Topological Band Theory III

I. Topological Insulators in 3D
   - Weak vs strong
   - Topological invariants from band structure

II. The surface of a topological insulator
   - Quantum Hall effect
   - $\theta$ term and topological magnetoelectric effect
   - Superconductivity
   - Surface topological order

III. Topological Superconductivity
   - 1D topological superconductor
   - Majorana chain
   - 2D topological superconductor
   - 10-fold way
   - Majorana modes and quantum information

Next Week same time, same place: Topological Mechanics

Lecture notes available at https://www.lorentz.leidenuniv.nl/lorentzchair/
There are 4 surface Dirac Points due to Kramers degeneracy.

$E(k) = \pm \frac{L}{a} k$ for $k = \Lambda_a, \Lambda_b$

$n_0 = 1$: Strong Topological Insulator

Fermi circle encloses odd number of Dirac points

Topological Metal: 1/4 graphene

$\pi$ Berry's phase

Robust to disorder: impossible to localize

$n_0 = 0$: Weak Topological Insulator

Fermi surface encloses even number of Dirac points

Related to layered 2D QSHI; $(n_1 n_2 n_3) \sim$ Miller indices

How do the Dirac points connect? Determined by 4 bulk $Z_2$ topological invariants $\nu_0 : (\nu_1 \nu_2 \nu_3)$
Bi$_{1-x}$Sb$_x$  

**Theory:** Predict Bi$_{1-x}$Sb$_x$ is a topological insulator by exploiting inversion symmetry of pure Bi, Sb (Fu,Kane PRL’07)  

**Experiment:** ARPES (Hsieh et al. Nature ’08)  

- Bi$_{1-x}$Sb$_x$ is a Strong Topological Insulator $n_0; (n_1,n_2,n_3) = 1; (111)$  
- 5 surface state bands cross $E_F$ between $\Gamma$ and $M$  

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Bi$_2$Se$_3$  


- $n_0; (n_1,n_2,n_3) = 1; (000)$ : Band inversion at $\Gamma$  
- Energy gap: $\Delta \sim 0.3$ eV : A room temperature topological insulator  
- Simple surface state structure : Similar to graphene, except only a single Dirac point  

Control $E_F$ on surface by exposing to NO$_2$  

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Topological Invariants in 3D

1. \(2D \rightarrow 3D\) : Time reversal invariant planes

The 2D invariant

\[
(-1)^\nu = \prod_{a=1}^{4} \delta(\Lambda_a) \quad \delta(\Lambda_a) = \frac{\text{Pf}[w(\Lambda_a)]}{\sqrt{\text{det}[w(\Lambda_a)]}}
\]

Each of the time reversal invariant planes in the 3D Brillouin zone is characterized by a 2D invariant.

Weak Topological Invariants (vector):

\[
(-1)^{\nu_i} = \prod_{a=1}^{4} \delta(\Lambda_a) \bigg|_{k_i=0} \quad \mathbf{G}_v = \frac{2\pi}{a} (\nu_1, \nu_2, \nu_3)
\]

“mod 2” reciprocal lattice vector indexes lattice planes for layered 2D QSHI

Strong Topological Invariant (scalar)

\[
(-1)^{\nu_o} = \prod_{a=1}^{8} \delta(\Lambda_a)
\]
Add an extra parameter, $k_4$, that smoothly connects the topological insulator to a trivial insulator (while breaking time reversal symmetry)

$H(k, k_4)$ is characterized by its second Chern number

$$n = \frac{1}{8\pi^2} \int d^4 k \text{Tr}[F \wedge F]$$

$n$ depends on how $H(k)$ is connected to $H_0$, but due to time reversal, the difference must be even.

$$\nu_0 = n \mod 2$$

Express in terms of Chern Simons 3-form:

$$\nu_0 = \frac{1}{4\pi^2} \int d^3 k Q_3(k) \mod 2$$

$$Q_3(k) = \text{Tr}[A \wedge dA + \frac{2}{3} A \wedge A \wedge A]$$

Gauge invariant up to an even integer.
Unique Properties of Topological Insulator Surface States

