, $\mathbf{V}$; $\vec{\mathbf{E}}$, $\vec{\mathbf{H}}$; $\vec{\mathbf{B}}$, $\mathbf{W}$; $\mathbf{Y}$ of the generalized meson field; $\overline{\Psi}$ is the neutrettor describing the neutretto field. (Thus $\overline{\Psi}^{\dagger}$ and $\overline{\Psi}$ cannot be varied independently). $\mathrm{B} = \boldsymbol{\beta}^{(1)}\boldsymbol{\beta}^{(2)}$, $\Gamma_{\mu} = \boldsymbol{\gamma}_{\mu}^{(1)} + \boldsymbol{\gamma}_{\mu}^{(2)}$; $\boldsymbol{\gamma}_{\mu} = \boldsymbol{\beta}\boldsymbol{\alpha}_{\mu}$; $\boldsymbol{\alpha}_0 = -1$. $\sum\limits_{(P,\ldots)}$ is a summation over the wave-functions ψ of the particles mentioned under the summation sign. $\mathfrak{H}_{\mu\nu}$ and $\mathfrak{A}^{\nu}$ describe the electromagnetic field in the usual way ($\mathfrak{H}_{12} = \mathfrak{H}_z$, $\mathfrak{H}_{10} = \mathfrak{E}_x$, $\mathfrak{A}_1 = \mathfrak{A}_x$, $\mathfrak{A}^0 = -\mathfrak{A}_0 = \mathfrak{V}$); $\mathfrak{S}$ is the Fermi variable.

Z is an undor of the second rank

$$Z_{k_1k_2} = 2f_{op}\psi^{\mathfrak{C}}_{N_{k_1}}\psi_{P_{k_2}} + 2f'_{op}\psi^{\mathfrak{C}}_{\nu_{k_1}}\psi_{\pi_{k_2}} \tag{38}$$

representing according to [M.F. (10)] the following tensors (compare [M.F. (11) — (12)]):

$$\begin{aligned}
\vec{\mathbf{a}} &= g_b\psi_N^{\dagger}\vec{\boldsymbol{\alpha}}\psi_P + g'_b\psi_\nu^{\dagger}\vec{\boldsymbol{\alpha}}\psi_\pi, & \mathbf{v} &= g_b\psi_N^{\dagger}\psi_P + g'_b\psi_\nu^{\dagger}\psi_\pi;\\
\vec{\mathbf{e}} &= -f_b\psi_N^{\dagger}i\boldsymbol{\beta}\vec{\boldsymbol{\alpha}}\psi_P - f'_b\psi_\nu^{\dagger}i\boldsymbol{\beta}\vec{\boldsymbol{\alpha}}\psi_\pi, & \vec{\mathbf{h}} &= -f_b\psi_N^{\dagger}\boldsymbol{\beta}\vec{\boldsymbol{\sigma}}\psi_P - f'_b\psi_\nu^{\dagger}\boldsymbol{\beta}\vec{\boldsymbol{\sigma}}\psi_\pi;\\
\mathbf{w} &= -g_d\psi_N^{\dagger}\boldsymbol{\gamma}_5\psi_P - g'_d\psi_\nu^{\dagger}\boldsymbol{\gamma}_5\psi_\pi, & \vec{\mathbf{b}} &= -g_d\psi_N^{\dagger}\vec{\boldsymbol{\sigma}}\psi_P - g'_d\psi_\nu^{\dagger}\vec{\boldsymbol{\sigma}}\psi_\pi;\\
\mathbf{y} &= f_d\psi_N^{\dagger}i\boldsymbol{\gamma}_5\boldsymbol{\beta}\psi_P + f'_d\psi_\nu^{\dagger}i\boldsymbol{\gamma}_5\boldsymbol{\beta}\psi_\pi, & \mathbf{s} &= f_o\psi_N^{\dagger}\boldsymbol{\beta}\psi_P + f'_o\psi_\nu^{\dagger}\boldsymbol{\beta}\psi_\pi.
\end{aligned} \tag{39}$$

$$(\boldsymbol{\gamma}_5 = -i\boldsymbol{\alpha}_{\mathbf{x}}\boldsymbol{\alpha}_{\mathbf{y}}\boldsymbol{\alpha}_{\mathbf{z}}.)$$

In a similar way $\overline{Z}$ represents the real tensors

$$\vec{\overline{\mathbf{a}}} = \vec{\overline{\mathbf{a}}}^{*} = \tfrac{1}{2}g_b(\psi_P^{\dagger}\vec{\boldsymbol{\alpha}}\psi_P - \psi_N^{\dagger}\vec{\boldsymbol{\alpha}}\psi_N) + \tfrac{1}{2}g'_b(\psi_\pi^{\dagger}\vec{\boldsymbol{\alpha}}\psi_\pi - \psi_\nu^{\dagger}\vec{\boldsymbol{\alpha}}\psi_\nu);\ \text{etc.} \tag{39a}$$

We assume that all constants f_o, g_b, etc., are real (16).

The scalar operators C_{op} and $\overline{C}_{op}$ multiply the tensors $\mathbf{s}$; $\vec{\mathbf{a}}$, $\mathbf{v}$; $\vec{\mathbf{e}}$, $\vec{\mathbf{h}}$; $\vec{\mathbf{b}}$, $\mathbf{w}$ and $\mathbf{y}$ in Z, and $\overline{\mathbf{s}}$; $\vec{\overline{\mathbf{a}}}$, $\overline{\mathbf{v}}$; etc. in $\overline{Z}$, by the constant factors C_0, C_1, C_2, C_3, C_4, and $\overline{C}_0$, $\overline{C}_1$, etc., respectively. These constants can still be arbitrarily chosen. It is a special assumption, that in the terms with C_{op} and $\overline{C}_{op}$ only products of the combinations (39)—(39*a*) occur. This assumption is not essential for the theory. For instance, one might have introduced the products of the types $(\psi_P^{\dagger}\psi_N)(\psi_N^{\dagger}\psi_P)$, $\{(\psi_P^{\dagger}\psi_N)(\psi_\nu^{\dagger}\psi_\pi) + (\psi_\pi^{\dagger}\psi_\nu)(\psi_N^{\dagger}\psi_P)\}$ and $(\psi_\pi^{\dagger}\psi_\nu)(\psi_\nu^{\dagger}\psi_\pi)$ with three different coefficients. For the sake of simplicity we have not done it here.

K in (37) is the normalization constant of the heavy quantum field. In the literature this constant is not always chosen in the same way. We shall make here a definite choice and put this constant according to Kemmer [10]) equal to $(c/2)$ in the following.

The Lagrangian function (37) can be written easily in vector notation. This should be done in order to remove at once the superfluous quantity $\overline{\Psi}^{\dagger}$. Integrating the terms, in which derivatives of $\vec{\overline{\mathbf{E}}}$, $\vec{\overline{\mathbf{H}}}$ or $\overline{\mathbf{Y}}$ occur, by parts, $L = \int L\, dx\, dy\, dz\, dt$ takes the following form (fletches over the vectors are here omitted; rot $\equiv$ curl; K is put equal to $c/2$):

$$L = \int L'\, dx\, dy\, dz\, dt; \tag{40}$$

$$\begin{aligned}
-L' = {} & c\{\mathbf{A}^* \cdot (\varkappa \mathbf{A} + \mathrm{rot}\,\mathbf{H} - \dot{\mathbf{E}}/c) - \mathbf{V}^*(\varkappa \mathbf{V} + \mathrm{div}\,\mathbf{E}) + \\
& + \mathbf{E}^* \cdot (\varkappa \mathbf{E} + \nabla \mathbf{V} + \dot{\mathbf{A}}/c) \quad - \mathbf{H}^* \cdot (\varkappa \mathbf{H} - \mathrm{rot}\,\mathbf{A}) + \\
& + \mathbf{W}^*(\varkappa \mathbf{W} + \dot{\mathbf{Y}}/c) - \mathbf{B}^* \cdot (\varkappa \mathbf{B} - \nabla \mathbf{Y}) + \\
& + \mathbf{Y}^*(\varkappa \mathbf{Y} - \mathrm{div}\,\mathbf{B} - \dot{\mathbf{W}}/c) - \varkappa \mathbf{S}^* \mathbf{S}\} + \\
& + 2c\{\overline{\mathbf{E}} \cdot (\nabla \overline{\mathbf{V}} + \dot{\overline{\mathbf{A}}}/c) + \overline{\mathbf{H}} \cdot \mathrm{rot}\,\overline{\mathbf{A}} - \overline{\mathbf{Y}}\,(\mathrm{div}\,\overline{\mathbf{B}} + \dot{\overline{\mathbf{W}}}/c)\} + \\
& + c\varkappa\,(\overline{\mathbf{A}}^2 - \overline{\mathbf{V}}^2 + \overline{\mathbf{E}}^2 - \overline{\mathbf{H}}^2 + \overline{\mathbf{W}}^2 - \overline{\mathbf{B}}^2 + \overline{\mathbf{Y}}^2 - \overline{\mathbf{S}}^2) + \\
& + (1/4\pi)\{\mathfrak{E} \cdot (\nabla \mathfrak{V} + \dot{\mathfrak{A}}/c) + \mathfrak{H} \cdot \mathrm{rot}\,\mathfrak{A} + \mathfrak{S}\,(\mathrm{div}\,\mathfrak{A} + \dot{\mathfrak{V}}/c)\} + \\
& + (1/8\pi)\,(\mathfrak{E}^2 - \mathfrak{H}^2 - \mathfrak{S}^2) + \\
& + \sum_{(P,N,\pi,\nu)} \psi^{\dagger}\,(mc^2 \boldsymbol{\beta} - i\hbar c\, \boldsymbol{\alpha} \cdot \nabla - i\hbar\, \partial/\partial t)\,\psi - e \sum_{(P,\pi)} \psi^{\dagger}\,(\boldsymbol{\alpha} \cdot \mathfrak{A} - \mathfrak{V})\psi + \\
& + (e/i\hbar)\{\mathbf{E}^* \cdot \mathfrak{A}\mathbf{V} - \mathbf{V}^*\mathfrak{A} \cdot \mathbf{E} + \mathbf{A}^* \cdot [\mathfrak{A}, \mathbf{H}] + \mathbf{H}^* \cdot [\mathfrak{A}, \mathbf{A}] - \\
& \qquad - \mathbf{Y}^*\mathfrak{A} \cdot \mathbf{B} + \mathbf{B}^* \cdot \mathfrak{A}\mathbf{Y} - \mathbf{E}^* \cdot \mathfrak{V}\mathbf{A} + \mathbf{A}^* \cdot \mathfrak{V}\mathbf{E} - \\
& \qquad\qquad - \mathbf{W}^*\mathfrak{V}\mathbf{Y} + \mathbf{Y}^*\mathfrak{V}\mathbf{W}\} + \\
& + c\varkappa\{\mathbf{A}^* \cdot \mathbf{a} - \mathbf{V}^*\mathbf{v} + \mathbf{E}^* \cdot \mathbf{e} - \mathbf{H}^* \cdot \mathbf{h} + \mathbf{W}^*\mathbf{w} - \mathbf{B}^* \cdot \mathbf{b} + \\
& \qquad\qquad + \mathbf{Y}^*\mathbf{y} - \mathbf{S}^*\mathbf{s} + \text{conj. compl.}\} + \\
& + 2c\varkappa\{\overline{\mathbf{A}} \cdot \overline{\mathbf{a}} - \overline{\mathbf{V}}\overline{\mathbf{v}} + \overline{\mathbf{E}} \cdot \overline{\mathbf{e}} - \overline{\mathbf{H}} \cdot \overline{\mathbf{h}} + \overline{\mathbf{W}}\overline{\mathbf{w}} - \overline{\mathbf{B}} \cdot \overline{\mathbf{b}} + \overline{\mathbf{Y}}\overline{\mathbf{y}} - \overline{\mathbf{S}}\overline{\mathbf{s}}\} + \\
& + c\varkappa\,\{C_1(\mathbf{a}^* \cdot \mathbf{a} - \mathbf{v}^*\mathbf{v}) + C_2(\mathbf{e}^* \cdot \mathbf{e} - \mathbf{h}^* \cdot \mathbf{h}) + \\
& \qquad + C_3\,(\mathbf{w}^*\mathbf{w} - \mathbf{b}^* \cdot \mathbf{b}) + C_4\,\mathbf{y}^*\mathbf{y} - C_0\,\mathbf{s}^*\mathbf{s} + \\
& + \overline{C}_1(\overline{\mathbf{a}}^2 - \overline{\mathbf{v}}^2) + \overline{C}_2(\overline{\mathbf{e}}^2 - \overline{\mathbf{h}}^2) + \overline{C}_3(\overline{\mathbf{w}}^2 - \overline{\mathbf{b}}^2) + \overline{C}_4\,\overline{\mathbf{y}}^2 - \overline{C}_0\overline{\mathbf{s}}^2\}.
\end{aligned}$$

Now using (40) instead of (37) as the Lagrangian function we can regard

$$\mathbf{A}, \mathbf{E}, \mathbf{W}, \mathbf{Y};\ \overline{\mathbf{A}}, \overline{\mathbf{W}};\ \mathfrak{A}, \mathfrak{V};\ \psi_P, \psi_N;\ \psi_\pi \text{ and } \psi_\nu \tag{41}$$

as the canonical co-ordinates $q(x)$; the canonical conjugated momenta $p(x)$ are then given (on account of $K = c/2$) by

$$-\mathbf{E}^*, \mathbf{A}^*, \mathbf{Y}^*, -\mathbf{W}^*; \; -2\overline{\mathbf{E}},\; 2\overline{\mathbf{Y}};$$

$$-\mathfrak{E}/4\pi c, -\mathfrak{S}/4\pi c; \; i\hbar\psi_P^*, i\hbar\psi_N^*; \; i\hbar\psi_\pi^* \text{ and } i\hbar\psi_\nu^*. \tag{41a}$$

The commutation relations between these *canonical variables* are now given by

$$\begin{aligned}[q_i(x);\; q_j(x')]_- &\equiv q_i(x)\, q_j(x') - q_j(x')\, q_i(x) = 0; \\ [p_i(x);\; p_j(x')]_- = 0; \quad [q_i(x);\; p_j(x')]_- &= i\hbar\,\delta_{ij}\,\delta(x - x')\end{aligned} \tag{42.E-B}$$

for $\mathbf{A}, \mathbf{E}, \mathbf{W}, \mathbf{Y}$; $\overline{\mathbf{A}}, \overline{\mathbf{W}}$; $\mathfrak{A}, \mathfrak{B}$, and their canonical conjugates; by

$$\begin{aligned}[q_i(x);\; q_j(x')]_+ &\equiv q_i(x)\, q_j(x') + q_j(x')\, q_i(x) = 0; \\ [p_i(x);\; p_j(x')]_+ = 0; \quad [q_i(x);\; p_j(x')]_+ &= i\hbar\,\delta_{ij}\,\delta(x - x')\end{aligned} \tag{42.F-D}$$

for $\psi_P, \psi_N, \psi_\pi, \psi_\nu$ and their canonical conjugates. Each of the *canonical* Einstein-Bose variables is assumed to be commutative with each of the *canonical* Fermi-Dirac variables.

The quantities

$$\mathbf{S}, \mathbf{S}^*, \mathbf{V}, \mathbf{V}^*, \mathbf{H}, \mathbf{H}^*, \mathbf{B}, \mathbf{B}^*, \overline{\mathbf{S}}, \overline{\mathbf{V}}, \overline{\mathbf{H}}, \overline{\mathbf{B}} \text{ and } \mathfrak{H} \tag{43}$$

must be regarded as *derived variables*. They do not possess canonical conjugates, nor are they canonical conjugates of other variables. Varying these derived variables in

$$\delta \int L \, dx\, dy\, dz\, dt = 0 \tag{44}$$

we find the following "identities", which may be regarded as the *definitions of the derived variables*:

$$\varkappa(\mathbf{V} + \mathbf{v}) + \operatorname{div} \mathbf{E} + (e/i\hbar c)\,(\mathfrak{A}\,.\,\mathbf{E}) = 0,$$

$$\varkappa(\mathbf{H} + \mathbf{h}) - \operatorname{rot} \mathbf{A} - (e/i\hbar c)\,[\mathfrak{A}, \mathbf{A}] = 0,$$

$$\varkappa(\mathbf{B} + \mathbf{b}) - \nabla \mathbf{Y} - (e/i\hbar c)\,\mathfrak{A}\mathbf{Y} = 0, \quad \varkappa(\mathbf{S} + \mathbf{s}) = 0,$$

and conjugate complex equations; (45)

$$\varkappa(\overline{\mathbf{V}} + \overline{\mathbf{v}}) + \operatorname{div} \overline{\mathbf{E}} = 0, \qquad \varkappa(\overline{\mathbf{B}} + \overline{\mathbf{b}}) - \nabla\overline{\mathbf{Y}} = 0,$$

$$\varkappa(\overline{\mathbf{H}} + \overline{\mathbf{h}}) - \operatorname{rot} \overline{\mathbf{A}} = 0, \qquad \varkappa(\overline{\mathbf{S}} + \overline{\mathbf{s}}) = 0;$$

$$\mathfrak{H} = \operatorname{rot} \mathfrak{A}.$$

From (42) and (45) the commutation relations for the derived variables follow.

The Hamiltonian function is now given by

$$\Sigma\, p\dot{q} - L' = H'.$$

This H' should be expressed in terms of the p and q, that is, the other variables should be eliminated by means of (45). First, however, we shall give a short expression for H', in which this elimination has not yet taken place. Since (40) is linear in the $\dot{q}$ and does no more contain any $\dot{p}$, we find H' by omitting from (40) all terms containing derivatives with respect to the time. Now making use of (45) and integrating by parts we can write the result in the following form *):

$$H' = \int H\, dx\, dy\, dz; \qquad (46)$$

$$\begin{aligned} H = c\varkappa\{&\mathbf{S}^*\mathbf{S} + \mathbf{A}^*.\mathbf{A} + \mathbf{V}^*\mathbf{V} + \mathbf{E}^*.\mathbf{E} + \mathbf{H}^*.\mathbf{H} + \mathbf{W}^*\mathbf{W} + \mathbf{B}^*.\mathbf{B} + \mathbf{Y}^*\mathbf{Y} + \\ &+ \bar{\mathbf{S}}^2 + \bar{\mathbf{A}}^2 + \bar{\mathbf{V}}^2 + \bar{\mathbf{E}}^2 + \bar{\mathbf{H}}^2 + \bar{\mathbf{W}}^2 + \bar{\mathbf{B}}^2 + \bar{\mathbf{Y}}^2\} + \\ &+ (1/8\pi)\{\mathfrak{E}^2 + \mathfrak{H}^2\} + (1/4\pi)\{(\operatorname{div}\mathfrak{A} - \tfrac{1}{2}\mathfrak{S})\,\mathfrak{S} + \mathfrak{V}\,(4\pi e\rho - \operatorname{div}\mathfrak{E})\} + \\ &+ \sum_{(N,\nu)} \psi^\dagger(mc^2\,\beta + c\alpha.\mathcal{P}_{op})\,\psi + \sum_{(P,\pi)} \psi^\dagger(mc^2\,\beta + c\alpha.\mathcal{P}^{kin}_{op})\,\psi + \\ &+ c\varkappa\,\{\mathbf{A}^*.\mathbf{a} + \mathbf{E}^*.\mathbf{e} + \mathbf{W}^*\mathbf{w} + \mathbf{Y}^*\mathbf{y} + \text{conj. compl.}\} + \\ &+ 2c\varkappa\,\{\bar{\mathbf{A}}.\bar{\mathbf{a}} + \bar{\mathbf{E}}.\bar{\mathbf{e}} + \bar{\mathbf{W}}\,\bar{\mathbf{w}} + \bar{\mathbf{Y}}\,\bar{\mathbf{y}}\} + \\ &+ c\varkappa\,\{C_1\,(\mathbf{a}^*.\mathbf{a} - \mathbf{v}^*\,\mathbf{v}) + C_2\,(\mathbf{e}^*.\mathbf{e} - \mathbf{h}^*.\mathbf{h}) + \\ &\qquad + C_3\,(\mathbf{w}^*\mathbf{w} - \mathbf{b}^*.\mathbf{b}) + C_4\,\mathbf{y}^*\mathbf{y} - C_0\,\mathbf{s}^*\mathbf{s} + \\ &+ \bar{C}_1\,(\bar{\mathbf{a}}^2 - \bar{\mathbf{v}}^2) + \bar{C}_2\,(\bar{\mathbf{e}}^2 - \bar{\mathbf{h}}^2) + \bar{C}_3\,(\bar{\mathbf{w}}^2 - \bar{\mathbf{b}}^2) + \bar{C}_4\,\bar{\mathbf{y}}^2 - \bar{C}_0\,\bar{\mathbf{s}}^2\}. \end{aligned}$$

Here we have put

$$\mathcal{P}_{op} = -i\hbar\,\nabla; \quad \mathcal{P}^{kin}_{op} = -i\hbar\,\nabla - (e/c)\,\mathfrak{A}, \qquad (47)$$

and

$$e\rho = e\sum_{(P,\pi)} \psi^\dagger\psi + (e/i\hbar)\,(\mathbf{A}^*.\mathbf{E} - \mathbf{E}^*.\mathbf{A} + \mathbf{Y}^*\mathbf{W} - \mathbf{W}^*\mathbf{Y}). \qquad (48)$$

The physical situation is described by a situation function χ, which can be regarded — like this was originally done for instance by Fermi [35] [36]) — as a functional depending on the actual c-number values of the field components; but it is simpler, to regard it as a function of an infinite set of partition numbers („*Besetzungszahlen*") N_1, N_2, ... , denoting the numbers of particles or quanta in different states 1, 2, ...

*) $H \neq \Sigma\, p\dot{q} - L$; $\int H = \int(\Sigma\, p\dot{q} - L')$.

According to F e r m i [35]) [36]) the situation function satisfies the special condition

$$\mathfrak{S}(x, y, z, t)\,\chi = 0, \tag{49}$$

so that

$$\nabla\mathfrak{S}\chi = \Delta\mathfrak{S}\chi = 0; \qquad \mathfrak{S}\chi = \dot{\mathfrak{S}}\chi = \ddot{\mathfrak{S}}\chi = \ldots\ldots = 0. \tag{50}$$

The derivatives of the q-number field components with respect to the time can be expressed in terms of the field components themselves and their gradients, by means of the canonical field equations, which can be obtained either from the L a g r a n g i a n variational principle, or by

$$i\hbar\,\dot{\mathcal{F}} = [\mathcal{F}; H]_{-}, \tag{51}$$

making use of the commutation relations (42). In both ways we find

$$\begin{aligned} \operatorname{div} \mathfrak{E} \qquad &= 4\pi e\rho \;\; - \dot{\mathfrak{S}}/c, \\ \operatorname{rot} \mathfrak{H} - \dot{\mathfrak{E}}/c &= 4\pi e j/c + \nabla\mathfrak{S}, \end{aligned} \tag{52}$$

and

$$\begin{aligned} \mathfrak{E} &= -\nabla\mathfrak{B} - \dot{\mathfrak{A}}/c, \\ \mathfrak{S} &= \operatorname{div}\mathfrak{A} + \dot{\mathfrak{B}}/c. \end{aligned} \tag{53}$$

In (52) we have put

$$ej/c = e\sum_{(P,\pi)} \psi^{\dagger}\boldsymbol{\alpha}\psi + (e/i\hbar)\,\{[\mathbf{H}^*,\mathbf{A}] + [\mathbf{A}^*,\mathbf{H}] + $$
$$+ \mathbf{V}^*\mathbf{E} - \mathbf{E}^*\mathbf{V} + \mathbf{Y}^*\mathbf{B} - \mathbf{B}^*\mathbf{Y}\}. \tag{48a}$$

The continuity equation

$$\dot{\rho} + \operatorname{div} j = 0 \tag{54}$$

follows directly from (48), (48a) and the field equations for the wave-functions of protons, electrons and mesons, as shown by B h a b h a. From (52), (53) and (54) we deduce:

$$\begin{aligned} \dot{\mathfrak{S}}/c &= 4\pi e\rho - \operatorname{div}\mathfrak{E}, \\ \ddot{\mathfrak{S}}/c^2 &= \Delta\mathfrak{S}, \qquad \dddot{\mathfrak{S}}/c^3 = \Delta\dot{\mathfrak{S}}/c, \text{ etc.} \end{aligned} \tag{55}$$

so that

$$\Box\mathfrak{S} = 0. \tag{56}$$

From (50), (52) and (53) we find

$$\mathfrak{S}\chi = (\operatorname{div} \mathfrak{A} + \dot{\mathfrak{V}}/c)\chi = 0, \tag{57}$$

and

$$\begin{aligned}(-\dot{\mathfrak{S}}/c)\chi &= (\operatorname{div} \mathfrak{E} - 4\pi e\rho)\chi = 0,\\ \nabla\mathfrak{S}\chi &= (\operatorname{rot} \mathfrak{H} - \dot{\mathfrak{E}}/c - 4\pi e j/c)\chi = 0.\end{aligned} \tag{58}$$

From (55) and (42) we conclude that $\mathfrak{S}$ and its derivatives with respect to time are commutative, so that the conditions (50) are compatible with each other. From (55) we see that the relations (50) do not impose other conditions on the situation function than the relations (57) and (58).

The expectancy value of an observable $\mathcal{F}(t)$ is given by

$$\bar{\mathcal{F}}(t) = \sum_{N_1, N_2, \ldots} \chi^* \mathcal{F}(t)\, \chi. \tag{59}$$

(Summation over all possible values of the partition numbers.)

For actual calculations it is often more convenient to regard the field components as matrices, which do not depend on the time; then $\bar{\mathcal{F}}(t)$ is given by

$$\bar{\mathcal{F}}(t) = \sum_{N_1, N_2, \ldots} \chi^*(t)\, \mathcal{F} \chi(t), \tag{60}$$

where $\chi(t)$ is determined by

$$i\hbar\dot{\chi}(t) = \mathcal{H} \chi(t). \tag{61}$$

The condition (49) now takes the form of

$$\mathfrak{S}(x, y, z)\, \chi(t) = 0. \tag{62}$$

In order to find out, how $\mathcal{H}$ operates on the function $\chi(t)$ of the arguments $t, N_1, N_2, \ldots$ in (61), we must express $\mathcal{H}$ in terms of the canonical variables $q(x)$ and $p(x)$, and express these variables in terms of Jordan-Wigner and Jordan-Klein matrices operating on $\chi(N_1, N_2, \ldots)$. This has been done explicitly for Kemmer's case (*b*) by Bhabha[18]) and by Kobayasi and Okayama[45]).

Before we proceed to this treatment of the Hamiltonian, however, we shall first eliminate from it the longitudinal electromagnetic field and the help-quantity $\mathfrak{S}$.

§ 5. *Elimination of the longitudinal electromagnetic field.* F e r m i [38]) [36]) has shown how to eliminate the longitudinal electromagnetic field from the H a m i l t o n i a n. Here, we shall apply his method to the H a m i l t o n i a n given by (46).

For this purpose it is convenient to introduce the following notation. Let the operator $(1/\nabla)$, operating on an irrotational (longitudinal) vector field $\vec{\mathfrak{A}}_{long}(x)$, be defined by

$$\frac{1}{\nabla} \cdot \vec{\mathfrak{A}}_{long}(x) = X(x) = \int_0^x \vec{\mathfrak{A}}_{long} \cdot d\vec{\mathfrak{s}} \longrightarrow \vec{\nabla} X = \vec{\mathfrak{A}}_{long}. \quad (63)$$

Let in the same way (1/div), operating on a scalar field $\rho(x)$, be determined by

$$\frac{1}{\text{div}}\, \rho(x) = \vec{\mathfrak{X}}(x) \longrightarrow \text{div}\, \vec{\mathfrak{X}} = \rho, \ \text{rot}\, \vec{\mathfrak{X}} = 0. \quad (64)$$

The operator (1/div) is identical with the operator $-(1/4\pi)$ New of G i b b s [46]). Finally we put

$$\frac{1}{\nabla} \cdot \frac{1}{\text{div}} = \frac{1}{\Delta}. \quad (65)$$

This operator is identical with the operator $-(1/4\pi)$ Pot of G i b b s.

