Lorentz linearization and its application in the study of the closure of the Zuiderzee

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Physics of Coasts

1.Introduction

In 1918 a committee was established in the Netherlands to study the effect of the closure of the Zuiderzee, a large bay of the North Sea, on tidal motion and storm surges in the western Wadden Sea, figure 2. The famous physician H.A. Lorentz was asked to be the chairman. Lorentz was already retired at that time. He was not familiar with the matter of fluid dynamics but he accepted the assignment. Although much was known about water and land reclamation, a project of this scale had not been carried out before, hence a thorough investigation was believed to be necessary. Some specialists, e.g. Ir. C. Lely politician and promoter of the project, believed that the construction of the 30 km long dam would cause minor changes in tidal patterns and height of the water elevation during storms in the Wadden Sea. While others, e.g. H.E. de Bruijn, argued that the height of the high tide would double, however this was no result of any computation it was more 'based on experience gained elsewhere and on relevant research.' In 1918, the Dutch parliament approved to proceed with the project under the condition that a committee would be appointed to investigate the consequences. After 8 years of hydrodynamical research the State committee Zuiderzee, or committee Lorentz as it was soon called, published a report bearing conclusions that cleared the way for the actual building of the dam.
In this report the focus will be on the method of analysing the influence of the bottom stress and on the way of calculating in a time where no (digital) computers where available. The basic (shallow water) equations of motion, applicable to this problem, contain the quadratic bottom stress and it was Lorentz who found an elegant way of avoiding this complicated term by replacing it by an effective linear one. This method is described in the final report of the State committee Zuiderzee (1926) but also in an article by Lorentz (1922). Later this method was revisited by Zimmerman (1982,1992) and applied in many theoretical studies on tides. Recently the success of Lorentz linearization was reconfirmed by Terra et al. (2004) who compared the results of laboratory experiments with theoretical predictions using the Lorentz method.

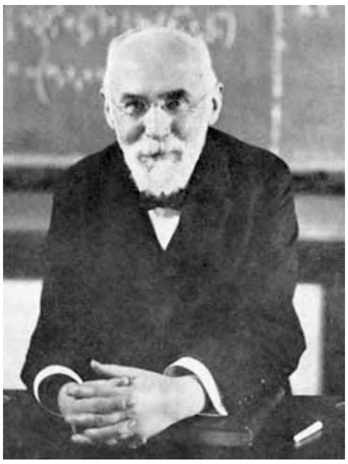
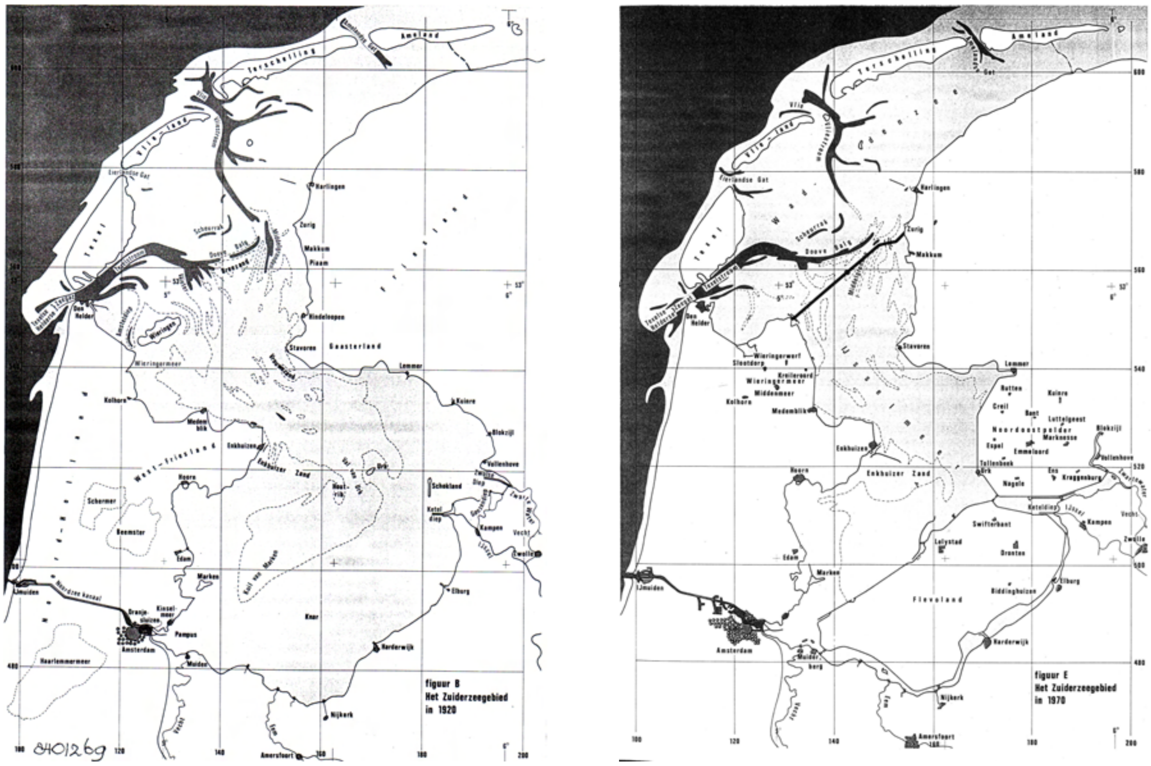