“Half” an ordinary 2DEG; ¼ Graphene

Spin polarized Fermi surface
- Charge Current ~ Spin Density
- Spin Current ~ Charge Density

Robust to disorder
- $\pi$ Berry’s phase
- Weak Antilocalization
- Impossible to localize

Broken symmetry (or strong interactions) leads to exotic gapped states
- Quantum Hall state, topological magnetoelectric effect
  Fu, Kane ‘07; Qi, Hughes, Zhang ‘08, Essin, Moore, Vanderbilt ‘09
- Superconducting state
  Fu, Kane ‘08
- Surface topological order: symmetry preserving gapped state
  Metlitski, Kane, Fisher ‘13, Bonderson, Nayak, Qi ‘13, Chen, Fidkowski, Vishwanath ‘13, Wang, Potter, Senthil ‘14
**Surface Quantum Hall Effect**

**Orbital QHE:** E=0 Landau Level for Dirac fermions. “Fractional” IQHE

\[
\sigma_{xy} = \frac{e^2}{2h} \left( n + \frac{1}{2} \right)
\]

**Anomalous QHE:** Induce a surface gap by depositing magnetic material

\[
H_0 = \psi^\dagger \left( -i v \vec{\sigma} \cdot \vec{\nabla} - \mu + \Delta_M \sigma_z \right) \psi
\]

Mass due to Exchange field

\[
\sigma_{xy} = \text{sgn}(\Delta_M) \frac{e^2}{2h}
\]

\[E_{\text{gap}} = 2|\Delta_M|\]

**Chiral Edge State at Domain Wall:** \(\Delta_M \leftrightarrow -\Delta_M\)
**Topological Magnetoelectric Effect**

Qi, Hughes, Zhang ’08; Essin, Moore, Vanderbilt ’09

Consider a solid cylinder of TI with a magnetically gapped surface

\[ J = \sigma_{xy} E = \frac{e^2}{h} \left( n + \frac{1}{2} \right) E = M \]

\[ M = \alpha E \quad \alpha = \frac{e^2}{h} \left( n + \frac{1}{2} \right) \]

**Magnetoelectric Polarizability**

\[ \Delta L = \alpha \mathbf{E} \cdot \mathbf{B} \]

\[ \alpha = \theta \frac{e^2}{2\pi h} \]

TR sym. : \( \theta = 0 \) or \( \pi \) mod \( 2\pi \)

The fractional part of the magnetoelectric polarizability is determined by the bulk, and independent of the surface (provided there is a gap)

Analogous to the electric polarization, \( P \), in 1D.

<table>
<thead>
<tr>
<th>( \Delta L )</th>
<th>formula</th>
<th>“uncertainty quantum”</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P \cdot \mathbf{E} )</td>
<td>( \frac{e}{2\pi} \int_{BZ} \text{Tr}[\mathbf{A}] )</td>
<td>( e ) (extra end electron)</td>
</tr>
<tr>
<td>( \alpha \mathbf{E} \cdot \mathbf{B} )</td>
<td>( \frac{e^2}{4\pi^2 h} \int_{BZ} \text{Tr}[\mathbf{A} \wedge d\mathbf{A} + \frac{2}{3} \mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A}] )</td>
<td>( e^2 / h ) (extra surface quantum Hall layer)</td>
</tr>
</tbody>
</table>
Superconducting proximity effect on a topological insulator

Proximity to superconductor introduces energy gap by breaking gauge symmetry

\[ H = \psi^\dagger (-iv \vec{\sigma} \cdot \vec{\nabla} - \mu) \psi \]
\[ + \Delta_S \psi^\dagger \psi^\dagger + \Delta_s^\ast \psi \downarrow \psi \uparrow \]

- Half an ordinary superconductor
- Similar to spinless $p_x + ip_y$ superconductor, except:
  - Does not violate time reversal symmetry
  - s-wave singlet superconductivity
  - Required minus sign is provided by $\pi$ Berry’s phase due to Dirac Point
- Nontrivial ground state supports Majorana fermion bound states at vortices
Strong Interactions

Topological Insulator coupled to compact U(1) gauge field, A

For a compact gauge field, magnetic monopoles are excitations in the theory. Useful diagnostic for strongly interacting theories.