Splitting up $\mathfrak{E}$ into a longitudinal and a transversal field, we can write

$$\frac{1}{8\pi}\int \mathfrak{E}^2 = \frac{1}{8\pi}\int \mathfrak{E}_{tr}^2 + \frac{1}{8\pi}\int \mathfrak{E}_{long}^2. \quad (66)$$

Since from (52) follows

$$\mathfrak{E}_{long} + \frac{1}{\text{div}}\frac{\dot{\mathfrak{S}}}{c} = \frac{4\pi e}{\text{div}}\rho, \quad (67)$$

we derive by an "integration by parts":

$$-\tfrac{1}{2}\int e\rho \frac{4\pi}{\Delta} e\rho = \frac{1}{8\pi}\int \left(\mathfrak{E}_{long} + \frac{1}{\text{div}}\frac{\dot{\mathfrak{S}}}{c}\right)^2 =$$

$$= \frac{1}{8\pi}\int \mathfrak{E}_{long}^2 + \frac{1}{4\pi}\int \mathfrak{E}_{long}\frac{1}{\text{div}}\frac{\dot{\mathfrak{S}}}{c} - \frac{1}{8\pi}\int \frac{\dot{\mathfrak{S}}}{c}\frac{1}{\Delta}\frac{\dot{\mathfrak{S}}}{c}. \quad (68)$$

Here, we have made use of the commutativity of

$$\dot{\mathfrak{S}}/c = 4\pi e\rho - \text{div}\,\mathfrak{E} \quad \text{with} \quad \mathfrak{E}_{long}.$$

From (68) we find

$$\frac{1}{8\pi}\int \mathfrak{E}_{long}^2 = \frac{1}{8\pi}\int\left(\frac{1}{\Delta}\frac{\dot{\mathfrak{S}}}{c} + \frac{2}{\nabla}\mathfrak{E}_{long}\right)\frac{\dot{\mathfrak{S}}}{c} + \frac{e^2}{2}\iint\frac{\rho(\vec{x})\,\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}. \qquad (69)$$

On account of (50) the first term of the right hand member of this equation, if operating on the situation function χ, does not give any contribution. Such a term we shall call a *zero-term*. Now, all *real* *) zero-terms O in the Hamiltonian are of no physical interest, since the contributions given by such terms O to the commutators of any observable $\mathcal{F}$ with $\mathcal{H}$, have no matrix elements between states satisfying $O\chi$ = zero, which do not vanish. This follows from

$$\sum_{N_1,N_2\ldots}\chi_1^*\,[\mathcal{F};O]\,\chi_2 = \sum_{N_1,N_2,\ldots}\{\chi_1^*\,\mathcal{F}(O\chi_2) - (O\chi_1)^*\mathcal{F}\chi_2\} = \text{zero}. \qquad (70)$$

For this reason we may change $(1/8\pi)\int\mathfrak{E}_{long}^2$ in the Hamiltonian into the static Coulomb interaction

$$\frac{e^2}{2}\iint\frac{\rho(\vec{x})\,\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}. \qquad (69a)$$

For the same reason the term $(1/4\pi)\,(\mathrm{div}\,\mathfrak{A} - \tfrac{1}{2}\mathfrak{S})\mathfrak{S}$ may be omitted from the Hamiltonian (46). Indeed these zero-terms are of importance only in the Lagrangian function, where they must serve for the construction of a consistent theory, but for practical purposes they are of no interest.

The electrostatic potential occurs in the Hamiltonian in the terms

$$(1/4\pi)\int\mathfrak{V}(4\pi e\rho - \mathrm{div}\,\mathfrak{E}) = \int\mathfrak{V}\dot{\mathfrak{S}}/4\pi c. \qquad (71)$$

Also these terms can be omitted.

In order to remove the longitudinal vector-potential from the Hamiltonian we perform a *canonical transformation*, which was indicated by Fermi [38]) [36]). This transformation is given by

$$\chi = e^{iF}\,\hat{\chi}, \qquad \chi^* = \hat{\chi}^*\,e^{-iF}, \qquad (72)$$

so that

$$\hat{\mathcal{F}} = e^{-iF}\,\mathcal{F}\,e^{iF} = \sum_{n=0}^{\infty}(1/n!)\,[\{\mathcal{F}\};iF]^{(n)}. \qquad (73)$$

*) A q-number is called *real*, if it is a self-adjoint (Hermitian) matrix with respect to the partition numbers $N_1, N_2, \ldots\ldots$

Here we have put

$$\mathcal{F} \equiv [\{\mathcal{F}\};\ iF]^{(0)}, \qquad [\mathcal{F};\ iF]_- \equiv [\{\mathcal{F}\};\ iF]^{(1)},$$

generally (74)

$$[[\{\mathcal{F}\};\ iF]^{(n)};\ iF]_- \equiv [\{\mathcal{F}\};\ iF]^{(n+1)}.$$

Then

$$\tilde{\mathcal{F}} = \sum_{N_1,N_2,\ldots} \chi^* \mathcal{F} \chi = \sum_{N_1,N_2,\ldots} \hat{\chi}^* \hat{\mathcal{F}} \hat{\chi}, \tag{75}$$

so that the physical situation can be described by $\hat{\mathcal{F}}$ and $\hat{\chi}$ as well as by $\mathcal{F}$ and χ. According to F e r m i [38]) [36]) one must put

$$F = F_1 + F_2; \tag{76}$$

$$iF_1 = \ (ie/\hbar c) \int \rho \, \frac{1}{\nabla} \cdot \mathfrak{A}_{long} = \ (e/i\hbar c) \int \mathfrak{A}_{long} \cdot \frac{1}{\mathrm{div}} \rho, \tag{76a}$$

$$iF_2 = (i/4\pi\hbar c) \int \mathfrak{B} \,\mathrm{div}\, \mathfrak{A}_{long} = (1/4\pi i\hbar c) \int \mathfrak{A}_{long} \cdot \nabla \mathfrak{B}. \tag{76b}$$

In order to eliminate $\mathfrak{A}_{long}$ from the H a m i l t o n i a n, the transformation with F_1 is sufficient. We shall calculate, how the variables still occurring in the H a m i l t o n i a n are transformed by it.

For this purpose we must calculate the high-order commutators occurring in the last member of (73). We remark that, among the variables occurring in the H a m i l t o n i a n, only the wave-functions of charged particles and mesons do not commute with (76*a*). The longitudinal electric field strength, which does not commute with $\mathfrak{A}_{long}$, has already been eliminated from the H a m i l t o n i a n in the foregoing.

Now, from [40])

$$e\rho = (e/i\hbar) \sum_{q(P,\pi,mes)} \bar{p}q \tag{77}$$

(summation over all canonical co-ordinates $q(x)$ describing the annihilation of protons, positons or positive mesons, or the creation of antiprotons, negatons or negative mesons), we deduce with the help of the commutation relations (42):

$$\begin{aligned} [q_e(x);\ \rho(x')]_- &= \ \delta(x - x')\, q_e(x'), \\ [p_e(x);\ \rho(x')]_- &= -\,\delta(x - x')\, p_e(x'), \end{aligned} \tag{78}$$

for all canonical co-ordinates q_e describing a decrease of the total electric charge by e, and their canonical conjugates p_e describing an increase by e. Thus, we find the recursion formulae

$$\begin{aligned} [\{q_e(x)\};\ iF_1]^{(n+1)} &= -\left(\frac{e}{i\hbar c}\frac{1}{\nabla}\cdot\mathfrak{A}_{long}\right)[\{q_e(x)\};\ iF_1]^{(n)}, \\ [\{p_e(x)\};\ iF_1]^{(n+1)} &= +\left(\frac{e}{i\hbar c}\frac{1}{\nabla}\cdot\mathfrak{A}_{long}\right)[\{p_e(x)\};\ iF_1]^{(n)}, \end{aligned} \tag{79}$$

so that (73) yields

$$\begin{aligned} \hat{q}_e &= e^{il}\, q_e; \qquad \hat{p}_e = p_e\, e^{-il}; \\ l &\equiv (e/\hbar c)\,(1/\nabla)\,.\,\mathfrak{A}_{long}\,. \end{aligned} \tag{80}$$

These formulae can be applied to ψ_P, ψ_π, **A**, **E**, **Y**, **W** and their canonical conjugates. From the fact that the wave-functions of the neutrons and neutrinos are not transformed at all it can be deduced that **s**, **v**, **a**, ..., as defined by (39), transform like the q_e according to (80), whereas **s***, **v***, ..., transform like the p_e.

The derived variables were defined by the "identities" (45), which contain the gradient operator and the vector potential in the combinations

$$\{\nabla + (e/i\hbar c)\,\mathfrak{A}\}\,q_e \text{ and } \{\nabla - (e/i\hbar c)\,\mathfrak{A}\}\,p_e \tag{81}$$

only. By the transformation (72), (76*a*) these expressions change on account of (80) into

$$\begin{aligned} \{\nabla + (e/i\hbar c)\,\hat{\mathfrak{A}}\}\,\hat{q}_e &= e^{il}\,\{\nabla + (e/i\hbar c)\,\mathfrak{A}\}\,q_e + (\nabla e^{il})\,.\,q_e = \\ &= e^{il}\,\{\nabla + (e/i\hbar c)\,\mathfrak{A}_{tr}\}\,q_e; \\ \{\nabla - (e/i\hbar c)\,\hat{\mathfrak{A}}\}\,\hat{p}_e &= (\{\nabla - (e/i\hbar c)\,\mathfrak{A}_{tr}\}\,p_e)\,e^{-il}. \end{aligned} \tag{82}$$

Thus the longitudinal vector-potential $\mathfrak{A}_{long}$ is eliminated from these expressions. In a similar way it disappears from the expression $\mathcal{P}^{kin}_{op}\,\psi$.

The *transformed Hamiltonian*

$$\hat{H} = \int c\varkappa\,\{\hat{\mathbf{S}}^*\hat{\mathbf{S}} + \hat{\mathbf{A}}^*.\,\hat{\mathbf{A}} + \ldots\ldots, \tag{46a}$$

in which the derived variables are "defined" by the *transformed identities*

$$\varkappa(\hat{\mathbf{V}} + \hat{\mathbf{v}}) + (\nabla + (e/i\hbar c)\,\hat{\mathfrak{A}}\,.\,\hat{\mathbf{E}}) = 0, \text{ etc.} \tag{45a}$$

can now be *expressed in terms of the original matrices* ψ, $\mathbf{A}$, $\mathbf{E}$, ..., by inserting the expressions (80) and (82) for the transformed variables into (46*a*)—(45*a*). Then all factors e^{-il} and e^{il} arising from (80) and (82) cancel each other. *The longitudinal vector potential disappears entirely.* Thus we find the *transformed* Hamiltonian H' expressed in terms of the *original canonical* variables, but the matrices $\mathfrak{E}_{long}$, $\mathfrak{B}$, $\mathfrak{S}$ and $\mathfrak{A}_{long}$ do no longer occur in it.

It must be pointed out, however, that, if we want to calculate the matrix element of some observable by means of the transformed situation function $\hat{\chi}$, we must use, according to (75), the *transformed* q-numbers, and not the original matrices occurring now in the Hamiltonian.

For instance, $\mathfrak{E}$ is changed by the transformation (72)—(73)—(76*a*) into *) $\hat{\mathfrak{E}} = \mathfrak{E} + (4\pi e/\text{div})\,\rho$. Since from this expression the original matrix of the longitudinal field $\mathfrak{E}_{long}$ has not yet been eliminated, it is impossible to calculate the expectancy value or a matrix element of the electric field, if (1°) the dependence of the situation function on the partition numbers denoting the numbers of longitudinal "photons", or (2°) the way, in which $\mathfrak{E}_{long}$ operates on this situation function, is not known. This means that, though the longitudinal field does no more occur in the Hamiltonian, it *has not yet been eliminated entirely from the theory.*

For this purpose, the transformation with e^{iF_2} *was introduced* by Fermi [38]) [36]). It is easily seen from (73) and (76*b*) that among the canonical variables only $\mathfrak{E}_{long}$ and $\mathfrak{S}$ are changed by this second transformation. It turns out that by the combined transformation (76) $\mathfrak{E}_{long}$ and $\mathfrak{S}$ are changed into *)

$$\hat{\mathfrak{E}}_{long} = \mathfrak{E}_{long} + \nabla\mathfrak{B} + (4\pi e/\text{div})\,\rho = (4\pi e/\text{div})\,\rho - \dot{\mathfrak{A}}_{long}/c. \tag{83}$$

$$\hat{\mathfrak{S}} = \mathfrak{S} - \text{div}\,\mathfrak{A} = \dot{\mathfrak{B}}/c. \tag{84}$$

From (83) and (52) we find

$$\dot{\hat{\mathfrak{S}}} = \text{div}\,\dot{\mathfrak{A}}_{long}. \tag{85}$$

Now, from (72)—(73) and (50) follows

$$\hat{\mathfrak{S}}\hat{\chi} = \dot{\hat{\mathfrak{S}}}\hat{\chi} = 0, \tag{86}$$

*) Compare the foot-note on page 72.

so that we find from (84) and (85)

$$\dot{\mathfrak{B}}\hat{\chi} = 0, \tag{87}$$

$$\text{div}\ \dot{\mathfrak{A}}_{long}\hat{\chi} = 0. \tag{88a}$$

Since rot $\dot{\mathfrak{A}}_{long} = 0$, we can write — always on account of (49) — :

$$\dot{\mathfrak{A}}_{long}\hat{\chi} = 0. \tag{88}$$

This means that by a description of the physical situation by means of the *transformed* situation function, the *original* matrices of the scalar potential and the longitudinal vector-potential, if operating on the new situation function, multiply it by a quantity, which does not depend on the time. Now the electromagnetic field strengths are given by (compare (45), (83))

$$\begin{aligned} \hat{\mathfrak{H}} &= \text{rot}\ \hat{\mathfrak{A}}_{tr} = \text{rot}\ \mathfrak{A}_{tr}, \\ \hat{\mathfrak{E}} &= (4\pi e/\text{div})\ \rho + \dot{\mathfrak{A}}_{tr}/c, \end{aligned} \tag{89}$$

where we have omitted from (83) *the term with* $\dot{\mathfrak{A}}_{long}$, *which on account of* (88) *has only vanishing matrix elements between states satisfying* (49) *or* (86). If only $\mathfrak{A}_{tr}$, $\dot{\mathfrak{A}}_{tr}$ and ρ are expressed in terms of Jordan-Wigner and Jordan-Klein matrices operating on a *situation function, the dependence of which on the partition numbers of the longitudinal field is not known,* we still can compute the total electromagnetic field strengths from (89), so that we may say that we have succeeded in *eliminating the longitudinal "photons" completely from our calculations* *).

This was possible only since we had the extra condition (49) *on the situation function at our disposal.* Since such an extra condition does not exist for any of the field components of the meson field, a *complete* elimination of some part of the *meson* field seems to be *impossible* [11]) [37]) (compare § 4).

*) That is to say, for the calculation of the matrix elements of $\mathfrak{B}$ and $\mathfrak{A}_{long}$ themselves — which are not transformed at all by the Fermi transformation (72)–(76) — it would be necessary to know the dependence of $\hat{\chi}$ on the numbers of longitudinal photons. However, these quantities, which according to (87) and (88) are constants, are of no interest for physical problems.

§ 6. *Discussion of the Hamiltonian.* Now we can insert (45) and (69) into the Hamiltonian $\mathcal{H}$ (46), omit all real zero-terms (compare (70)), transform by (72) — (76) and express the transformed q-numbers in terms of the original ones by means of (80) — (82). In this way we find the transformed Hamiltonian expressed in terms of the original canonical variables. The result can be written in the following form (we write $\mathcal{H}$ instead of $\hat{\mathcal{H}}$ in the following):

$$\hat{\mathcal{H}} = \int H\, dx\, dy\, dz; \qquad H = H_0 + H_C + H_e + H_{ee} + H_g + H_{gg} + H_{eg}. \tag{90}$$

$$H_0 = c\varkappa\{\mathbf{A}^*.\mathbf{A} + \mathbf{E}^*.\mathbf{E} + \mathbf{W}^*\mathbf{W} + \mathbf{Y}^*\mathbf{Y} + \overline{\mathbf{A}}^2 + \overline{\mathbf{E}}^2 + \overline{\mathbf{W}}^2 + \overline{\mathbf{Y}}^2\} + (c/\varkappa)\{(\mathrm{rot}\,\mathbf{A}^*.\,\mathrm{rot}\,\mathbf{A}) + \mathrm{div}\,\mathbf{E}^*.\,\mathrm{div}\,\mathbf{E} + (\nabla\mathbf{Y}^*.\,\nabla\mathbf{Y}) + (\mathrm{rot}\,\overline{\mathbf{A}})^2 + (\mathrm{div}\,\overline{\mathbf{E}})^2 + (\nabla\overline{\mathbf{Y}})^2\} + (1/8\pi)\{\mathfrak{E}_{tr}^2 + (\mathrm{rot}\,\mathfrak{A}_{tr})^2\} + \sum_{(P,N,\pi,\nu)} \psi^\dagger(mc^2\boldsymbol{\beta} + c\boldsymbol{\alpha}.\mathcal{P}_{op})\,\psi.$$

$$H_C = \frac{e^2}{2}\int\int \frac{\rho(\vec{x})\,\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}; \quad (e\rho \text{ is given by (48)}).$$

$$H_e = -e\,\mathfrak{A}_{tr}\,.\,j^{(0)}/c;$$

$$ej^{(0)}/c = e\sum_{(P,\pi)}\psi^\dagger\boldsymbol{\alpha}\psi + (e/i\hbar\varkappa)\{[\mathbf{A}^*, \mathrm{rot}\,\mathbf{A}] + [\mathrm{rot}\,\mathbf{A}^*, \mathbf{A}] + \mathbf{E}^*\,\mathrm{div}\,\mathbf{E} - \mathrm{div}\,\mathbf{E}^*.\mathbf{E} + \mathbf{Y}^*\nabla\mathbf{Y} - \nabla\mathbf{Y}^*.\mathbf{Y}\}.$$

$$H_{ee} = (e^2/c\varkappa\hbar^2)\{(\mathfrak{A}.\mathbf{E}^*)(\mathfrak{A}.\mathbf{E}) - (\mathfrak{A}.\mathbf{A}^*)(\mathfrak{A}.\mathbf{A}) + \mathfrak{A}^2(\mathbf{A}^*.\mathbf{A} + \mathbf{Y}^*\mathbf{Y})\},$$

$$H_g = c\varkappa\{\mathbf{A}^*.\mathbf{a} + \mathbf{E}^*.\mathbf{e} + \mathbf{W}^*\mathbf{w} + \mathbf{Y}^*\mathbf{y} + \mathbf{a}^*.\mathbf{A} + \mathbf{e}^*.\mathbf{E} + \mathbf{w}^*\mathbf{W} + \mathbf{y}^*\mathbf{Y}\} + c\{\mathrm{div}\,\mathbf{E}^*.\,\mathbf{v} - (\mathrm{rot}\,\mathbf{A}^*.\,\mathbf{h}) - (\nabla\mathbf{Y}^*.\,\mathbf{b}) + \mathbf{v}^*\,\mathrm{div}\,\mathbf{E} - (\mathbf{h}^*.\,\mathrm{rot}\,\mathbf{A}) - (\mathbf{b}^*.\nabla\mathbf{Y})\} + 2c\varkappa\{\overline{\mathbf{A}}.\overline{\mathbf{a}} + \overline{\mathbf{E}}.\overline{\mathbf{e}} + \overline{\mathbf{W}}\overline{\mathbf{w}} + \overline{\mathbf{Y}}\overline{\mathbf{y}}\} + 2c[\overline{\mathbf{v}}\,\mathrm{div}\,\overline{\mathbf{E}} - (\overline{\mathbf{h}}.\,\mathrm{rot}\,\overline{\mathbf{A}}) - (\overline{\mathbf{b}}.\nabla\overline{\mathbf{Y}})\}.$$

$$H_{gg} = c\varkappa\{(1-C_0)\mathbf{s}^*\mathbf{s} + C_1\mathbf{a}^*.\mathbf{a} + (1-C_1)\mathbf{v}^*\mathbf{v} + C_2\mathbf{e}^*.\mathbf{e} + (1-C_2)\mathbf{h}^*.\mathbf{h} + C_3\mathbf{w}^*\mathbf{w} + (1-C_3)\mathbf{b}^*.\mathbf{b} + C_4\mathbf{y}^*\mathbf{y} + (1-\overline{C}_0)\overline{\mathbf{s}}^2 + \overline{C}_1\overline{\mathbf{a}}^2 + (1-\overline{C}_1)\overline{\mathbf{v}}^2 + \overline{C}_2\overline{\mathbf{e}}^2 + (1-\overline{C}_2)\overline{\mathbf{h}}^2 + \overline{C}_3\overline{\mathbf{w}}^2 + (1-\overline{C}_3)\overline{\mathbf{b}}^2 + \overline{C}_4\overline{\mathbf{y}}^2\}.$$

$$H_{eg} = -e\,\mathfrak{A}_{tr}.\,j_g/c;$$

$$ej_g/c = -(e/i\hbar)\{[\mathbf{A}^*, \mathbf{h}] + [\mathbf{h}^*, \mathbf{A}] - \mathbf{E}^*\mathbf{v} + \mathbf{v}^*\mathbf{E} + \mathbf{Y}^*\mathbf{b} - \mathbf{b}^*\mathbf{Y}\}.$$

For a discussion of this Hamiltonian we expand all canonical variables in series of plane waves. For this purpose we introduce for every Fourier component of the field a set of complex unit vectors $\vec{\mathfrak{c}}^{\mu}_{\vec{p}}$ defined by

$$\vec{\mathfrak{c}}^{0}_{\vec{p}} = \vec{p}/p; \quad \vec{\mathfrak{c}}^{\mu *}_{\vec{p}} = \vec{\mathfrak{c}}^{-\mu}_{\vec{p}}; \quad (\vec{\mathfrak{c}}^{\mu *}_{\vec{p}} . \vec{\mathfrak{c}}^{\mu'}_{\vec{p}}) = \delta_{\mu\mu'}; \quad [\vec{\mathfrak{c}}^{0}_{\vec{p}}, \vec{\mathfrak{c}}^{\eta}_{\vec{p}}] = - i\eta\, \vec{\mathfrak{c}}^{\eta}_{\vec{p}};$$
$$[\vec{\mathfrak{c}}^{\eta *}_{\vec{p}}, \vec{\mathfrak{c}}^{\eta}_{\vec{p}}] = i\eta\, \vec{\mathfrak{c}}^{0}_{\vec{p}}; \quad \vec{\mathfrak{c}}^{-\mu}_{-\vec{p}} = - \vec{\mathfrak{c}}^{\mu}_{\vec{p}}; \quad (\mu = -1, 0, 1; \quad \eta = \pm 1). \tag{91}$$

As usual we shall expand the field components in a hypothetical cube (volume $= \Omega$), in which the fields are assumed to be periodic:

$$\vec{\mathbf{A}}(x) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{p}} \sum_{\mu=-1}^{1} \mathsf{A}_{\vec{p},\mu}\, \vec{\mathfrak{c}}^{\mu}_{\vec{p}}\, e^{(i/\hbar)\, \vec{p}.\vec{x}} \quad ;$$
$$\vec{\mathbf{A}}^{*}(x) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{p}} \sum_{\mu=-1}^{1} \mathsf{A}^{*}_{\vec{p},\mu}\, \vec{\mathfrak{c}}^{\mu *}_{\vec{p}}\, e^{-(i/\hbar)\, \vec{p}.\vec{x}};$$
$$\mathbf{W}(x) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{p}} \mathsf{W}_{\vec{p}}\, e^{(i/\hbar)\, \vec{p}.\vec{x}} \quad ;$$
$$\mathbf{W}^{*}(x) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{p}} \mathsf{W}^{*}_{\vec{p}}\, e^{-(i/\hbar)\, \vec{p}.\vec{x}} \quad . \tag{92}$$

In a similar way the amplitudes of $\vec{\mathbf{E}}$ and $\mathbf{Y}$ will be called $\mathsf{E}_{\vec{p},\mu}$ and $\mathsf{Y}_{\vec{p}}$. The real fields $\vec{\overline{\mathbf{A}}}, \vec{\overline{\mathbf{E}}}, \overline{\mathbf{W}}, \overline{\mathbf{Y}}, \mathfrak{A}_{tr}, \mathfrak{E}_{tr}$ will be expanded according to

$$\vec{\overline{\mathbf{A}}}(x) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{p}} \sum_{\mu=-1}^{1} \overline{\mathsf{A}}_{\vec{p},\mu}\, \vec{\mathfrak{c}}^{\mu}_{\vec{p}}\, e^{(i/\hbar)\, \vec{p}.\vec{x}}; \quad (\overline{\mathsf{A}}^{*}_{\vec{p},\mu} \equiv - \overline{\mathsf{A}}_{-\vec{p},\mu});$$
$$\overline{\mathbf{W}}(x) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{p}} \overline{\mathsf{W}}_{\vec{p}}\, e^{(i/\hbar)\, \vec{p}.\vec{x}} \quad ; \quad (\overline{\mathsf{W}}^{*}_{\vec{p}} \equiv + \overline{\mathsf{W}}_{-\vec{p}}); \tag{93}$$
$$\vec{\mathfrak{A}}_{tr}(x) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{p}} \sum_{\eta=\pm 1} \mathfrak{a}_{\vec{p},\eta}\, \vec{\mathfrak{c}}^{\eta}_{\vec{p}}\, e^{(i/\hbar)\, \vec{p}.\vec{x}}; \quad (\mathfrak{a}^{*}_{\vec{p},\eta} \equiv - \mathfrak{a}_{-\vec{p},\eta}); \text{ etc.}$$

From the commutation relations (42.E-B) between these canonical field components, the commutation relations between the amplitudes can be easily derived, since *)

$$\delta_{jj'}\,\delta(\vec{x}-\vec{x}')=\frac{1}{\Omega}\sum_{\vec{p}}\sum_{\mu=-1}^{1}(\mathfrak{c}^{\mu *}_{\vec{p}})_{j'}\,(\mathfrak{c}^{\mu}_{\vec{p}})_{j}\,e^{(i/\hbar)\,\vec{p}.(\vec{x}-\vec{x}')},$$
$$\frac{1}{\Omega}\int(\vec{\mathfrak{c}}^{\mu' *}_{\vec{p}'}\,.\,\vec{\mathfrak{c}}^{\mu}_{\vec{p}})\,e^{(i/\hbar)\,(\vec{p}-\vec{p}').\vec{x}}\,d\vec{x}=\delta_{\mu\mu'}\,.\,\delta(\vec{p}-\vec{p}')\quad. \tag{94}$$

Thus we find:

$$\begin{aligned}
&[\mathsf{E}_{\vec{p},\mu}\,;\,\mathsf{A}^*_{\vec{p}',\mu'}]_-=[\mathsf{E}^*_{\vec{p},\mu}\,;\,\mathsf{A}_{\vec{p}',\mu'}]_-=i\hbar\,\delta_{\mu\mu'}\,\delta(\vec{p}-\vec{p}');\\
&[\mathsf{W}_{\vec{p}};\,\mathsf{Y}^*_{\vec{p}'}]_-=[\mathsf{W}^*_{\vec{p}}\,;\,\mathsf{Y}_{\vec{p}'}]_-=i\hbar\,\delta(\vec{p}-\vec{p}');\\
&[\bar{\mathsf{E}}_{\vec{p},\mu}\,;\,\bar{\mathsf{A}}^*_{\vec{p}',\mu'}]_-\equiv[\bar{\mathsf{A}}_{-\vec{p}',\mu'}\,;\,\bar{\mathsf{E}}_{\vec{p},\mu}]_-=(i\hbar/2)\,\delta_{\mu\mu'}\,\delta(\vec{p}-\vec{p}');\\
&[\bar{\mathsf{W}}_{\vec{p}};\,\bar{\mathsf{Y}}^*_{\vec{p}'}]_-\equiv[\bar{\mathsf{W}}_{\vec{p}}\,;\,\bar{\mathsf{Y}}_{-\vec{p}'}]_-=(i\hbar/2)\,\delta(\vec{p}-\vec{p}');\\
&[\mathfrak{e}_{\vec{p},\eta}\,;\,\mathfrak{a}^*_{\vec{p}',\eta'}]_-\equiv[\mathfrak{a}_{-\vec{p}',\eta'}\,;\,\mathfrak{e}_{\vec{p},\eta}]_-=4\pi i\hbar c\,\delta_{\eta\eta'}\,\delta(\vec{p}-\vec{p}');
\end{aligned} \tag{95}$$

all other pairs being commutative with each other.