Figure 1: H.A. Lorentz (1853-1928)

The structure of the rest this paper is as follows: in sections 2 and 3 basic equations and Lorentz linearization will be explained, section 4 describes the set up of the actual calculations the committee performed and in section 5 an example is given of such a calculation. Finally in section 6 the conclusions are presented. Throughout the whole report the original notation of the committee is used, this means that equations are similarly found in the final report of the State committee Zuiderzee (1926).



(a) (b)

Figure 2: The system of shoals and channels in 1920 (a) and after closure in 1970 (b)

2. Basic equations of motion

The first assumption Lorentz made was that he could schematize the area of Wadden Sea and Zuiderzee as a network of tidal channels see figures 2 and 3. These channels were given a depth consistent with the real world average depth. In cases where a channel had a cross sectional depth profile the solution was found to have several channels next to each other with different depths. In every channel the along-channel velocity and the sea surface elevation were calculated. The dynamics were described by the the standard equations of motion, assuming that the width of the channel is small compared both to the Rossby deformation radius as to the channel length. This excludes variation over the width of the channel. At first hand the influence of the wind was neglected, but recall that one of the tasks the commission had was to see what the effects of storms on tidal motion in the Wadden Sea where after construction of the dyke. The calculations were based on the one-dimensional shallow water equations. In this formulation the continuity equation becomes

$$\frac{\partial s}{\partial x} = -b\frac{\partial h}{\partial t}, \tag{1}$$

where b is the width of the channel, h the height above a certain reverence level so that, if q is the mean depth, the total water depth is $q + h$. This means that s represents the volume flux through a surface:

$$s = b(q + h)v, \tag{2}$$

where v is the cross-section a averaged velocity. Note that h and s are functions of space and time and that generally $h/q \ll 1$, so in calculations it was used that $s = bqv$. The momentum equation reads

$$\rho\frac{\partial v}{\partial t} = -g\rho\frac{\partial h}{\partial x} + W, \tag{3}$$

where ρ is the density of water, g the acceleration due to gravity and W a force per unit volume due to bottom friction. It is this latter term that caused the trouble since it is quadratic in the velocity:

$$W = -\rho\frac{g}{qC^2}|v|v. \tag{4}$$

The constant C is nowadays known Chezy's constant (in that time constant of Eytelwein) and the ratio $\frac{g}{c^2} = C_d$, the so called drag coefficient. This C_d has often the value of 0.0025.