Low energy theory for A: \( \theta \) term

\[
S = i \theta N \quad N = \frac{1}{32\pi^2} \int d^3 x d\tau \varepsilon_{\mu
\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} \in \mathbb{Z}
\]

• Time reversal symmetry: \( \theta = 0 \text{ or } \pi \mod 2\pi \). \( \theta = \pi \mod 2\pi \) for electron TI

• Witten Effect: Magnetic monopoles are charged \( Q = \frac{\theta}{2\pi} e \)

• Monopoles (or dyons) are bosons with half integer charge \( Q = e \left( n + \frac{1}{2} \right) \)
Can the surface of a TI be gapped without breaking symmetry?

Pass monopole from inside to outside of TI:

\[ Q = \frac{e}{2} \text{ boson} \]

Charge \( e/2 \) must be deposited at surface

Break \( T \) at surface:

\[ \sigma_{xy} = \frac{e^2}{2h} : \text{ charge } e/2 \text{ flows away on surface} \]

Superconductor at surface:

Charge conservation is violated at surface

Keep \( U(1) \) and \( T \) at surface:

Charge \( e/2 \) stays at surface: Requires a topologically ordered surface state with \( e/2 \) quasiparticle
Requirements for a Topological Surface Phase on fermion TI

It should be impossible in 2D if symmetry is preserved, but if symmetry is broken there should be a 2D state with the same topological order

Broken T: Topo – M slab

A 2D Non-Abelian quantum Hall state with

• Hall conductance $\nu \frac{e^2}{h}$; $\nu = 1/2$
• Thermal Hall cond. $c \pi^2 k_B^2/6h$; $c = 1/2$

Theories of symmetry preserving gapped state:

Related to Moore-Read (Pfaffian) state of FQHE at $\nu=1/2$

• “T-Pfaffian state” Bonderson, Nayak, Qi ’13 ; Chen, Fidkowski, Vishwanath ‘13
• “Moore-Read/antisemion state” Metlitski, Kane, Fisher ’13 ; Wang, Potter, Senthil ‘13
BCS Theory of Superconductivity

mean field theory: \[ \Psi^\dagger \Psi^\dagger \Psi \Rightarrow \langle \Psi^\dagger \Psi^\dagger \rangle \Psi \Psi = \Delta^* \Psi \Psi \]

\[ H = \frac{1}{2} \sum_k (\Psi^\dagger \Psi) H_{BdG} \left( \begin{array}{c} \Psi \\ \Psi^\dagger \end{array} \right) \]

Bogoliubov de Gennes Hamiltonian

\[ H_{BdG} = \begin{pmatrix} H_0 & \Delta \\ \Delta^* & -H_0 \end{pmatrix} \]

Intrinsic anti-unitary particle – hole symmetry

\[ \Xi H_{BdG} \Xi^{-1} = -H_{BdG} \]

\[ \Xi \Phi = \tau_x \Phi^* \quad \tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ \Xi^2 = +1 \]

Particle – hole redundancy

\[ \Phi_{-E} = \Xi \Phi_E \Rightarrow \gamma_{E}^\dagger = \gamma_{-E} \]

Bloch - BdG Hamiltonians satisfy

\[ \Xi H_{BdG}(k) \Xi^{-1} = -H_{BdG}(-k) \]

Topological classification problem similar to time reversal symmetry
1D $\mathbb{Z}_2$ Topological Superconductor: $\nu = 0, 1$ (Kitaev, 2000)

Bulk-Boundary correspondence: Discrete end state spectrum

Majorana Fermion: Particle = Antiparticle $\gamma = \gamma^\dagger$

Real part of a Dirac fermion:

$$\begin{align*}
\gamma_1 &= \Psi + \Psi^\dagger \\
\gamma_2 &= -i(\Psi - \Psi^\dagger)
\end{align*}$$

$\gamma_i^2 = 1$; $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$

“Half a state”