Following the method of Pauli and Weisskopf [15]) we put

$$\begin{aligned}
&\mathsf{W}_{\vec{p}}=i\mathfrak{f}_{\vec{p}}\sqrt{\frac{\hbar}{2}}(\boldsymbol{a}_{\vec{p}}+\boldsymbol{c}^*_{-\vec{p}}), &&\bar{\mathsf{W}}_{\vec{p}}=i\mathfrak{f}'_{\vec{p}}\sqrt{\frac{\hbar}{4}}(\boldsymbol{m}_{\vec{p}}-\boldsymbol{m}^*_{-\vec{p}}),\\
&\mathsf{Y}_{\vec{p}}=\frac{1}{\mathfrak{f}_{\vec{p}}}\sqrt{\frac{\hbar}{2}}(\boldsymbol{a}_{\vec{p}}-\boldsymbol{c}^*_{-\vec{p}}), &&\bar{\mathsf{Y}}_{\vec{p}}=\frac{1}{\mathfrak{f}'_{\vec{p}}}\sqrt{\frac{\hbar}{4}}(\boldsymbol{m}_{\vec{p}}+\boldsymbol{m}^*_{-\vec{p}}),\\
&\mathsf{A}_{\vec{p},0}=\mathfrak{f}_{\vec{p},0}\sqrt{\frac{\hbar}{2}}(\boldsymbol{b}_{\vec{p},0}-\boldsymbol{d}^*_{-\vec{p},0}), &&\bar{\mathsf{A}}_{\vec{p},0}=\mathfrak{f}'_{\vec{p},0}\sqrt{\frac{\hbar}{4}}(\boldsymbol{n}_{\vec{p},0}-\boldsymbol{n}^*_{-\vec{p},0}),\\
&\mathsf{A}_{\vec{p},\eta}=\frac{1}{\mathfrak{f}_{\vec{p},\eta}}\sqrt{\frac{\hbar}{2}}(\boldsymbol{b}_{\vec{p},\eta}-\boldsymbol{d}^*_{-\vec{p},\eta}), &&\bar{\mathsf{A}}_{\vec{p},\eta}=\frac{1}{\mathfrak{f}'_{\vec{p},\eta}}\sqrt{\frac{\hbar}{4}}(\boldsymbol{n}_{\vec{p},\eta}-\boldsymbol{n}^*_{-\vec{p},\eta}),\\
&\mathsf{E}_{\vec{p},0}=\frac{i}{\mathfrak{f}_{\vec{p},0}}\sqrt{\frac{\hbar}{2}}(\boldsymbol{b}_{\vec{p},0}+\boldsymbol{d}^*_{-\vec{p},0}), &&\bar{\mathsf{E}}_{\vec{p},0}=\frac{i}{\mathfrak{f}'_{\vec{p},0}}\sqrt{\frac{\hbar}{4}}(\boldsymbol{n}_{\vec{p},0}+\boldsymbol{n}^*_{-\vec{p},0}),\\
&\mathsf{E}_{\vec{p},\eta}=i\mathfrak{f}_{\vec{p},\eta}\sqrt{\frac{\hbar}{2}}(\boldsymbol{b}_{\vec{p},\eta}+\boldsymbol{d}^*_{-\vec{p},\eta}), &&\bar{\mathsf{E}}_{\vec{p},\eta}=i\mathfrak{f}'_{\vec{p},\eta}\sqrt{\frac{\hbar}{4}}(\boldsymbol{n}_{\vec{p},\eta}+\boldsymbol{n}^*_{-\vec{p},\eta}),\\
&\mathfrak{a}_{\vec{p},\eta}=\frac{1}{\mathfrak{g}_{\vec{p},\eta}}\sqrt{2\pi\hbar c}\,(\boldsymbol{l}_{\vec{p},\eta}-\boldsymbol{l}^*_{-\vec{p},\eta}), &&\mathfrak{e}_{\vec{p},\eta}=i\mathfrak{g}_{\vec{p},\eta}\sqrt{2\pi\hbar c}\,(\boldsymbol{l}_{\vec{p},\eta}+\boldsymbol{l}^*_{-\vec{p},\eta}).
\end{aligned} \tag{96}$$

*) For calculations like that of $\hat{\mathfrak{E}}$ (83) and $\hat{\mathfrak{S}}$ (84) in the preceding section, it is convenient to introduce the *longitudinal* and the *transversal* δ-functions defined by

$$\delta^{long}_{jj'}(\vec{x}-\vec{x}')=\frac{1}{\Omega}\sum_{\vec{p}}(\mathfrak{c}^{0}_{\vec{p}})_{j'}\,(\mathfrak{c}^{0}_{\vec{p}})_{j}\,e^{(i/\hbar)\,\vec{p}.(\vec{x}-\vec{x}')}\quad\text{and}$$
$$\delta^{long}_{jj'}(\vec{x}-\vec{x}')+\delta^{tr}_{jj'}(\vec{x}-\vec{x}')=\delta_{jj'}\,\delta(\vec{x}-\vec{x}').$$

Compare Novobatzky, loc. cit. [47]), formula (28).

Here $f_{\vec{p}}$, $f_{\vec{p},0}$, $f_{\vec{p},\eta}$, $f'_{\vec{p}}$, $f'_{\vec{p},0}$, $f'_{\vec{p},\eta}$ and $g_{\vec{p},\eta}$ $(\eta = \pm 1)$ are real constants depending on $\vec{p}$ (and η respectively) and satisfying

$$f'_{\vec{p}} = f'_{-\vec{p}}, \quad f'_{\vec{p},\mu} = f'_{-\vec{p},\mu}; \quad g_{\vec{p},\eta} = g_{-\vec{p},\eta}. \tag{97}$$

We shall choose them afterwards in a convenient way. On account of (95)—(96) the matrices $\boldsymbol{a}_{\vec{p}}$, $\boldsymbol{b}_{\vec{p},\mu}$, $\boldsymbol{c}_{\vec{p}}$, $\boldsymbol{d}_{\vec{p},\mu}$, $\boldsymbol{l}_{\vec{p},\eta}$, $\boldsymbol{m}_{\vec{p}}$ and $\boldsymbol{n}_{\vec{p},\mu}$ satisfy the following commutation relations:

$$[\boldsymbol{a}_{\vec{p}}\,;\boldsymbol{a}^*_{\vec{p}'}]_- = [\boldsymbol{c}_{\vec{p}}\,;\boldsymbol{c}^*_{\vec{p}'}]_- = [\boldsymbol{m}_{\vec{p}}\,;\boldsymbol{m}^*_{\vec{p}'}]_- = \delta(\vec{p}-\vec{p}'),$$
$$[\boldsymbol{b}_{\vec{p},\mu}\,;\boldsymbol{b}^*_{\vec{p}',\mu'}]_- = [\boldsymbol{d}_{\vec{p},\mu}\,;\boldsymbol{d}^*_{\vec{p}',\mu'}]_- = [\boldsymbol{n}_{\vec{p},\mu}\,;\boldsymbol{n}^*_{\vec{p}',\mu'}]_- = \delta_{\mu\mu'}\,\delta(\vec{p}-\vec{p}'), \tag{98}$$
$$[\boldsymbol{l}_{\vec{p},\eta}\,;\boldsymbol{l}^*_{\vec{p}',\eta'}]_- = \delta_{\eta\eta'}\,\delta(\vec{p}-\vec{p}'); \quad (\mu,\mu' = -1,0,1; \quad \eta,\eta' = \pm 1);$$

all other pairs commuting with each other.

So $\boldsymbol{a}^*_{\vec{p}}\boldsymbol{a}_{\vec{p}}$, $\boldsymbol{b}^*_{\vec{p},\mu}\boldsymbol{b}_{\vec{p},\mu}$, $\boldsymbol{c}^*_{\vec{p}}\boldsymbol{c}_{\vec{p}}$, $\boldsymbol{d}^*_{\vec{p},\mu}\boldsymbol{d}_{\vec{p},\mu}$, $\boldsymbol{l}^*_{\vec{p},\eta}\boldsymbol{l}_{\vec{p},\eta}$, $\boldsymbol{m}^*_{\vec{p}}\boldsymbol{m}_{\vec{p}}$ and $\boldsymbol{n}^*_{\vec{p},\mu}\boldsymbol{n}_{\vec{p},\mu}$ possess the eigenvalues 0, 1, 2, 3, ... and the q-numbers introduced by (96) can be regarded as ordinary Jordan-Klein matrices.

We shall now expand the particle wave fields ψ in series of plane waves according to [48])

$$\psi(\vec{x}) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{p}} \sum_{\sigma=\pm 1} (\boldsymbol{v}_{\vec{p},\sigma}\, u_{\vec{p},\sigma} + \boldsymbol{w}^*_{-\vec{p},\sigma}\, u^{\mathfrak{C}}_{-\vec{p},\sigma})\, e^{(i/\hbar)\,\vec{p}.\vec{x}}, \tag{99}$$

where the $u_{\vec{p},\sigma}$ are normalized four-component "spin-functions" (undors of the first rank [31])), which satisfy

$$\begin{aligned} \{mc\boldsymbol{\beta} + (\vec{p}\,.\,\vec{\boldsymbol{\alpha}})\}\, u_{\vec{p},\sigma} &= +\sqrt{(mc)^2 + p^2}\,.\,u_{\vec{p},\sigma}\,; \\ (\vec{p}\,.\,\vec{\boldsymbol{\sigma}})\; u_{\vec{p},\sigma} &= \sigma p\, u_{\vec{p},\sigma} \qquad ;\ (\sigma = \pm 1); \end{aligned} \tag{100}$$

so that the charge-conjugated [48]) [31]) of $u_{-\vec{p},\sigma}$ satisfies

$$\begin{aligned} \{mc\boldsymbol{\beta} + (\vec{p}\,.\,\vec{\boldsymbol{\alpha}})\}\, u^{\mathfrak{C}}_{-\vec{p},\sigma} &= -\sqrt{(mc)^2 + p^2}\,.\,u^{\mathfrak{C}}_{-\vec{p},\sigma}\,; \\ (\vec{p}\,.\,\vec{\boldsymbol{\sigma}})\; u^{\mathfrak{C}}_{-\vec{p},\sigma} &= \sigma p\, u^{\mathfrak{C}}_{-\vec{p},\sigma}\,. \end{aligned} \tag{100$\mathfrak{C}$}$$

The matrices $\boldsymbol{v}_{\vec{p},\sigma}$ and $\boldsymbol{w}_{\vec{p},\sigma}$ are ordinary Jordan-Wigner matrices, since from (42.F-D) follows

$$\begin{aligned} [\boldsymbol{v}_{\vec{p},\sigma}\,;\boldsymbol{v}^*_{\vec{p}',\sigma'}]_+ &\equiv \boldsymbol{v}_{\vec{p},\sigma}\,\boldsymbol{v}^*_{\vec{p}',\sigma'} + \boldsymbol{v}^*_{\vec{p}',\sigma'}\,\boldsymbol{v}_{\vec{p},\sigma} = \delta_{\sigma\sigma'}\,\delta(\vec{p}-\vec{p}')\,; \\ [\boldsymbol{w}_{\vec{p},\sigma}\,;\boldsymbol{w}^*_{\vec{p}',\sigma'}]_+ &= \delta_{\sigma\sigma'}\,\delta(\vec{p}-\vec{p}')\,; \end{aligned} \tag{101}$$

all other pairs of Fermi-Dirac amplitudes being anticommutative with each other.

From the Lagrangian function (37)—(40) we can derive the total linear momentum, the (orbital) angular momentum, the spin angular momentum and the total electric charge in the ordinary way [40]. Since the terms in the Lagrangian (37) describing the interactions do not contain derivatives, the expressions for these quantities in terms of the canonical variables do not contain interaction terms. Inserting into these expressions [40] the expansions (92), (93), (99) of the wave-functions and substituting (96), we find the following expression for the total momentum of the field:

$$\begin{aligned}\vec{P} = \sum_{\vec p} \vec p\, \{ & a^*_{\vec p} a_{\vec p} + \sum_{\mu=-1}^{1} b^*_{\vec p,\mu} b_{\vec p,\mu} + c_{\vec p} c^*_{\vec p} + \sum_{\mu=-1}^{1} d_{\vec p,\mu} d^*_{\vec p,\mu} + \\ & + \tfrac{1}{2}(m^*_{\vec p} m_{\vec p} + m_{\vec p} m^*_{\vec p}) + \sum_{\mu=-1}^{1} \tfrac{1}{2}(n^*_{\vec p,\mu} n_{\vec p,\mu} + n_{\vec p,\mu} n^*_{\vec p,\mu}) + \\ & + \sum_{\eta=\pm 1} \tfrac{1}{2}(l^*_{\vec p,\eta} l_{\vec p,\eta} + l_{\vec p,\eta} l^*_{\vec p,\eta}) + \sum_{(P,N,\pi,\nu)} \sum_{\sigma=\pm 1} (v^*_{\vec p,\sigma} v_{\vec p,\sigma} - w_{\vec p,\sigma} w^*_{\vec p,\sigma}) \}.\end{aligned} \tag{102}$$

In a similar way the total electric charge is found to be equal to

$$\begin{aligned} e = e \sum_{\vec p} \{ & a^*_{\vec p} a_{\vec p} + \sum_{\mu=-1}^{1} b^*_{\vec p,\mu} b_{\vec p,\mu} + \sum_{(P,\pi)} \sum_{\sigma=\pm 1} v^*_{\vec p,\sigma} v_{\vec p,\sigma} \} - \\ - e \sum_{\vec p} \{ & c_{\vec p} c^*_{\vec p} + \sum_{\mu=-1}^{1} d_{\vec p,\mu} d^*_{\vec p,\mu} - \sum_{(P,\pi)} \sum_{\sigma=\pm 1} w_{\vec p,\sigma} w^*_{\vec p,\sigma} \}. \end{aligned} \tag{103}$$

The total spin angular momentum can be written as a sum

$$\vec{S} = \sum_{\vec p} \sum_{\mu=-1}^{1} \vec{c}^{\,\mu}_{\vec p} S_{\vec p,\mu}\,, \tag{104}$$

where $S_{\vec p,\mu}$ can be expressed in terms of the amplitudes $v_{\vec p,\sigma}$, $w_{\vec p,\sigma}$, $a_{\vec p}$, $b_{\vec p,0}$, $b_{\vec p,\eta}$, etc., $v^*_{\vec p,\sigma}$, etc., belonging to the momentum $\vec p$ only. We shall calculate only the spin-component parallel to the momentum

$$\vec{S}_{\parallel} = \sum_{\vec p} \vec{c}^{\,0}_{\vec p} S_{\vec p,0}\,; \tag{105}$$

$$\begin{aligned} S_{\vec p,0} = \sum_{\mu=-1}^{1} \mu\hbar\, \{ & b^*_{\vec p,\mu} b_{\vec p,\mu} + d_{\vec p,\mu} d^*_{\vec p,\mu} + \tfrac{1}{2}(n^*_{\vec p,\mu} n_{\vec p,\mu} + n_{\vec p,\mu} n^*_{\vec p,\mu}) \} + \\ & + \sum_{\eta=\pm 1} \eta\hbar\, \{ \tfrac{1}{2}(l^*_{\vec p,\eta} l_{\vec p,\eta} + l_{\vec p,\eta} l^*_{\vec p,\eta}) \} + \\ & + \sum_{(P,N,\pi,\nu)} \sum_{\sigma=\pm 1} (\sigma\hbar/2)\, \{ v^*_{\vec p,\sigma} v_{\vec p,\sigma} - w_{\vec p,\sigma} w^*_{\vec p,\sigma} \}. \end{aligned}$$

Finally we shall compute the contribution to the total energy from the term $\hat{H}_0$ of the Hamiltonian (90). We find

$$\begin{aligned}\hat{H}_0 = c\varkappa \sum_{\vec p} \{ & (\mathsf{W}^*_{\vec p} \mathsf{W}_{\vec p} + \mathsf{A}^*_{\vec p,0} \mathsf{A}_{\vec p,0} + \sum_{\eta=\pm 1} \mathsf{E}^*_{\vec p,\eta} \mathsf{E}_{\vec p,\eta} + \bar{\mathsf{W}}^2_{\vec p} + \bar{\mathsf{A}}^2_{\vec p,0} + \\ & + \sum_{\eta=\pm 1} \bar{\mathsf{E}}^2_{\vec p,\eta}) + (1 + p^2/\hbar^2\varkappa^2)(\mathsf{Y}^*_{\vec p} \mathsf{Y}_{\vec p} + \mathsf{E}^*_{\vec p,0} \mathsf{E}_{\vec p,0} + \sum_{\eta=\pm 1} \mathsf{A}^*_{\vec p,\eta} \mathsf{A}_{\vec p,\eta} + \\ & + \bar{\mathsf{Y}}^2_{\vec p} + \bar{\mathsf{E}}^2_{\vec p,0} + \sum_{\eta=\pm 1} \bar{\mathsf{A}}^2_{\vec p,\eta})\} + \\ + (1/8\pi) \sum_{\vec p} \sum_{\eta=\pm 1} & \{\mathfrak{e}^2_{\vec p,\eta} + (p^2/\hbar^2)\, \mathfrak{a}^2_{\vec p,\eta}\} + \\ + c \sum_{(P,N,\pi,\nu)} \sum_{\vec p} \sum_{\sigma=\pm 1} & (\boldsymbol{v}^*_{\vec p,\sigma} \boldsymbol{v}_{\vec p,\sigma} - \boldsymbol{w}_{\vec p,\sigma} \boldsymbol{w}^*_{\vec p,\sigma}) \sqrt{(mc)^2 + p^2}.\end{aligned} \tag{106}$$

Inserting (96) we find a simple expression, if we choose

$$\begin{aligned} \mathfrak{f}_{\vec p} = \mathfrak{f}_{\vec p,\mu} = \mathfrak{f}'_{\vec p} = \mathfrak{f}'_{\vec p,\mu} &= \sqrt[4]{1 + p^2/\hbar^2\varkappa^2}\ (\equiv \sqrt{\varepsilon_p/\mathfrak{m}c^2}); \\ \mathfrak{g}_{\vec p,\eta} &= \sqrt{p/\hbar}. \end{aligned} \tag{107}$$

We put

$$\begin{aligned} \varepsilon_p &= c\sqrt{(\mathfrak{m}c)^2 + p^2}; & W_p^{(P,N)} &= c\sqrt{(Mc)^2 + p^2}; \\ W_p^{(\pi)} &= c\sqrt{(mc)^2 + p^2}; & W_p^{(\nu)} &= c\sqrt{(\mu c)^2 + p^2}; \end{aligned} \tag{108}$$

where M, $\mathfrak{m}$, m and μ denote the masses of nuclons, heavy quanta, electrons and neutrinos respectively. Then H_0 takes the following form:

$$\begin{aligned}\hat{H}_0 = \sum_{\vec p} \varepsilon_p \{ & \boldsymbol{a}^*_{\vec p} \boldsymbol{a}_{\vec p} + \sum_{\mu=-1}^{1} \boldsymbol{b}^*_{\vec p,\mu} \boldsymbol{b}_{\vec p,\mu} + \boldsymbol{c}_{\vec p} \boldsymbol{c}^*_{\vec p} + \sum_{\mu=-1}^{1} \boldsymbol{d}_{\vec p,\mu} \boldsymbol{d}^*_{\vec p,\mu} + \\ & + \tfrac{1}{2}(\boldsymbol{m}^*_{\vec p} \boldsymbol{m}_{\vec p} + \boldsymbol{m}_{\vec p} \boldsymbol{m}^*_{\vec p}) + \sum_{\mu=-1}^{1} \tfrac{1}{2}(\boldsymbol{n}^*_{\vec p,\mu} \boldsymbol{n}_{\vec p,\mu} + \boldsymbol{n}_{\vec p,\mu} \boldsymbol{n}^*_{\vec p,\mu})\} + \\ + \sum_{\vec p} cp \sum_{\eta=\pm 1} & \tfrac{1}{2}(\boldsymbol{l}^*_{\vec p,\eta} \boldsymbol{l}_{\vec p,\eta} + \boldsymbol{l}_{\vec p,\eta} \boldsymbol{l}^*_{\vec p,\eta}) + \\ + \sum_{(P,N,\pi,\nu)} \sum_{\vec p} W_p & \sum_{\sigma=\pm 1} (\boldsymbol{v}^*_{\vec p,\sigma} \boldsymbol{v}_{\vec p,\sigma} - \boldsymbol{w}_{\vec p,\sigma} \boldsymbol{w}^*_{\vec p,\sigma}).\end{aligned} \tag{109}$$

From (98), (101), (102), (103), (105) and (109) follows the usual interpretation of the operators $\boldsymbol{a}^*_{\vec p}\boldsymbol{a}_{\vec p}$, etc. as numbers of particles in different states.*) The matrices $\boldsymbol{a}_{\vec p}$, ... and $\boldsymbol{v}_{\vec p,\sigma}$, ... describe the annihilation of a quantum or a particle; the matrices $\boldsymbol{a}^*_{\vec p}$, ... and $\boldsymbol{v}^*_{\vec p,\sigma}$, ... its creation. ν $\hat N$

Now the Hamiltonian (90) can be interpreted. H_0 is the

*) We remark that the transformed operators of the numbers of photons, mesons, etc. ($\hat N$) are built up by means of the components a, m, etc. of the old variables A, 𝔄, etc. (without a ^). In the following all ^ signs are omitted. FJB

energy when neglecting all interactions. H_C is the static Coulomb interaction between protons, electrons and mesons. It contains the infinite electrostatic self-energies of the particles; for instance:

$$\frac{e^2}{2}\iint \frac{\psi_P^*(\vec{x})\,\psi_P(\vec{x})\,\psi_P^*(\vec{x}')\,\psi_P(\vec{x}')}{|\vec{x}-\vec{x}'|} =$$

$$= \frac{e^2}{2}\iint \frac{\psi_P^*(\vec{x})\,\psi_P^*(\vec{x}')\,\psi_P(\vec{x}')\,\psi_P(\vec{x})}{|\vec{x}-\vec{x}'|} + \frac{e^2}{2}\iint \frac{\psi_P^*(\vec{x})\delta(\vec{x}-\vec{x}')\,\psi_P(\vec{x}')}{|\vec{x}-\vec{x}'|}. \quad (110)$$

The first term of the right hand member represents the ordinary Coulomb forces together with the electrostatic exchange forces; the second term is the infinite self-energy. H_e is well known from radiation theory. Here some terms are added to it, representing the radiation of the moving mesons. This part of the Hamiltonian gives rise to the creation or the annihilation of a photon ($\mathfrak{A}_{tr}$) under transition of a charged particle or a charged heavy quantum from one quantum state into another or under creation or annihilation of a *pair* of charged particles or heavy quanta ($\psi^*\psi$ or $\Psi^*\Psi$).

The term H_{ee} gives rise to direct two-photon effects, by which a meson jumps from one state into another, or by which a pair of mesons is created or annihilated. In the Dirac theory of electrons such a term does not occur.

H_g describes the interaction between heavy quanta and matter. It gives rise to the following processes (Y^+ denotes a *theticon* or positive meson; Y^- an *arneticon* or negative meson; Y^0 a neutretto; P^+ denotes a proton, P^- an anti-proton (*hystaton*), N^+ a neutron and N^- an antineutron; π a positon, ε a negaton, ν an "antineutrino" (compare § 4) and o a "neutrino"; finally $h\nu$ a photon):

Annihilation or creation of: $(P^-+N^++Y^+)$ or $(P^++N^-+Y^-)$ or $(P^++P^-+Y^0)$ or $(N^++N^-+Y^0)$ or $(\varepsilon+\nu+Y^+)$ or $(\pi+o+Y^-)$ or $(\pi+\varepsilon+Y^0)$ or $(\nu+o+Y^0)$. Further:

$$\begin{array}{lll}
P^+ \rightleftarrows N^+ + Y^+; & N^+ \rightleftarrows P^+ + Y^-; & Y^+ \rightleftarrows P^+ + N^-; \\
P^- \rightleftarrows N^- + Y^-; & N^- \rightleftarrows P^- + Y^+; & Y^- \rightleftarrows P^- + N^+; \\
\pi \rightleftarrows \nu + Y^+; & \nu \rightleftarrows \pi + Y^-; & Y^+ \rightleftarrows \pi + o\ ; \\
\varepsilon \rightleftarrows o + Y^-; & o \rightleftarrows \varepsilon + Y^+; & Y^- \rightleftarrows \varepsilon + \nu\ .
\end{array}$$

Further: emission or absorption of a neutretto by a nuclon or by a light particle jumping from one state into another without changing its charge; and creation of a pair of nuclons or of light particles from a neutretto or annihilation of such a pair to a neutretto.

Especially the interaction between the nuclons and the heavy quanta is of much interest, since the corresponding coefficients are so large.

H_{gg} describes a direct interaction of nuclons with nuclons, of nuclons with light particles and of light particles with light particles. To some extent it can be compared with H_C, since both terms in the Hamiltonian do not give rise to the creation or annihilation of a photon or a heavy quantum. This term H_{gg} gives a first order contribution to the β-disintegration of instable nuclei:

$$P^+ \rightarrow N^+ + \pi + o, \qquad N^+ \rightarrow P^+ + \varepsilon + \nu.$$

The term H_{eg} is of much importance. This term was given explicitly for the first time by Bhabha [18]), though also Kemmer drew attention to it in a foot-note [10]). It gives a first order contribution [45]) to the matrix element for the "photomesic" processes

$$Y^+ + N^+ \rightleftarrows P^+ + h\nu, \qquad Y^- + P^+ \rightleftarrows N^+ + h\nu,$$

which couple the "soft" and the "penetrating" components of cosmic radiation [49]). According to unpublished calculations the effect of this first order term seems to be to compensate for a good deal the strong second order transitions, in which first a photon is emitted by the heavy quantum, and only "afterwards" the latter is absorbed by the nucleon (compare § 12).