3. Lorentz linearization

Lorentz decided to linearize the problem in respect to the bottom friction and replaced the quadratic bottom friction by a tidally averaged linear one. So he proposed an alternative model which contains the momentum equation:

$$\frac{\partial v}{\partial t} = -g\frac{\partial h}{\partial x} - kv. \tag{5}$$

This is possible by choosing k so that it approximates the magnitude of the 'true' friction. This value had to be estimated for each channel separately. A complicating factor is that the velocity is not constant over the channel in time while dealing with tides. The friction coefficient was determined using the condition that the mean energy dissipation (mean as in tidal mean) is the same for the 'true' and for the linearized friction. Lorentz considered only the positive part of the velocity during one tidal evolution (the negative part will be the same; only the sign is reversed) and modelled it as sinusoidal with amplitude v_{max} and radian frequency n:

$$v = v_{max}\cos nt. \tag{6}$$

The work done is

$$\int_{-\pi/2n}^{\pi/2n} Wv dt. \tag{7}$$

The condition that Lorentz imposed was that the two formulations for the frictional force, i.e. (4) and the last term in (5) should yield the same dissipation of energy over one tidal period, thus by the fictional force during half a tidal period is:

$$k\int_{-\pi/2n}^{\pi/2n} v^2 dt = \frac{g}{c^2 q}\int_{-\pi/2n}^{\pi/2n} v^3 dt. \tag{8}$$

So if v is inserted as in given in (6)

$$k\int_{-\pi/2n}^{\pi/2n} (\cos nt)^2 dt = \frac{g v_{max}}{C^2 q}\int_{-\pi/2n}^{\pi/2n} (\cos nt)^3 dt. \tag{9}$$

Development of these integrals yields an expression for k:

$$k = \frac{8}{3\pi} * \frac{gv_{max}}{C^2 q}. \quad (10)$$

This expression uses the maximum value of the velocity in the channel, a value that is not known a priory. Still, there is the problem that for the determination of k in a particular channel one needs at least an estimation of the maximum velocity. This value was partly obtained from measurements, but also derived by making a first guess and corrected later in the calculations. Strictly speaking, it is not even possible to define a constant maximum velocity along a channel, this was accounted for by splitting the channel in two in cases where the values of k differed too much. As mentioned before, in case of a cross-sectional depth profile they choose several channels along each other with different depths.

4. Selection of the tidal channels

Apart from all the theoretical work, the committee initiated many field campaigns. During the whole time observations were made in the Zuiderzee and were later used. In order to calculate the tide and see if it represented the measured values prior to closure well enough, the committee divided the complete Zuiderzee in a system consisting of main channels, see fig (3). Only the M2 constituent of the tide ('the semi diurnal tide') was used in the first calculations. Every channel had as unknown values the volume flux s and the sea surface elevation h at one end using the equations of motion, the values at the other end were expressed in the same unknowns. At the connection points the condition was that surface elevation and volume flux must be continues. Thereby at the land borders s had to be zero and at the sea side h was imposed.

5. Lorentz method

To facilitate the calculations, Lorentz introduced some other variables. This section will follow his original way of calculating the tides in the Wadden Sea and Zuiderzee and will adopt the same symbols he used in the original report.

Starting point are the two shallow water equations for a channel as in eq. (1) and (5):

$$\begin{aligned} \frac{\partial v}{\partial t} &= -g\frac{\partial h}{\partial x} - kv, \\ \frac{\partial s}{\partial x} &= -b\frac{\partial h}{\partial t}, \\ s &= bqv. \end{aligned}$$

Lorentz converted this into a system that can be solved for s and h:

$$\frac{\partial s}{\partial t} + ks = -gbq\frac{\partial h}{\partial x} \quad (11)$$