Two Majorana fermions define a single two level system

$$H = 2i\varepsilon_0\gamma_1\gamma_2 = \varepsilon_0\Psi^\dagger\Psi$$
Kitaev Model for 1D p wave superconductor

\[ H - \mu N = \sum_i t(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) - \mu c_i^\dagger c_i + \Delta(c_i c_{i+1} + c_{i+1}^\dagger c_i^\dagger) \]

\[ = \sum_k (c_k^\dagger c_{-k}) H_{BdG}(k) \left( \begin{array}{c} c_k \\ c_{-k}^\dagger \end{array} \right) \]

\[ H_{BdG}(k) = \tau_z (2t \cos k - \mu) + \tau_x \Delta \sin k = \mathbf{d}(k) \cdot \mathbf{\bar{\tau}} \]

|μ| > 2t : Strong pairing phase
trivial superconductor

|μ| < 2t : Weak pairing phase
topological superconductor

Similar to SSH model, except different symmetry:

\[(d_x, d_y, d_z)_k = (-d_x, -d_y, d_z)_{-k}\]
Majorana Chain

\[ c_i \rightarrow \gamma_{1i} + i\gamma_{2i} \]

\[ H = 2i \sum_i t_1 \gamma_{1i} \gamma_{2i} + t_2 \gamma_{2i} \gamma_{1i+1} \]

\[ \mu c_i^\dagger c_i \rightarrow 2i \mu \gamma_{1i} \gamma_{2i} \]
\[ t(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) \rightarrow 2it(\gamma_{1i} \gamma_{2i+1} - \gamma_{2i} \gamma_{1i+1}) \]
\[ \Delta(c_i c_{i+1} + c_{i+1}^\dagger c_i^\dagger) \rightarrow 2i\Delta(\gamma_{1i} \gamma_{2i+1} + \gamma_{2i} \gamma_{1i+1}) \]

For \( \Delta = t \): nearest neighbor Majorana chain

\[ t_1 = \mu, \ t_2 = 2t \]

For \( t_1 > t_2 \):
- **trivial SC**: Nearest neighbor Majorana chain.

For \( t_1 < t_2 \):
- **topological SC**: Nearest neighbor Majorana chain.

Unpaired Majorana Fermion at end.
2D \(\mathbb{Z}\) topological superconductor (broken T symmetry)

Bulk-Boundary correspondence: \(n = \#\) Chiral Majorana Fermion edge states

[Diagram showing the bulk-boundary correspondence with \(\gamma_k\) and \(\gamma_{-k} = \gamma_k^\dagger\) relations.]

Examples
- Spinless \(p_x + ip_y\) superconductor (n=1)
- Chiral triplet \(p\) wave superconductor (eg \(\text{Sr}_2\text{RuO}_4\)) (n=2)

Read Green model:
\[
H = \sum_k \left( \frac{k^2}{2m} - \mu \right) c_k^\dagger c_k + (\Delta(k)c_k c_{-k} + c.c.)
\]
\[
\Delta(k) = \Delta_0(k_x + ik_y)
\]

Lattice BdG model:
\[
H_{\text{BdG}}(k) = \tau_z \left( 2t \left[ \cos k_x + \cos k_y \right] - \mu \right) + \Delta \left( \tau_x \sin k_x + \tau_y \sin k_y \right) = \mathbf{d}(k) \cdot \bar{\tau}
\]

- \(|\mu| > 4t\): Strong pairing phase, trivial superconductor, Chern number 0
- \(|\mu| < 4t\): Weak pairing phase, topological superconductor, Chern number 1
Periodic Table of Topological Insulators and Superconductors

Anti-Unitary Symmetries:

- Time Reversal: \( \Theta H(k)\Theta^{-1} = +H(-k) ; \quad \Theta^2 = \pm 1 \)
- Particle-Hole: \( \Xi H(k)\Xi^{-1} = -H(-k) ; \quad \Xi^2 = \pm 1 \)

Unitary (chiral) symmetry:
\( \Pi H(k)\Pi^{-1} = -H(k) ; \quad \Pi \propto \Theta \Xi \)