If (96) and (107), (108) are inserted into (92) and (93), one finds:

$$\vec{\mathbf{A}}(x)=\sqrt{\frac{\hbar}{2\Omega}}\sum_{\vec{p}}\{e^{(i/\hbar)\vec{p}.\vec{x}}(\vec{\mathfrak{c}}^{0}_{\vec{p}}\sqrt{\frac{\varepsilon_p}{\mathfrak{m}c^2}}\,\boldsymbol{b}_{\vec{p},0}+\sum_{\eta}\vec{\mathfrak{c}}^{\eta}_{\vec{p}}\sqrt{\frac{\mathfrak{m}c^2}{\varepsilon_p}}\,\boldsymbol{b}_{\vec{p},\eta})+$$
$$+e^{-(i/\hbar)\vec{p}.\vec{x}}(\vec{\mathfrak{c}}^{0}_{\vec{p}}\sqrt{\frac{\varepsilon_p}{\mathfrak{m}c^2}}\,\boldsymbol{d}^*_{\vec{p},0}+\sum_{\eta}\vec{\mathfrak{c}}^{\eta *}_{\vec{p}}\sqrt{\frac{\mathfrak{m}c^2}{\varepsilon_p}}\,\boldsymbol{d}^*_{\vec{p},\eta})\}, \tag{111}$$

$$\text{rot}\,\vec{\mathbf{A}}(x)=\sqrt{\frac{1}{2\hbar\Omega}}\sum_{\vec{p}}|p|\sqrt{\frac{\mathfrak{m}c^2}{\varepsilon_p}}\sum_{\eta}\eta\{\boldsymbol{b}_{\vec{p},\eta}\,\vec{\mathfrak{c}}^{\eta}_{\vec{p}}\,e^{(i/\hbar)\vec{p}.\vec{x}}+\boldsymbol{d}^*_{\vec{p},\eta}\,\vec{\mathfrak{c}}^{\eta *}_{\vec{p}}\,e^{-(i/\hbar)\vec{p}.\vec{x}}\},$$

$$\vec{\mathbf{E}}(x)=i\sqrt{\frac{\hbar}{2\Omega}}\sum_{\vec{p}}\{e^{(i/\hbar)\vec{p}.\vec{x}}(\vec{\mathfrak{c}}^{0}_{\vec{p}}\sqrt{\frac{\mathfrak{m}c^2}{\varepsilon_p}}\,\boldsymbol{b}_{\vec{p},0}+\sum_{\eta}\vec{\mathfrak{c}}^{\eta}_{\vec{p}}\sqrt{\frac{\varepsilon_p}{\mathfrak{m}c^2}}\,\boldsymbol{b}_{\vec{p},\eta})-$$
$$-e^{-(i/\hbar)\vec{p}.\vec{x}}(\vec{\mathfrak{c}}^{0}_{\vec{p}}\sqrt{\frac{\mathfrak{m}c^2}{\varepsilon_p}}\,\boldsymbol{d}^*_{\vec{p},0}+\sum_{\eta}\vec{\mathfrak{c}}^{\eta *}_{\vec{p}}\sqrt{\frac{\varepsilon_p}{\mathfrak{m}c^2}}\,\boldsymbol{d}^*_{\vec{p},\eta})\},$$

$$\operatorname{div} \vec{\mathbf{E}}(x) = -\sqrt{\frac{1}{2\hbar\Omega}} \sum_{\vec{p}} |p| \sqrt{\frac{\mathfrak{m}c^2}{\varepsilon_p}} \{ \boldsymbol{b}_{\vec{p},0}\, e^{(i/\hbar)\vec{p}.\vec{x}} + \boldsymbol{d}^*_{\vec{p},0}\, e^{-(i/\hbar)\vec{p}.\vec{x}} \},$$

$$\mathbf{W}(x) = i\sqrt{\frac{\hbar}{2\Omega}} \sum_{\vec{p}} \sqrt{\frac{\varepsilon_p}{\mathfrak{m}c^2}} \{ \boldsymbol{a}_{\vec{p}}\, e^{(i/\hbar)\vec{p}.\vec{x}} + \mathbf{c}^*_{\vec{p}}\, e^{-(i/\hbar)\vec{p}.\vec{x}} \}, \tag{111}$$

$$\mathbf{Y}(x) = \sqrt{\frac{\hbar}{2\Omega}} \sum_{\vec{p}} \sqrt{\frac{\mathfrak{m}c^2}{\varepsilon_p}} \{ \boldsymbol{a}_{\vec{p}}\, e^{(i/\hbar)\vec{p}.\vec{x}} - \mathbf{c}^*_{\vec{p}}\, e^{-(i/\hbar)\vec{p}.\vec{x}} \},$$

$$\vec{\nabla}\mathbf{Y}(x) = i\sqrt{\frac{1}{2\hbar\Omega}} \sum_{\vec{p}} |p| \sqrt{\frac{\mathfrak{m}c^2}{\varepsilon_p}} \{ \boldsymbol{a}_{\vec{p}}\, \vec{\mathfrak{c}}^{\,0}_{\vec{p}}\, e^{(i/\hbar)\vec{p}.\vec{x}} + \mathbf{c}^*_{\vec{p}}\, \vec{\mathfrak{c}}^{\,0}_{\vec{p}}\, e^{-(i/\hbar)\vec{p}.\vec{x}} \};$$

and in a similar way:

$$\vec{\overline{\mathbf{A}}}(x) = \sqrt{\frac{\hbar}{4\Omega}} \sum_{\vec{p}} \{ e^{(i/\hbar)\vec{p}.\vec{x}} (\vec{\mathfrak{c}}^{\,0}_{\vec{p}} \sqrt{\frac{\varepsilon_p}{\mathfrak{m}c^2}}\, \boldsymbol{n}_{\vec{p},0} + \sum_{\eta} \vec{\mathfrak{c}}^{\,\eta}_{\vec{p}} \sqrt{\frac{\mathfrak{m}c^2}{\varepsilon_p}}\, \boldsymbol{n}_{\vec{p},\eta}) +$$
$$+ e^{-(i/\hbar)\vec{p}.\vec{x}} (\vec{\mathfrak{c}}^{\,0}_{\vec{p}} \sqrt{\frac{\varepsilon_p}{\mathfrak{m}c^2}}\, \boldsymbol{n}^*_{\vec{p},0} + \sum_{\eta} \vec{\mathfrak{c}}^{\,\eta *}_{\vec{p}} \sqrt{\frac{\mathfrak{m}c^2}{\varepsilon_p}}\, \boldsymbol{n}^*_{\vec{p},\eta}) \}, \tag{112}$$

$$\overline{\mathbf{W}}(x) = i\sqrt{\frac{\hbar}{4\Omega}} \sum_{\vec{p}} \sqrt{\frac{\varepsilon_p}{\mathfrak{m}c^2}} \{ \boldsymbol{m}_{\vec{p}}\, e^{(i/\hbar)\vec{p}.\vec{x}} - \boldsymbol{m}^*_{\vec{p}}\, e^{-(i/\hbar)\vec{p}.\vec{x}} \}, \text{ etc.,}$$

and:

$$\vec{\mathfrak{A}}_{tr}(x) = \hbar\sqrt{\frac{2\pi c}{\Omega}} \sum_{\vec{p}} \sqrt{\frac{1}{|p|}} \sum_{\eta} \{ \boldsymbol{l}_{\vec{p},\eta}\, \vec{\mathfrak{c}}^{\,\eta}_{\vec{p}}\, e^{(i/\hbar)\vec{p}.\vec{x}} + \boldsymbol{l}^*_{\vec{p},\eta}\, \vec{\mathfrak{c}}^{\,\eta *}_{\vec{p}}\, e^{-(i/\hbar)\vec{p}.\vec{x}} \}. \tag{113}$$

The expansions (112) for the neutretto field are obtained from those for the meson field (111) by changing $\sqrt{\hbar/2\Omega}$ into $\sqrt{\hbar/4\Omega}$ and by changing $\boldsymbol{a}$ into $\boldsymbol{m}$, $\mathbf{c}$ into $-\boldsymbol{m}$, and $\boldsymbol{b}$ and $\boldsymbol{d}$ both into $\boldsymbol{n}$.

The conjugate complex of (111) are obtained in a similar way by changing $\boldsymbol{b}$ into $\boldsymbol{d}$ and $\boldsymbol{d}^*$ into $\boldsymbol{b}^*$; $\boldsymbol{a}$ into $-\mathbf{c}$ and $\mathbf{c}^*$ into $-\boldsymbol{a}^*$. *Nothing else should be changed.*

The expressions (112) and (113) are real themselves.

Cross sections Φ for any of the cosmic-ray reactions discussed in the foregoing are calculated in the usual way [50]):

$$\Phi_{a\to f} = \frac{2\pi}{\hbar v} \cdot \rho(\mathcal{E}) \cdot \Sigma\, |Q|^2, \tag{114}$$

(summation over all admitted final states);

$$Q = H_{fa} + \sum_i \frac{H_{fi} H_{ia}}{\mathcal{E}_a - \mathcal{E}} + \ldots\ldots \tag{115}$$

Here v is the relative velocity of the impacting particle or quantum and the "targets"; H_{fa}, etc., are matrix elements of the perturbations H_C, H_e, H_{ee}, H_g, H_{gg} and H_{ge} (B o r n approximation); a, i and f denote the initial, intermediate and final states. $\mathcal{E}_n$ denotes the energy of the situation n.

$\rho(\mathcal{E})d(\mathcal{E}_f - \mathcal{E}_a)$ is the number of final states with an energy in the interval $d\mathcal{E}_f$, which can be reached directly or through the intermediate states i from the initial state a. We have written $d(\mathcal{E}_f - \mathcal{E}_a)$ in order to indicate that it may be necessary to vary the initial state together with the final state in order to ensure a non vanishing matrix element between both, if the creation of an antiparticle is described as the vanishing of a particle from the so-called *continuum of states of negative energy.*

The ratio between the contributions to the "matrix element" Q for a given process from the n^{th} and the $(n+2)^{th}$ approximation has the order of magnitude

$$\sum_{i_{n+1}} \sum_{i_n} \frac{H_{i_{n+1} i_n} \cdot H_{i_n i_{n-1}}}{(\mathcal{E}_a - \mathcal{E}_{i_{n+1}})(\mathcal{E}_a - \mathcal{E}_{i_n})}. \tag{116}$$

The summation over i_n and i_{n+1} will often have the form of an integral over a continuum of intermediate states i_n. In that case it gives rise to a factor

$$\int_0^\infty (4\pi\Omega/c^2h^3)p\,\varepsilon\,d\varepsilon, \tag{117}$$

where ε and p represent the energy and momentum of an extra quantum emitted and absorbed again in the higher order process. The factor Ω will be cancelled by a factor $(1/\Omega)$ from $H_{i_{n+1} i_n} H_{i_n i_{n-1}}$. The dependence of the matrix elements $H_{i'i}$ on the energy ε will generally not be sufficient to ensure convergency of the integral over the intermediate states, which will diverge on account of the factors in (117). Therefore *high order calculations will often diverge, if a continuum of intermediate states is possible.*

The ratio between the probability of a multiple process, in which $(n + 1)$ quanta are created, and the probability of a multiple process, in which only n quanta are created, can be estimated, if one assumes that the lowest order of approximation giving rise to such processes will give a result of the right order of magnitude. Then, the creation of $(n + 1)$ quanta will be found by an approximation, which is one order higher than that for the creation of n quanta. The intermediate states are now determined (apart from their order of sequence and the polarizations) by the final state, and the ratio of probability is only of the order of magnitude

$$\mathsf{M} \, . \int \frac{4\pi p \varepsilon \, d\varepsilon}{h^3 c^2} \left| \frac{H\sqrt{\Omega}}{\Delta \mathcal{E}} \right|^2 . \tag{118}$$

The factor M denotes the number of possible polarizations of the $(n+1)^{th}$ emitted quantum; the integral has to be taken over all energies ε of this extra quantum that are *allowed by the conservation laws*. The momentum of the quantum corresponding to ε is denoted by p. $\Delta\mathcal{E}$ is the difference between the energies of the additional intermediate state and the initial (or final) state. In most cases $\int \varepsilon \, d\varepsilon / |\Delta\mathcal{E}|^2$ will be of the order of magnitude 1. Then the ratio in question is given by

$$\Phi_{n+1}/\Phi_n \approx |H|^2 \Omega \, . \, p\mathsf{M}/2\pi^2 \hbar^3 c^2. \tag{119}$$

For the emission of photons we find from (90) and (113):

$$|H_e \sqrt{\Omega}| \approx e\hbar c\sqrt{2\pi/pc}, \tag{120}$$

so that

$$\Phi_{n+1}/\Phi_n \approx (\mathsf{M}/\pi)\,(e^2/\hbar c) \approx 2/137\pi. \tag{121}$$

For the emission of heavy quanta, however, we find from (90) and (111) that the matrix elements $|H_g \sqrt{\Omega}|$ are not all of the same order. Those arising from $\vec{\mathbf{A}}_{tr}$, $\vec{\mathbf{E}}_{long}$ and $\mathbf{Y}$ are only of the order of magnitude

$$g\hbar c\sqrt{2\pi/\varepsilon}, \tag{122}$$

where g is one of the constants g_1, g_2, g_3 defined by (24). So these terms would yield a ratio

$$\Phi_{n+1}/\Phi_n \approx (3/2\pi)\,(g^2/\hbar c), \tag{123}$$

if we take $\varepsilon \approx 2pc$, $\mathsf{M} = 3$. If we insert into (123) the values of $(g^2/\hbar c)$ found in § 3, we find in this way

$$\Phi_{n+1}/\Phi_n \approx \tfrac{1}{10} \sim \tfrac{1}{40}.$$

The terms, however, arising from $\vec{\mathbf{A}}_{long}$, $\vec{\mathbf{E}}_{tr}$ and $\mathbf{W}$, have matrix elements $|H_g \sqrt{\Omega}|$, which are larger by a factor $(\varepsilon_p/\mathfrak{m}c^2)$, and the matrix elements of the terms with rot $\vec{\mathbf{A}}$, div $\vec{\mathbf{E}}$ and $\vec{\nabla}\mathbf{Y}$ are larger than (123) by a factor $(p/\mathfrak{m}c)$. We conclude that *these terms may give* *rise to the production of an appreciable number of showers of heavy quanta of high energy* ($\varepsilon_p - mc^2 \geq 10^8$ eV), if the available amount of energy is large enough [49]).

Finally it is useful to compare the matrix elements in (111) describing the annihilation (creation) of a positive meson with a momentum $\vec{p}$, and those describing the creation (annihilation) of a negative meson with a momentum $-\vec{p}$, since in a summation over different intermediate states both effects will often occur in terms, which must be added together. It must be remarked that generally both terms will possess a different denominator $(\mathcal{E}_a - \mathcal{E}_i)$, but in approximative calculations these denominators may sometimes be put equal to each other. Then we remark that in $\vec{\mathbf{A}}$, rot $\vec{\mathbf{A}}$ and their conjugate complex the matrix elements given by the terms with $\boldsymbol{b}_{\vec{p},\mu}$ and $\boldsymbol{d}^*_{-\vec{p},\mu}$ (or with $\boldsymbol{b}^*_{\vec{p},\mu}$ and $\boldsymbol{d}_{-\vec{p},\mu}$) are (on account of $\vec{c}^{\,\mu *}_{-p} = -\vec{c}^{\,\mu}_{p}$) opposite equal to each other, whereas they are exactly equal in $\vec{\mathbf{E}}$, div $\vec{\mathbf{E}}$ and their conjugate complex. In a similar way the matrices $\boldsymbol{a}_{\vec{p}}$ and $\boldsymbol{c}^*_{-\vec{p}}$ (or $\boldsymbol{a}^*_{\vec{p}}$ and $\boldsymbol{c}_{-\vec{p}}$) occur with the same coefficient in $\mathbf{W}$ and $\mathbf{W}^*$, but with opposite coefficients in $\mathbf{Y}$, $\overrightarrow{\nabla \mathbf{Y}}$ and their conjugate complex.

Thus the product of the matrix elements describing creation and subsequent annihilation of a theticon with a momentum $\vec{p}$ and a polarization μ will exactly be equal to the corresponding product describing the creation and subsequent annihilation of an arneticon with the same polarization but with a momentum $-\vec{p}$. Only the sign can be different. A different sign appears only, if the creation and the annihilation are described by J o r d a n-K l e i n matrices, of which one originates in (111) from $\vec{\mathbf{E}}$ (or its divergence) and the other from $\vec{\mathbf{A}}^*$ (or its curl), or one from $\vec{\mathbf{A}}$ (or rot $\vec{\mathbf{A}}$) and the other from $\vec{\mathbf{E}}^*$ (or div $\vec{\mathbf{E}}^*$). For instance, if the creation of the arneticon is described by $\vec{\mathbf{A}}$ and its annihilation by $\vec{\mathbf{E}}^*$, the product of the matrix elements is opposite equal to that corresponding to the creation of a theticon described by $\vec{\mathbf{E}}^*$, and its subsequent annihilation described by $\vec{\mathbf{A}}$. If, however, creation and annihilation are described for instance by div $\vec{\mathbf{E}}$ and $\vec{\mathbf{E}}^*$ (and $\vec{\mathbf{E}}^*$ and div $\vec{\mathbf{E}}$ respectively), the pro-

ducts of matrix elements for both processes are exactly equal.
A similar rule holds for **Y** and **W**.

These rules will enable us to avoid some frequent but substantial *errors in sign* [49] [45] in calculations like that of the second order contribution to the matrix element Q of the photomesic and the mesophotic effects:

$$Y^-(\vec{p})+P^+(0) \begin{array}{c} \nearrow\!\!\!\swarrow\ Y^-(\vec{p}') + h\nu(\vec{k}) + P^+(0)\ \searrow\!\!\!\nwarrow \\ \searrow\!\!\!\nwarrow\ Y^-(\vec{p})+Y^+(-\vec{p}')+N(\vec{p}')\ \nearrow\!\!\!\swarrow \end{array} N(\vec{p}')+h\nu(\vec{k}), \quad (124)$$

$$(\vec{p} = \vec{p}' + \vec{k})$$

where the difference between the products $H_{fI}H_{Ia}$ and $H_{fII}H_{IIa}$ is of the above-mentioned nature and where the denominators $(\mathcal{E}_a - \mathcal{E}_i)$ are equal in a non-relativistic approximation.

§ 7. *The heavy quanta interaction between nuclons.* In the preceding sections we have developed a quantum theory of the field of heavy quanta. We shall now apply it to a number of important problems. The first one we shall deal with is the force between proton-neutrons.

There are mainly two methods leading to the purpose. The first one is that of a *perturbation calculus*; it was performed by several authors [10] [19] [18]. In these calculations the "recoil" of the nuclons by the emission and absorption of heavy quanta was neglected. Assuming Kemmer's "symmetrical" theory of mesons and neutrettos [30] an attraction between nuclons was found in the second approximation. A fourth order calculation [19] yielded a strong repulsion at small distances ($r < 1/2\varkappa$). It is not certain, however, that calculations of the successive higher order effects, if they give converging results at all, will not yield still stronger interactions, the sign of which is problematic. That much is certain that for small values of the distance between the nuclons the result of a perturbation calculus of finite order is not trustworthy.

The other method was used by Yukawa [21]. This method has almost the form of a classical calculation, in which only "static" interactions of the "static" parts of the nuclon fields are taken into account. This method can also be used for the derivation of the Breit interaction between electrons [51]. There, the "static" interaction through the medium of photons takes the place of a similar interaction through the heavy quantum field in our case.

From the H a m i l t o n i a n H the equation of motion for the quantized nuclon wave-function is derived according to (51). (It can also be obtained directly from the L a g r a n g i a n by variation of its canonical conjugate). The H a m i l t o n i a n contains the nuclon variables in the terms H_0, H_g and H_{gg}. These terms are of the form of

$$H = H_0 + H_g + H_{gg} =$$
$$= \int \psi^\dagger H^0_{op} \psi + \int \sum_n g_n \{\Psi_n^* (\psi^\dagger \boldsymbol{\omega}_n \psi) + (\psi^\dagger \boldsymbol{\omega}_n^\dagger \psi) \Psi_n\} +$$
$$+ \int \sum_n g_n^2 (\psi^\dagger \boldsymbol{\omega}_n^\dagger \psi)(\psi^\dagger \boldsymbol{\omega}_n \psi). \quad (125)$$

Here $\boldsymbol{\omega}_n$ are some matrices operating on the undor-index and the isotopic spin co-ordinate of ψ, whereas Ψ_n denotes the components of the heavy quanta field and their derivatives with respect to the spatial co-ordinates.

Now, it is well known that generally a superquantized H a m i l t o n i a n of the form of

$$H = \int \psi^\dagger(1) H^0(1)_{op} \psi(1) + \tfrac{1}{2} \int\int \psi^\dagger(1) \psi^\dagger(2) W(1,2)_{op} \psi(2) \psi(1) \quad (126)$$

yields an equation of motion for the quantized wave function

$$i\hbar\dot{\psi}(1) = [\psi(1); H]_- = \{H^0(1)_{op} + \int \psi^\dagger(2) W(1,2)_{op}^{sym} \psi(2)\} \psi(1), \quad (127)$$

whereas the equation of motion (61) of the situation function χ now is equivalent to a "S c h r ö d i n g e r equation" for the n particle problem of the form of

$$i\hbar\dot{\psi}(1,2, \ldots, n) = \{\sum_{k=0}^{n} H^0(k)_{op} + \sum_{k>l} W(k,l)_{op}^{sym}\} \psi(1,2, \ldots, n). \quad (128)$$

Here we have put

$$W(k,l)_{op}^{sym} = \tfrac{1}{2}\{W(k,l)_{op} + W(l,k)_{op}\}; \quad (129)$$

$\psi(1,2, \ldots, n)$ denotes the antisymmetric situation-(wave)function of the n body problem; so it is a c-number, contrary to the q-number $\psi(1)$ in (126)—(127).

Though the actual H a m i l t o n i a n (125) has the form of (126) with $W(1,2)_{op} = \sum_n g_n^2 \boldsymbol{\omega}_n^{\dagger(1)} \boldsymbol{\omega}_n^{(2)} \delta(\vec{r}_{12})$ — if we neglect an infinite self-energy $\sum_n g_n^2 \int\int \psi^\dagger(1) \boldsymbol{\omega}_n^{\dagger(1)} \delta(1,2) \boldsymbol{\omega}_n^{(2)} \psi(2) \delta(\vec{r}_{12})$— we cannot identify $H_0 + H_g$ with the first term of the right hand member of (126) and conclude to (128). For the field Ψ_n occurring in H_g is not a given external field, but is generated again by the nuclon field, as can be seen from the equations (2), or better from the second order equations, which are obtained from them by iteration. These second order equations read, if all interactions with the electromagnetic field are neglected:

$$
\begin{aligned}
(\Box - \varkappa^2)\,\mathbf{A} &= -\nabla(\mathrm{div}\ \mathbf{a} + \dot{\mathbf{v}}/c) - \varkappa(\mathrm{rot}\,\mathbf{h} - \dot{\mathbf{e}}/c) + \varkappa^2\mathbf{a},\\
(\Box - \varkappa^2)\,\mathbf{E} &= \ \mathrm{rot}(\mathrm{rot}\ \mathbf{e} + \dot{\mathbf{h}}/c) - \varkappa(\nabla\ \mathbf{v} + \dot{\mathbf{a}}/c) + \varkappa^2\mathbf{e},\\
(\Box - \varkappa^2)\mathbf{W} &= -\,\mathrm{div}(\nabla\,\mathbf{w} + \dot{\mathbf{b}}/c) - \varkappa\dot{\mathbf{y}}/c \quad + \varkappa^2\mathbf{w},\\
(\Box - \varkappa^2)\,\mathbf{Y} &= \ \varkappa(\mathrm{div}\ \mathbf{b} + \dot{\mathbf{w}}/c) + \varkappa^2\mathbf{y};
\end{aligned}
\qquad (130)
$$

and:

$$(\Box - \varkappa^2)\,\overline{\mathbf{A}} = -\nabla(\mathrm{div}\,\overline{\mathbf{a}} + \dot{\overline{\mathbf{v}}}/c) - \varkappa(\mathrm{rot}\,\overline{\mathbf{h}} - \dot{\overline{\mathbf{e}}}/c) + \varkappa^2\overline{\mathbf{a}},\ \text{etc.} \qquad (130a)$$

From these equations the heavy quanta fields $\Psi_n(x, y, z, t)$ can be solved, if the nuclon field $\psi(x, y, z, t)$ is given [52]). Following the method of Y u k a w a, however, we shall now neglect all derivatives with respect to the time in (130) and solve these equations only for the static case. Then, these equations take the form of

$$(\Delta - \varkappa^2)\Psi = -\,4\pi U, \qquad (131)$$

where U denotes combinations of **s**, **v**, **a**, **e**, **h**, etc., $\overline{\mathbf{s}}$, $\overline{\mathbf{v}}$, etc., and their first and second order derivatives with respect to the spatial co-ordinates. The solution of (131) is given by

$$\Psi(1) = \int d2\ .\ U(2)e^{-\kappa r_{12}}/r_{12}, \qquad (132)$$

where the gradients occurring in U can be eliminated by an integration by parts

$$\int d2\ .\ \{\nabla_2 U'(2)\}e^{-\kappa r_{12}}/r_{12} = -\int d2\ .\ U'(2)\,\nabla_2(e^{-\kappa r_{12}}/r_{12}). \qquad (133)$$

Here ∇_n denotes differentiation with respect to the set of co-ordinates (n) or x_n, y_n, z_n.

The expressions (132)—(133) can be substituted into (125). Then, however, it must be remembered that in (125) Ψ was essentially commutative with ψ and ψ^*, whereas the expression (132) does not commute with them, since U contains both ψ^* and ψ. This is a result of the omission of the derivatives with respect to the time from (130).

After the substitution of (132)—(133) into (125), the latter expression seems to take again the form of (126). Now, by the substitution, H_g seems to become a part of the term with $W(1,2)_{op}$ in (126). We should make an error of a factor 2 in H_g, however, if we would conclude from this new expression to the corresponding Schrödinger equation (128). The cause of this error is again the fact that by the substitution of (132) into (125) the commutator of H with ψ, which was essential for the derivation of (127) (thus of (61)—(128)) from H, is affected.

In order to avoid this difficulty it is more convenient to make use of the commutation relations *before* substituting (132) into the Hamiltonian [21]). Then (132) is inserted into the field equation (127) instead of into the Hamiltonian (125). By this substitution a part (H_g) of the first term of (127) becomes a part of the interaction term with $W(1,2)_{op}^{sym}$ again. The sum of this new contribution to $W(1,2)_{op}^{sym}$ (from H_g) and the original interaction term $W(1,2)_{op}^{sym}$ then can be regarded as the *effective interaction operator* $W(1,2)_{op}^{eff}$.

The Hamiltonian (126) can be regarded as a convenient expression, from which can be derived the wave equation (127) describing the motion of ψ in interaction with the field of heavy quanta. This field of heavy quanta, however, does not interest us for the present. We want to know only the motion of ψ in interaction (through the field of heavy quanta) with itself. This is described by our new equation (127*eff*) of the form of (127), from which, however, Ψ was eliminated, so that W_{op}^{sym} was replaced by W_{op}^{eff}.

This new equation (127*eff*), on the other hand, can be obtained directly from another *effective* "Hamiltonian", differing from (125) since H_g does no more occur in it and since W_{op} has been replaced by W_{op}^{eff}. We remark that the transition from (127*eff*) to this effective Hamiltonian (126*eff*), in analogy to the transition from (127) to (126), takes place by adding the factor $\int \psi^{\dagger}(1)$ to the first term, but a factor $\frac{1}{2} \int \psi^{\dagger}(1)$ to the last term of (127*eff*). Since this last term was obtained from H_g by omitting the factor $\int \psi^{\dagger}(1)$ and by substituting (132)—(133) for Ψ, we find that *the effective Hamiltonian differs from the expression obtained from the terms* (125) *of the original Hamiltonian by only inserting* (132)—(133), *by an additional factor* $\frac{1}{2}$ *to the term* H_g.

The "physical meaning" of such a factor $\frac{1}{2}$ is that, if the action of one particle through the field on the other particle has been taken into account by a term of the form of a direct interaction between both particles to the Hamiltonian, it is no more necessary to take into account the action of the second particle on the first, since this reaction is already contained in the term describing the first action *).

The effective Hamiltonian has been chosen in such a way that (51) remains valid for the nuclon field. So we can proceed from (51) to (61) and from (61) to the Schrödinger equation (128), in which now the *effective interaction potential* takes the place of W_{op}^{sym}.

We must still remark that the procedure here discussed is allowed only if the expression for W_{op}^{eff}, obtained from (127) by (132)—(133), is *automatically symmetric* in the co-ordinates of the two particles, since an *asymmetrical* expression in the equation of motion can never be *interpreted* as an *effective* interaction operator taking the place of the *automatically symmetric* operator W_{op}^{sym} in (127)—(128). It is not allowed to symmetrize the effective operator afterwards, since $\psi(1)$ and $\psi(2)$ play a different part in the equation (127).