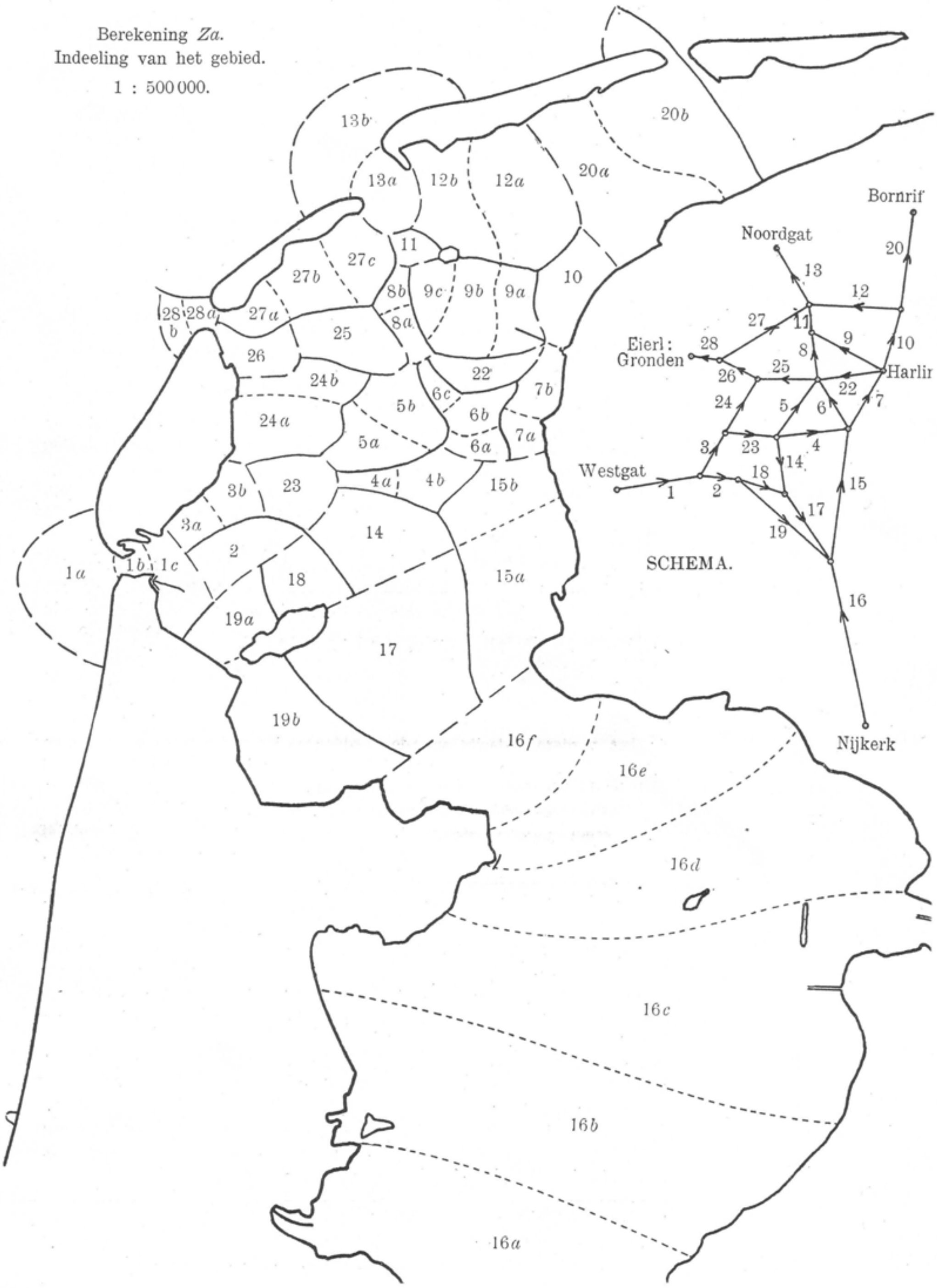


Figure 3: The schematized representation of the Zuiderzee by a network of channels.

and

$$\frac{\partial s}{\partial x} = -b\frac{\partial h}{\partial t}. \tag{12}$$

This system allows for solutions that describe travelling waves in the x-direction with damping-factor u:

$$h = ae^{int+ux},$$
$$s = ce^{int+ux}.$$

Note that these are complex solutions, the real part of the r.h.s. is the solution. Substitution of these solutions in equations (11) and (12) yields expressions for the complex amplitudes u and c:

$$u^2 = \tfrac{n}{gq}(-n+ik), \tag{13}$$
$$c = -\tfrac{ibn}{u}a. \tag{14}$$

The real part of u can be seen as the spatial damping factor of the tidal wave due to the bottom friction, the imaginary part as the wavenumber. When taking account of the waves travelling in the opposite (negative) x-direction, similar expressions can be derived. A superposition of these results gives:

$$h = ae^{int+ux} + a'e^{int-ux}, \tag{15}$$
$$s = \tfrac{ibn}{u}(-ae^{int+ux} + a'e^{int-ux}). \tag{16}$$