8 antiunitary symmetry classes

<table>
<thead>
<tr>
<th>Symmetry</th>
<th>( \Theta )</th>
<th>( \Xi )</th>
<th>( \Pi )</th>
<th>( d )</th>
</tr>
</thead>
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<tr>
<td>A</td>
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<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
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<td>0</td>
<td>1</td>
<td>2</td>
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<td>1</td>
<td>1</td>
<td>4</td>
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<tr>
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<td>0</td>
<td>5</td>
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<td>1</td>
<td>1</td>
<td>6</td>
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<td>0</td>
<td>7</td>
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<td>-1</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>CI</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>10</td>
</tr>
</tbody>
</table>

Bott Periodicity \( d \to d+8 \)

Altland-Zirnbauer Random Matrix Classes

Kitaev, 2008
Schnyder, Ryu, Furusaki, Ludwig 2008
Majorana zero mode at a vortex

Boundary condition on fermion wavefunction

\[ \psi(L) = (-1)^{p+1} \psi(0) \]

\[ \psi(x) \propto e^{iq_mx} \quad ; \quad q_m = \frac{\pi}{L} \left( 2m + 1 + p \right) \]

Hole in a topological superconductor threaded by flux

Without the hole: Caroli, de Gennes, Matricon theory ('64)

\[ \Delta \varepsilon \sim \frac{2\pi v}{L} \]

\[ \Delta \varepsilon \sim \frac{\Delta^2}{E_F} \]
Majorana Bound States on Topological Insulators

1. $\frac{h}{2e}$ vortex in 2D superconducting state

2. Superconductor-magnet interface at edge of 2D QSHI

Quasiparticle Bound state at $E=0$

Majorana Fermion $\gamma_0$ “Half a State”

$$\gamma_0^\dagger = \gamma_0$$

$$\gamma_{-E}^\dagger = \gamma_E$$

$$|m| = \Delta_S - \Delta_M$$

$E_{\text{gap}} = 2|m|$
1D Majorana Fermions on Topological Insulators

1. 1D Chiral Majorana mode at superconductor-magnet interface

\[ \gamma_k = \gamma_{-k}^\dagger : \text{“Half” a 1D chiral Dirac fermion} \]

\[ H = -i\hbar v_F \gamma \partial_x \gamma \]

2. S-TI-S Josephson Junction

Gapless non-chiral Majorana fermion for phase difference \( \phi = \pi \)

\[ H = -i\hbar v_F \left( \gamma_L \partial_x \gamma_L - \gamma_R \partial_x \gamma_R \right) + i\Delta \cos(\phi/2) \gamma_L \gamma_R \]
Majorana Fermions and Topological Quantum Computing

(Kitaev ’03)

The degenerate states associated with Majorana zero modes define a topologically protected quantum memory

- 2 Majorana separated bound states = 1 fermion
  - 2 degenerate states (full/empty) = 1 qubit
- 2N separated Majoranas = N qubits
- Quantum Information is stored non locally
  - Immune from local decoherence

\[ \Psi = \gamma_1 + i\gamma_2 \]

Braiding performs unitary operations

Non-Abelian statistics

Interchange rule (Ivanov 03)

\[ \gamma_i \rightarrow \gamma_j \]
\[ \gamma_j \rightarrow -\gamma_i \]

These operations, however, are not sufficient to make a universal quantum computer
Potential condensed matter hosts for Majorana modes

- Quasiparticles in fractional Quantum Hall effect at $v=5/2$  Moore Read ’91

- Unconventional superconductors
  - Sr$_2$RuO$_4$  Das Sarma, Nayak, Tewari ‘06
  - Fermionic atoms near feshbach resonance  Gurarie ‘05
  - Cu$_x$Bi$_2$Se$_3$  ?

- Proximity Effect Devices using ordinary superconductors
  - Topological Insulator devices  Fu, Kane ‘08
  - 2D Semiconductor/Magnet devices  Sau, Lutchyn, Tewari, Das Sarma ’09, Lee ’09
  - 1D Semiconductor devices:
    - eg In As quantum wires  Oreg, von Oppen, Alicea, Fisher ’10
    - Expt : Maurik et al. (Kouwenhoven) ‘12
  - 1D Ferromagnetic atomic chains on superconductors  Nadj-Perg et al. (Yazdani) ‘14