The actual way of calculation is now the following: In $\mathcal{H}_g$ we insert the solutions (132)—(133) of (131); we verify whether the operator operating in the resulting expression on $\psi(1)\psi(2)$, can be written in a symmetrical form. It will turn out that, if derivatives with respect to the time are neglected not only in the left hand members, but also in the right hand members of (130), this is possible indeed. Then we multiply this operator by $\frac{1}{2}$, and add it to the corresponding operator in $\mathcal{H}_{gg}$, which is of the form of $\sum_n g_n^2 \boldsymbol{\omega}_n^{\dagger(1)} \boldsymbol{\omega}_n^{(2)} \delta(\vec{r}_{12})$ (see above). The result represents the (effective) interaction operator †) to be inserted at once in the Schrödinger equation (128).

As regards the infinite self-energy neglected from $\mathcal{H}_{gg}$ in the foregoing, it would give an infinite additional term to $\mathcal{H}_0$, which does not interest us for the moment. It represents the static part of the "*mesic self-energy*" of the nucleons. In a complete theory, this term is hoped to explain the heavy mass of the nuclons, like the mass of the electron is hoped to be explained as representing the electromagnetic self-energy of an electron. However, it is not clear, then,

*) Compare H. A. Kramers, loc. cit. [53]), p. 301.

†) The same result can also be obtained by inserting (132)—(133) into the *total* Hamiltonian including the meson terms in $\mathcal{H}_0$. Then no factor $\frac{1}{2}$ is required, since exactly half the term $\mathcal{H}_g$ is cancelled by the terms of $\mathcal{H}_0$ describing the heavy quanta. The fact that both methods yield the same result indicates that by transition from (125) to (126*eff*) the total energy of the meson field generated by the nuclons is accounted for as interaction energy of the nuclons themselves.

why the mass of a neutron should be larger than that of a proton.

We shall now give the result of the discussed calculation, if it is performed for the Hamiltonian given by (90). Here and in the following we shall put (compare (24)):

$$g_0 = \mathsf{K} f_o, \quad g_1 = \mathsf{K} g_b, \quad g_2 = \mathsf{K} f_b, \quad g_3 = \mathsf{K} g_d, \quad g_4 = \mathsf{K} f_d; \qquad \mathsf{K} = +\sqrt{c\varkappa^3/4\pi}. \tag{134}$$

Then, making use of

$$(\varkappa^2 - \Delta)(e^{-\kappa r}/r) = 4\pi\delta(\vec{r}), \tag{135}$$

we can write the resulting *effective potential* in the following form:

$$\begin{aligned} W(1,2)_{op} = \tfrac{1}{2}(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})[g_1^2\{1 - (\boldsymbol{\alpha}^{(1)}\cdot\boldsymbol{\alpha}^{(2)}) - (1/\varkappa^2)(\boldsymbol{\alpha}^{(1)}\cdot\nabla_1)(\boldsymbol{\alpha}^{(2)}\cdot\nabla_2)\} + \\ + (g_1g_2/\varkappa)\{\boldsymbol{\beta}^{(1)}([\boldsymbol{\sigma}^{(1)},\boldsymbol{\alpha}^{(2)}]\cdot\nabla_1) + \boldsymbol{\beta}^{(2)}([\boldsymbol{\sigma}^{(2)},\boldsymbol{\alpha}^{(1)}]\cdot\nabla_2) + \\ + i\boldsymbol{\beta}^{(1)}(\boldsymbol{\alpha}^{(1)}\cdot\nabla_2) + i\boldsymbol{\beta}^{(2)}(\boldsymbol{\alpha}^{(2)}\cdot\nabla_1)\} + \\ + g_2^2\{\boldsymbol{\beta}^{(1)}\boldsymbol{\beta}^{(2)}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}) + (1/\varkappa^2)\boldsymbol{\beta}^{(1)}\boldsymbol{\beta}^{(2)}(\boldsymbol{\sigma}^{(1)}\cdot\nabla_1)(\boldsymbol{\sigma}^{(2)}\cdot\nabla_2) - \\ - (1/\varkappa^2)\boldsymbol{\beta}^{(1)}\boldsymbol{\beta}^{(2)}(\boldsymbol{\alpha}^{(1)}\cdot\nabla_1)(\boldsymbol{\alpha}^{(2)}\cdot\nabla_2)\} - \\ - (g_3^2/\varkappa^2)(\boldsymbol{\sigma}^{(1)}\cdot\nabla_1)(\boldsymbol{\sigma}^{(2)}\cdot\nabla_2) - (g_3g_4/\varkappa)\{i\boldsymbol{\gamma}_5^{(1)}\boldsymbol{\beta}^{(1)}(\boldsymbol{\sigma}^{(2)}\cdot\nabla_2) + \\ + i\boldsymbol{\gamma}_5^{(2)}\boldsymbol{\beta}^{(2)}(\boldsymbol{\sigma}^{(1)}\cdot\nabla_1)\} + \\ + g_4^2\boldsymbol{\gamma}_5^{(1)}\boldsymbol{\beta}^{(1)}\boldsymbol{\gamma}_5^{(2)}\boldsymbol{\beta}^{(2)}](e^{-\kappa r_{12}}/r_{12}) + \\ + \tfrac{1}{2}(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})(4\pi/\varkappa^2)[g_0^2(1 - C_0)\boldsymbol{\beta}^{(1)}\boldsymbol{\beta}^{(2)} - \\ - g_1^2C_1\{1 - (\boldsymbol{\alpha}^{(1)}\cdot\boldsymbol{\alpha}^{(2)})\} - \\ - g_2^2\{C_2\boldsymbol{\beta}^{(1)}\boldsymbol{\beta}^{(2)}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}) + (C_2 - 1)\boldsymbol{\beta}^{(1)}\boldsymbol{\beta}^{(2)}(\boldsymbol{\alpha}^{(1)}\cdot\boldsymbol{\alpha}^{(2)})\} + \\ + g_3^2(1 - C_3)\{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}) - \boldsymbol{\gamma}_5^{(1)}\boldsymbol{\gamma}_5^{(2)}\} - \\ - g_4^2C_4\boldsymbol{\gamma}_5^{(1)}\boldsymbol{\beta}^{(1)}\boldsymbol{\gamma}_5^{(2)}\boldsymbol{\beta}^{(2)}]\delta(\vec{r}_{12}). \end{aligned} \tag{136}$$

The symmetry of this expression is obvious. Comparing it with the expressions obtained by Kemmer [10]) by perturbation calculus we observe that in his expressions the terms with g_1g_2 and with g_3g_4 are lacking.

In order to ensure charge-independency of the nuclear forces (compare § 3) even with regard to the δ-function interactions, we have put

$$C_{op} = \overline{C}_{op} \tag{137}$$

in (136). The choice of these operators made by Kemmer [10]) and by Bhabha in the first section of his paper [18]), was $C_1 = 0$,

$C_2 = -1$. Later on, B h a b h a changed his choice [18]) into that of Y u k a w a [21]), viz. $C_1 = C_2 = 0$, hoping to avoid in this way the δ-functions entirely. From the formalistic point of view of (37) the latter choice seems to be the more natural one. (Also $C_{op} = 1$ would seem a reasonable choice from this point of view). It must be pointed out, however, that even in the non-relativistic approximation

$$\beta^{(1)} = \beta^{(2)} = 1; \quad \alpha^{(1)} = \alpha^{(2)} = \gamma_5^{(1)} = \gamma_5^{(2)} = 0 \tag{138}$$

this choice ($C_1 = C_2 = 0$) does not eliminate all δ-functions, since (compare (5*a*))

$$(1/\varkappa^2)(\sigma^{(1)} . \nabla_1)(\sigma^{(2)} . \nabla_2)(e^{-\kappa r_{12}}/r_{12}) =$$
$$= \{G(\sigma^{(1)}, \sigma^{(2)}) - \tfrac{1}{3}(\sigma^{(1)} . \sigma^{(2)})\}e^{-\kappa r}/r + (4\pi/3\varkappa^2)(\sigma^{(1)} . \sigma^{(2)})\delta(\vec{r}), \tag{139}$$

where we have put

$$G(\vec{a}, \vec{b}) \equiv \{\tfrac{1}{3}(\vec{a} . \vec{b}) - (\vec{a} . \vec{r})(\vec{b} . \vec{r})/r^2\}\{1 + (3/\varkappa r) + (3/\varkappa^2 r^2)\}. \tag{140}$$

Inserting (139) and

$$-(1/\varkappa)\vec{\nabla}_1(e^{-\kappa r}/r) = (1/\varkappa)\vec{\nabla}_2(e^{-\kappa r}/r) = \{1 + (1/\varkappa r)\}(\vec{r}/r)(e^{-\kappa r}/r), \quad (\vec{r} = \vec{r}_1 - \vec{r}_2) \tag{141}$$

into (136), we find

$$\begin{aligned} W(1,2)_{op} = &\tfrac{1}{2}(\tau^{(1)}.\tau^{(2)})\,[g_1^2\{1 - \tfrac{2}{3}(\alpha^{(1)} . \alpha^{(2)}) - G(\alpha^{(1)}, \alpha^{(2)})\} - \\ &- g_1 g_2\{\beta^{(1)}([\sigma^{(1)}, \alpha^{(2)}].r) - \beta^{(2)}([\sigma^{(2)}, \alpha^{(1)}].r) - \\ &- i\beta^{(1)}(\alpha^{(1)}.r) + i\beta^{(2)}(\alpha^{(2)}.r)\}(1/r)\{1 + (1/\varkappa r)\} + \\ &+ g_2^2\beta^{(1)}\beta^{(2)}\{\tfrac{2}{3}(\sigma^{(1)}.\sigma^{(2)}) + G(\sigma^{(1)},\sigma^{(2)}) + \tfrac{1}{3}(\alpha^{(1)}.\alpha^{(2)}) - G(\alpha^{(1)},\alpha^{(2)})\} + \\ &+ g_3^2\{\tfrac{1}{3}(\sigma^{(1)} . \sigma^{(2)}) - G(\sigma^{(1)}, \sigma^{(2)})\} - \\ &- g_3 g_4\{i\gamma_5^{(1)}\beta^{(1)}(\sigma^{(2)}.r) - i\gamma_5^{(2)}\beta^{(2)}(\sigma^{(1)}.r)\}(1/r)\{1 + (1/\varkappa r)\} + \\ &+ g_4^2\gamma_5^{(1)}\beta^{(1)}\gamma_5^{(2)}\beta^{(2)}](e^{-\kappa r}/r) + \\ &+ \tfrac{1}{2}(\tau^{(1)}.\tau^{(2)})(4\pi/\varkappa^2)[g_0^2(1 - C_0)\beta^{(1)}\beta^{(2)} - g_1^2 C_1\{1 - (\alpha^{(1)}.\alpha^{(2)})\} - \\ &- g_2^2\{C_2\beta^{(1)}\beta^{(2)}(\sigma^{(1)}.\sigma^{(2)}) + (C_2 - 1)\beta^{(1)}\beta^{(2)}(\alpha^{(1)}.\alpha^{(2)})\} + \\ &+ g_3^2(1 - C_3)\{(\sigma^{(1)}.\sigma^{(2)}) - \gamma_5^{(1)}\gamma_5^{(2)}\} - g_4^2 C_4\gamma_5^{(1)}\beta^{(1)}\gamma_5^{(2)}\beta^{(2)}]\delta(\vec{r}) - \\ &- \tfrac{1}{6}(\tau^{(1)} . \tau^{(2)})(4\pi/\varkappa^2)[g_3^2(\sigma^{(1)}.\sigma^{(2)}) + g_1^2(\alpha^{(1)}.\alpha^{(2)}) + \\ &+ g_2^2\beta^{(1)}\beta^{(2)}((\alpha^{(1)}.\alpha^{(2)}) - (\sigma^{(1)}.\sigma^{(2)}))]\delta(\vec{r}). \end{aligned} \tag{142}$$

*) Hier staat bij elkaar:

$$\begin{aligned} &+\tfrac{1}{2}(\tau^{(1)}.\tau^{(2)})(4\pi/\kappa^2)[g_0^2(1 - C_0)\beta^{(1)}\beta^{(2)} - g_1^2\{C_1 + (\tfrac{1}{3} - C_1)(\alpha^{(1)}.\alpha^{(2)})\} + \\ &+ g_2^2\beta^{(1)}\beta^{(2)}\{(\tfrac{1}{3} - C_2)(\sigma^{(1)}.\sigma^{(2)}) + (\tfrac{1}{3} - C_2)(\alpha^{(1)}.\alpha^{(2)})\} + \\ &+ g_3^2\{(\tfrac{2}{3} - C_3)(\sigma^{(1)}\sigma^{(2)}) + (C_3 - 1)\gamma_5^{(1)}\gamma_5^{(2)}\} - \\ &- g_4^2 C_4\gamma_5^{(1)}\beta^{(1)}\gamma_5^{(2)}\beta^{(2)}]\ \delta(\vec{r}). \end{aligned}$$

R.

No choice of C_0, C_1, C_2, C_3, C_4 will eliminate the δ-functions from this expression. If we choose

$$C_0 = 1, \quad C_1 = 0, \quad C_2 = \tfrac{1}{3}, \quad C_3 = \tfrac{2}{3}, \quad C_4 = \text{arbitrary}, \tag{143}$$

the *ordinary* δ-functions disappear at least from the non-relativistic approximation (138). This choice, however, ~~does not eliminate the terms with the longitudinal δ-functions~~. seems to be an artificial one.

The expression given by Breit[51]) for the interaction between charged particles through the electromagnetic field was a reasonable approximation since only retardation effects and non-secular high-frequency effects are neglected in its derivation. Since *we* have neglected not only the $(\partial^2/\partial t^2)$-term of the left hand member of (130), but also the derivatives with respect to the time of $\mathbf{s}$, $\mathbf{v}$, $\vec{\mathbf{a}}$, $\vec{\mathbf{e}}$, $\vec{\mathbf{h}}$, etc., the approximation of the *nuclon* interaction given by (142) is *much worse*. Indeed, the "velocity"-dependence given by the terms with $\vec{\boldsymbol{\alpha}}$ can hardly be regarded as well justified by such a derivation. The velocity-independent part of (142), obtained by (138), will be a good approximation for the effective potential between slow nuclons, however. This *non-relativistic approximation* yields, [if we choose C_{op} (137) according to (143), or, better,] *if we ignore the δ-functions*, which — according to Kemmer[22]) — give "only" an infinite contribution to the levels of the deuteron:

$$W(1,2)_{op} \approx \tfrac{1}{2}(\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) \{A + B'(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) + C'\, G(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)})\}\, e^{-\kappa r}/r;$$
$$A = g_1^2\ (\geq 0), \qquad B' = \tfrac{2}{3}\, g_2^2 + \tfrac{1}{3}\, g_3^2\ (\geq 0), \tag{144}$$
$$C' = g_2^2 - g_3^2\ (\gtrless 0).$$

It should be hoped that it will be possible to explain by this interaction the experiments on scattering of neutrons by protons and the binding energy, the magnetic moment and the electric quadrupole moment[54]) of the deuteron. If this shall be possible, the term with $G(\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)})$, which couples the ^{3}S state with the $^3D_1$ state (compare § 2), cannot be neglected. It should be possible to calculate the strength of the coupling directly from the measured electric quadrupole moment and the magnetic moment of the deuteron. Compare, however, the following section.

It cannot be expected that the force derived in this section will give an exact explanation of the binding energies of other nuclei than the deuteron, since the triple and multiple forces arising from a "multiple exchange" of heavy quanta between nuclons[19]) may be of considerable importance there.

It is hoped by Bhabha[13]) and Iwanenko[52]) that an entirely *classical* treatment of the heavy quanta field will yield an expression for the interaction between nuclons, in which the velocity of the nuclons will have been taken into account.

The solution of equations like

$$(\square - \varkappa^2)\,\Psi = -\,4\pi\,U \tag{145}$$

is *not* given by (132) with "retardation" of $U(2)$. The equation (145) has been solved by Iwanenko[52]).

§ 8. *The deuteron problem.* Inserting (144) into (128) we find the (non-relativistic) Schrödinger equation of the deuteron:

$$\{\mathcal{E} + (\hbar^2/2M^{(1)}_{op})\,\Delta_1 + (\hbar^2/2M^{(2)}_{op})\,\Delta_2 - W(1,2)_{op}\}\,\psi(1,2) = 0. \tag{146}$$

Here M_{op} is given by

$$M_{op} = M_P\,\frac{1+\boldsymbol{\tau}_z}{2} + M_N\,\frac{1-\boldsymbol{\tau}_z}{2}\,, \tag{147}$$

so that

$$1/M_{op} = (1/M_P)\,\frac{1+\boldsymbol{\tau}_z}{2} + (1/M_N)\,\frac{1-\boldsymbol{\tau}_z}{2}\,. \tag{148}$$

We shall introduce the *relative co-ordinates*

$$\vec{r} = \vec{r}_1 - \vec{r}_2,\ \{x = x_1 - x_2,\ y = y_1 - y_2,\ z = z_1 - z_2\}, \tag{149}$$

and the *co-ordinates of the centre of gravity*

$$\vec{r}_0(M^{(1)}_{op} + M^{(2)}_{op}) = \vec{r}_1\,M^{(1)}_{op} + \vec{r}_2\,M^{(2)}_{op}, \tag{150}$$

so that

$$\vec{r}_0 = \frac{\vec{r}_1 + \vec{r}_2}{2} - \frac{M_N - M_P}{M_N + M_P}\cdot\frac{\vec{r}_1 - \vec{r}_2}{2}\cdot\frac{\boldsymbol{\tau}_z^{(1)} - \boldsymbol{\tau}_z^{(2)}}{2}\,. \tag{151}$$

Differentiation with respect to these new co-ordinates is defined in the usual way (as if $M^{(1)}_{op}$ and $M^{(2)}_{op}$ were constants). We shall put

$$\frac{2}{M} = \frac{1}{M_P} + \frac{1}{M_N};\quad \delta_M = \frac{M_N - M_P}{M_N + M_P}\,. \tag{152}$$

Then

$$\frac{\partial}{\partial x_{\frac{1}{2}}} = \frac{\partial x_0}{\partial x_{\frac{1}{2}}}\,\frac{\partial}{\partial x_0} + \frac{\partial x}{\partial x_{\frac{1}{2}}}\,\frac{\partial}{\partial x} \tag{153}$$

yields

$$p_{\frac{1}{2}} = \tfrac{1}{2}\,p_0\left(1 \mp \delta_M\,\frac{\boldsymbol{\tau}_z^{(1)} - \boldsymbol{\tau}_z^{(2)}}{2}\right) \pm p. \tag{154}$$

Putting

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad \Delta_0 = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y_0^2} + \frac{\partial^2}{\partial z_0^2} \tag{155}$$

we can write (146) in the following form:

$$\{\mathcal{E} + \frac{\hbar^2}{M}\left(1 + \delta_M \frac{\boldsymbol{\tau}_z^{(1)} + \boldsymbol{\tau}_z^{(2)}}{2}\right)(\Delta + \tfrac{1}{4}\Delta_0) - W(1,2)_{op}\}\,\psi(1,2) = 0. \tag{156}$$

Since the wave-function of a deuteron is an eigenfunction of the operator $(\boldsymbol{\tau}_z^{(1)} + \boldsymbol{\tau}_z^{(2)})$:

$$(\boldsymbol{\tau}_z^{(1)} + \boldsymbol{\tau}_z^{(2)})\,\psi(1,2) = 0, \tag{157}$$

the isotopic spin operators vanish from the kinetic energy operator in (156) and we can separate the co-ordinates of the centre of gravity:

$$\Big\{\mathcal{E} + \frac{\hbar^2}{M}\Delta - \tfrac{1}{2}(\vec{\boldsymbol{\tau}}^{(1)} \cdot \vec{\boldsymbol{\tau}}^{(2)})\,[A + B'(\vec{\boldsymbol{\sigma}}^{(1)} \cdot \vec{\boldsymbol{\sigma}}^{(2)}) +$$

$$+ C'\,\mathcal{G}(\vec{\boldsymbol{\sigma}}^{(1)}, \vec{\boldsymbol{\sigma}}^{(2)})]\frac{e^{-\kappa r}}{r}\Big\}\,\psi = 0, \tag{158}$$

where

$$\psi = T(\tau_1, \tau_2).\ \Psi(x, y, z, \sigma_1, \sigma_2). \tag{159}$$

By σ_n and τ_n we denote the ordinary and the isotopic spin co-ordinate of the n^{th} particle. In the $^3S - {}^3D_1$ state of the deuteron the isotopic spin function T is given by the antisymmetrical "singlet spin function" $^1\chi_0(\tau_1, \tau_2)$; in the ^1S state of the deuteron it is given by the symmetric "triplet spin function" $^3\chi_0(\tau_1, \tau_2)$. The singlet and triplet spin functions $^M\chi_\mu$ are given by

$$\textit{Singlet:}\ {}^1\chi_0(\sigma_1, \sigma_2) = \frac{1}{\sqrt{2}}\{\alpha(\sigma_1)\,\beta(\sigma_2) - \alpha(\sigma_2)\,\beta(\sigma_1)\},$$

$$\textit{Triplet:}\ {}^3\chi_0(\sigma_1, \sigma_2) = \frac{1}{\sqrt{2}}\{\alpha(\sigma_1)\,\beta(\sigma_2) + \alpha(\sigma_2)\,\beta(\sigma_1)\}, \tag{160}$$

$$^3\chi_1(\sigma_1, \sigma_2) = \alpha(\sigma_1)\,\alpha(\sigma_2), \qquad {}^3\chi_{-1}(\sigma_1, \sigma_2) = \beta(\sigma_1)\,\beta(\sigma_2),$$

where

$$\alpha(\sigma) = \delta_{1,\sigma}, \qquad \beta(\sigma) = \delta_{-1,\sigma}, \tag{161}$$

so that $\alpha(\tau)$ denotes the isotopic spin function of a single proton and $\beta(\tau)$ that of a neutron.

In the ^1S state of the deuteron the wave-function (159) takes the form of

$$\psi^{(^1S)} = {}^3\chi_0(\tau_1, \tau_2) \cdot {}^1\chi_0(\sigma_1, \sigma_2)\ {}^1\Psi(r). \tag{162}$$

Making use of

$$\begin{aligned}(\vec{\boldsymbol{\tau}}^{(1)} \cdot \vec{\boldsymbol{\tau}}^{(2)})\, {}^3\chi_0(\tau_1, \tau_2) &= +{}^3\chi_0(\tau_1, \tau_2), \\ (\vec{\boldsymbol{\sigma}}^{(1)} \cdot \vec{\boldsymbol{\sigma}}^{(2)})\, {}^1\chi_0(\sigma_1, \sigma_2) &= -3\, {}^1\chi_0(\sigma_1, \sigma_2), \quad G(\vec{\boldsymbol{\sigma}}^{(1)}, \vec{\boldsymbol{\sigma}}^{(2)})\, {}^1\chi_0(\sigma_1, \sigma_2) = 0,\end{aligned} \tag{163}$$

we can reduce the Schrödinger equation for this ^{1}S state of the deuteron to

$$\left\{ {}^1\mathcal{E} + \frac{\hbar^2}{M}\Delta - \tfrac{1}{2}(A - 3B')\frac{e^{-\kappa r}}{r} \right\} {}^1\Psi(r) = 0. \tag{164}$$

Comparing this equation with (8) we find

$${}^1\mathcal{K} = \tfrac{3}{2}B' - \tfrac{1}{2}A \tag{22a}$$

again. We have mentioned (§ 2) that the Schrödinger equation (164) has been solved numerically [24] [25] for those values of $^1\mathcal{K}$, which yield a real level. Then, it is usually assumed that the actual value of $^1\mathcal{K}$ is approximately equal to that value, for which the energy of the level becomes zero. However, it is not impossible (see § 2) that this assumption introduces an error, which may amount to perhaps 10% of the value of $^1\mathcal{K}$ (compare (13) with (13*a*)).

The eigenfunctions for the ^{3}S — ^{3}D$_1$ state of the deuteron [23] are linear combinations of those of a "pure" ^{3}S state:

$$\Psi_\mu^{(^3S)} = \frac{u(r)}{r}\, {}^3\chi_\mu(\sigma_1, \sigma_2), \tag{165}$$

and those of a "pure" ^{3}D$_j$ state ($j = 1$):

$$\begin{aligned}\Psi_\mu^{(^3D_j)} &= \frac{v(r)}{r}\, \Phi_\mu^{(j)}(\vartheta, \varphi, \sigma_1, \sigma_2), \\ \Phi_\mu^{(j)} &= \sum_{\mu'=-1}^{1} a_{\mu\mu'}^{(j)}\, Y_2^{\mu-\mu'}(\vartheta, \varphi)\, {}^3\chi_{\mu'}(\sigma_1, \sigma_2).\end{aligned} \tag{166}$$

Here the functions Y_l^m (with $|m| \leq l$) are eigenfunctions of the orbital angular momentum operators $\mathcal{M}^2$ and $\mathcal{M}_z$ belonging to the eigenvalues $\hbar^2 l(l+1)$ and $\hbar m$ respectively:

$$\Delta Y_l^m = -\frac{l(l+1)}{r^2} Y_l^m, \quad \Delta\Phi_\mu^{(j)} = -\frac{6}{r^2}\Phi_\mu^{(j)}. \tag{167}$$

If they are normalized according to

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta\, d\vartheta\, |Y_l^m|^2 = 4\pi, \tag{168}$$

they are given by [55] [56])

$$Y_2^2 = (Y_2^{-2})^* = \sqrt{\frac{15}{8}} \sin^2 \vartheta \, . \, e^{2i\phi},$$
$$Y_2^1 = -(Y_2^{-1})^* = -\sqrt{\frac{15}{2}} \sin \vartheta \cos \vartheta \, . \, e^{i\phi}, \qquad (169)$$
$$Y_2^0 = \sqrt{\frac{5}{4}} (3 \cos^2 \vartheta - 1).$$

The triplet spin functions ${}^3\chi_\mu$ are given by (160). They are eigenfunctions of the spin angular momentum operators $\mathcal{S}^2$ and $\mathcal{S}_z$ belonging to the eigenvalues $2\hbar^2$ and $\mu\hbar$ respectively. From (166) we conclude that the ${}^3D_j$-functions $\Phi_\mu^{(j)}$ as well as the triplet spin functions ${}^3\chi_\mu$ satisfy the relations

$$(\vec{\sigma}^{(1)} \, . \, \vec{\sigma}^{(2)}) \, {}^3\chi_\mu = {}^3\chi_\mu, \quad (\vec{\sigma}^{(1)} \, . \, \vec{\sigma}^{(2)}) \, \Phi_\mu^{(j)} = \Phi_\mu^{(j)}. \qquad (170)$$

The coefficients $a_{\mu\mu'}^{(j)}$ in (166) are now chosen in such a way that the $\Phi_\mu^{(j)}$ are eigenfunctions not only of $\mathcal{J}_z = \mathcal{M}_z + \mathcal{S}_z$ belonging (on account of (166)) to the eigenvalue $\hbar\mu$, but also of

$$\mathcal{J}^2 = \mathcal{M}^2 + \mathcal{S}^2 + 2(\vec{\mathcal{M}} \, . \, \vec{\mathcal{S}})$$

belonging to the eigenvalue $\hbar^2 j(j+1)$. For $j = 1$ this choice of the coefficients $a_{\mu\mu'}^{(j)}$ is given by:

$\mu' =$	-1	0	1
$\mu = -1$	$+\sqrt{\frac{1}{10}}$	$-\sqrt{\frac{3}{10}}$	$+\sqrt{\frac{6}{10}}$
$\mu = 0$	$+\sqrt{\frac{3}{10}}$	$-\sqrt{\frac{4}{10}}$	$+\sqrt{\frac{3}{10}}$
$\mu = 1$	$+\sqrt{\frac{6}{10}}$	$-\sqrt{\frac{3}{10}}$	$+\sqrt{\frac{1}{10}}$

$= a_{\mu\mu'}^{(1)}$. (171)

We have "normalized" these coefficients in such a way that the $\Phi_\mu^{(1)}$ are normalized, just as the ${}^3\chi_\mu$, according to

$$\sum_{\sigma_1, \sigma_2} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin \vartheta \, d\vartheta \, \Phi_{\mu'}^{(1)*} \Phi_\mu^{(1)} = 4\pi \, \delta_{\mu\mu'}. \qquad (172)$$

The spin functions ${}^3\chi_\mu$ and the spin-angular functions $\Phi_\mu^{(1)}$ satisfy the relations

$$\{\tfrac{1}{3}(\vec{\sigma}^{(1)} \, . \, \vec{\sigma}^{(2)}) - (\vec{\sigma}^{(1)} \, . \, \vec{r}) \, (\vec{\sigma}^{(2)} \, . \, \vec{r})/r^2\} \, {}^3\chi_\mu = -\tfrac{2}{3}\sqrt{2} \, . \, \Phi_\mu^{(1)},$$
$$\{\tfrac{1}{3}(\vec{\sigma}^{(1)} \, . \, \vec{\sigma}^{(2)}) - (\vec{\sigma}^{(1)} \, . \, \vec{r}) \, (\vec{\sigma}^{(2)} \, . \, \vec{r})/r^2\} \, \Phi_\mu^{(1)} = \tfrac{2}{3}\Phi_\mu^{(1)} - \tfrac{2}{3}\sqrt{2} \, . \, {}^3\chi_\mu. \qquad (173)$$

Now, the triplet solution of (158) will be a linear combination of (165) and (166):

$$^3\psi_\mu = {}^1\chi_0(\tau_1, \tau_2) \cdot {}^3\Psi_\mu; \qquad {}^3\Psi_\mu = \Psi_\mu^{(^3S)} + \Psi_\mu^{(^3D_1)}. \tag{174}$$

Inserting (174) with (165) and (166) into (158) and making use of (167,) (170) and (173) we find the following equations for the radial functions $u(r)$ and $v(r)$ [23]:

$$\frac{\hbar^2}{M}\frac{d^2u}{dr^2} + {}^3\mathcal{E}u + {}^3\mathcal{K}\frac{e^{-\kappa r}}{r}u - C'\left(1 + \frac{3}{\varkappa r} + \frac{3}{\varkappa^2 r^2}\right)\frac{e^{-\kappa r}}{r}v\sqrt{2} = 0,$$

$$\frac{\hbar^2}{M}\left(\frac{d^2v}{dr^2} - \frac{6v}{r^2}\right) + {}^3\mathcal{E}v + {}^3\mathcal{K}\frac{e^{-\kappa r}}{r}v + \tag{175}$$

$$+ C'\left(1 + \frac{3}{\varkappa r} + \frac{3}{\varkappa^2 r^2}\right)\frac{e^{-\kappa r}}{r}(v - u\sqrt{2}) = 0.$$

Here we have put again

$$^3\mathcal{K} = \tfrac{3}{2}B' + \tfrac{3}{2}A. \tag{22b}$$

Since the last terms of (175) are proportional to $1/r^3$ these equations cannot be solved in the usual way by expansion in powers of r. This does not necessarily mean that the eigenvalue problem (175) has no solution at all. But if first the potential is "cut off" at some distance r_0, the ground level will tend to $-\infty$ for $r_0 \to 0$.