Here, the only remaining unknown variables are a and a', assuming that k and u are known at that time in the calculation. At one boundary s and h are prescribed, this is the connection with the open sea. The values can be measured and used in the calculations. At $x = 0$:

$$h_0 = ae^{int} + a'e^{int}, \tag{17}$$
$$s_0 = \frac{ibn}{u}(-ae^{int} + a'e^{int}). \tag{18}$$

This can be solved for ae^{int} and $a'e^{int}$. Now at the other end of the channel h and s can be expressed in h_0 and s_0.
At $x = l$, $(l = L/100km)$ is the dimension of length that Lorentz used:

$$h_l = H_h h_0 + H_s s_0, \tag{19}$$
$$s_l = S_h h_0 + S_s s_0, \tag{20}$$

with

$$\begin{aligned} H_h &= \tfrac{1}{2}(e^{ul} + e^{-ul}), \\ H_s &= \tfrac{iu}{2bn}(e^{ul} - e^{-ul}), \\ S_h &= \tfrac{ibn}{2u}(e^{ul} - e^{-ul}), \\ S_s &= \tfrac{1}{2}(e^{ul} + e^{-ul}). \end{aligned} \tag{21}$$

Before computing h_l and s_l with (21) realise that only the real parts contribute. There was the possibility of an extra computational check on correctness in the long calculation since $H_h S_s - H_s S_h = 1$. In other words the real part of quantity $H_h S_s - H_s S_h$ has to be 1 and the imaginary part has to be zero.

Thus far no mathematical problems are encountered and the problem is solved with the above equations. However, in the time of Lorentz and his committee there were no digital computers. The calculations had to be done by hand and were done in duplicate on different calculating machines to avoid errors. One of the team leaders, Dr J.P. Thijsse, later stated that only the calculations of the existing tide took one month for two persons. It has to be realized that this was only a small part of the total amount of calculations. For this reason some more variables were introduced that made the calculations better to comprehend. Remind that the problem in principle is solved, now there just some convenient modifications that will be made. For this were introduced: $\vartheta = bg \tan k/n$ and $f = bg$. From here u is rewritten:

$$\begin{aligned} u^2 &= \frac{-b_0 n^2/g}{\Sigma f \cos^2 \vartheta + \Sigma f \cos^2 \vartheta \tan \vartheta_0} \\ &= \frac{-b_0 n^2/g}{G(\cos \vartheta_0 + i \sin \vartheta_0)}, \end{aligned} \tag{22}$$

with $G = \Sigma f \cos^2 \vartheta$, $\vartheta_0 = f \tan K/G$ and $K = \Sigma f \cos^2 \vartheta \tan \vartheta_0$. This results in the following expression for u:

$$u = i\sqrt{\frac{b_0 n^2 \cos\vartheta_0}{Gg}}(\cos\frac{\vartheta_0}{2} - i\sin\frac{\vartheta_0}{2}) = \sigma + ir, \tag{23}$$

where $r = R\cos\varphi$ and $\sigma = R\sin\varphi$ with $R = \sqrt{\frac{b_0 n^2 \cos\vartheta_0}{Gg}}$ and $\varphi = \frac{1}{2}\vartheta_0$. Recalling the formulations for the water elevation and volume transport (21), we have another expression for u now, therefore we rewrite $\frac{1}{2}(e^{ul} + e^{-ul})$ and $\frac{1}{2}(e^{ul} - e^{-ul})$

$$\begin{aligned}\frac{1}{2}(e^{ul} + e^{-ul}) = \frac{1}{2}e^{\sigma l}(\cos rl + i\sin rl) + \frac{1}{2}e^{-\sigma l}(\cos rl - i\sin rl)\\ = \cos rl\cosh\sigma l + i\sin rl\sinh\sigma l,\end{aligned} \tag{24}$$

and

$$\frac{1}{2}(e^{ul} - e^{-ul}) = \cos rl\sinh\sigma l + i\sin rl\cosh\sigma l.$$