If (175) is not to be regarded as the limiting case for $r_0 \to 0$ of a potential, which is cut off at r_0, it would perhaps be possible to ignore entirely the solutions of (175), which do not converge for $r \to 0$. Then, in the ground state of the deuteron the terms with $1/r^3$ should necessarily have the character of a strong *repulsion*, making the wave-function and its derivatives zero for $r \to 0$. Though it *may* be possible to remove the difficulty entirely in this way, the procedure seems to be not very satisfactory since there are many reasons why the interaction potential (144) is questionable for small distances.

First of all we have neglected all high order interactions, since we have solved the meson field acting on the nuclons from (131), but we have ignored the derivatives with respect to the time in the right hand members of the original equations (130). Now, we know from perturbation calculus [19]) that the higher approximations yield interactions, which for $r \to 0$ become much stronger than the first order interaction, so that the theory cannot be trusted at all in this region. Further it is quite possible that quantum mechanics, in their present form, are not competent for the discussion of problems, in which distances $r < r_0$ ($< 1/\varkappa$) are involved [57].

Thus one can hardly trust any conclusion from the theory, for which the form of the potential for small values of r is essential. Then, it seems to be reasonable to "cut off" the potentials entirely at some distance $r_0 (< 1/\varkappa)$ and to regard the ground level of the in this way corrected equations (175) as the actual ground state of the deuteron (though this state disappears for $r_0 \to 0$).

The cutting off radius r_0 then should be chosen of the order of magnitude of $\frac{1}{4}$ or $\frac{1}{2}$ of the "range" $1/\varkappa$ of the nuclear forces, since in this region the high order effects become strong. In the ground state the interaction with $1/r^3$ may now have the character of an attraction.

In the way indicated here the deuteron problem was solved by B e t h e [32]) for the special case

$$g_1 = g_3 = 0. \tag{176}$$

Then, the effective potential (144) takes a simple form. The two remaining parameters g_2 and r_0 were chosen by B e t h e in such a way as to adjust the singlet and the triplet level of the deuteron. The calculation was performed for K e m m e r's "symmetrical theory" of mesons and neutrettos as well as for the "neutral theory" discussed in § 3. Further the cutting off was made in two different ways, viz. $W(r) = 0$ or $W(r) = W(r_0)$ for $r < r_0$. The mass of the heavy quantum was assumed to be equal to 177 electron masses, the triplet level of the deuteron to be $-2{\cdot}17$ MeV. The ^1S level was not put equal to zero, but calculated from the cross section for the scattering of slow neutrons by protons, which was assumed to be equal to $18{\cdot}3 \times 10^{-24}$ cm^2. The results are the following [32]):

Cut off:	Neutral theory		Symmetrical theory	
	zero	straight	zero	straight
$g_2^2/\hbar c$ *)	$0{\cdot}162_4$	$0{\cdot}160_0$	0·500	0·308
$\varkappa r_0$	0·320	0·436	1·679	1·733
Q	2·73	2·67	—24·7	—17·8
$\bar{\alpha}$	6·88	6·68	23·46	18·52

(177)

*) The constant g_2 of B e t h e corresponds to $g_2/\sqrt{2}$ in our notation.

Here $\bar{\alpha}$ is the percentage of ^{3}D wave-function contained in the eigenfunction $^3\Psi$ of the triplet level; Q is the electric quadrupole moment of the deuteron in units 10^{-27} cm^2. From experimental data was calculated $Q = +2{\cdot}5$, but this value [32]) is uncertain since use was made of approximate wave-functions of the hydrogene molecule [54]).

Comparing the figures in (177) we remark that $Q < 0$ for the symmetrical theory, $Q > 0$ for the neutral theory. Since Q is defined as the average value of $(3z^2 - r^2)$, if the z-axis has the direction of $\vec{J}$ in the $^3S - {}^3D_1$ state, $Q > 0$ means a "cigar shape" and $Q < 0$ a "pill box shape" [32]). These signs of Q can be understood by a simple consideration about the region where the character of the potential will be attractive and where the wave-function will be large (compare Bethe, loc. cit. [32])). This region turns out to be more concentrated at small values of r for the neutral theory, so that it can be understood why this theory yields smaller values of Q. Essential for these considerations is the assumption $B' = \frac{4}{3}C' > 0$, introduced by (176).

We remark that, if C' had been chosen negative, just the neutral theory would have yielded a deuteron of a pronounced "pill box" character, whereas the symmetrical theory then would lead to a "cigar" shape. *Introducing the field of spinless heavy quanta*, that is, choosing a convenient value for the constant g_3, *we can change the sign as well as the value of the electric quadrupole moment predicted by the theory*. In this way theory and experiment can be fitted even in the symmetrical theory. This seems to be the main advantage of the generalized meson theory proposed by Møller and Rosenfeld [11]). For, indeed, the symmetrical theory seems to be preferable to the neutral theory, in view of the cosmic ray phenomena.

It is reasonable to expect that the percentage of $^3D_1$-wave-function in the triplet state can be calculated from the surplus magnetic moment of the deuteron due to this D-state. The orbital magnetic moment $\vec{\mu}^{orb}$ can be expressed in terms of the mechanical moment $\vec{M}$. Neglecting the difference of M_N and M_P we find for a deuteron by a simple consideration

$$\vec{\mu}^{orb} = \tfrac{1}{2}\,\frac{e}{2M_P c}\,\vec{M}. \tag{178}$$

The extra factor $\frac{1}{2}$ is a consequence of the fact that in the system, in which the centre of gravity is at rest, the radius vector of the charged particle is only half the distance r of the nuclons. Generally we have, on account of (148), (149), (151) and (154):

$$\vec{\mathcal{M}} \equiv [\vec{r}_1, \vec{p}_1] + [\vec{r}_2, \vec{p}_2] = [\vec{r}_0, \vec{p}_0] + [\vec{r}, \vec{p}] \tag{179}$$

and

$$\vec{\mu}^{orb} = \frac{1}{2c}\left\{\left(e \cdot \frac{1+\tau_z^{(1)}}{2}\right)\frac{1}{M_{op}^{(1)}}[\vec{r}_1, \vec{p}_1] + \left(e \cdot \frac{1+\tau_z^{(2)}}{2}\right)\frac{1}{M_{op}^{(2)}}[\vec{r}_2, \vec{p}_2]\right\} =$$

$$= \frac{e}{2M_P c}\left\{\frac{1+\tau_z^{(1)}}{2}[\vec{r}_1, \vec{p}_1] + \frac{1+\tau_z^{(2)}}{2}[\vec{r}_2, \vec{p}_2]\right\} =$$

$$= \tfrac{1}{2} \cdot \frac{e}{2M_P c}\left\{\left(1 + \frac{\tau_z^{(1)}+\tau_z^{(2)}}{2}\right)\vec{\mathcal{M}} + \delta_M \frac{1-\tau_z^{(1)}\tau_z^{(2)}}{2}([\vec{r}, \vec{p}] - [\vec{r}_0, \vec{p}_0]) + \right.$$

$$\left. + \frac{\tau_z^{(1)}-\tau_z^{(2)}}{2}([\vec{r}_0, \vec{p}] - \tfrac{1}{4}\delta_M^2 \overrightarrow{[r, p_0]})\right\}. \tag{180}$$

The expectancy value of this expression for the ground state of the deuteron is given exactly by (178).

The magnetic moment in the direction of the total angular momentum $\vec{\mathcal{J}}$ is now easily calculated putting the quantum number m_j equal to $j = 1$. From (166) — (171) we find for the total magnetic moment of the $^3D_1$ state of the deuteron in the direction of $\vec{\mathcal{J}}$:

$$\mu_D^{(^3D_1)} = \tfrac{3}{4}\frac{e\hbar}{2M_P c} - \tfrac{1}{2}(\mu_N + \mu_P). \tag{181}$$

If now the wave-function of the ground state is expressed in terms of *normalized* ^3S and $^3D_1$ functions:

$$^3\Psi = (\underline{\Psi}^{(^3S)} + \alpha\underline{\Psi}^{(^3D_1)})/\sqrt{1+\alpha^2}, \tag{182}$$

$$\alpha = \bar{\alpha}/(100 - \bar{\alpha}),$$
$$\alpha^2 = \int_0^\infty v^2 dr : \int_0^\infty u^2 dr, \tag{183}$$

then the effective magnetic moment μ_D of this state is given, in units of $(e\hbar/2M_P c)$, by

$$\mu_D = \frac{1}{1+\alpha^2}\left[\mu_N + \mu_P + \alpha^2\{\tfrac{3}{4} - \tfrac{1}{2}(\mu_N + \mu_P)\}\right]. \tag{184}$$

Taking $\mu_P = 2{\cdot}78$ and $\mu_D = 0{\cdot}85$ [58]), we calculate from (184):

$$\tfrac{1}{2}\alpha^2 = \frac{\mu_N + \mu_P - \mu_D}{\mu_N + \mu_P + 2\mu_D - 1\tfrac{1}{2}} = \frac{\mu_N + 1{\cdot}93}{\mu_N + 2{\cdot}98}. \tag{185}$$

Taking for μ_P the value of Estermann, Simpson and Stern[59]) ($\mu_P = 2{\cdot}46$), and the value of Farkas[19]) for μ_P/μ_D ($= 3{\cdot}8$), we find $\mu_D = 0{\cdot}65$ and

$$\tfrac{1}{2}\alpha^2 = \frac{\mu_N + 1{\cdot}81}{\mu_N + 2{\cdot}26}. \tag{185a}$$

The magnetic moment μ_N of the neutron was measured by the method due to Bloch[60]) and improved by Frisch, Von Halban and Koch[61]). Thus, μ_N was found[62]) to be in the neighbourhood of -2. So if the formulae (184)—(185) can be trusted, we must conclude that α is small and that the actual value of $|\mu_N|$ lies a little below 1·93 (or below 1·81). *)

Then, it should be hoped that by a convenient choice of the constants g_1, g_2 and g_3 in (144) and of the cutting off radius r_0 not only the energy levels, but also the electric quadrupole moment Q and the magnetic moment of the deuteron can be fitted with experiment, and that the cutting off radius r_0, determined in this way, will turn out to have a value between $(1/\varkappa)$ and $(1/10\varkappa)$. For this purpose calculations of the deuteron states for a choice of g_1, g_2 and g_3 different from (176) will be of great interest. Calculations of this kind exist for potentials of the the form of a "square well" [63]).

Even if it is possible to fit in this way theory with experiment, we must bear in mind that, if some of the calculated quantities ${}^3\mathcal{E}$, ${}^1\mathcal{E}$, Q, $\bar{\alpha}$, ... will turn out to be sensitive to the value of r_0 within the interval $1/\varkappa \sim 1/10\varkappa$, the value of such a quantity following from the theory can hardly be believed to be reliable, *if no physical meaning is given to the parameter* r_0.

It is not certain, however, whether the fitting of the *magnetic moment* will be possible at all. If it is true that new measurements of $|\mu_N|$ yield a value, which is still higher than 2, there must be some error in the derivation of (185).

In this connection it is of interest to remark that the *charge density* and the *charge current* distribution [M.F.(33)] of the *mesic field generated by the nuclons* on account of (130)—(132) *vanishes in the case of the deuteron*. Generally, this charge distribution is given by the q-number

$$\rho(1) = \tfrac{1}{2}\iint d2\, d3\,.\, \psi^\dagger(2)\,\psi^\dagger(3)\,\Omega\,(1;\,2,\,3)\,\psi(3)\,\psi(2), \tag{186}$$

$\mu_P = 2{,}79$
$\mu_N = -1{,}935 \pm 0{,}02$
$\mu_D = 0{,}856$

*) Nieuwere experimenten geven $\alpha = 0$. N!

where Ω is given in a non-relativistic approximation by

$$\Omega(1; 2, 3) = - e \frac{g_1 g_2}{\hbar c} \frac{\varkappa}{8\pi} \frac{e^{-\varkappa(r_{12}+r_{13})}}{r_{12}^2 r_{13}^2} \left(1 + \frac{1}{\varkappa r_{12}}\right)\left(1 + \frac{1}{\varkappa r_{13}}\right) \times$$

$$\times \left(\{\vec{\sigma}^{(2)} + \vec{\sigma}^{(3)}\} \cdot [\vec{r}_{12}, \vec{r}_{13}]\right) \left(\tau_x^{(2)} \tau_y^{(3)} - \tau_y^{(2)} \tau_x^{(3)}\right) = \Omega(1; 3, 2). \quad (187)$$

The expectancy value (59) of this q-number $\rho(1)$ for the N body problem is given by [64])

$$\tilde{\rho}(1) = \int \ldots \int \psi^\dagger(1, 2, \ldots, N) \,.\, \sum_{m<n} \Omega(1; m, n) \,.\, \psi(1, 2, \ldots, N). \quad (188)$$

For the deuteron it is easily seen that $(\tau_x^{(1)}\tau_y^{(2)} - \tau_y^{(1)}\tau_x^{(2)})$ possesses non-vanishing matrix elements only for transitions from antisymmetrical states to symmetrical states and *vice versa*. Thus (188) does not contribute anything to the charge distribution (and to the electric quadrupole moment) of the deuteron in the ground state.

In a similar way it can be proved that the electric *current density* of the meson field generated by a *deuteron in the ground state* must vanish.

§ 9. *The neutron-proton scattering*. The cross section for the scattering of neutrons by protons [65]) [29]) is related to the situation of the energy levels of the deuteron by the generalized formula [66]) of Bethe and Peierls [67]). This formula, however, does not give the right dependence of the cross section on the energy of the impacting neutrons.

If the form of the effective interaction potential is known, the scattering cross section can be calculated directly. For potentials of the form of a square well or a Gauss error function this has been done by several authors [63]) [68]). A meson potential of the form of $(-\mathcal{K} \,.\, e^{-\kappa r}/r)$ for the singlet state was used only for a comparison of the scattering of protons and of neutrons by protons [28]).

By a direct perturbation calculus the cross section for neutron-proton scattering for slow as well as for high energy neutrons was calculated by Bhabha [18]). Here, g_3 was put equal to zero. In the extreme relativistic case he finds the following differential cross section for scattering through the angle ϑ_0 in the system of co-ordinates, in which the centre of gravity of the proton and neutron is at rest:

$$d\Phi^{(E.R.)} \approx \frac{\pi}{2} \frac{g_1^4 + 2g_2^4}{\hbar^2 c^2} \frac{p_0^2}{\mathfrak{m}^2 c^2} \left(\frac{1}{\varkappa}\right)^2 (1 + \cos\vartheta_0)^2 \, d\cos\vartheta_0. \quad (189)$$

Here p_0 denotes the momentum of each of the collising particles in the system of the centre of gravity, $\mathfrak{m}$ is the mass of the mesons. The calculation has been performed for an "asymmetrical" meson theory (*without neutrettos*), so that the particle scattered through an angle ϑ_0 is changed from a neutron into a proton.

The factor p_0^2 of (184) ensures large cross sections for high energies. We shall see in the following, however, that for large energies the cross sections for many other processes increase. This is connected with the phenomenon of showers, which was discussed in § 6. If quantum-mechanics must not be modified [57]) at high energies ($\gg 10^8$ eV), we can only say that, according to the theory, high energy particles will give rise to a large number of very probable effects, which manifest themselves as showers and nuclear explosions.

§ 10. *The spontaneous disintegration of heavy quanta.* In § 6 we have mentioned that the term H_g' of the H a m i l t o n i a n possesses non-vanishing matrix elements for the following transitions:

$$Y^+ \rightarrow \pi + o, \quad Y^- \rightarrow \varepsilon + \nu, \quad Y^0 \rightarrow \nu + o, \quad Y^0 \rightarrow \pi + \varepsilon. \quad (190)$$

Here o is a neutrino, ε is a negative electron and ν and π are the neutral and the charged state of the corresponding light antiparticles (antineutrino and positon). Y^+, Y^- and Y^0 are a theticon, an arneticon and a neutretto respectively.

The transitions (190) are allowed by the conservation laws of momentum and energy, so that we must expect that there exists a transition probability for these first order effects. The probabilities per unit time are easily calculated according to

$$w = \frac{2\pi}{h}\,\rho(\mathcal{E})\,.\,\Sigma\,|\,Q\,|^2, \qquad (114a)$$

where the "matrix element" Q is obtained directly from (90) with (111).

For P r o c a-K e m m e r quanta the calculation was performed by Y u k a w a in his third paper [21]). The probability is calculated in the system, in which the meson is at rest, so that the terms with derivatives with respect to the spatial co-ordinates in H_g' can be omitted. In (114*a*) a summation is performed over both directions of the spin of each of the created light particles of positive energy. In the expression for the energy of these light particles the mass terms can be neglected since $\frac{1}{2}\mathfrak{m}c^2 \gg mc^2$. Then the calculation becomes extremely simple.

The density of final states $\rho(\mathcal{E})$ is easily calculated. The number of states for one of the emitted light Dirac particles (with a momentum p of about $\frac{1}{2}\mathfrak{m}c$ and an energy cp approximately), the momentum of which is situated in the interval between p and $p + dp$ and has a direction within a solid angle $d\omega$, is given by

$$\rho(\mathcal{E})d\mathcal{E} = \Omega \frac{p^2}{h^3} dp\, d\omega = \Omega \frac{\mathfrak{m}^2c^2}{32\pi^3\hbar^3} dp\, d\omega. \tag{191}$$

Now, the *differential of the energy of the final state is two times that of one of the emitted particles* (since the conservation of momentum requires that an increase of the momentum of one of the light particles is coupled with exactly the same increase of the momentum of the other emitted particle), so that

$$d\mathcal{E} = 2c\,dp. \tag{192}$$

From (191) and (192) we obtain

$$\rho(\mathcal{E}) = \Omega \frac{\mathfrak{m}^2c}{64\pi^3\hbar^3} d\omega. \tag{193}$$

In this way we find for the probability of disintegration per unit time:

for a Proca-Kemmer meson at rest:

$$\frac{1}{\tau_0} \equiv w_0 = \frac{2g_1'^2 + g_2'^2}{6\hbar c} \cdot \frac{\mathfrak{m}c^2}{\hbar}; \tag{194a}$$

for a spinless meson at rest:

$$\frac{1}{\tau_0} \equiv w_0 = \frac{g_4'^2}{2\hbar c} \cdot \frac{\mathfrak{m}c^2}{\hbar}. \tag{194b}$$

For neutrettos the probabilities of disintegration have exactly the same value; the probabilities for each of the processes $Y^0 \rightarrow \nu + o$ or $Y^0 \rightarrow \pi + \varepsilon$ apart have half the value of (194).

In (194) we have introduced the notation

$$g_0' = \mathsf{K}f_o', \quad g_1' = \mathsf{K}g_b', \quad g_2' = \mathsf{K}f_b', \quad g_3' = \mathsf{K}g_a', \quad g_4' = \mathsf{K}f_a';$$
$$(\mathsf{K} = +\sqrt{c\varkappa^3/4\pi}). \tag{134a}$$

If the heavy quantum is moving with a velocity v with respect to a system A, the probability per unit time with respect to (an observer in) A is given by the Lorentz transformation of the time coordinate, so that

$$\frac{1}{\tau} \equiv w = \frac{w_0}{\sqrt{1 - v^2/c^2}} \tag{195}$$

(*probability per unit time proportional to the kinetic energy of the decaying heavy quantum*).

We remark that the formulae (194*a*) and (195) for a Proca-Kemmer meson were given by Yukawa *in his third paper* [21]). In his fourth paper [23]) Yukawa has considered it necessary to change (194*a*) by adding a factor 2. This is due to an error in the interpretation of $\rho(\mathcal{E})$. If we describe the process like Yukawa by saying that by annihilation of a meson one light particle with a momentum p and a negative energy $-cp$ is changed into another light particle with the same momentum p, but with a different isotopic spin co-ordinate and with a positive energy $+cp$, the law of conservation of momentum requires that the energy of the initial state ($\mathcal{E}_a$), (which, from this point of view, is one among a continuum of states, like the final state), is varied together with the energy of the final state ($\mathcal{E}_f$). So in this description we have (compare § 6):

$$\rho(\mathcal{E})\,d(\mathcal{E}_f - \mathcal{E}_a) = 2\rho(\mathcal{E})\,d\mathcal{E}_f \tag{196}$$

with

$$d\mathcal{E}_f = c\,dp. \tag{197}$$

The difference of a factor *two* between (192) and (197) is thus compensated by an additional factor 2 in (196), which was overlooked by Yukawa. This error is continued in the publication of Yukawa on the mass and mean life time of the meson [69]).

Instead of the "*Fermi-Ansatz*" of § 4 for the interaction between heavy quanta and light Dirac particles, Yukawa [23]) has also investigated the consequences of a "*Konopinski-Uhlenbeck Ansatz*", in which derivatives of the neutrino wave-function with respect to the spatial and time co-ordinates occur. Though the formula given by Yukawa [23]) for the disintegration probability of a meson in consequence of such an interaction is not entirely correct *), it remains true that the expression for this probability

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*) If again the mass term in the energy of the light particles is neglected, the equation (68) of the paper of Yukawa [23]) should read:

$$w = \frac{g'^2}{2\hbar c}\,\frac{\mathfrak{m}c^2}{\hbar}\,\frac{\mathfrak{m}c^2}{\mathcal{E}}\left\{\tfrac{2}{3}\left|\lambda_1 + i\zeta\lambda_2 - i\,\frac{\mathfrak{m}}{2m}\,\mu_2\right|^2 + \tfrac{1}{3}\left|\mu_1 - i\zeta\mu_2 + i\,\frac{\mathfrak{m}}{2m}\,\lambda_2\right|^2\right\}.$$

If the mass term is taken into account, the „*K.U.*" interaction is corrected by a term of the order of magnitude of the uncorrected term resulting from the „Fermi" interaction. I am indebted to Dr. Podolanski for these results.

possesses a factor $(\mathfrak{m}/m)^2$ in addition to factors of the kind of those appearing in (194). This means that, if the constants g' are chosen of such an order of magnitude that the theoretical expression (194) for the disintegration probability is in agreement with the experimental data on this spontaneous decay of the heavy quanta [70] [71]), which yield a value of about $\tau_0 \approx 2 \times 10^{-6}$ sec, the constants g' must be chosen much smaller, if a "*K.U.*" interaction is assumed, than if the "Fermi" interaction of Kemmer is assumed. The difference corresponds to a factor 3×10^{-5}. This means that the value of g' resulting from a "*K.U.*"-*Ansatz* would yield a probability for the β-disintegration of instable nuclei (compare § 11), which is too small by a factor at least [72]) of the same order of magnitude [23]). Thus it seems that it can hardly have any sense to introduce this complicate interaction into the *meson* theory *), since the original K.U.-*Ansatz* [42]) was introduced only as a possible explanation of the phenomena of β-disintegration. For this reason we shall not discuss this interaction in the following section, but refer to the paper of Yukawa [23]).

If we put $\tau_0 \approx 2 \times 10^{-6}$ sec. and $\mathfrak{m} = 100\ \xi\ m$, we find from (194):

$$(2g_1'^2 + g_2'^2)/3\hbar c \approx (1{\cdot}3/\xi) \times 10^{-17} \tag{198a}$$

and

$$g_4'^2/\hbar c \approx (1{\cdot}3/\xi) \times 10^{-17}, \tag{198b}$$

if we assume that the disintegration probabilities of *spinless* and of *Proca-Kemmer* heavy quanta are of the same order of magnitude (an assumption, which does not necessarily follow from the experimental data!). Thus we find, taking $\mathfrak{m} \approx 175\ m$:

$$g'^2/\hbar c \approx \tfrac{3}{4} \times 10^{-17};\quad g' \approx 1\tfrac{1}{2} \times 10^{-17}. \tag{199}$$

§ 11. *The β-disintegration of instable nuclei.* Like the nuclon-nuclon interaction, the β-disintegration is partly a first order effect due to H_{gg}, partly a second order effect due to H_g. In the same way as in the discussion of the nuclon interaction (§ 7) we can according to Yukawa [23]) replace again the second order interaction between heavy and light particles by an *effective* Hamiltonian term of the same type as H_{gg}. Then, the first and second order interactions are described *together* by means of an operator, which can be

*) Of course, the arguments given here do not exclude the possibility of introducing the K.U.-*Ansatz* in the terms H_{gg} only.

transformed into an expression of the same type as the ordinary interaction term of Fermi [41]) (see below).

The calculation was performed by Yukawa [23]) for a combined K.-U. and Fermi interaction, but we shall confine ourselves to the latter (see § 10). The calculation runs exactly in the same way as in § 7. Only we now must take into account the interaction between mesons and light particles as well as that between mesons and nuclons. Thus

$$\tfrac{1}{2}\iint f(1,2)\, g_m \psi^\dagger(1)\, \boldsymbol{\omega}_m^\dagger(1)\, \psi(1)\,.\, g_n \psi^\dagger(2)\, \boldsymbol{\omega}_n(2)\, \psi(2) + \text{conj. compl.}$$

of (125) and (126*eff*) of § 7 (where $f(1,2) = f(2,1)$) is now replaced by (compare (38))

$$\tfrac{1}{2}\iint f(1,2)\{g_m \psi^\dagger(1)\, \boldsymbol{\omega}_m^\dagger \psi(1) + g'_m \psi'^\dagger(1)\, \boldsymbol{\omega}_m^\dagger \psi'(1)\}\{g_n \psi^\dagger(2)\, \boldsymbol{\omega}_n \psi(2) + $$
$$+ g'_n \psi'^\dagger(2)\, \boldsymbol{\omega}_n \psi'(2)\} + \text{conj. compl.} \qquad (200)$$

where ψ denotes the wave-function of the nuclons and ψ' that of the light particles.

We are now interested in matrix elements describing for instance the reaction

$$N^+ \rightarrow P^+ + \varepsilon + \nu. \qquad (201)$$

These matrix elements arise from the terms

$$\tfrac{1}{2}\iint f(1,2)\Big\{g_m \psi^\dagger(1)\, \frac{\tau_x + i\tau_y}{2}\, \tilde{\boldsymbol{\omega}}_m \psi(1)\,.\, g'_n \psi'^\dagger(2)\, \frac{\tau_x - i\tau_y}{2}\, \tilde{\boldsymbol{\omega}}_n \psi'(2) + $$
$$+ g_n \psi^\dagger(2)\, \frac{\tau_x + i\tau_y}{2}\, \tilde{\boldsymbol{\omega}}_n \psi(2)\,.\, g'_m \psi'^\dagger(1)\, \frac{\tau_x - i\tau_y}{2}\, \tilde{\boldsymbol{\omega}}_m \psi'(1)\Big\} \qquad (202)$$

of (200). Here $\tilde{\boldsymbol{\omega}}_m$ and $\tilde{\boldsymbol{\omega}}_n$ are self-adjoint Dirac matrices operating on the *undor* indices of the wave-functions only. Now, it is easily seen that, on account of the symmetry of $W(1,2)_{op}^{eff}$ in § 7 the required matrix elements can be deduced directly from (136) by omitting the factor $\tfrac{1}{2}(\vec{\tau}^{(1)}.\vec{\tau}^{(2)})$ and by taking the matrix element of the operators there denoted by $^{(1)}$ between the states of the vanishing neutron N and the created proton P; and of the operators denoted by $^{(2)}$ between the charge-conjugated of the state of the created negaton ε (that is, a positonic state of negative energy; compare (100)) and the state of the created antineutrino ν.

We shall now make use of the relation (135) in order to eliminate from (136) all terms without a gradient operator, which possess a factor $(e^{-\kappa r}/r)$. In that way we achieve that only two types of terms remain: those containing a gradient operator ∇ or Δ acting on a factor $(e^{-\kappa r}/r)$, and those containing a δ-function. All matrix elements are integrals over the co-ordinates of the heavy $^{(1)}$ and the light $^{(2)}$ particles. All gradient operators ∇_1 can still be replaced by $-\nabla_2$, since they operate on a function of r_{12} only. Then integrating by parts over the co-ordinates of the light Dirac particles $^{(2)}$, the integrand can be changed into one, in which all gradient operators operate on the wave-function of the involved light particles only.

Now, if the terms with gradient operators are compared with those without a gradient operator but still proportional to $(e^{-\kappa r}/r)$, (i.e. those terms, which were eliminated by means of (135)), we first remark that each of the gradient operators is accompanied by a factor $(1/\varkappa) = \hbar/\mathfrak{m}c$. The gradient itself multiplies the matrix element by the momentum p of one of the emitted light particles divided by $\hbar$, so that each factor $(1/\varkappa)\,\nabla$ is equivalent to a factor $p/\mathfrak{m}c$. Now, the momenta of the light particles actually emitted by β-active elements are of the order of magnitude of at most 10 mc. Since $\mathfrak{m}c$ is at least 100 mc (170 or 180 mc seems to be more probable), the factor $p/\mathfrak{m}c$ is at most about 1/10 (or 1/20). We shall neglect these terms.

This means that we assume that the wave-functions of the emitted light particles are nearly constant in a region of the order of magnitude $(1/\varkappa)$, that is, inside the nucleus. Thus we can replace these wave-functions by a constant.

Since the neutrino can be considered to be free, its wave-function can be assumed to be a normalized plane wave. Then this wave-function is, apart from the four-component "spin-function", inside the nucleus equal to the normalization factor $1/\sqrt{\Omega}$ only, where Ω is the volume of a large cube, in which all wave-functions are assumed to be periodic.

The emitted electron cannot be regarded to be free, on account of the Coulomb field of the nucleus (charge Ze). A reasonable assumption is [41]) that the wave-function inside the nucleus (where the electric field decreases towards the centre) is equal to the value, which the wave-function of an electron in the Coulomb field of a hypothetical point charge Ze would have at a nuclear radius distance from this point charge. The wave-function should be one of the continuum of states with an energy $W > mc$. If these wave-functions are normalized in such a way that the density-in-phase of states is again equal to Ω/h^3, the value of these wave-functions will, apart from the "spin-function", be a function of the energy (W) of the emitted electron and of the charge (Ze) of the nucleus after the emission. This function $\sqrt{f(Z, W)/\Omega}$ can be taken from one of the papers on the β-decay theory [41] [42] [73] [74]).

The terms remaining in the matrix element derived from (136) are now of the form of

$$g_m g_n' \iint d\vec{r}_1\, d\vec{r}_2\, \delta(\vec{r}_{12})\, (\psi_P^\dagger(\vec{r}_1)\, \tilde{\boldsymbol{\omega}}_m\, \psi_N(\vec{r}_1))\, \frac{1}{\Omega}\, \sqrt{f(Z, W)}\, (u_\nu^\dagger\, \tilde{\boldsymbol{\omega}}_n u_\epsilon^{\mathfrak{C}}) = \\ = (g_m g_n'/\Omega)\, \sqrt{f(Z, W)} \int d\vec{r}_1\, (\psi_P^\dagger(\vec{r}_1)\, \tilde{\boldsymbol{\omega}}_m\, \psi_N(\vec{r}_1)) \cdot (u_\nu^\dagger\, \tilde{\boldsymbol{\omega}}_n u_\epsilon^{\mathfrak{C}}). \tag{203}$$

Here u_ν is the four-component spin function of an antineutrino of positive energy, $u_\epsilon^{\mathfrak{C}}$ that of a positon of negative energy (that is, the charge-conjugated of the spin function u_ϵ of a negaton of positive energy; compare (99)—(100)).

Since the heavy particles in the nucleus can be treated in non-relativistic approximation, we may in the factor with $\tilde{\boldsymbol{\omega}}_m$ replace $\boldsymbol{\beta}$ by 1; $\boldsymbol{\gamma}_5$, $\vec{\boldsymbol{\alpha}}$ and $\boldsymbol{\beta}\vec{\boldsymbol{\alpha}}$ by 0; $\boldsymbol{\beta}\vec{\boldsymbol{\sigma}}$ by $\vec{\boldsymbol{\sigma}}$ (compare (138)).

The probability per unit time of a disintegration, by which the electron is emitted with a momentum between p and $p + dp$ in a solid angle $d\omega$, and the antineutrino with a momentum between p' and $p' + dp'$ within a solid angle $d\omega'$, is given by (114a). Here we must put:

$$\rho(\mathcal{E})\, d\mathcal{E} = (\Omega/c^2 h^3)^2\, p\, W\, dW\, d\omega\, p'\, W'\, dW'\, d\omega' \tag{204}$$

where

$$(W/c)^2 = (mc)^2 + p^2; \qquad (W'/c)^2 = (\mu c)^2 + p'^2, \tag{108a}$$

and

$$d\mathcal{E} = d(W + W'), \text{ (so that } dW\, dW' = dW\, d\mathcal{E}). \tag{205}$$

Further, on account of (203) and (136)—(135):

$$Q = (4\pi/\varkappa^2\Omega)\, \sqrt{f(Z, W)} \cdot \{(1 - C_0)\, g_0 g_0' (u_\nu^\dagger \boldsymbol{\beta} u_\epsilon^{\mathfrak{C}}) \int \psi_P^\dagger \psi_N + \\ + (1 - C_1)\, g_1 g_1'\, (u_\nu^\dagger\, u_\epsilon^{\mathfrak{C}}) \int \psi_P^\dagger \psi_N + \\ + (1 - C_2) g_2 g_2' (u_\nu^\dagger \boldsymbol{\beta}\vec{\boldsymbol{\sigma}} u_\epsilon^{\mathfrak{C}}) \cdot \int \psi_P^\dagger \vec{\boldsymbol{\sigma}} \psi_N + (1 - C_3) g_3 g_3' (u_\nu^\dagger \vec{\boldsymbol{\sigma}} u_\epsilon^{\mathfrak{C}}) \cdot \int \psi_P^\dagger \vec{\boldsymbol{\sigma}} \psi_N\}. \tag{206}$$

We remark that just those matrix elements, which yielded a potential of the form of a δ-function for the nucleon-interaction, and which were omitted in § 7, since they gave "only" an infinite contribution to the binding energy of the deuteron [22]), correspond to the terms, which give rise to a β-disintegration *). Thus, by a convenient choice of the constants C_0, C_1, C_2 and C_3 it will always be possible to fit the total disintegration probability with the experimental data without influencing the deuteron problem.

It is interesting to remark that the term in the Hamiltonian with $g_0 g_0'$ (which usually does not enter into calculations, as a consequence of the special part played in the theory by **S**) yields a contribution here to the β-decay. It is a term of the same type as the original Fermi-interaction [41]), which appears in our theory with the constant $(4\pi g_1 g_1'/\varkappa^2)$.

The probability per unit time $P(Z, W)\, dW$ for a disintegration, by which the electron has an energy between W and $W + dW$, is now calculated in the ordinary way. The sum must be taken over both directions of the spin of the emitted light Dirac particles; over the solid angles $d\omega$ and $d\omega'$ we have to integrate independently. The summation over the spins brings in a new factor depending on the energy of the emitted electron. Especially the (non-vanishing) cross products of different terms of (206) are of interest, since they can make the *β-spectrum a little more asymmetric* than the Fermi-distribution either for positon-emitters, or for negaton-emitters. Putting

$$W/mc^2 = w, \quad (1 - C_n) g_n g_n'/\hbar c = g_n'' \text{ and } M_{\boldsymbol{\omega}} = \int \psi_P^\dagger \boldsymbol{\omega} \psi_N \tag{207}$$

we obtain the following expression for $P(Z, W)$:

$$P(Z, W)\, dW =$$

$$= (mc^2/\hbar)\, G^2 |\mathfrak{M}(w)|^2 w \sqrt{w^2 - 1}\, (w_0 - w) \sqrt{(w_0 - w)^2 - (\mu/m)^2}\, dw, \tag{208}$$

where

$$G^2\, |\mathfrak{M}(w)|^2 = \frac{8}{\pi} \left(\frac{m}{\mathfrak{m}}\right)^4 \Bigg[\{ (g_0''^2 + g_1''^2)\, |M_{\mathbf{1}}|^2 + $$

$$+ (g_2''^2 + g_3''^2)\, |\vec{M}_{\boldsymbol{\sigma}}|^2 \} \left\{ 1 - \frac{\mu/m}{w(w_0 - w)} \right\} +$$

$$+ 2\eta\, \{ g_0'' g_1''\, |M_{\mathbf{1}}|^2 + g_2'' g_3''\, |\vec{M}_{\boldsymbol{\sigma}}|^2 \} \left\{ \frac{1}{w} - \frac{\mu/m}{w_0 - w} \right\} \Bigg]. \tag{209}$$

*) It should be remarked that it is possible to choose the constants C different in the terms with $\psi_P^* \psi_N \psi_N^* \psi_P$ and with $\psi_P^* \psi_N \psi_\nu^* \psi_\pi$ in Lagrangian and Hamiltonian.

Here m and μ are the masses of electron and neutrino respectively; w_0 is the total energy available for the emission of particles, expressed in units of mc^2. Finally η is equal to $\pm$ 1:

$$\begin{aligned} \eta &= +1 \text{ for positon-emitters,} \\ \eta &= -1 \text{ for negaton-emitters.} \end{aligned} \tag{210}$$

As for the relative sign of g_0'' and g_1'', and of g_2'' and g_3'', we refer to the equations (36). Thus, if in (39) $g\psi_N^\dagger \boldsymbol{\omega}\psi_P + g'\psi_\nu^\dagger \boldsymbol{\omega}\psi_\pi$ is changed into $g\psi_N^\dagger \boldsymbol{\omega}\psi_P + g'\psi_\epsilon^\dagger \boldsymbol{\omega}\psi_o$, the signs of $g_0'g_1'$ and of $g_2'g_3'$ are changed. Then, however, we must also reverse the signs of η in (210); so that the result is exactly the same, as it should be.

If we assume $\mu = 0$, the factor $|\,\mathfrak{M}(w)\,|^2$ in (208)—(209) may be written as

$$|\,\mathfrak{M}(w)\,|^2 = (\mathcal{A} + \eta\mathcal{B}/w), \tag{211}$$

where $w \geq 1$ and $|\,\mathcal{B}\,| \leq \mathcal{A}$. From (211) we conclude that, if $g_0''g_1''$ and $g_2''g_3''$ are both positive, the spectrum becomes a little more asymmetric and the total disintegration probability a little higher for *all positon*-emitters ($\mathcal{B} > 0$), whereas these effects will be just inverse for all *negaton*-emitters; or *vice versa* for $g_0''g_1'' < 0 > g_2''g_3''$ ($\mathcal{B} < 0$). If, however, $g_0''g_1''$ and $g_2''g_3''$ have different signs, the sense of the deviation of the spectrum from the original Fermi spectrum will depend on the relative value of the matrix elements M_1 and $\vec{M}_\sigma$.

In order to investigate whether it is necessary to make use of the possibility of choosing the constants C different from zero, we now calculate

$$G_0^2 = (8/\pi)\,(m/\mathfrak{m})^4\,(g^2/\hbar c)\,(g'^2/\hbar c). \tag{212}$$

From

$$\begin{aligned} g'^2/\hbar c &\approx (9/7\,\xi) \times 10^{-17}, \\ (\mathfrak{m} &= 100\,\xi\,m), \end{aligned} \tag{198c}$$

we find

(213)

$g^2/\hbar c =$	1/5	1/10	1/16
$10^{25} \,.\, G_0^2 \approx$	$2/(3\,\xi^5)$	$1/(3\,\xi^5)$	$1/(5\,\xi^5)$

From the decay constants of light elements one can deduce [73])

$$G^2\,|\,\mathfrak{M}(w)\,|^2_{exp} \approx 12 \times 10^{-25}. \tag{214}$$

Since in (209) several matrix elements occur, we conclude that, roughly,

$$G^2_{exp} \approx 6 \times 10^{-25}. \tag{214a}$$

This means that only for small values of ξ ($\mathfrak{m} \approx 55\ m$) G_0^2 is of the right order of magnitude [69]). For $\xi \approx 1\frac{3}{4}$, however, we find from (213): $G_0^2 \approx 0{\cdot}02 \times 10^{-25}$. In this case, which seems to be the more probable one [75]), the spontaneous meson disintegration in the nucleus is not sufficient [72]) in order to explain the order of magnitude of the β-radioactivity, and it seems to be necessary to add "direct Fermi terms" by a convenient choice of the constants C in the Lagrangian and the Hamiltonian (H_{gg}). This argument, however, is not conclusive since, after all, the radioactivity may be due to a large value of one of the constants g_0'' or g_3', which were of no interest in the discussion of the deuteron problem and do not enter into the expression for the probability of the spontaneous meson disintegration.

Summarizing we can say that any serious disagreement between the theory and reliable experimental data can at present be avoided by a convenient choice of the constants.

§ 12. *Scattering and absorption of mesons by nuclei.* In this section we shall briefly mention some processes of "scattering" or absorption of mesons by nuclear particles.

Passing through the Coulomb field of a nucleus a meson can emit photons (*Bremsstrahlung*) or be deflected (Rutherford scattering). The theoretical cross section for *Bremsstrahlung* is smaller by a factor 10^{-4} or 10^{-5} than the corresponding effect of electrons [2]). The effect is calculated from the terms H_e' and H_C of the Hamiltonian; it can be regarded as an ordinary Rutherford scattering coupled with the emission of a photon.

The cross section for *Rutherford scattering* is obtained in a similar way as the corresponding expression for electrons. A difference arises from the fact that the expression for the electric charge density of the field of the scattered particle, which enters into the formulae, is more complicate for mesons than for electrons (compare (48)). Laporte [76]) has shown that as a consequence of this fact already in the *first Born approximation* an *azimuth-dependence of the differential cross section* for Rutherford scattering of *transversal linearly polarized mesons* into a given direction appears. This effect, however, is very small for slow mesons (fourth order in v/c).

The meson can be virtually absorbed itself by a nuclon and be re-emitted. The cross section for this "*anomalous scattering*" [49]) or

"Compton scattering of mesons" [2]) was calculated by Heitler [49]) and Bhabha [18]). The former simplified the calculation by computing the cross section only for momenta, which are small in comparison with Mc ($\approx$ 10 $\mathfrak{m}c$). For a *longitudinal* meson impacting with a momentum p and an energy ε on a nucleon at rest, by which it can be absorbed, (that is, a proton for the scattering of arneticons, and a neutron for the scattering of theticons), Heitler obtained in this "non-relativistic" approximation the following differential cross section for scattering into a given solid angle $d\omega$:

$$d\Phi_{Heitler}^{(N.R.)} = (1/\varkappa)^2 \, (g_1^2/\hbar c) \, \{(g_1^2 + 2g_2^2)/\hbar c\} \, (p^2/\mathfrak{m}\varepsilon)^2 \, d\omega. \qquad (215)$$

This result was obtained after summation over the three possible directions of polarization of the scattered meson. If the incident mesons are transversal, the factor $(g_1^2/\hbar c)$ must be replaced by $(g_2^2/\hbar c)$.

A relativistic formula was derived by Bhabha [18]). The complicate formula, which was found by him for scattering of *unpolarized* mesons through an angle ϑ_0 *in the system of the centre of gravity* into a given solid angle $d\omega_0$, tends according to him to

$$d\Phi_{Bhabha}^{(N.R.)} = 6(1/\varkappa)^2 \, \{(g_1 + g_2)^2/\hbar c\}^2 \, \{M\mathfrak{m}/(M+\mathfrak{m})(2M+\mathfrak{m})\}^2 \, d\omega_0 \qquad (216)$$

for the *non-relativistic* case $p_0 \ll \mathfrak{m}c$, and to

$$d\Phi_{Bhabha}^{(E.R.)} = \tfrac{1}{3} \, (1/\varkappa)^2 \, \{(g_1^2 + 2g_2^2)/\hbar c\}^2 \, (p_0/\mathfrak{m}c)^2 \, (1 + \cos\vartheta_0) \, d\omega_0 \qquad (217)$$

for the *extreme-relativistic* case $p_0 \gg Mc$; both expressions are differential cross sections for scattering *by a nuclon capable of absorbing the meson* (proton or neutron). Here p_0 is smaller than the corresponding p in (215), since Bhabha takes the momentum with respect to the centre of gravity.

Comparing (216) with (215) we remark that the expressions do not agree with each other, so that there must be some error. The *extreme relativistic* equation of Bhabha (217) shows more similarity to the formula of Heitler (215) than Bhabha's non-relativistic approximation (216).

Taking $\mathfrak{m} \approx 175\, m$ (so that $1/\varkappa \approx 2{\cdot}2 \times 10^{-13}$ cm) and $g^2/\hbar c \approx \frac{1}{8}$, we find by integration over angles the following total cross sections:

$$\Phi_{Heitler}^{(N.R.)} \approx 3 \times 10^{-26} \times (p^2/\mathfrak{m}\varepsilon)^2 \text{ cm}^2. \qquad (215a)$$

$$\Phi_{Bhabha}^{(N.R.)} \approx 1\tfrac{1}{2} \times 10^{-27} \text{ cm}^2. \qquad (216a)$$

$$\Phi_{Bhabha}^{(E.R.)} \approx 3 \times 10^{-26} \times (p/\mathfrak{m}c)^2 \text{ cm}^2. \qquad (217a)$$

Bhabha himself states that his non-relativistic cross section (216*a*) is of the order of magnitude 10^{-28} cm^2. Heitler states that his expression yields a cross section corresponding to an average range of $2 \sim 5$ cm of lead for $p \approx \mathrm{m}c$, $\varepsilon \approx 2\mathrm{m}c^2$. This result is wrong by a factor $1\frac{1}{3} \sim 2\frac{1}{2}$ on account of too high a value of $(g^2/\hbar c)$ used by him in that early stage of the theory (before Kemmer's neutretto-hypothesis, compare § 3) and still by a factor 8 by a slip in the calculation. According to (215*a*) the range in lead for $p \approx \mathrm{m}c$ is about

$$\lambda_+ \approx \frac{207}{207-82} \cdot \frac{1{\cdot}67 \times 10^{-24}}{11} \cdot \frac{1}{\Phi} \approx 35 \text{ cm} \qquad (218)$$

for theticons (positive mesons), which are scattered by neutrons; and for arneticons (negative mesons), which are scattered by protons:

$$\lambda_- \approx \frac{207}{82} \cdot \frac{1{\cdot}67 \times 10^{-24}}{11} \cdot \frac{1}{\Phi} \approx 50 \text{ cm (of lead)}. \qquad (218a)$$

According to the ("non-relativistic") equation of Heitler or the (extreme relativistic) equation of Bhabha the range would decrease for high energies proportional to $1/\varepsilon^2$. Then, however, according to § 6 the probability of the creation of showers would increase and the range would decrease even more strongly. The probability of a third order effect

$$Y^+ + N^+ (\rightarrow P^+ \rightarrow N^+ + Y^+) \rightarrow P^+ + Y^- + Y^+ \qquad (219)$$

was estimated by Heitler [49]). According to a "non-relativistic" calculation ($p \ll Mc \approx 10\,\mathrm{m}c$) the ratio of the cross sections for this process (219) and for the single "anomalous scattering" (215*a*) would be equal to

$$\Phi_{(219)}/\Phi_{(215)} \approx (3/5\pi)\,(g^2/\hbar c)\,(\varepsilon/\mathrm{m}c^2)^2. \qquad (220)$$

Taking $g^2/\hbar c \approx \frac{1}{8}$ we find

$$\Phi_{(219)}/\Phi_{(215)} \approx (\varepsilon/6\tfrac{1}{2}\,\mathrm{m}c^2)^2. \qquad (220a)$$

For high energies, however, the calculation is not reliable.

The cross section for the "Compton scattering" of the original "*yukons*" (*scalar* mesons) was calculated by Yukawa [2]). For these particles the cross section tended to zero for increasing energy. For this reason it would be of interest to calculate the cross section for high energy *spinless mesons* (case *d* of Kemmer).

Slow mesons may be very well absorbed by an atomic nucleus with subsequent emission of a nuclon (the analogon of the *photo-electric effect*). The calculation was performed by Yukawa [2]) for yukons and by Sakata and Tanikawa [77]) for Proca-Kemmer mesons. The cross section for the latter process is given by [77])

$$\Phi \approx 64\pi \cdot \left(\frac{1}{\varkappa}\right)^2 \cdot \frac{g_1^2 + 2g_2^2}{3\hbar c} \cdot \frac{\mathfrak{m}c^2}{\varepsilon} \cdot \left(\frac{p}{\mathfrak{m}c} + 2\frac{\mathfrak{m}c}{p} \cdot \frac{\varepsilon}{Mc^2}\right)\left(\frac{I}{\varepsilon}\right)^{5/2}. \quad (221)$$

Here p is the momentum of the impacting meson, and its energy $\varepsilon = \sqrt{(\mathfrak{m}c^2)^2 + (cp)^2}$ is assumed to be small in comparison with the rest energy of a nuclon ($\mathfrak{m}c^2 \leq \varepsilon \ll Mc^2$). I ($\approx 10^7$ eV $\ll \varepsilon$) is the binding energy in the nucleus of the nuclon emitted in the process. For *slow* mesons we can write

$$\varepsilon \approx \mathfrak{m}c^2, \quad p/\mathfrak{m}c \approx v/c \ll \sqrt{2\mathfrak{m}/M}, \quad (222)$$

so that we find (taking $g^2/\hbar c = \frac{1}{8}$, $\mathfrak{m} = 175\, m$ and $I = 10^7$ eV):

$$\Phi \approx 10^{-27} . (c/v)\ \text{cm}^2\ (\textit{per absorbing nuclon in the nucleus}). \quad (223)$$

Thus we find an absorption probability proportional to $(1/v)$. For $v/c = 1/50$ the range in lead for theticons has decreased to only about 5 cm. *Very slow mesons are absorbed quickly*. The arneticons are again a little more penetrating than theticons in heavy elements, since the latter contain less protons than neutrons.

Another important effect is the *mesophotic effect* and its reverse, the *photomesic effect* (124). The possible intermediate states are $(\vec{p} = \vec{p}' + \vec{k})$:

$$P^+(0) + Y^-(\vec{p}) \left\{ \begin{array}{l} \longleftrightarrow N^+(\vec{p}') + Y^+(-\vec{p}') + Y^-(\vec{p}) \longleftrightarrow \\ \longleftrightarrow P^+(0) + Y^-(\vec{p}') + h\nu(\vec{k}) \longleftrightarrow \\ \longleftarrow\!\!\!\longrightarrow \\ \longleftrightarrow h\nu(\vec{k}) + P^+(-\vec{k}) + Y^-(\vec{p}) \longleftrightarrow \\ \longleftrightarrow P^+(0) + P^-(\vec{k}) + N^+(\vec{p}') \longleftrightarrow \end{array} \right\} N^+(\vec{p}') + h\nu(\vec{k})$$

and (224)

$$N^+(0) + Y^+(\vec{p}) \left\{ \begin{array}{l} \longleftrightarrow P^+(\vec{p}') + Y^-(-\vec{p}') + Y^+(\vec{p}) \longleftrightarrow \\ \longleftrightarrow N^+(0) + Y^+(\vec{p}') + h\nu(\vec{k}) \longleftrightarrow \\ \longleftarrow\!\!\!\longrightarrow \\ \longleftrightarrow \quad P^+(\vec{p}) \quad \longleftrightarrow \\ N^+(0) + Y^+(\vec{p}) + P^-(-\vec{p}) + P^+(p') + h\nu(\vec{k}) \end{array} \right\} P^+(\vec{p}') + h\nu(\vec{k})$$

The first order contributions (from H_{eg}) to the matrix elements Q for these processes (115) are of the same order of magnitude as the second order contributions, but of *opposite sign*, so that some of the terms in the differential cross sections tending to infinity for $\varepsilon \to \infty$ are cancelled.

The calculation was performed in non-relativistic approximation by Heitler[49]). Then in (224) the lower two intermediate states can be neglected in both cases. The first order effect, however, was overlooked by Heitler, so that his results cannot be trusted. Moreover, there was a slip in the calculation, so that in the matrix element given by the formula (5*b*) of his publication[49]) the vector $\vec{p}''$ in the first term should change its sign. Then, in the first formula on page 534 the terms with $(p/p')\ (\vec{p}' \cdot \vec{e})$ do no longer cancel.

It is noticeable that the *same* error in sign slipped into the calculation of the photomesic effect performed by Kobayasi and Okayama[45]), though they started from the right matrix element. The considerations of § 6 (page 81—82), however, show that the terms in the matrix element Q arising from $\mathbf{E}^*$ div $\mathbf{E}$ in $H_e H_g$ cannot cancel each other by summation over the two first intermediate states in (224). — Kobayasi and Okayama took into account the first order contribution from H_{eg} [45]).