So finally we find for (21)

$$H_h = S_s = \cos rl\cosh\sigma l + i\sin rl\sinh\sigma l, \tag{25}$$

$$\begin{aligned}H_s &= \frac{iu}{2bn}(e^{ul} - e^{-ul})\\ &= \frac{i}{2bn}(\sigma + ir)(\cos rl\sinh\sigma l + i\sin rl\cosh\sigma l)\\ &= \frac{R}{b_0 n}((-\cos\varphi\cos rl\sinh\sigma l - \sin\varphi\sin rl\cosh\sigma l)\\ &\quad +i(+\sin\varphi\cos rl\sinh\sigma l) - \cos\varphi\sin rl\cosh\sigma l),\end{aligned} \tag{26}$$

$$\begin{aligned}\text{and } S_h &= \frac{b_0 n}{R}((-\cos\varphi\cos rl\sinh\sigma l + \sin\varphi\sin rl\cosh\sigma l)\\ &\quad +i(-\sin\varphi\cos rl\sinh\sigma l) - \cos\varphi\sin rl\cosh\sigma l).\end{aligned} \tag{27}$$

These were the terms the committee painstakingly calculated. An example of such an activity is given in figure 4. In this scheme the calculations are done for channel Westgat, see figure 3. The notation here is: $H_h = S_s = (1) + (2)i$, $H_s = (3) + (4)i$ and $S_h = (5) + (6)i$.

Geul.	Lengte.	Dwarsprofiel.				Maximum snelheid i/d diepste geul.	
	L	$b_1 \times q_1$	$b_2 \times q_2$	$b_3 \times q_3$	$b_4 \times q_4$	Z	A
1	47	2,0 × 20,0	1,5 × 8,0	3,0 × 4,0	7,0 × 0	80	98
2*a*	28	1,5 × 10,0	3,5 × 5,0	19,5 × 0	—	75	100
2*b*	19	3,3 × 16,0	7,3 × 0	—	—	87	100
3*a*	63	41,8 × 3,8	—	—	—	17	—
3*b*	35	7,5 × 6,5	16,0 × 3	—	—	45	—

Berekening van geul 1. $l = 0{,}47$

	deel 1	deel 2	deel 3	deel 4	totaal
b	2	1,5	3	7	$13{,}5 = b_0$
q	20	8	4	0	
f	40	12	12	0	64
v_m	80	57,54	42,54	0	
$tg\, \vartheta$ . . .	0,84680	1,52275	2,25180	∞	$1{,}02755 = tg\, \vartheta_0$
ϑ	0,70263	0,98972	1,15287	$\frac{\pi}{2}$	$0{,}79899 = \vartheta_0$
$\cos \vartheta$ . . .	0,76315	0,54893	0,40587	0	$0{,}69743 = \cos \vartheta_0$
$tg\, \vartheta : \cos \vartheta$.	1,10961	2,77402	5,54805	∞	
$f \cos^2 \vartheta$ . .	23,2959	3,6159	1,9768	0	$28{,}8886 = G$
$f \cos^2 \vartheta\, tg\, \vartheta$.	19,7270	5,5061	4,4513	0	$29{,}6844 = K$

$$R = \sqrt{20{,}124 \frac{13{,}5}{28{,}8886} 0{,}69743} = 2{,}56101 \qquad \varphi = 0{,}39950$$

$$\cos \varphi = 0{,}92126 \qquad \sin \varphi = 0{,}38896 \qquad \frac{R}{b_0\, n} = 1{,}350026$$

$$r\, l = 1{,}10890 \qquad \sigma\, l = 0{,}46818$$

$$\cos r\, l = 0{,}44565 \qquad \cosh \sigma\, l = 1{,}11162$$

$$\sin r\, l = 0{,}89521 \qquad \sinh \sigma\, l = 0{,}48547$$

$$(1) = \underline{+0{,}49539} \qquad (2) = \underline{+0{,}43460}$$

$$(3) = 1{,}350026\,(-0{,}19931 - 0{,}38707) = \underline{-0{,}79163}$$

$$(4) = 1{,}350026\,(+0{,}08415 - 0{,}91678) = \underline{-1{,}12407}$$

$$(5) = \frac{1}{1{,}350026}(-0{,}19931 + 0{,}38707) = \underline{+0{,}13908}$$

$$(6) = \frac{1}{1{,}350026}(-0{,}08415 - 0{,}91678) = \underline{-0{,}74141}$$

Contrôle:

$$\begin{array}{rlrl} (1)^2 & = +0{,}245411 & & \\ -(2)^2 & = -0{,}188877 & 2 \times (1) \times (2) & = +0{,}430592 \\ -(3) \times (5) & = +0{,}110100 & -(3) \times (6) & = -0{,}586922 \\ (4) \times (6) & = +0{,}833397 & -(4) \times (5) & = +0{,}156336 \\ \hline & +1{,}000031 & & +0{,}000006 \end{array}$$