It is interesting to calculate separately the cross sections of *longitudinal* and of *transversal* Proca-Kemmer mesons for the mesophotic effect, since the non-relativistic cross sections for these two polarizations of the meson ($p \ll Mc$) depend on the energy in a different way. Making the same approximations as in the calculations of Heitler[49]) and of Kobayasi and Okayama[45]) (for instance neglection of the recoil of the nuclon) one finds for the cross section for longitudinal mesons a non-relativistic expression, which for increasing energies ε of the impacting meson increases proportional to ε^2; but for transversal mesons the non-relativistic cross section increases only with the logarithmus of ε. In the former case the term with ε^2 is due to a contribution of the first order effect, from which the high powers in ε are *not* cancelled by a corresponding contribution of the second order effect to the matrix element Q. In the case of transversal mesons the "high powers" of ε are cancelled, and the logarithmical increase is due to the contribution of scattering through angles $\vartheta \to 0$. It must still be mentioned that, in order to get a not too complicate result, one should in the latter (transversal) case average the cross section over both possible transversal polarizations of the impacting meson.

The non-relativistic cross section of photons for a photomesic effect increases always quadratically with the energy, since here the sum must be taken over all possible polarizations of the emitted meson; but it is plausible to draw from the preceding considerations the conclusion that, if high energy mesons are created in the atmosphere of the earth by the photomesic effect, longitudinal mesons must be preponderant in the region, where they are created; though at sea level this may be different, on account of a stronger absorption of longitudinal than of transversal mesons.

These conclusions, however, are not certain, since nothing can be said about the predictions of the theory on the high energy photomesic and mesophotic effects before the laborious calculation of the relativistic cross sections has been performed. Then, the lower two intermediate states of (224) must necessarily be taken into account (even when the angle between the momenta of photon and meson is small!), and the terms with **A*.a** and **E*.e** in H_g, which are neglected in a non-relativistic calculation, must no longer be forgotten. It will be a good policy to take into account even the recoil of the nuclon, which may be considerable at high energies. For the photomesic effect the creation of spinless mesons must be taken into account, if g_3 and g_4 do not vanish; for the mesophotic effect the spinless mesons can be treated separately.

Some relativistic calculation has been performed by K o b a y a s i and O k a y a m a [45], but from their publication it is not clear as to how far these calculations were approximative. — Anyhow, it would be of interest to know *exactly* the dependence on the energy of the cross sections following from the *unaltered* theory (without introduction of a "fundamental length") for this process as well as for the anomalous scattering of mesons (compare § 13).

It should still be added that it *should be hoped* that the theoretical cross section of photons for a photomesic effect is *not too small* for high energy photons, since the creation of a sufficient number of mesons is only in this way understood. Other effects, by which a meson can be created, seem to be far less probable. Thus the introduction of a fundamental length, which according to K o b a y a s i and O k a y a m a [45]) makes the cross section of the photomesic effect nearly constant for high energies, *might* endanger the explanation of the large number of mesons observed in the cosmic radiation; even if only those effects are "cut off", in which the change in momentum of the nuclon is only "smaller than", instead of "very small in comparison with" the quantity $(\hbar/r_0)$, where r_0 denotes the fundamental length [57] [78] [45].

In this connection it should be remembered that the original *scalar* theory of Yukawa yielded a photomesic cross section that *decreased* with increasing energy of the photon [23] [79]. This was then regarded as an argument *against* the theory [79].

Another interesting effect is the so-called *meson-neutretto chain* [9])

$$Y^+ + N^+ \rightarrow P^+ + Y^0 \quad \text{or} \quad Y^- + P^+ \rightarrow N^+ + Y^0$$

with

$$Y^0 + P^+ \rightarrow N^+ + Y^+ \quad \text{or} \quad Y^0 + N^+ \rightarrow P^+ + Y^-. \tag{225}$$

This process makes it possible for a heavy quantum to travel through matter partly in the shape of a meson, partly in the shape of a neutretto *). Measurements of Maass [80] gave evidence of this meson-neutretto chain, as discussed by Arley and Heitler [81]. The *anti-coincidence arrays* recently developed in France [82] seem to furnish an adequate method for new experimental investigations in this direction. The question arises whether a beam of neutrettos is actually more penetrating than a beam of mesons, if the processes (225) are probable.

§ 13. *Discussion of the limits and the value of the theory.* In the foregoing (pages 95 and 100) we have already mentioned the question whether a fundamental length [57] must be introduced into the theory in order to make it fit with the experimental data. This is assumed, indeed, by several authors [49] [78] [45]. They assume that processes, in which a nuclon changes its momentum by an amount, which is much larger than $\hbar/r_0 \approx \hbar\varkappa$, are forbidden. Wentzel [78] has pointed out that this cutting off would explain the narrow angular spread of hard cosmic-ray showers, which should exist according to the measurements of Schmeiser and Bothe [83]. This procedure makes it also possible to avoid [19] the infinite [10] mesic self-energy of the nuclon. For the anomalous

*) In a similar way, the mesophotic and the photomesic effects together give rise to a meson-photon chain.

magnetic moment of the nuclons, due to the magnetic moment of the meson field generated by a single nuclon, Fröhlich, Heitler and Kemmer[19]) find in this way values, which are at least of the right order of magnitude. It is noticeable, however, that the finite binding energy obtained by such a cutting off at momenta $\hbar\varkappa = \mathfrak{m}\bar{c}$ is only small in comparison with Mc^2, so that the mass of the nuclons cannot be "explained" in this way[19]).

In the present stage of the theory and experimental data a conclusion can hardly be drawn as regards the necessity or the impossibility of such a cutting off procedure. Bhabha[13]) has pointed out that a great part of the present difficulties of the theory may rather be due to the insufficiency of the methods of perturbation calculus, which are generally used and which break down as soon as perturbations are computed, which are too large. For instance by the application of the method of the variation of constants, which is used in the derivation of the equation (114), which we have used for all calculations of cross sections, it is necessary to postulate (1°) that the probability per unit time for the transition $a \rightarrow f$ is calculated only for a time, in which the total probability of any transition is still very small in comparison with 1; (2°) that this same time is long in comparison with the period of the frequency $e^{\mathcal{E} t/i\hbar}$ Though in general the latter condition is satisfied, the former one is not always compatible with this second condition.

According to Bhabha[13]) the divergence of some results obtained by a perturbation calculus, in which changes of momenta of the order of magnitude $\mathfrak{m}c$ are involved, does not mean at all that in other *similar* cases, where the calculation yields at least a convergent result, those momenta should be cut off. To this argument may be added that it is perhaps still a little premature to argue that the results of the theory, in which such changes of momenta are involved, are not trustworthy, as long as these results have not yet been evaluated theoretically and verified experimentally. In the case of the photomesic effect and the mesophotic effect, for instance, the inapplicability of the theoretical results can be understood from the approximative character of the calculations without introducing the idea of a fundamental length, at which the theory breaks down. As regards the increase of the cross sections for high energy heavy quanta discussed in the foregoing, it must be remarked that, from a theoretical point of view, the question is not yet settled (1°) *at what* *energy* the heavy quanta become *really less penetrating* than they seem to be according to the experimental data, (2°) whether this limit is the same for longitudinal and transversal mesons and for spinless mesons; whereas, from an experimental point of view, the dependence of the penetrating power of mesons on the energy is not yet known with certainty. Even it is not yet excluded entirely that the penetrating power of very high energy Proca-Kemmer mesons does not really exist at all; for a large cross section of the photomesic effect some arguments can be adduced (see § 12), and this may be connected with a large cross section for the mesophotic effect.

Anyhow, it is interesting to investigate the *possibilities* of a quantum theory, in which a fundamental length is introduced. Such an altered quantum theory, however, should not be imagined as an ordinary quantum theory, in which only some prescription is given restricting the validity of the theory. This can only be an early stage of the theory. The change that should be made would, indeed, be a more revolutionary one.

For instance, one can imagine that the infinite static self-energy of the point electron is removed by changing the δ-function in (110) into a D-function, which is derived from a "relativistic $\mathfrak{D}$-function" in a similar way as the δ-function is derived from the relativistic Δ-function depending on $(r \pm ct)$ only. This would mean that the δ-functions in the commutation relations are altered in such a way that by an expansion of the wave-functions in series of plane waves the amplitudes of states of high momentum would no longer satisfy the usual commutation relations.

Then putting

$$i\hbar\dot{\mathcal{F}} = [\mathcal{F}; \mathcal{H}]_{-}, \tag{51}$$

the equations of motion, which in the present theory take the form of the field equations following from a variational principle, would no longer be differential equations in the altered theory, but would take the form of integro-differential equations. The laws of conservation of energy, momentum and angular momentum [40]) would no longer be strictly valid, but would still be reasonable approximations as long as no large momenta are involved.

The relativistic invariance of such a theory would be a problem in itself.

One should note, however, that then there would no longer be a reason for ignoring the self-energies in the expression for the total energy, and care must be taken that the energy of matter does not appear twice in the Hamiltonian, for instance once in the terms describing the fields of Dirac particles, and again in the terms describing the (electromagnetic, mesic and neutrettic) quantum fields [84]) [85]).

We must consider the possibility that the δ-functions, which we have used in the theory of the β-decay but have omitted in the theory of the nuclon interaction, will obtain in such an altered theory some finite value and will be of importance for the levels of the deuteron. Thus, *if* one assumes that it is necessary indeed to alter the theory, one should not be too certain about the present deuteron theory and about the *determination of the order of magnitude of the constant* $g^2/\hbar c$, which again appears in all effects calculated by means of the meson theory.

On the other hand it must be borne in mind that for the time being it does not yet follow from the experiment that such a revolutionary change in the theory is absolutely necessary. Therefore it may be prudent not only to investigate the *fundaments* of quantum-mechanics, but also to review its methods of calculation. Even more needed are detailed *experimental investigations on the energy-dependence of the cross sections* for the numerous processes, which are possible according to the theory of mesons and neutrettos.

Finally it must be remarked that, until now, the meson theory has one very unpleasant feature (apart from the divergencies and the δ-functions); viz. the *enormous number of constants*, which must be chosen in a convenient way in the hope to make the theory fit the experiment. In a "pleasant" theory one should for instance expect that the constants C_0, C_1, C_2, C_3, C_4 are all equal to zero (or perhaps to 1), that the constants g_0, g_1, g_2, g_3 and g_4 are all equal to one value $\pm g$ or to zero, and the constants g' in the same line. We have seen that the possibility of such a simplification is at least questionable, if we want to explain for instance the experimental data on the deuteron by means of the methods discussed in § 7—8. For instance, the attempt of Bethe [32]) ($g_0 = g_1 = g_3 = g_4 = 0$) did not succeed very well (§ 8). This may be a consequence of our methods, or of the incertitude of the experimental data, which are perhaps not all as reliable as one should wish, partly on account of the indirect way, in which they are obtained. It may also be in the nature of things that so many constants are involved. In that case the theory will become satisfactory only, if the number of effects explained numerically by the theory will appreciably surpass the number of constants.

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SAMENVATTING

I. *Undor-rekening en lading-conjugatie.*

Onder "*undoren* van den n-den trap" verstaan wij grootheden, gekenmerkt door 4^n complexe getallen, die zich bij Lorentz-transformaties lineair transformeeren als de 4^n producten van de componenten van n Diracsche golffuncties (§ 1). Indien de spiegelingstransformatie van zulk een Diracsche golffunctie zoodanig wordt gedefinieerd, dat een dubbele spiegeling overeenkomt met een omkeering van het teeken van de golffunctie, bestaat er een lineaire operator, die uit den complex-geconjugeerde van een undor Ψ van den n-den trap wederom zulk een undor vormt, den "*lading-geconjugeerde*" ($\Psi^{\pounds}$) van Ψ, en wel zoo, dat $\Psi^{\pounds\pounds} = \Psi$ is (§ 2—3).

In het bijzonder onderzoeken wij undoren van den eersten (§ 1—4) en van den tweeden (§ 5) trap. Een undor van den tweeden trap vertegenwoordigt een vijftal antisymmetrische tensoren (van den nulden, eersten, tweeden, derden en vierden trap). Deze tensoren zijn reëel (afgezien van een willekeurigen constanten factor), indien de undor $\Psi_{k_1k_2}$ gelijk is aan zijn lading-geconjugeerde $\Psi^{\pounds}_{k_1k_2}$ of aan zijn *lading-geadjungeerde* $\Psi^{\mathfrak{L}}_{k_1k_2}$, waaronder wij $\Psi^{\pounds}_{k_2k_1}$ verstaan (§ 5). Undoren, die gelijk zijn aan hun lading-geadjungeerde, noemen wij "*neutrettoren*" (§ 4—5).

Tenslotte leiden wij een "*metrischen undor*" af, waardoor aan elken gewonen ("*covarianten*") undor een "*contravarianten*" undor wordt toegevoegd (§ 6). Wij definieeren een *gradient-undor* en kunnen nu undor-vergelijkingen in "covariante notatie" schrijven.

II. *De undor-vergelijking van het mesonenveld.*

Wij schrijven de Proca-vergelijkingen voor het mesonenveld in undor-notatie (§ 1). Het Proca-veld wordt voorgesteld door een symmetrischen undor van den tweeden trap. De vergelijkingen laten zich uitbreiden tot die voor een veld, beschreven door een *niet*-symmetrischen undor van den tweeden trap (§ 2). Deze uitbreiding komt neer op het invoeren van een nieuw veld van spinlooze mesonen. — Het veld van neutrale mesonen (*"neutretto's"*) kan worden beschreven door een neutrettorveld (§ 2).

Bij elke oplossing van de veldvergelijkingen bestaat er een "lading-geconjugeerde" oplossing van de z.g. "lading-inverse" vergelijkingen (§ 3). Bij deze *lading-inversie* dient niet alleen de electrische lading e van teeken te worden omgekeerd; indien men rekening houdt met de anticommutativiteit van de golffuncties van nuclonen *) en lichte deeltjes, moeten ook de *mesische* ladingen f en g van teeken omdraaien. — Een beschrijving van het veld door middel dezer lading-geconjugeerde veldgrootheden noemen wij de *lading-geconjugeerde beschrijving* van het veld.

De electrische ladings-stroom-dichtheid laat zich op eenvoudige wijze uitdrukken met behulp van den undor, die het mesonenveld beschrijft (§ 4).

Door iteratie van de meson-vergelijking vindt men voor vrije mesonen een Klein-Gordon vergelijking. Houdt men rekening met de wisselwerking der mesonen met het electromagnetische veld, dan treden in deze vergelijking extra termen op, waarvan de belangrijkste kunnen worden opgevat als de beschrijving van een magnetisch moment van het meson, dat dan $(e/2mc)$ maal het spin-impulsmoment $(\hbar)$ van het meson blijkt te zijn (§ 4).

Wanneer men aanneemt, dat de quantum-mechanica een *"lading-invariante"* theorie is, d.w.z. dat bij een lading-geconjugeerde beschrijving van het veld alle waarneembare grootheden op dezelfde wijze kunnen worden berekend als bij een gewone beschrijving van het veld, behalve dan dat alle ladingen met het andere teeken moeten worden genomen, dan volgen uit deze veronderstelling een aantal betrekkingen, die volledig bepalen, van welke deeltjes men moet aannemen, dat zij aan de Fermi-Dirac-statistiek voldoen, en welke deeltjes aan de Einstein-Bose-statistiek moeten gehoorzamen. (Andere soorten van statistiek zijn in beginsel echter niet uitgesloten). Het blijkt, dat deeltjes met geheeltalligen spin aan de Einstein-Bose-statistiek moeten voldoen, en deeltjes met heelplus-halftalligen spin aan de Fermi-Dirac-statistiek (§ 5). Bovendien kan men van de lading-invariantie van de theorie gebruik maken om de uitdrukking voor de energie van het veld in een dus-

*) D.w.z. protonen, neutronen en hun antideeltjes.

danigen vorm te schrijven, dat het duidelijk is dat alle vrije elementaire deeltjes een positieve energie bezitten.

III. *Toepassing van de theorie der zware quanta op problemen der kern-physica en der cosmische straling.*

Wij geven een critisch overzicht van een aantal publicaties, die de laatste jaren over de theorie der *zware quanta* (mesonen en neutretto's) zijn verschenen. Na een korte inleiding (§ 1) wordt de theorie der zware quanta geschetst, zooals deze door Kemmer en andere schrijvers is ontwikkeld (§ 2—3). De grootte-orde van de in de theorie optredende constanten wordt besproken.

Vervolgens worden de in zwang zijnde methoden, volgens welke men het veld pleegt te quantiseeren, aan een critiek onderworpen (§ 4). Het blijkt, dat het bewijs van de relativistische invariantie der theorie nog moet worden gegeven. Van niet-relativistisch standpunt uit bekeken, zijn de gangbare methodes niet zeer consequent, en wij hebben aan een eenigszins afwijkende behandeling de voorkeur gegeven.

De wisselwerking van het mesonenveld met het longitudinale Maxwell-veld kan worden beschreven als een statische Coulombsche krachtwerking tusschen de electrische ladingen (§ 5). De Hamiltoniaan wordt afgeleid, die de wisselwerkingen van de uitgebreide mesonen- en neutretto-velden met alle andere velden beheerscht (§ 6). De aard dezer wisselwerkingen wordt aangeduid. Van bijzonder belang is het feit, dat de grootte van zekere matrixelementen van de gequantiseerde Hamiltoniaan de waarschijnlijkheid van veel-quanta-processen doet verwachten.

Vervolgens wordt de wisselwerking tusschen nuclonen door tusschenkomst van het veld der zware quanta besproken (§ 7). Hoewel de afleiding van de uitdrukking voor deze wisselwerking veel overeenkomst vertoont met de afleiding van de Breitsche wisselwerking tusschen electronen, blijkt de "effectieve potentiaal" in het laatste geval een betere benadering te zijn dan in het geval der wisselwerking tusschen nuclonen.

De Schrödinger-vergelijking voor het deuteron wordt opgeschreven, de optredende moeilijkheden worden besproken en het magnetische moment van het deuteron wordt ter sprake gebracht (§ 8). Een kort overzicht wordt gegeven van den huidigen stand der theoretische onderzoekingen omtrent het probleem der strooiing van nuclonen door nuclonen (§ 9).

Na een beschouwing over het spontane uiteenvallen der zware quanta (§ 10) wordt de theorie van de β-radioactiviteit behandeld in het kader van de mesonentheorie (§ 11). Eenige belangwekkende bijzonderheden treden op door de uitbreiding van het mesonenveld met spinlooze zware quanta. Het vraagstuk van de overeenstemming tusschen theorie en experiment wordt nader in oogenschouw genomen.

Vervolgens worden de bestaande theoretische onderzoekingen omtrent verschijnselen van "strooiing" en absorptie van mesonen door atoomkernen aan een critische beschouwing onderworpen (§ 12). Deze verschijnselen zijn van belang voor de bestudeering der cosmische straling. Longitudinaal en transversaal gepolariseerde zware quanta blijken niet even sterk "als photon" te worden "gestrooid".

Tenslotte worden argumenten vóór, bezwaren tegen en mogelijke gevolgen van de invoering van een fundamenteele lengte in de quantum-theorie besproken, en wordt de wenschelijkheid naar voren gebracht van nadere experimenteele onderzoekingen omtrent de energie-afhankelijkheid van de werkzame doorsneden voor de verschillende processen, die volgens de theorie mogelijk zijn (§ 13). Het groote aantal constanten, dat in de theorie in zijn algemeenen vorm optreedt, wordt als een bezwaar gevoeld.

THEORY OF HEAVY QUANTA - errata et addenda:

page 6, line 4, s a scalar, READ: is a scalar.

,, 69, ,, 13, $\hat{\mathcal{E}} = (4\pi e/\mathrm{div})\rho + \dot{\mathfrak{A}}_n/c$, READ: $\hat{\mathcal{E}}\hat{\chi} = \{(4\pi e/\mathrm{div})\rho - \dot{\mathfrak{A}}_n/c\}\hat{\chi}$

,, 92, ,, 21, $Y_2^{\mu-\mu'}(\vartheta,\varphi), {}^3\chi_{\mu'}(\sigma_1,\sigma_2)$ READ: $Y_2^{\mu-\mu'}(\vartheta,\varphi)\,{}^3\chi_{\mu'}(\sigma_1,\sigma_2)$.

page 65 sqq (chap. III, § 5 - 6), corrigenda:

The transformation with F_2 was needed by F e r m i, since in his paper $-\dot{\mathfrak{A}}_{long}/c$ and $\dot{\mathcal{H}}/c$ take the part of $\mathcal{E}_{long}$ and $\mathfrak{S}$ in our case (as canonical conjugated momenta to $\mathfrak{A}_{long}$ and $\mathcal{H}$). For us, the transformation (76a) with F_1 only suffices. In § 6, the matrices a, B, c, d, l, m, n, v, w are components of the original (not of the transformed) field. Now, it can be shown that the usual expressions (compare B e l i n f a n t e, Physica 7, 449, 1940) for the total energy, momentum, total angular momentum, spin component parallel to momentum (see (105)), charge density and total electric charge can be written as:

$$H = H' + H'', \quad \vec{P} = \vec{P}' + \vec{P}'', \quad \vec{J} = \vec{J}' + \vec{J}'', \quad \vec{S}_{\parallel} = \hat{S}_{\parallel}, \quad \rho = \hat{\rho}, \quad e = \hat{e}.$$

Here, H'', $\vec{P}''$ and $\vec{J}''$ are zero-terms (see page 65), whereas the TRANSFORMED operators (compare (73))

$$\hat{H}', \quad \hat{\vec{P}}', \quad \hat{e} \quad \text{and} \quad \hat{S}_{\parallel}$$

are given by the formulae (90) (see (109)), (102), (103) and (105). (For this reason, a^*a etc. (without transformation!) can be regarded as the TRANSFORMED operators of numbers of particles or quanta) $(= \hat{N})$.

The equation (70) is not correct for instance for $F = \partial/\partial 0$. Instead of the considerations of page 65, therefore, we must observe that straight-forward calculation from (41), (42), (49), (50) and the explicit usual expressions for H, $\vec{P}$, $\vec{J}$, etc. yields:

$$H'\chi = H\chi;$$

$$\vec{P}'\chi = \vec{P}\chi; \quad [\vec{P}'; H']\chi = [\vec{P}; H]\chi \ (=0),$$

$$\vec{J}'\chi = \vec{J}\chi; \quad [\vec{J}'; H']\chi = [\vec{J}; H]\chi \ (=0).$$

$$[\rho; H'] = [\rho; H]; \quad [[\rho; H']; H'] = [[\rho; H]; H]; \quad \text{and} \quad [\{\rho\}; H']^{(n)}\chi = [\{\rho\}; H]^{(n)}\chi$$

Further, with $\hat{N}$ defined by a^*a, ..., w^*w, we find in this way:

$$[\hat{N}; \hat{H}'']\hat{\chi} = [\hat{N}; \hat{H}]\hat{\chi}; \quad [[\hat{N}; \hat{H}']; \hat{H}']\hat{\chi} = [[\hat{N}; \hat{H}]; \hat{H}]\hat{\chi}; \quad [\{\hat{N}\}; \hat{H}']^{(n)}\hat{\chi} = [\{\hat{N}\}; \hat{H}]^{(n)}\hat{\chi}; \text{etc.}$$

Thus, the dashed quantities can replace the original quantities.

We remark that in (76) all field components must be taken at some given time $t = t_0$ in order to have F constant in the expressions for dF/dt, $d\hat{\chi}/dt$ etc. The equations of § 5 - 6 , the formulae given above and the expression (90) for the Hamiltonian $\hat{H}'$ are, therefore, valid at $t = t_0$ only, but this suffices if the situation is described by $\hat{\chi}(t) = e^{-iF}\chi(t)$ (see (60)-(62)) with $\chi(t_0) = \chi$ (compare (59) with (60)). Thus, ONLY AT $t = t_0$, we have ($\dot{\mathcal{H}}\hat{\chi} = \dot{\mathfrak{A}}_{long}\hat{\chi} =$ = 0 with (76 a+b), or:) $\mathfrak{S}\hat{\chi} = \mathcal{E}_{long}\hat{\chi}$ = 0 with(76a) without (76b).

Het kan veel eleganter. Publicatie ± 1944-45 (na de oorlog) in Physica tegemoet te zien. ~~[illegible]~~ ("On the longitudinal and the transversal delta function", § 4, bewezen
".. On the supplementary condition in quantum electrodynamics".

LITTERATURE:

To chap. I:

F.J.Belinfante, Physica 7, 305, 1940.

To chap. II, § 5:

W.Pauli and F.J.Belinfante, Physica 7, 177, 1940.

To chap. III, § 4:

F.J.Belinfante, Physica 7, 449 and 765, 1940.

STELLINGEN

I

De door Heisenberg en Pauli gegeven definitie van differentiatie van een veelterm in q-getallen naar een dezer q-getallen is onbruikbaar voor de ontwikkeling van een algemeene quantumtheorie voor golfvelden.

Z. Phys. **56**, 1, 1929.

II

Ten onrechte meent Schönberg, dat een uitbreiding van de kanonieke theorie der gequantiseerde golfvelden van Heisenberg en Pauli tot het geval van velden van deeltjes, die de Fermi-Dirac statistiek volgen, onmogelijk is.

Physica **5**, 961, 1938.

III

De wijze, waarop Novobatzky het electromagnetische veld quantiseert, verschilt *in wezen* niet van de methode van Fermi, doch om de nevenvoorwaarden te vermijden, waaraan volgens den laatste de situatiefunctie zou moeten voldoen, wordt de relativistische covariantie van het formalisme opgegeven.

Z. Phys. **111**, 292, 1938.

IV

Er zijn redenen om aan te nemen, dat het voorschrift, volgens hetwelk in de golfmechanica uit den operator F_{op} van een waarneembare grootheid de gemiddelde waarde dezer grootheid wordt berekend, bij z.g. tweede quantiseering der golfvelden slechts dan tot den juisten operator voor de gemiddelde waarde dezer grootheid zal voeren, indien de lading-getransformeerde en de lading-inverse van F_{op} elkanders tegengestelde zijn.

F. J. Belinfante, proefschr. Leiden 1939, blz. 34.

V

Het is mogelijk, een algemeen voorschrift te geven, volgens hetwelk men uit de Lagrangiaansche functie een symmetrischen energie-impuls-dichtheidstensor kan afleiden zonder gebruik te maken van de methode van Hilbert.

Göttinger Nachr. **1915**, 395.

VI

Het ware wenschelijk, proeven over adiabatische demagnetisatie ook met waterhoudende ceriumzouten uit te voeren.

VII

Het ware van belang, de temperatuur-afhankelijkheid van de ferromagnetische anisotropie van kubische kristallen ook in het temperatuurgebied van vloeibaar helium te kennen.

VIII

Het ware wenschelijk, bij het onderwijs in de natuurkunde aan de scholen voor voorbereidend hooger en middelbaar onderwijs de beginselen der quantum-theorie te bespreken aan de hand van eenige demonstratie-proeven.

IX

Het verdient aanbeveling, bij het onderwijs in de wiskunde aan de scholen voor voorbereidend hooger en middelbaar onderwijs meer aandacht te besteden aan vraagstukken met strijdige, overbodige of onvoldoende gegevens.

X

Voor niet te groote waarden van het natuurlijke getal N kan men op eenvoudige wijze door nacijferen de juistheid vaststellen van de formules $\sum_{j=0}^{N} \binom{2N-2j}{N-j} \binom{2j}{j} = 4^N$ en $\sum_{j=0}^{2N} (-1)^j \binom{4N-2j}{2N-j} \binom{2j}{j} = 4^N \binom{2N}{N}$. Indien deze formules algemeen geldig zijn, schijnen zij niet eenvoudig te kunnen worden bewezen door volledige inductie.

XI

Het feit, dat voor nieuw ontdekte deeltjes namen als "positron", "mesotron", "deuton" e.d. in de litteratuur opduiken, wijst op de wenschelijkheid van meer overleg tusschen physici en classici.

XII

Het gebruik van Esperanto voor wetenschappelijke doeleinden is in beginsel mogelijk en zou de doeltreffendheid van internationale wetenschappelijke congressen kunnen verhoogen.