De stroomen in de verschillende deelen van de geul vertoonen kleine phaseverschillen, zoodat de amplitude van den stroom in de geheele geul wordt bepaald door $\Sigma \frac{v_m f}{100} \cos(\vartheta - \vartheta_0)$.

Dit levert in dit geval: 43,431.

Figure 4: An example of a calculation as done by the State Committee Zuiderzee in 1926, see text for explanation.

6. Conclusions of the Committee

In 1926 the State Committee Zuiderzee published the comprehensive report bearing its conclusions concerning the new dam that was planned in the Zuiderzee. It had taken the committee 8 years of applied hydrodynamical research. One of the most striking methods used was the, somewhat anonymously called, third method, a new way a dealing with the bottom-friction in the shallow water equations of motion. This method turned out to be quite successful in predicting the situation without the dam, as the model results where extensively validated against measurements. In the same way the tidal characteristics where calculated for the situation after closure and it turned out that the tides would be enhanced in the new configuration. The basin would come closer to the resonance length of 1/4 wavelength and thereby would have much stronger currents. Figure 5 shown the ratio of the currents before and after the construction of the dam. This fact gave rise to one of the main recommendations: to build the dam further up north than originally planned. This advice has been followed. After completion of the dam on the 28th of may in 1932 the tide patterns corroborated fully with the predictions of the committee.

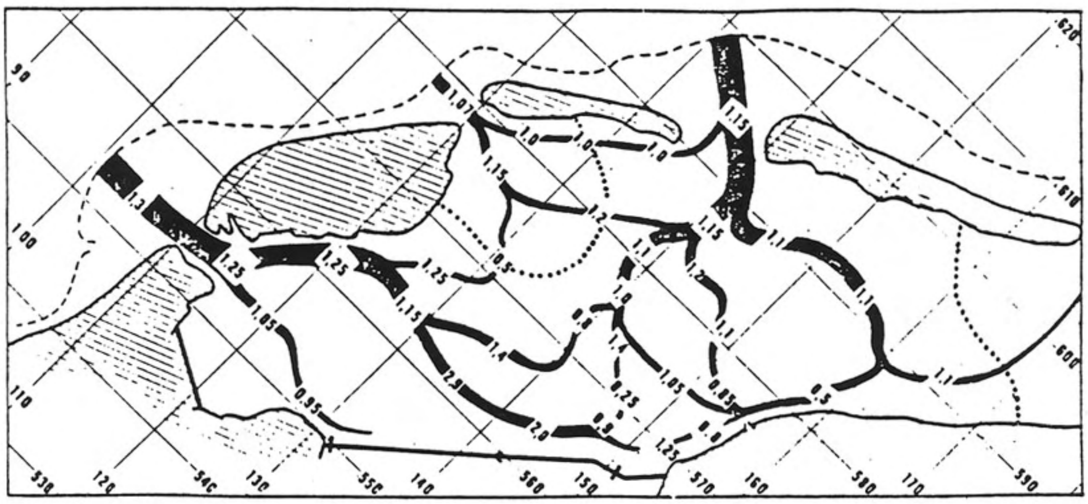


Figure 5: The ratio of the currents before and after closure of the Zuiderzee.

Not covered in this report is the way the committee dealt with the storm influence and the prevention of the land from it. Measurements later in the century learned that, where the tidal effect where accurate up to a few centimetres, the predictions on storm effects where less accurate. Thijsse (1972) concluded on basis of empirical data that the predictions of the committee had been 30 cm too low. By that time the dykes had been heightened as result of the storm in 1953. Despite the underestimation of the storm effects Lorentz and his committee succeeded in predicting changes at a convincing level of accuracy. The report formed an important handbook for hydraulic engineers for a long time and still is a milestone in the scientific history of tidal prediction.

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